Special Lagrangian Geometry in irreducible symplectic 4-folds

Alessandro Arsie *
S.I.S.S.A. - I.S.A.S.
Via Beirut 4 - 34013 Trieste, Italy

Abstract

We prove that, (having fixed a Kähler class and the unique corresponding hyperkähler metric), all special Lagrangian submanifolds of an irreducible symplectic 4-fold are obtained by complex submanifolds via a generalization of the so called "hyperkähler rotation trick"; thus they retain part of the rigidity of the complex submanifolds: indeed all special Lagrangian submanifolds of X turn out to be real analytic.

MSC (1991): Primary: 53C15, Secondary: 53A40, 51P05, 53C20
Keywords: special Lagrangian submanifolds, hyperkähler structures.

REF.: 75/99/FM/GEO

1 Introduction

Under the flourishing research activity on D-branes in string theory, the role of special Lagrangian submanifolds in physics has become more and more relevant (see for example [2]) untill it was eventually conjectured in [12] that they can be considered as the cornerstones of the mirror phenomenon. Indeed, D-branes are special Lagrangian submanifolds equipped with a flat $U(1)$ line bundle. In physical literature, special Lagrangian submanifolds of the compactification space are related to physical states which retain part of

*e-mail: arsie@sissa.it

the vacuum supersymmetry: for this reason they are often called supersymmetric cycles or BPS states.

Despite their importance, there are very few explicit examples of special Lagrangian submanifolds, especially in Calabi-Yau 3-folds. However, in an irreducible symplectic 4-fold (realized as a hyperkähler manifold) we have a complete control of the special Lagrangian geometry of its submanifolds, via a sort of "hyperkähler trick"; moreover this enables us to prove that special Lagrangian submanifolds retain part of the rigidity of complex submanifolds.

We first recall the following:

Definition 1.1: *A complex manifold X is called irreducible symplectic if it satisfies the following three conditions:*

1) X is compact and Kähler;

2) X is simply connected;

3) $H^0(X, \Omega^2_X)$ is spanned by an everywhere non-degenerate 2-form ω.

In particular, irreducible symplectic manifolds are special cases of Calabi-Yau manifolds (the top holomorphic form which trivializes the canonical line bundle is given by a suitable power of the holomorphic 2-form ω). In dimension 2, K3 surfaces are the only irreducible symplectic manifolds, and indeed irreducible symplectic manifolds appear as higher-dimensional analogues of K3 surfaces, as strongly suggested in [6]. Unfortunately, up to now there are very few explicit examples of irreducible symplectic manifolds. Indeed almost all known examples turn out to be birational to two standard series: Hilbert schemes of points on K3 surfaces and generalized Kummer variety (both series were first studied in [3]), but quite recently O'Grady has constructed irreducible symplectic manifolds which are not birational to any of the elements of the two groups (see [11]).

Finally, let us recall from [5] the following:

Definition 1.2: *Let X be a Calabi-Yau n-fold, with Kähler form ω and holomorphic nowhere vanishing n-form Ω. A (real) n-dimensional submanifold $j : \Lambda \hookrightarrow X$ of X is called special Lagrangian if the following two conditions are satisfied:*

1) Λ is Lagrangian with respect to ω, i.e. $j^*\omega = 0$;

2) there exists a multiple Ω' of Ω such that $j^*\mathrm{Im}(\Omega')=0$; one can prove (see [5]) that both conditions are equivalent to:

1') $j^*\mathrm{Re}(\Omega') = Vol_g(\Lambda)$.

The condition 1') in the previous definition means that the real part of Ω' restricts to the volume form of Λ, induced by the Calabi-Yau Riemannian metric g. In this way special Lagrangian submanifolds are considered as a type of calibrated submanifolds (see [5] for further details on this point).

2 Characterization of special Lagrangian submanifolds

In this section we will describe all special Lagrangian submanifolds of an irreducible symplectic 4-fold X (having fixed a Kähler class $[\omega]$ in the Kähler cone). The key result is the following:

Theorem 2.1: *Every connected special Lagrangian submanifold of an irreducible symplectic 4-fold is also bi-Lagrangian, in the sense that it is Lagrangian with respect to two different symplectic structures.*

Proof: Let us a fix a Kähler class on the irreducible symplectic 4-fold X. By Yau's Theorem this determines a unique hyperkähler metric g. Choose a hyperkähler structure (I, J, K) compatible with the metric g (notice that the triple (I, J, K) is not uniquely determined) and consider the associated symplectic structures $\omega_I(.,.) := g(I.,.)$, $\omega_J(.,.) := g(J.,.)$ and $\omega_K(.,.) := g(K.,.)$.

Consider a special Lagrangian submanifold Λ in the complex structure K (this is not restrictive, since (I, J, K) is not uniquely determined); that is assume that Λ is calibrated by the real part of the holomorphic (in the structure K) 4-form:

$$\Omega_K := \frac{1}{2!}(\omega_I + i\omega_J)^2. \tag{1}$$

Notice that the real and immaginary part of Ω_K are then given by:

$$\mathrm{Re}(\Omega_K) = \frac{1}{2}(\omega_I^2 - \omega_J^2) \quad \mathrm{Im}(\Omega_K) = \omega_I \wedge \omega_J. \tag{2}$$

Obviously, by the property of being special Lagrangian we have that Λ is Lagrangian with respect to ω_K. We will prove that having fixed the calibration, if Λ is not Lagrangian also with respect to ω_I, then it is necessarily Lagrangian with respect to ω_J. First we work locally and consider $V := T_p\Lambda$ $(p \in \Lambda)$, spanned by (w_1, w_2, w_3, w_4). Since Λ is assumed not to be Lagrangian with respect to ω_I, we have to deal with two cases.

First case: V is a symplectic vector space for the structure ω_I. In this case we can choose a symplectic basis for V and this can always be chosen to be of the form v_1, Iv_1, v_2, Iv_2. Then V is Lagrangian in the symplectic structure ω_J; indeed $\omega_J(v_1, Iv_1) = g(Jv_1, Iv_1) = g(IJv_1, -v_1) = -\omega_K(v_1, v_1) = 0$; analogously for $\omega_J(v_2, Iv_2)$; $\omega_J(v_1, Iv_2) = g(Jv_1, Iv_2) = -\omega_K(v_1, v_2) = 0$ since v_1, v_2 belong to a Lagrangian subspace of ω_K, and analogously for $\omega_J(v_2, Iv_1) = -\omega_K(v_2, v_1) = 0$. Thus V is also Lagrangian for the symplectic structure ω_J.

Second case: V is neither symplectic nor Lagrangian for the structure ω_I. Notice V can not be symplectic with respect to ω_J, otherwise by the first case it would be Lagrangian in the strucutre ω_I; moreover we can assume that V is not Lagrangian with respect to ω_J, otherwise there is nothing to prove. So in this case V is neither Lagrangian nor symplectic in the structure ω_I *and* in the structure ω_J. This means that V contains a symplectic 2-plane π with respect to ω_I and a symplectic 2-plane ρ with respect to ω_J. We prove that this can not happen, since it violates the calibration condition. We have to distinguish three different subcases according to the intersection of π with ρ.

First subcase: π and ρ have zero intersection. If this happens we can always choose a basis of V of the form (v_1, Iv_1, v_2, Jv_2). Write π for the 2-plane spanned by v_1, Iv_1 and ρ for that spanned by v_2, Jv_2, so that $V = \pi \oplus \rho$. Indeed, since V is not Lagrangian with respect to ω_I, it has to contain a symplectic 2-plane like π, and similarly for ρ and ω_J. Moreover, since V is not symplectic with respect to ω_I, it turns out that the symplectic 2-plane π can not be completed to a symplectic basis of V, so that V has to contain an isotropic 2-plane for ω_I, which is ρ. The same reasoning (with the roles reversed) applies obviously to the symplectic structure ω_J. Hence, in this case we have:

$$2\mathrm{Re}(\Omega_K)|_V = (\omega_I^2 - \omega_J^2)(v_1, Iv_1, v_2, Jv_2) = \omega_I(v_1, Iv_1)\omega_I(v_2, Jv_2) -$$
$$\omega_I(v_1, v_2)\omega_I(Iv_1, Jv_2) + \omega_I(v_1, Jv_2)\omega_I(Iv_1, v_2) - \omega_J(v_1, Iv_1)\omega_J(v_2, Jv_2) +$$
$$\omega_J(v_1, v_2)\omega_J(Iv_1, Jv_2) - \omega_J(v_1, Jv_2)\omega_J(Iv_2, v_2) = 0,$$

using the defining relations of $\omega_I, \omega_J, \omega_K$, the quaternionic relation $IJ = K$, the invariance of g and the fact that V is Lagrangian with respect to ω_K. So this subcase is not consistent with the calibration property.

Second subcase: π and ρ have a 1-dimensional intersection spanned by a vector v_1. In this case we can choose a basis of V of the form (v_1, Iv_1, Jv_1, w) (π is spanned by (v_1, Iv_1), while ρ is spanned by (v_1, Jv_1)). Again by the same computation of the previous subcase one shows that this configuration is not compatible with the calibration.

Third subcase: Finally $\pi{=}\rho$ can not clearly happen, since otherwise one can choose a basis of π equal to (v_1, Iv_1), but then, in this basis ω_J is identically vanishing, contrary to the assumption that $\rho = \pi$ is a symplectic 2-plane also for ω_J.

Since the second case can never happen V has to be Lagrangian also with respect to ω_J.

Up to now, we have worked only locally; to conclude the proof it is necessary to show that if $T_p\Lambda$ is Lagrangian with respect to ω_J, then it can not be possible that $T_q\Lambda$ is Lagrangian with respect to ω_I, for a different $q \in \Lambda$. Notice that any tangent space to Λ can not be Lagrangian with respect to *both* ω_I and ω_J, otherwise it would violates the calibration condition. Since the maps $\alpha_{I,J} : \Lambda \ni p \to \omega_{I,J}|_{T_p\Lambda}$ are continous and everything is supposed to be smooth, it turns out that $\Lambda = \alpha_I^{-1}(0) \cup \alpha_J^{-1}(0)$. But this is clearly impossible, since Λ is connected and so can not be the union of two proper closed disjoint subsets. □

The previous theorem is important in view of the following:

Corollary 2.1: *Every (connected, compact and without border) special Lagrangian submanifold Λ of a hyperkähler 4-fold X can be realized as a complex submanifold, via hyperkähler rotation of the complex structure of X.*

Proof: Let Λ be a special Lagrangian submanifold of X in the complex structure K. Then by definition $\mathrm{Re}(\Omega_K)_{|\Lambda} = \mathrm{Vol}_g(\Lambda)$, but by the previous theorem, since $\omega_J|_\Lambda = 0$ this means:

$$\mathrm{Vol}_g(\Lambda) = \frac{1}{2}\int_\Lambda \omega_I^2. \tag{3}$$

By Wirtinger's theorem, since Λ is assumed to be compact and without border, condition (3) is equivalent to say that Λ is a complex submanifold of X, in the complex structure I, that is performing a hyperkähler rotation. Notice that in the complex structure I, Λ is still a Lagrangian submanifold with respect to ω_K and ω_I, so it is Lagrangian with respect to the holomorphic (in the structure I) 2-form $\Omega_I := \omega_J + i\omega_K$. □

Collecting the results so far proved, we can show that special Lagrangian submanifolds of X are particularly "rigid":

Proposition 2.1: *Any (connected, compact and without border) special Lagrangian submanifold Λ of a hyperkähler 4-fold X is real analytic.*

Proof: Let Λ be a special Lagrangian submanifold of X, having fixed some complex structure on X, let us say K; then, by Corollary 2.1 there exists a new complex structure, let us say I, in which Λ is holomorphic, that is, it is locally given by:

$$f_1(z_1,\ldots,z_4)=0 \quad \text{and} \quad f_2(z_1,\ldots,z_4)=0.$$

Now observe that coming back to the original complex structure K, we induce an *analytic* change of coordinates from the holomorphic coordinates z^i ($I\frac{\partial}{\partial z^i}=i\frac{\partial}{\partial z^i}$) to new holomorphic coordinates w^i ($K\frac{\partial}{\partial w^i}=i\frac{\partial}{\partial w^i}$) such that locally:

$$z^i=c_1w^i+c_2\bar{w}^i \quad \bar{z}^i=d_1w^i+d_2\bar{w}^i, \tag{4}$$

for some complex constants c_j, d_j. Thus in the complex structure K the special Lagrangian submanifold Λ is given by $f_j(c_1w^i+c_2\bar{w}^i, d_1w^i+d_2\bar{w}^i)=0$ which is again the zero locus of a set of functions analytic in $w^i, \bar{w}^i$. □

Quite obviously, the action of the hyperkähler rotation can be extended also to the holomorphic functions defined on complex submanifolds S of X; in particular we have an action of the hyperkähler rotation on the structure sheaf $\mathcal{O}_S$ (here, as always, we identify $\mathcal{O}_S$ with its direct image $j_*\mathcal{O}_S$, where $j : S \hookrightarrow X$ is the holomorphic embedding). We are thus led to give the following:

Definition 2.2: *Let Λ be a special Lagrangian submanifold of a hyperkähler 4-fold X (in the complex structure K). Then we define the special Lagrangian structure sheaf $\mathcal{L}_\Lambda$ as the subsheaf of the (complexified) real analytic structure sheaf $\mathcal{A}_\Lambda$, obtained by the action of the hyperkähler rotation on the structure sheaf $\mathcal{O}_\Lambda$ of Λ, as a complex Lagrangian submanifold of X (in the structure I).*

The previous definition will be important in implementing a form of homological mirror symmetry for irreducible symplectic 4-folds (see [1]).

3 Concluding remarks

It is important to remark that all previous results are true also for special Lagrangian submanifolds of K3 surfaces, but their proof is completely trivial in that case.

Another observation is related to *singular* Lagrangian submanifolds: indeed, by the previous results, it turns out that we can also give examples of special Lagrangian *subvarieties*, obtained via hyperkähler rotation of Lagrangian complex subvarieties. On the other hand, contrary to the case of the corresponding submanifolds, we can not expect that *all* special Lagrangian subvarieties are obtained in this way, and consequently we can not expect that all special Lagrangian subvarieties are real analytic. Indeed, there are examples (compare [5]) of singular special Lagrangian submanifold in $\mathbb{C}^n$ which are only smooth, but not real analytic.

The discussion about singular Lagrangian submanifolds leads us to comment on the mirror symmetry construction suggested in [12]. Indeed, according to the recipe of [12], any Calabi-Yau X, admitting a mirror $\hat{X}$, has a peculiar fibre space structure: on a physical ground it is argued that X can be realized as the total space of a fibration in special Lagrangian tori. Unfortunately, there are very few examples of such realization: in particular, as far as we know, there is only one example for Calabi-Yau 3-folds of the so called Borcea-Voisin type (see [4]). Instead, in the case of irreducible symplectic projective manifolds the situation is completely different. Indeed, a recent result of Matsushita (see [9] and [10]) shows that for any fibre space structure $f : X \to B$ of a projective irreducible symplectic manifold X, with projective base B, the generic fibre $f^{-1}(b)$ is an Abelian variety (up to finite unramified cover), and it is also Lagrangian with respect to the non degenerate holomorphic 2-form Ω; moreover, in the case of 4-folds one can prove that the generic fibre is an Abelian surface and f is equidimensional, (i.e. all irreducible components of the fibres have the same dimension). By Corollary 2.1 it turns out that this fibre space structure can also be realized as a special Lagrangian torus fibration; moreover, in this case all special Lagrangian fibres, even the singular ones, are analytic, since they are obtained by performing a hyperkähler rotation starting from Lagrangian Abelian surfaces. So, in these cases, we have special Lagrangian torus fibration in which all fibres are analytic: one can hope to understand the degeneration types of singular special Lagrangian tori, moving from these constructions.

Explicit examples of projective irreducible symplectic 4-folds, fibered over a projective base have been constructed by Markuschevich in [7] and [8]. One of this constructions is the following: consider a double cover $\pi : S \to \mathbb{P}^2$ of the projective plane, ramified along a smooth sextic $C \hookrightarrow \mathbb{P}^2$ (S is then realized as a K3 surfaces in a weighted projective space). Since any line in $\mathbb{P}^2$ will intersect generically the sextic C in six distinct point, we have that the covering $\pi : S \to \mathbb{P}^2$ determines a (flat) family of hyperelliptic curves over the dual projective plane $f : \mathcal{X} \to \mathbb{P}^2$. Then the Altmann-Kleiman compactification of the relative Jacobian of the family turns out to be a simplectic projective irreducible 4-folds, fibered over $\mathbb{P}^2$, and in fact all fibres are Lagrangian Abelian varieties.

Finally, we believe that our characterization of special Lagrangian submanifolds of irreducible symplectic 4-folds can be extended also to higher dimensional irreducible symplectic manifolds: to this aim notice that the proof we have given becomes longer and longer, since one has to deal with new cases and subcases. It would be nice, instead, to find out a sort of inductive argument, which works for all dimensions.

References

[1] Arsie A., *Categorial Mirror Symmetry for irreducible symplectic 4-folds*, in preparation.

[2] Becker K., Becker M., Strominger A., *Fivebranes, membranes and non-perturbative string theory*, hep-th/9507158.

[3] Beauville A., *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983), 755-782.

[4] Gross M., Wilson P.M.H., *Mirror Symmetry via 3-tori for a class of Calabi-Yau threefolds*, alg-geom/9608004.

[5] Harvey R., Lawson Jr., H.B., *Calibrated Geometries*, Acta Math., **148** (1982), 47-157.

[6] Huybrechts D., *Compact Hyperkähler manifolds: basic results*, alg-geom/9705025.

[7] Markushevich D., *Completely integrable projective symplectic 4-dimensional varieties*, Izvestiya: Mathematics 59 (1995), 159-187.

[8] Markushevich D., *Integrable symplectic structures on compact complex manifolds*, Math. USSR Sb. 59 (1988), 459-469.

[9] Matsushita D., *On fibre space structures of a projective irreducible symplectic manifold*, Topology 38 (1999), 79-83.

[10] Matsushita D., *Addendum to: On fibre space structures of a projective irreducible symplectic manifold*, math.ag/9903045.

[11] O'Grady K.G., *Desingularized moduli spaces of sheaves on a K3, I and II*, alg-geom/9708009, math.ag/9805099.

[12] Strominger A., Yau S.-T., Zaslow E., *Mirror Symmetry is T-duality*, Nucl. Phys. **B479**, (1996), 243-259.