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ζ-Determinant and the Quillen Determinant on the Grassmannian of Elliptic Self-Adjoint Boundary Conditions

Le ζ–déterminant et le déterminant de Quillen sur l'espace grassmannien de conditions au bord elliptiques autoadjointes

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Résumé – *Dans cette Note, nous annonçons la preuve de l'égalité du ζ–déterminant d'un opérateur de Dirac sur une variété à bord, compacte, de dimension impaire, avec la section déterminant de Quillen calculée dans une trivialisation canonique sur l'espace grassmannien de conditions au bord elliptiques autoadjointes.*

Abstract We announce a proof of the equality of the ζ–determinant of a Dirac operator over an odd–dimensional manifold with boundary with the Quillen determinant section computed in a canonical trivialization over the Grassmannian of elliptic self–adjoint boundary conditions.

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Version française abrégée – Soit $\mathcal{D} : C^\infty(M;\mathcal{S}) \longrightarrow C^\infty(M;\mathcal{S})$ un opérateur de Dirac compatible sur une variété M à bord, compacte, de dimension impaire, de bord Y. On suppose que les structures sont produits au voisinage de Y. On désigne par $Gr^*_\infty(\mathcal{D})$ l'espace grassmannien lisse réduit de toutes les projecteurs P de $L^2(Y;\mathcal{S}|Y)$, de manière que $P - P(\mathcal{D})$ est un opérateur de noyau lisse, où $P(\mathcal{D})$ est la projection de Calderón sur l'espace des données de Cauchy défini par $\mathcal{D}$ ([2]), et en plus que $-\Gamma P \Gamma = I - P$. Dans l'égalité dernière Γ désigne la multiplication de Clifford par le vecteur cotangent unité et normal au bord.

Soit $\gamma_0 : H^1(M;\mathcal{S}) \to H^{1/2}(Y;\mathcal{S}_{|Y})$ l'opérateur de trace défini par la restriction des sections de Sobolev au bord. Pour tout $P \in \mathcal{G}r^*_\infty(\mathcal{D})$ le problème à bord elliptique

$$\begin{cases} \mathcal{D}_P & = & \mathcal{D} : \mathrm{dom}(\mathcal{D}_P) \longrightarrow L^2(M;\mathcal{S}) \\ \mathrm{dom}(\mathcal{D}_P) & = & \{\psi \in H^1(M;\mathcal{S}) : P\gamma_0\psi = 0\}, \end{cases}$$

définit un opérateur de Fredholm autoadjoint avec un spectre réel et discret et avec $\mathrm{Ker}(\mathcal{D}_P) \subset C^\infty(M;\mathcal{S})$ ([2]).

Le ζ–déterminant $\det_\zeta \mathcal{D}_P \in \mathbf{C}$ est bien défini sur $Gr^*_\infty(\mathcal{D})$ ([9]). Sans faisant des choix, la 'fonction' déterminante sur $Gr^*_\infty(\mathcal{D})$ apparaît comme une section *det* du fibré en droites déterminant de Quillen associé à la famille $\{\mathcal{D}_P \mid P \in \mathcal{G}r^*_\infty(\mathcal{D})\}$. Toutefois, sur $Gr^*_\infty(\mathcal{D})$ ce fibré a une trivialisation canonique qui permet de réaliser *det* comme une fonction complexe $\det_{\mathcal{C}} : \mathcal{G}r^*_\infty(\mathcal{D}) \to \mathbf{C}$, appelée le 'déterminant canonique' ([6] ,[7]).

Le résultat principal de cette Note est l'égalité projective de ces deux régularisations du déterminant de $\mathcal{D}_P$:

Théorème 0.1 *Sur $\mathcal{G}r^*_\infty(\mathcal{D})$ on a*

$$\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_{\mathcal{C}} \mathcal{D}_P. \tag{1}$$

Remarque 0.2 Dans la monographie [2] il était donné un exposé détailé de la relation entre l'indice de l'opérateur non–autoadjoint $\mathcal{D}_P$, où P désigne un projecteur pseudo–différentiel quelconque de la manière que $P - P(\mathcal{D})$ est un opérateur lissant, et l'indice de l'opérateur correspondant sur le bord $PP(\mathcal{D})$:

$Ran\ P(\mathcal{D}) \to Ran\ P$. Il était montré que

$$\text{index } \mathcal{D}_P = \text{index } PP(\mathcal{D}) \tag{2}$$

(montré la première fois dans [1]), c'est à dire que l'indice d'un opérateur agissant sur toute la variété est égal à l'indice d'un opérateur agissant seulement sur le bord. Le Théorème 0.1 dit que le ζ-déterminant de l'opérateur $\mathcal{D}_P$ est, abstraction faite d'une constante naturelle multiplicative, égal au déterminant canonique de l'opérateur $\mathcal{D}_P$, qui est vraiment le déterminant de Fredholm de l'opérateur $PP(\mathcal{D})$. Dans ce sens, notre résultat nouveau étend (2), de façon un peu inattendu, à une correspondance entre leurs déterminants, même si le déterminant étant un invariant beaucoup plus subtil.

Pour montrer Théorème 0.1, nous étudions la variation de $\det_\zeta \mathcal{D}_P$ et $\det_{\mathcal{C}} \mathcal{D}_P$ produite par un changement de la condition au bord P. Nous fixons $P_1, P_2 \in \mathcal{G}r^*_\infty(\mathcal{D})$ et nous considérons des chemins $P_{i,r} = U_r P_i U_r^{-1}$, où $\{U_r\}$ est une famille d'opérateurs elliptiques et unitaires. Nous étudions la différence

$$\log \det_\zeta \mathcal{D}_{P_{2,r}} - \log \det_\zeta \mathcal{D}_{P_{1,r}}.$$

Le résultat décisif est:

Proposition 0.3

$$\frac{d}{dr}|_{r=0} \left\{ \log \det_\zeta \mathcal{D}_{P_{2,r}} - \log \det_\zeta \mathcal{D}_{P_{1,r}} \right\} = \frac{d}{dr}|_{r=0} \left\{ \log \det_{\mathcal{C}} \mathcal{D}_{P_{2,r}} - \log \det_{\mathcal{C}} \mathcal{D}_{P_{1,r}} \right\}. \tag{3}$$

Intégrant (3), nous obtenons

$$\frac{\det_\zeta \mathcal{D}_{UP_2U^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{UP_2U^{-1}}}{\det_\zeta \mathcal{D}_{P_2} / \det_{\mathcal{C}} \mathcal{D}_{P_2}} = \frac{\det_\zeta \mathcal{D}_{UP_1U^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{UP_1U^{-1}}}{\det_\zeta \mathcal{D}_{P_1} / \det_{\mathcal{C}} \mathcal{D}_{P_1}} := \alpha(U).$$

Ici $\alpha := \alpha_{P_1,P_2} : \mathcal{U}_\infty \longrightarrow \mathbf{C}^*$ est un charactère sur le groupe $\mathcal{U}_\infty$. Substituant $P_1 = P(\mathcal{D}), P_2 = P$, nous avons

$$\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_{\mathcal{C}} \mathcal{D}_P \cdot \alpha(U).$$

Plus d'inspection de $\alpha(U)$ montre que $\alpha = 1$.

1 Introduction and Statement of Results

The ζ–determinant of a self–adjoint elliptic operator has been of central importance in spectral geometry, topology, and quantum field theory (QFT). However, it has not been clear how close the relation is of this determinant, obtained via a sophisticated analytical regularization procedure, to the true operator determinant defined by means of standard algebra and functional analysis. In this paper we show that the ζ-determinant of an elliptic boundary problem on an odd-dimensional manifold is equal (up to a multiplicative constant) to the Fredholm determinant of a canonically associated operator over the boundary. This opens the door to further investigation of pasting formulae for the ζ-determinant, as suggested by recent studies in QFT.

Let M be a compact odd–dimensional Riemannian manifold with boundary $Y := \partial M$, and let S denote a vector bundle of Clifford modules over M . Assume that we have a compatible Dirac operator $\mathcal{D} : C^\infty(M;\mathcal{S}) \longrightarrow C^\infty(M;\mathcal{S})$ acting on sections of S . In this paper we consider only the case of product structures in a neighbourhood of the boundary Y . This means that there exists $\mathcal{N} = [0,1] \times Y$ a collar neighbourhood of the boundary $Y = \{0\} \times Y$, such that the Riemannian metric on M and Hermitian metric on $\mathcal{S}$ are pull-backs of the metric structures over the boundary. Over $\mathcal{N}$ the Dirac operator $\mathcal{D}$ then takes the form $\mathcal{D}|_{\mathcal{N}} = \Gamma(\partial_u + B)$, where $B : C^\infty(Y;\mathcal{S}|Y) \longrightarrow C^\infty(Y;\mathcal{S}|Y)$ is the corresponding Dirac operator over Y, and $\Gamma : \mathcal{S}|Y \to \mathcal{S}|Y$ is Clifford multiplication by the normal vector du. To avoid technical trivialities we assume that $\mathrm{Ker}(B) = \{0\}$. Both B and Γ are independent of the normal coordinate u and they satisfy the identities

$$\Gamma^2 = -I, \quad \Gamma^* = -\Gamma, \quad \Gamma \cdot B + B \cdot \Gamma = 0.$$

The boundary Clifford bundle splits $\mathcal{S}|Y = \mathcal{S}^+ \oplus \mathcal{S}^-$ into the $\pm i$-eigenspaces of the anti-involution Γ and $\mathcal{D}|\mathcal{N}$ has the representation

$$\mathcal{D}|\mathcal{N} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \left(\frac{\partial}{\partial u} + \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix} \right),$$

where $B^\pm : F^\pm \to F^\mp$ are the chiral Dirac operators over the boundary acting on

$F^{\pm} = C^{\infty}(Y; \mathcal{S}^{\pm})$.

Let $\Pi_{>}$ be the spectral projection on $L^2(Y; \mathcal{S}|Y)$ with range equal to the subspace H^+ spanned by the eigensections of B with positive eigenvalues. We consider elliptic self–adjoint boundary conditions for $\mathcal{D}$ parameterised by the smooth restricted Grassmannian $Gr^*_{\infty}(\mathcal{D})$ of all orthogonal projections P on $L^2(Y; \mathcal{S}|Y)$ such that $P - \Pi_{>}$ is an operator with a smooth kernel and such that $-\Gamma P \Gamma = I - P$. For each $P \in Gr^*_{\infty}(\mathcal{D})$ the associated elliptic boundary value problem

$$\mathcal{D}_P := \mathcal{D} : \mathrm{dom}(\mathcal{D}_P) \longrightarrow L^2(M; \mathcal{S})$$

with $\mathrm{dom}(\mathcal{D}_P) = \{\psi \in H^1(M; \mathcal{S}) : P(\psi|Y) = 0\}$ is a self–adjoint Fredholm operator with $\mathrm{Ker}(\mathcal{D}_P) \subset C^{\infty}(M; \mathcal{S})$ and real discrete spectrum (cf. [2]).

$Gr^*_{\infty}(\mathcal{D})$ is characterised by two important properties. First, it contains the Calderon projection $P(\mathcal{D})$ with range equal to $\mathcal{H}(\mathcal{D})$ the space of Cauchy data of the operator $\mathcal{D}$:

$$\mathcal{H}(\mathcal{D}) = \{f \in C^{\infty}(Y; \mathcal{S}_{|Y}) : \exists s \in C^{\infty}(M; \mathcal{S}),\, \mathcal{D}s = 0,\, s|Y = f\}.$$

Second, $H^+ = \mathrm{range}(\Pi_{>})$ is the graph of the elliptic unitary isomorphism $(B^+B^-)^{-1/2}B^+ : F^+ \to F^-$, the 'phase' of B^+ . Similarly, any other elliptic boundary condition $P \in \mathcal{G}r^*_{\infty}(\mathcal{D})$ is described precisely by the property that it is the graph of an elliptic unitary isomorphism $T : F^+ \to F^-$ such that $T - (B^+B^-)^{-1/2}B^+$ has a smooth kernel. In particular there is a specific unitary isomorphism K such that $\mathcal{H}(\mathcal{D}) = graph(K : F^+ \longrightarrow F^-)$. The graph characterization induces a canonical trivialization σ_{graph} of the determinant line bundle $\mathcal{L}$ over $Gr^*_{\infty}(\mathcal{D})$ associated to the family of elliptic boundary value problems $\{\mathcal{D}_P : P \in \mathcal{G}r^*_{\infty}(\mathcal{D})\}$ [6, 7]. Computing the ratio of the determinant section $P \mapsto \det \mathcal{D}_P$ of $\mathcal{L}$ (the Quillen determinant) to σ_{graph} provides us with a canonical regularization of the determinant of $\mathcal{D}_P$ as a number in $\det_{\mathcal{C}} \mathcal{D}_P \in \mathbf{C}$ called the canonical determinant. One finds [6, 7]

$$\det_{\mathcal{C}} \mathcal{D}_P = \det_{Fr} \left[\frac{1}{2}\left(I + KT^{-1}\right)\right], \tag{4}$$

where range$(P) = graph(T : F^+ \longrightarrow F^-)$. Here the right-hand side of (4) is the Fredholm determinant taken on F^- and since KT^{-1} has the form $I + trace-class$ it is well-defined.

On the other hand, for $P \in \mathcal{G}r^*_\infty(\mathcal{D})$ the self-adjoint elliptic operator $\mathcal{D}_P$ has a well-defined ζ-function regularised determinant

$$\det_\zeta \mathcal{D}_P := e^{i\frac{\pi}{2}(\eta_{\mathcal{D}_P}(0)-\zeta_{\mathcal{D}_P^2}(0))} \cdot e^{-\frac{1}{2}\zeta'_{\mathcal{D}_P^2}(0)}.$$

The existence of the meromorphic extensions of the functions $\zeta_{\mathcal{D}_P^2}(s)$ and $\eta_{\mathcal{D}_P}(s)$ to the whole complex plane, and their nice behaviour in a neighbourhood of $s = 0$ was established in the paper [9]. While the canonical determinant is a robust algebraic regularization which is relatively easy to compute in terms of the local graph coordinates, the ζ-determinant is, in contrast, a highly delicate analytic regularization which is notoriously difficult to compute and there is little *a priori* reason to suppose that the two regularizations might coincide. The main result of this announcement is the following remarkable fact:

Theorem 1.1 *The following equality holds over* $\mathcal{G}r^*_\infty(\mathcal{D})$

$$\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_{\mathcal{C}} \mathcal{D}_P. \tag{5}$$

That is,

$$\det_\zeta \mathcal{D}_P = c \cdot \det_{Fr}\left[\frac{1}{2}\left(I + KT^{-1}\right)\right], \tag{6}$$

where the constant c is the value of the ζ-determinant evaluated at the Cauchy data space (Calderon) boundary condition.

Note that $\det_{\mathcal{C}} \mathcal{D}_{P(\mathcal{D})} = 1$ and so one may equivalently think of (5) as expressing the equality of the ratios $\det_\zeta \mathcal{D}_P / \det_\zeta \mathcal{D}_{P(\mathcal{D})}$ and $\det_{\mathcal{C}} \mathcal{D}_P / \det_{\mathcal{C}} \mathcal{D}_{P(\mathcal{D})}$.

Remark 1.2 *In the monograph [2] a detailed account was given of the relation between the index of the operator $\mathcal{D}_P$ for any P in the smooth Grassmannian*

$\mathcal{G}r_\infty(\mathcal{D})$ of projections differing from $\Pi_>$ by a smoothing operator, and the index of the boundary operator $PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to Ran\ P$. Outside of the real submanifold $\mathcal{G}r^*_\infty(\mathcal{D})$ of $\mathcal{G}r_\infty(\mathcal{D})$ the Fredholm operator $\mathcal{D}_P$ is not self-adjoint and it was shown that

$$index\ \mathcal{D}_P = index\ PP(\mathcal{D}) \tag{7}$$

(see [1] for the original presentation). Hence index $\mathcal{D}_P$ is equal to the index of an operator which lives only on the boundary. Theorem 1.1 extends (7) somewhat unexpectedly to a correspondence between the determinants.

The proof of Theorem 1.1 is outlined in Section 2. Details will appear in the paper [8]. Note that (6) generalises well-known formulas for the ζ–determinant in dimension one (see [3] for a detailed derivation of the one-dimensional formula using the general methods described here).

2 Sketch of the Proof

The proof of Theorem 1.1 adapts ideas suggested in the paper [5] of Forman in which a general method was outlined for computing ζ–determinants of elliptic operators over manifolds with boundary in terms of a certain Fredholm determinant over the space of boundary sections. Forman obtained precise results in one and two dimensions. We show that this method generalises to compute $\det_\zeta \mathcal{D}_P$ over $\mathcal{G}r^*_\infty(\mathcal{D})$ in *any* odd dimension.

To establish Theorem 1.1 we study the variation of $\det_\zeta \mathcal{D}_P$ and $\det_\mathcal{C} \mathcal{D}_P$ under a change of the boundary condition P. To compute the variation we exploit the transitive action $P \mapsto UPU^{-1}$ on $\mathcal{G}r^*_\infty(\mathcal{D})$ of the group $\mathcal{U}_\infty$ of unitary isomorphisms of the form

$$U = \begin{pmatrix} I_{F^+} & 0 \\ 0 & g \end{pmatrix} \tag{8}$$

written relative to the grading $F^+ \oplus F^-$, where $g - I_{F^-}$ is a smoothing operator, with $I_{F^\pm}$ the identity operator on $F^\pm$. Using a 'unitary twist' this action is extended to

a unitary action on $C^\infty(M;\mathcal{S})$ which by spectral invariance allows us to compute the variation of $\det_\zeta \mathcal{D}_P$ with respect to P (see [4, 7, 9] for details). Fix $P_1, P_2 \in \mathcal{G}r^*_\infty(\mathcal{D})$ and consider the paths $P_{i,r} = U_r P_i U_r^{-1}$ defined by a smooth 1–parameter family $U_r \in \mathcal{U}_\infty$ with $-1 < r < 1$ and $U_0 = I$. We study the difference

$$\log \det_\zeta \mathcal{D}_{P_{2,r}} - \log \det_\zeta \mathcal{D}_{P_{1,r}}.$$

The crucial result is:

Proposition 2.1 *The following equality holds:*

$$\frac{d}{dr}|_{r=0} \left\{ \log \det_\zeta \mathcal{D}_{P_{2,r}} - \log \det_\zeta \mathcal{D}_{P_{1,r}} \right\} = \frac{d}{dr}|_{r=0} \left\{ \log \det_C \mathcal{D}_{P_{2,r}} - \log \det_C \mathcal{D}_{P_{1,r}} \right\}. \tag{9}$$

Outline proof of Proposition 2.1

We start with the discussion of the left side of the equation (9). We need the following result proved in [9].

Proposition 2.2 *(1). The value of the ζ-function of $\mathcal{D}_P$ at $s = 0$ is constant on $\mathcal{G}r^*_\infty(\mathcal{D})$.*

(2). The variation of the η-invariant of $\mathcal{D}_P$ is stable in the following sense. For any pair of projections $P_1, P_2 \in \mathcal{G}r^*_\infty(\mathcal{D})$

$$\frac{d}{dr}|_{r=0}(\eta_{\mathcal{D}_{P_{1,r}}}(0)) = \frac{d}{dr}|_{r=0}(\eta_{\mathcal{D}_{P_{2,r}}}(0)) \ .$$

where the families $\{P_{i,r} = U_r P_1 U_r^{-1}\}_{r\in(-1,1)}$ are constructed by using a family of the unitary operators $\{U_r\}$ of the form (8), such that g_r is a smooth family of unitary operators on F^- of the form $I+$ smoothing operator *with $g_0 = I$.*

It follows from Proposition 2.2 that we have only to study the variation of the modulus

$$\det_\zeta|\mathcal{D}_{P_{i,r}}| = e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2_{P_{i,r}}}(0)}$$

of the determinant $\det_\zeta \mathcal{D}_{P_{i,r}}$. We compute:

$$\frac{d}{dr}|_{r=0}(\frac{1}{2}\zeta'_{\mathcal{D}^2_{P_{i,r}}}(0)) = \lim_{\epsilon\to 0} \mathrm{Tr}\,(\frac{d}{dr}|_{r=0}\mathcal{D}_{P_{i,r}})\mathcal{D}_{P_i}^{-1}e^{-\epsilon\mathcal{D}^2_{P_i}}. \tag{10}$$

To evaluate the limit in (10) we have to study the inverse of the operator $\mathcal{D}_P$. The operator $\mathcal{D}_P$ is invertible if and only if the operator

$$PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \longrightarrow \mathrm{range}(P)$$

is invertible, and in this case we have:

Proposition 2.3 *The operator $\mathcal{D}_P^{-1} : L^2(M;\mathcal{S}) \to \mathrm{dom}(\mathcal{D}_P)$ inverse of the operator $\mathcal{D}_P$ is given by the formula*

$$\mathcal{D}_P^{-1} = \mathcal{D}^{-1} - \mathcal{D}^{-1}\gamma_0^*\Gamma(PP(\mathcal{D}))^{-1}P\gamma_0\mathcal{D}^{-1}, \tag{11}$$

where $\mathcal{D}^{-1}$ denotes the operator $r_M \cdot \tilde{\mathcal{D}}^{-1} \cdot e_{\tilde{M}}$. Here $\tilde{\mathcal{D}}$ is the (invertible) double operator of $\mathcal{D}$ over the closed double $\tilde{M}$ of M ; $e_{\tilde{M}} : H^s(M;\mathcal{S}) \to H^s(\tilde{M};\tilde{\mathcal{S}})$ and $r_M : H^s(\tilde{M};\tilde{\mathcal{S}}) \to H^s(M;\mathcal{S})$ are the corresponding extension and restriction operators; and γ_0^* is the dual to the operator $\gamma_0 : H^s(M;\mathcal{S}) \to H^{s-1/2}(Y;\mathcal{S}|Y)$ restricting sections to the boundary (see [2] for details).

Note that $(PP(\mathcal{D}))^{-1}$ is the restriction to range(P) of the elliptic pseudodifferential operator $((I-P)(I-P(\mathcal{D})) + PP(\mathcal{D}))^{-1}$. We observe that the difference

$$\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1} = \mathcal{D}^{-1}\gamma^*\Gamma(P_1P(\mathcal{D}))^{-1}P_1\gamma_0\mathcal{D}^{-1} - \mathcal{D}^{-1}\gamma^*\Gamma(P_2P(\mathcal{D}))^{-1}P_2\gamma_0\mathcal{D}^{-1}$$

is a smoothing operator, which implies the next Proposition.

Proposition 2.4 *For any pair of projections $P_1, P_2 \in \mathcal{G}r^*_\infty(\mathcal{D})$ and any 1-parameter family of unitary operators U_r of the form (8) the following identity holds:*

$$\frac{d}{dr}|_{r=0}\left\{\log\det{}_\zeta\mathcal{D}_{P_{2,r}} - \log\det{}_\zeta\mathcal{D}_{P_{1,r}}\right\} = \mathrm{Tr}\,\dot{\mathcal{D}}(\mathcal{D}_{P_2}^{-1} - \mathcal{D}_{P_1}^{-1}). \tag{12}$$

The determinant lines of the operators $PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to Ran(P)$ and $\mathcal{D}_P$ are canonically isomorphic and under the isomorphism their abstract Quillen determinant elements identified. Hence $\det_{\mathcal{C}}(PP(\mathcal{D}))$ is also defined by the right-side of (4), which gives us

$$\det{}_{\mathcal{C}}\mathcal{D}_{P_{2,r}}\left(\det{}_{\mathcal{C}}\mathcal{D}_{P_{1,r}}\right)^{-1} = \det{}_{\mathcal{C}}(P_{2,r}P(\mathcal{D}))(\det{}_{\mathcal{C}}(P_{1,r}P(\mathcal{D})))^{-1}\ . \tag{13}$$

Equation (13) and the fact that $P(\mathcal{D}) = \gamma_0\mathcal{D}^{-1}\gamma^*\Gamma$ allow us to study the right side of (9). Using the formula (11) for the operator $\mathcal{D}_P^{-1}$ we compute the logarithmic variation of the right side of (13) as:

Proposition 2.5

$$\frac{d}{dr}|_{r=0}\log\det{}_{\mathcal{C}}(P_{2,r}\gamma_0\mathcal{D}^{-1}\gamma_0^*\Gamma(P_{1,r}P(\mathcal{D}))^{-1}P_{1,r}) = \mathrm{Tr}\,\dot{\mathcal{D}}(\mathcal{D}_{P_2}^{-1} - \mathcal{D}_{P_1}^{-1}). \tag{14}$$

This completes the proof of Proposition 2.1. Now integrating (9) we obtain:

$$\frac{\det_\zeta\mathcal{D}_{UP_2U^{-1}}/\det_{\mathcal{C}}\mathcal{D}_{UP_2U^{-1}}}{\det_\zeta\mathcal{D}_{P_2}/\det_{\mathcal{C}}\mathcal{D}_{P_2}} = \frac{\det_\zeta\mathcal{D}_{UP_1U^{-1}}/\det_{\mathcal{C}}\mathcal{D}_{UP_1U^{-1}}}{\det_\zeta\mathcal{D}_{P_1}/\det_{\mathcal{C}}\mathcal{D}_{P_1}} := \alpha(U).$$

For $U, V \in \mathcal{U}_\infty$ we have $\alpha(UV) = \alpha(U)\alpha(V)$ and hence $\alpha : \mathcal{U}_\infty \longrightarrow \mathbf{C}^*$ is a character on the group $\mathcal{U}_\infty$. Substituting $P_1 = P(\mathcal{D})$, $P_2 = P = UP(\mathcal{D})U^{-1}$ we obtain

$$\det{}_\zeta\mathcal{D}_P = \det{}_\zeta\mathcal{D}_{P(\mathcal{D})}\cdot\det{}_{\mathcal{C}}\mathcal{D}_P\cdot\alpha(U).$$

Further inspection of $\alpha(U)$ as the metric in the collar neighbourhood $\mathcal{N}$ is blown-up in the normal direction to the boundary shows that $\alpha = 1$, thus completing the proof of Theorem 1.1.

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