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Jacobi groups, Hurwitz spaces and Frobenius structures

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Abstract. We give explicit formulae for the generators of the free polynomial algebra of Jacobi forms of type A_l, B_l and G_2. With the aid of suitable generating functions we compute their intersection elements; this provides an elliptic deformation of analogous formulae for the polynomial invariants of Coxeter groups.
In the A_l case we identify the generators with the moduli of Hurwitz spaces of meromorphic functions over elliptic curves; then we construct a Frobenius structure on it, computing the free energy and the flat coordinates of the corresponding solution to WDVV equations of associativity. Several examples are given.
In the G_2 case we find two independent Frobenius structures and we compute the corresponding free energies and flat coordinates.

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Jacobi groups, although without this name, arose in the theory of generalized root systems of the affine type studied by Looijenga [Lo80]. The latter belong to the wider class of *marked extended affine root systems* which were introduced by Saito [Sa85, Sa90]. They are generalized root systems of affine type which preserve a semidefinite positive bilinear pairing with two–dimensional kernel: the "marked" refers to a choice of a sublattice of rank one in the kernel, in order to extend the metric to a preserved hyperbolic metric, namely nondegenerate and indefinite.

Already Looijenga noticed that those Weyl groups carry a natural $SL(2,\mathbb{Z})$–action and hence one can define a semidirect product which we denote by $\mathbb{J}(\mathfrak{g})$ in correspondence with a simple lie algebra $\mathfrak{g}$: later Wirthmüller [Wi92] began the study of a suitable algebra of invariants under $\mathbb{J}(\mathfrak{g})$, which he named "Jacobi forms" because in the case $\mathfrak{g} = A_1$ they reduce to the Jacobi forms in the sense of Eichler and Zagier [EZ]. He proved a Chevalley's type theorem (except for the case E_8) which states that the invariants constitute a polynomial, freely generated, bigraded algebra over the graded ring of modular forms $\mathbf{M}_\bullet$: this algebra, which we denote with $J^{(\mathfrak{g})}_{\bullet,\bullet}$, plays the same role as the algebra of polynomial invariants of finite Coxeter groups or the exponential invariants for the affine Weyl groups [Bo].

Now it is known from [Du92] that the spectrum of these algebras (i.e. the underlying manifold structure of the quotient space) can be endowed with a very rich geometric structure called "Frobenius manifold" [Du93]; Frobenius manifolds arise in a different, –physical–, context as intrinsic formulation of the *Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of associativity* for a two dimensional topological field theory.
Since there are many points of contact between the polynomial case (i.e. Coxeter groups) and the elliptic one (i.e. Jacobi groups), it is natural to work out an analog structure in the context of these Jacobi invariants.
A generalization to the case of "extended" affine Weyl groups was already made in [DZ98]; the authors considered a particular extension of the affine Weyl groups of simple Lie algebras which endows the algebra of invariants with a grading operator analogous to the usual grading of Coxeter–invariant polynomials.

In this work we go in the same direction by constructing a Frobenius structure on (a suitable covering of) the orbit space of the Jacobi group of type A_l, $\forall l = 1...$ and G_2.

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We have worked out explicit analytic expressions for the generators of the Jacobi algebras of type A_l and B_l; they are given in terms of a generating function which can be regarded as *elliptic deformations* of the analog generating functions for the Coxeter invariant polynomials. This techique also allows us to generalize the work done by Saito, Yano and Sekiguchi in [SYS80], where they computed the matrix elements of the intersection form in the basis of invariant polynomials.

The explicit computations enable us to carry out the corresponding formulas for the generators and the intersection form in the case of G_2. This latter –in some respect– is more interesting than the series A_l because lies in the category of "codimension one cases" studied by Saito in [Sa90]; this means that there exists a unique generator of the algebra of theta invariants of maximal *index*. This implies that there exists a natural flat structure on the spectrum of the invariant algebra as described in [Sa90]. On the other hand, however, the study of the A_2 case will show that there exists a different and independent flat structure, which is described in example 2.2 and section 2.7: this structure comes from the fact that the algebra $J^{(G_2)}_{\bullet,\bullet}$ is embedded into $J^{(A_2)}_{\bullet,\bullet}$ as a consequence of the fact that the respective Jacobi groups are in the reversed inclusion.

Since the results are of two different types, we have divided the work into two parts.
The first part deals with the general definition of Jacobi group, of Jacobi forms and related objects, giving the suitable geometric interpretation of their algebra. Later we specify to the series A_l and B_l and construct the corresponding Jacobi forms usign a technique of generating function which is essential for the second part and for the computation of the intersection elements $\mathfrak{M}_{i,k}$ (see later); as we said, this is an elliptic deformation of the formulae in [SYS80] for the case of polynomial invariants, and we show how to recover them under suitable limits.

In the second part we explain the basic definitions and objects of Frobenius manifolds and Hurwitz spaces (following essentially [Du93]) and finally come to the construction of a natural (twisted) Frobenius structure over a suitable covering of the orbit space of $\mathbb{J}(A_l)$; this provides a series of solution of WDVV which are polynomials in $t_1, ..., t_l, t_{l+1}^{-1}, t_{l+1}$ and depend on t_0 via modular functions. Having a singularity at $\mathbf{t} = 0$ (due to the presence of the modular functions of t_0 and a pole in t_{l+1}), these solutions to the associativity equations do not saytisfy the "good analyticity properties" in the sense of [Du93], Appendix A, but do provide interesting examples of twisted structure.
Notice moreover that the flat coordinates $t_0, t_1, ..., t_{l+1}$ are flat theta invariants on a suitable covering of the orbit space of the complex crystallographic affine Weyl group (though not precisely in the sense of [Sa90]).

1 Jacobi groups and Jacobi Forms

Jacobi groups are particular extensions of complex crystallographic groups [BS86]; they are discrete groups acting on a vector space in such a way that the quotient is compact.
They can be considered in much generality in the setting of generalized root systems but for our purposes these groups are rather easily described and each of them is associated to one of the finite dimensional simple Lie algebras. It will be most effective to consider an explicit description of the group and an explicit faithful representation as an action on a cone.

1.1 The group $\mathbb{J}(\mathfrak{g})$

In this section all the objects we will refer to (Weyl groups, root lattices,...) are the objects corresponding to a complex finite dimensional simple Lie algebra $\mathfrak{g}$ of rank l.
Let W be the Weyl group, Λ the root lattice (as an abelian group).
Let H_R be the Heisenberg group obtained as a central extension of $\Lambda \times \Lambda$ by $\mathbb{Z}$ using the cocycle defined by the invariant Killing form $<,>$ normalized to 2 for the short roots, (this implies that $\forall \lambda \in \Lambda,\ <\lambda,\lambda> \in 2\mathbb{Z}$).
This group is obtained by definition of the product in $\Lambda \times \Lambda \times \mathbb{Z}$ as

$$\begin{aligned} &\forall(\lambda,\mu,k),\ (\lambda',\mu',k') \in \Lambda \times \Lambda \times \mathbb{Z} \\ &(\lambda,\mu,k)\cdot(\lambda',\mu',k') := (\lambda+\lambda',\mu+\mu',k+k'+<\mu,\lambda'>)\ . \end{aligned}$$

Since the Killing form $<,>$ is Weyl invariant and the Weyl group W of $\mathfrak{g}$ acts on Λ, we can take the semidirect product $W \rtimes H_R$: this way we obtain an (infinite) discrete group which we denote by $\mathcal{W}$ where the semidirect product is specified by the multiplication rule

$$\begin{aligned} &\mathcal{W} := W \rtimes H_R, \qquad s.t.\ \forall w,w' \in W, \mathbf{t}=(\lambda,\mu,k), \mathbf{t}'=(\lambda',\mu',k') \in H_R \\ &(w,\mathbf{t})\cdot(w',\mathbf{t}') := (ww', w\cdot\lambda'+\lambda, w\cdot\mu'+\mu, k+k'+<\mu,\lambda>)\ . \end{aligned}$$

This group is rather well known and its invariants are theta functions, as we will explain in due course.
We now can give the definition of Jacobi group

Definition 1.1 *The* **Jacobi group** $\mathbb{J}$ $(=\mathbb{J}(\mathfrak{g}))$ *is the semidirect product* $\mathcal{W} \rtimes SL(2,\mathbb{Z})$. *The action of* $SL(2,\mathbb{Z})$ *on the group* $W \rtimes H_R$ *is defined by*

$$\begin{aligned} &Ad_\gamma(w) := w \\ &Ad_\gamma(\mathbf{t}) := \left(a\lambda - b\mu, -c\lambda + d\mu, k + \frac{ac}{2}<\lambda,\lambda> - bc<\lambda,\mu> + \frac{bd}{2}<\mu,\mu>\right)\ , \end{aligned} \tag{1}$$

for $w \in W$, $\mathbf{t}=(\lambda,\mu,k) \in H_R$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \in SL(2,\mathbb{Z})$. *Then the multiplication rule is defined by* $(\forall(w,\mathbf{t},\gamma),\ (w',\mathbf{t}',\gamma') \in W \times H_R \times SL(2,\mathbb{Z}))$

$$(w,\mathbf{t},\gamma)\cdot(w',\mathbf{t}',\gamma') := (w\cdot w', \mathbf{t}\cdot Ad_\gamma(w\mathbf{t}'), \gamma\cdot\gamma')\ .$$

1.1.1 Faithful representation of $\mathbb{J}$

Let us consider the cone $\Omega := \mathbb{C} \oplus \mathfrak{h} \oplus \mathcal{H} \ni (u,\mathbf{x},\tau)$, where $\mathfrak{h}$ is the complex Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{H}$ denotes the Poincaré upper half plane.
In the literature it is often called the *Tits cone* [Lo80] and it is the union of all the images under $\mathcal{W}$ of the closure of a fundamental chamber $\mathcal{C}$; therefore it is an invariant cone for the action of the group $\mathcal{W}$ (and of $\mathbb{J}$ as well).

Let us consider $\tau \in \mathcal{H}$; we have an embedding of $\Lambda \times \Lambda$ in $\mathfrak{h}$ as the complex crystallographic lattice $\Lambda + \tau\Lambda$ and we can define the action as in the following

Proposition 1.1 The Jacobi group $\mathbb{J}$ can be represented on Ω by definition of the action of $w \in W$, $\mathbf{t} \in H_R$ and $\gamma \in SL(2,\mathbb{Z})$ as

$$\begin{aligned}
&w(u,\mathbf{x},\tau) := (u, w\mathbf{x}, \tau) \\
&\mathbf{t}(u,\mathbf{x},\tau) := \left(u + k - <\mathbf{x},\mu> - \frac{\tau}{2} <\mu,\mu>, \mathbf{x} + \lambda + \tau\mu, \tau\right) \\
&\gamma(u,\mathbf{x},\tau) := \left(u + \frac{c<\mathbf{x},\mathbf{x}>}{2(c\tau+d)}, \frac{\mathbf{x}}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) .
\end{aligned}$$

The proof is straightforward although rather long, and it is left to the reader.

Remark 1.1 The action of the Jacobi group is non linear, but it could be made such by realizing it as a discrete subgroup of the conformal group: in fact, as we will see, its action is a conformal action for the metric $du \otimes d\tau + d\tau \otimes du + <d\mathbf{x} \overset{\otimes}{,} d\mathbf{x}>$.

1.2 Jacobi forms

Since we want to consider the orbit space $\Omega/\mathbb{J}(\mathfrak{g})$, it is necessary to study the algebra of invariant functions; this is the analog of the study of polynomials over a vector space which are invariant under the action of a Coxeter group (which gives the orbit space a structure of weighted projective space[1]). In this case one studies better the $SL(2,\mathbb{Z})$–*equivariant* functions for the Jacobi group; it means that they are invariant under the action of the normal subgroup $\mathcal{W}$, and transform under some representation of the metaplectic group (as in Definition 1.4); in more geometrical terms one studies the invariant sections of a suitable line–bundle over the orbit space.
We will come back to this picture later, but for now we give the definitions and the results of the theory.

Following [Wi92] we will consider holomorphic functions on Ω with the further property that they are *locally bounded* (in $u, \mathbf{x}$) as $\Im(\tau) \to +\infty$; this is natural since τ is interpreted as modular parameter of an elliptic curve and the Jacobi forms will realize the holomorphic sections of a certain line bundle over it, which one wants to extend at the cusps of $SL(2,\mathbb{Z})$.
To be more definite, we will study the invariant modular forms, *Jacobi forms*, of weight k, $\mathcal{F}(u,\mathbf{x},\tau)(d\tau)^{k/2}$, which have the following definition

Definition 1.2 ([Wi92]) *The Jacobi forms of* **weight** k *and* **index** m *are holomorphic functions on the Tits cone* $\Omega = \mathbb{C} \oplus \mathfrak{h} \oplus \mathcal{H} \ni (u,\mathbf{x},\tau)$ *which satisfy*

$$\begin{aligned}
&\mathbf{E}\varphi(u,\mathbf{x},\tau) := \frac{1}{2i\pi}\partial_u\varphi(u,\mathbf{x},\tau) = m\,\varphi(u,\mathbf{x},\tau) \\
&\varphi(u,\mathbf{x},\tau) = \varphi(u, w\cdot\mathbf{x},\tau) \ ; \\
&\varphi(u,\mathbf{x},\tau) = \varphi\left(u - <\mathbf{t},\mathbf{x}> - \frac{\tau}{2} <\mathbf{t},\mathbf{t}>, \mathbf{x} + \tau\mathbf{t} + \lambda, \tau\right) \ ; \\
&\varphi(u,\mathbf{x},\tau) = (c\tau+d)^{-k}\varphi\left(u + \frac{c<\mathbf{x},\mathbf{x}>}{2(c\tau+d)}, \frac{\mathbf{x}}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) ,
\end{aligned} \tag{2}$$

[1]Recall that a weighted projective space $\mathbb{P}_{n_0,n_1,\ldots,n_l}$ with weights $n_0,\ldots,n_l \in \mathbb{N}$ is the quotient space of $\mathbb{C}^{l+1}/\{0\}$ with respect to the $\mathbb{C}^*$-action determined by the formula $\rho(t)(z_0,\ldots,z_l) = (t^{n_0}z_0,\ldots,t^{n_l}z_l)$.

and are locally bounded *functions of* $\mathbf{x}$ *as* $\Im(\tau) \to +\infty$.

The space of Jacobi forms of **weight** k *and* **index** m *is denoted by* $J_{k,m}$.

This means that the Jacobi forms are the invariants of the group $\mathcal{W} = W \rtimes H_R$ and transform as $d\tau^{k/2}$ under the modular group $SL(2,\mathbb{Z})$, being also eigenfunctions of the **Euler vector** $\mathbf{E} = \frac{1}{2i\pi}\partial_u$ with eigenvalue m.

Remark 1.2 Instead of thinking in terms of line–bundles, we could define equivalently a truly invariant algebra by adjoining to Ω a coordinate of the line–bundle as follows; consider the trivial line bundle $Y := \mathbb{C} \times \Omega$ with coordinates $(v, u, \mathbf{x}, \tau)$. Then define the action of the Jacobi group on Y as

$$\gamma(v, u, \mathbf{x}, \tau) := \left(\frac{v}{c\tau + d}, u + \frac{c < \mathbf{x}, \mathbf{x} >}{2(c\tau + d)}, \frac{\mathbf{x}}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) ,$$

and in the obvious way (leaving v unchanged) for the elements of $\mathcal{W}$.
It follows that we can associate to any Jacobi form $\varphi \in J_{k,m}$ an invariant function on Y as

$$\Phi(v, u, \mathbf{x}, \tau) := v^k \varphi(u, \mathbf{x}, \tau)$$

and hence define a second grading vector $\mathcal{K} := v\frac{\partial}{\partial v}$ which could be called the *weighting operator.*

Digression 1 The action of the group $\mathcal{W}$ is the well known action in the theory of theta functions, namely, the invariance under $\mathcal{W} = W \rtimes H_R$ can be rephrased by saying that the Jacobi forms of index m are in particular theta functions of "level m" for the affine Weyl group in consideration. We recall briefly some basic facts (but for a complete reference, see [KP84]). Adapting the notion to our setting and notations, if P is the lattice of weights (i.e. $< P, \Lambda >= \mathbb{Z}$),

Definition 1.3 (Section III in [KP84]) *Let* $m \in \mathbb{N}$ *be fixed. The space of theta functions* $\widetilde{Th}_m$ *is defined as the set of holomorphic functions of* $(u, \mathbf{x}, \tau)$ *which are invariant under* H_R *and of degree* m *for the Euler field* $\mathbf{E}$.
The theta functions of characteristic $\mathbf{p} \in P$ *and degree* $m \in \mathbb{N}$ *for* $\mathbf{p}$ *in* $P \bmod m\Lambda^\vee$ *(here* $\Lambda^\vee$ *is the lattice of coroots) are defined by*

$$\Theta_{\mathbf{p},m}(u, \mathbf{x}, \tau) := e^{-2i\pi\, m\, u} \sum_{\lambda \in \Lambda^\vee + \frac{1}{m}\mathbf{p}} e^{im\pi\tau\|\lambda\|^2 - 2im\pi < \lambda, \mathbf{x} >} .$$

The $\mathbb{C}$ *linear span of them inside* $\widetilde{Th}_m$ *is denoted by* Th_m.

It can be shown that

Proposition 1.2 [Lemma 3.12 and Prop. 3.13 in [KP84]] For $m \in \mathbb{Z}$ we have

(a) $\widetilde{Th}_0 = \mathcal{O}(\mathcal{H})$ (where $\mathcal{O}(\mathcal{H})$ denotes the holomorphic functions of τ);

(b) $\widetilde{Th}_m = \{0\}$ for $m < 0$;

(c) the space $\widetilde{Th}_m$ is a $\mathcal{O}(\mathcal{H})$ module over Th_m and hence the (graded) ring $\widetilde{Th}_\bullet := \bigoplus_{m\in\mathbb{N}} \widetilde{Th}_m$ is a free module over $\mathcal{O}(\mathcal{H})$ with basis $\{\Theta_{\mathbf{p},m},\ m \in \mathbb{N},\ m > 0,\ \mathbf{p} \in P \bmod m\Lambda^\vee\} \cup \{1\}$.

Let us denote by $\widetilde{Th}_\bullet^W$ the Weyl–invariant part of this ring (which is clearly also an algebra); regarding its structure we have

Theorem 1.1 [Thm 2.7 in [Lo76]] The algebra $\widetilde{Th}_\bullet^W$ is a (graded) polynomial $\mathcal{O}(\mathcal{H})$–algebra freely generated by $l+1$ W–invariant theta functions $\theta_0, ..., \theta_l$. These generators are given by

$$\theta_j := W \cdot \Theta_{\mathbf{p_j}, m_j} \ ;$$

here $W \cdot \Theta$ denotes the symmetrization w.r.t. the action of the (finite) Weyl group, $\mathbf{p_j}$ are the fundamental weights (we have set $\mathbf{p_0} := 0$) and m_j are the integers appearing in the decomposition of the dual of the highest coroot $\widetilde{\alpha^\vee}^\vee$ in the basis of roots, namely $\widetilde{\alpha^\vee}^\vee = \sum_{j=1}^l m_j \alpha_j$, (setting for brevity $m_0 := 1$).

We now go back to the Jacobi forms.
The set of Jacobi forms of any weight and index has a natural structure of bi–graded algebra where the two gradings are the *weight* and the *index*; the following theorem is the the analog of Chevalley's theorem for invariant polynomials of a Coxeter group

Theorem 1.2 [Thm. (3.6) in [Wi92]] Given the Jacobi group associated to any finite dimensional simple lie algebra $\mathfrak{g}$ (possibly except E_8)

1. the bigraded algebra of Jacobi forms $J_{\bullet,\bullet} := \bigoplus_{k,m} J_{km}$ is freely generated by $l+1$ fundamental Jacobi forms $\{\varphi_0, .., \varphi_l\}$ over the graded ring of modular forms $\bigoplus_k \mathbf{M}_k$

$$J_{\bullet\bullet} = \mathbf{M}_\bullet \left[\varphi_0, .., \varphi_l\right] \ ; \tag{3}$$

2. The generator φ_j has weight $-k_j \leq 0$ and index $m_j > 0$ (for $j = 0...l$); the indices m_j are the integers appearing in

$$\tilde{\alpha}^\vee = \sum_{j=1}^{l} m_j \alpha_j^\vee \ ,$$

$\widetilde{\alpha^\vee}^\vee$ being the dual of the highest *coroot* $\widetilde{\alpha^\vee}$.

3. the weights are $k_0 = 0$, and k_j are the Coxeter exponents plus one namely, if c is the product of the fundamental reflections of the Weyl group, then $\left\{\frac{-2i\pi(k_1-1)}{h}, ..., \frac{-2i\pi(k_l-1)}{h}\right\}$ are its eigenvalues: notice that k_j, $j = 1..l$, equal the degrees of the invariant polynomials that generate the invariant algebra $\mathbb{C}[V_0]^W$.

Remark 1.3 The second statement of Thm. 1.2 about the indices of the generators is not surprising if we recall that the bigraded algebra $J_{\bullet,\bullet}$ is a subalgebra of the Weyl–invariant part of the (graded) $\mathcal{O}(\mathcal{H})$ algebra of theta functions $\widetilde{Th}_\bullet$.
This means that the generators are graded–linear (w.r.t. the grading induced by the index) combinations of the generators $\theta_0, ..., \theta_l$, with coefficients which depend holomorphically on τ only. This dependence is forced because the generators θ_j transform as a multiplet of modular forms under $SL(2,\mathbb{Z})$. There is only one case in which a theta function is also a Jacobi form: it happens for the theta function of characteristic zero in the simply–laced cases for which the lattice of (co)roots is self dual, namely $\Lambda = \Lambda^\vee = P$. This occurs only for E_8, for which the theorem has not been proved.

The index zero subspace $J_{\bullet,0} \subset J_{\bullet,\bullet}$ is the graded algebra of *modular forms* $\mathbf{M}_\bullet$. To show this we observe that $J_{\bullet,0}$ is spanned –by definition– by Jacobi forms independent of u and hence they cannot depend on $\mathbf{x}$ either; indeed they are (for τ fixed) *bounded* functions on $\mathfrak{h}/(\Lambda+\tau\Lambda)$, which is compact, and hence they are constant w.r.t. $\mathbf{x}$.

Before going on, it is useful to remind some definitions and facts on the modular group and forms.

Definition 1.4 (see [KP84]) *The* **metaplectic group** $Mp(2,\mathbb{R})$ *is defined as the set*

$$Mp(2,\mathbb{R}) := \{(A, j_A) \in SL(2,\mathbb{R}) \times \{f : \mathcal{H} \to \mathbb{C}\} \ s.t. \ j_A(\tau)^2 = c\tau + d\}$$

endowed with the multiplication rule

$$(A, j_A) \cdot (B, j_B) := (A \cdot B, j_A(B\tau) j_B(\tau))$$

Clearly we can consider the discrete subgroup $Mp(2,\mathbb{Z})$ and give the following

Definition 1.5 *Let Γ be a finite index subgroup of $Mp(2,\mathbb{Z})$ and a group homomorphism $\chi : \Gamma \to S^1$; a function $f : \mathcal{H} \to \mathbb{C}$ is called a* **modular form of weight k and multiplier system χ for Γ** *if it is holomorphic and for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have*

$$f(A\tau) = \chi(A, j_A)(c\tau + d)^k f(\tau) \ .$$

An important example of modular form with multiplier system is the **Dedekind's η function** (which we will use in the following)

$$\eta(\tau) := e^{\frac{i\pi\tau}{12}} \prod_1^\infty \left(1 - e^{2i\pi\tau n}\right) \ . \tag{4}$$

It has the following transformations properties under the metaplectic group;

$$\begin{aligned} \eta(\tau+1) &= e^{\frac{i\pi}{12}} \eta(\tau) \ ; \\ \eta\left(-\frac{1}{\tau}\right) &= \sqrt{(-i\tau)}\eta(\tau) \ , \end{aligned}$$

where the determination of the square-root is in the right half plane [Ba]: notice that the η function is a modular form of weight 1/2 with multiplier system such that $\chi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{i\pi/12}$ and $\chi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = e^{-i\pi/4}$.

We finally recall a fundamental theorem for modular forms which can be found in [Se].

Theorem 1.3 [Corollary 2, Ch. VII in [Se]] The algebra of holomorphic modular forms $\mathbf{M}_\bullet$ is a free graded algebra over $\mathbb{C}$ generated by the two *Eisenstein series*

$$G_2(\tau) := \sum_{m^2+n^2\neq 0} \frac{1}{(m+n\tau)^4} \in \mathbf{M}_4 \ ; \qquad G_3(\tau) := \sum_{m^2+n^2\neq 0} \frac{1}{(m+n\tau)^6} \in \mathbf{M}_6 \ ,$$

of weights, respectively, 4 and 6.
Consequently the subspace of modular forms of weight k is spanned by the monomials $G_2^a G_3^b$ with $4a+6b=k$, namely

$$\mathbf{M}_k := \mathbb{C}\left[G_2^a G_3^b, \ \forall a,b, \in \mathbb{N} \ s.t. \ 4a+6b=k\right] \ .$$

It will be useful to introduce the following special notations

$$\begin{aligned} g_2(\tau) &:= 60 G_2(\tau) \\ g_3(\tau) &:= 140 G_3(\tau) \\ g_1(\tau) &:= \frac{\eta'(\tau)}{\eta(\tau)} \ . \end{aligned}$$

The last function is not a modular function as it has the transformation rules

$$g_1(\tau+1) = g_1(\tau) \ , \qquad g_1\left(-\frac{1}{\tau}\right) = \tau^2 g_1(\tau) + \frac{1}{2} \ .$$

There is a natural connection ∇_τ on modular forms given in the following theorem–definition.

Theorem 1.4 Given a modular function F of weight k, ($F \in \mathbf{M}_k$), the **modular connection** ∇_τ is defined by the formula

$$(\nabla_\tau F)(\tau) := \eta^{2k}(\tau) \frac{d}{d\tau}\left(\eta^{-2k}(\tau) F(\tau)\right) \in \mathbf{M}_{k+2} \ ,$$

and maps $\mathbf{M}_k$ in $\mathbf{M}_{k+2}$.

Example 1.1 For example, one can compute the following covariant derivatives of the fundamental Eisenstein series

$$\begin{aligned} \left[\frac{\eta''}{\eta} - 3\left(\frac{\eta'}{\eta}\right)^2\right] &= \frac{5}{8\pi^2} G_2 = \frac{g_2}{96\pi^2} \\ \nabla_\tau g_2 \equiv \partial_\tau g_2 - 8\frac{\eta'}{\eta} g_2 &= \frac{3}{i\pi} g_3 \\ \nabla_\tau g_3 \equiv \partial_\tau g_3 - 12\frac{\eta'}{\eta} g_3 &= \frac{1}{6i\pi}(g_2)^2 \end{aligned}$$

where, as above, $g_2(\tau) := 60 G_2(\tau)$ and $g_3(\tau) := 140 G_3(\tau)$.

Remark 1.4 While the holomorphic modular forms are of *positive weight* and are generated by modular forms of degrees 4, 6, the Jacobi forms can be of *negative weight* and are generated by Jacobi forms of *negative weight*.

Geometric interpretation .

Jacobi forms are holomorphic sections of a certain line bundle over the quotient space $\Omega/\mathbb{J}$; care should be taken in considering the orbifold points and cusps for the action of $SL(2,\mathbb{Z})$, but here we shall be slightly cavalier over these (important) details.
We now describe the line–bundle by displaying the transition functions.

Let $\mathfrak{h}$ be the Cartan subalgebra of the lie algebra $\mathfrak{g}$ and consider the fibration (over $\mathcal{H}$) of complex crystallographic lattices $\Lambda + \tau\Lambda \subset \mathfrak{h}$.
Consider the quotient $E^l_\tau := \mathfrak{h}/(\Lambda + \tau\Lambda)$: it is isomorphic to a product of l copies of elliptic curves of modular parameter τ (consider the coordinates of $\mathfrak{h}$ induced by the fundamental roots). We regard E^l_τ naturally as a fibration of elliptic curves over $\mathcal{H}$.
Now a linear fractional transformation of τ with integer coefficients induces an isomorphism on the fibres of this fibration: namely if $\tau' = \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ then the fibres E^l_τ and $E^l_{\tau'}$ are isomorphic. The explicit isomorphism is given by

$$\begin{aligned} \Phi_{\tau,\tau'} : \mathfrak{h}/(\Lambda + \tau\Lambda) &\to \mathfrak{h}/(\Lambda + \tau'\Lambda) \\ \mathbf{x} &\mapsto \Phi_{\tau,\tau'}(\mathbf{x}) = \frac{\mathbf{x}}{c\tau + d} \ . \end{aligned}$$

The group $SL(2,\mathbb{Z})$ acts as a discrete group on the fiber bundle $\mathcal{E} := \begin{array}{c} E^l_\tau \\ \downarrow \\ \mathcal{H} \end{array}$. Over this space we consider the family of line bundles $\mathcal{L}_{k,m}$ indexed by $(k,m) \in \mathbb{Z} \times \mathbb{N}$, whose transition functions are described hereafter.
This line bundle is defined over the fibration $\mathcal{E}$ previously introduced. The open cover on which the transition functions are defined is constructed as follows; let C_e be the usual fundamental domain of the action of $PSL(2,\mathbb{Z})$ on $\mathcal{H}$, namely $C_e := \{\tau \in \mathcal{H} \ : |\tau| > 1, \ -\frac{1}{2} < \Re(\tau) < \frac{1}{2}\}$, and let $i_e := i, \rho_e := e^{i\pi/3}$ and $\rho'_e := e^{2i\pi/3}$ the orbifold points at the boundary of C_e: let us denote, for any $\gamma \in SL(2,\mathbb{Z})$ the transformed fundamental region by C_γ (where the point $i\infty$ is mapped either in itself or on some rational), and accordingly the boundary orbifold points by $i_\gamma, \rho_\gamma, \rho'_\gamma$.
Let $\mathcal{A}_{0,0}(\tau)$ be the fundamental polymesh of $\mathfrak{h}/(\Lambda + \tau\Lambda)$ namely

$$\mathcal{A}_{0,0}(\tau) := \{\mathbf{x} \in \mathfrak{h} \ : \forall j = 1..l, \ < \mathbf{x}, \mathbf{p}_j > \in \text{ fundamental mesh of } \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})\} \ ,$$

where the fundamental mesh is defined as containing the segments $[0,1)$ and $[0,\tau)$. We consequently define the translated fundamental polymeshes as $\mathcal{A}_{\lambda,\mu}(\tau) := \lambda + \tau\mu + \mathcal{A}_{0,0}(\tau)$. The trivializing charts of the line bundle $\mathcal{L}_{k,m}$ are the sets

$$U_{\gamma,\lambda,\mu} := \pi \left\{ (\mathbf{x},\tau) \in \bigcup_{\tau \in C_\gamma} \mathcal{A}_{\lambda,\mu}(\tau) \right\} \ ,$$

where $\pi : \mathfrak{h} \oplus \mathcal{H} \to (\mathfrak{h} \oplus \mathcal{H})/\mathbb{J}(\mathfrak{g})$ is the canonical projection. We define the transition functions for the line bundle $\mathcal{L}_{k,m}$ between the two charts of the orbit space corresponding to $(1) := (e,0,0)$ and $(2) := (\gamma,\lambda,\mu)$ by

$$g^{(k,m)}_{(1),(2)}(\mathbf{x},\tau) := (c\tau + d)^{-k} \exp\left(m\frac{c\|\mathbf{x}\|^2}{2(c\tau + d)} - m < \mu, x > -m\frac{\tau}{2}\|\mu\|^2 \right) \ .$$

Notice that they are tensor products of an appropriate power of the pull–back of the canonical bundle of $\mathcal{H}$ and of the line–bundle of classical theta functions over the fibre bundle $\mathcal{E}$ previously introduced.

Now one realizes that there is a one–to–one correspondence between the W–invariant sections of this bi–graded family of line bundles and the Jacobi forms of the corresponding index and weight: the correspondence is given simply taking a Jacobi form $\varphi \in J_{k,m}$ and setting $u \equiv 0$. Indeed it follows from the definitions of Jacobi forms and of the above line bundles that if $\varphi \in J_{k,m}$, then $\varphi(0, \mathbf{x}, \tau) \in \Gamma(\mathcal{L}_{k,m})$.

1.3 The generators of the algebra of Jacobi forms for $\mathbb{J}(A_l)$ and $\mathbb{J}(B_l)$

We are going to build some explicit analytic form of the generators of the algebra of Jacobi forms for the series A_l and B_l, but some preliminary formulae do apply also in the general case. Contrarily to what happens in the study of polynomial invariants for Coxeter groups, the generators we are to build will be essentially unique up to weighted *linear* transformations; this is a feature only of the Jacobi forms of type A_l and B_l.
During the construction in the A_l case it will appear a natural extension of the representation cone to a larger one, $\Omega' \supset \Omega$, and a natural vector field $\mathcal{Z}$ which will realize a recursive construction of the remaining generators starting from the lightest: this has to do with the weighting operator of Rem. 1.2. In both cases A_l, B_l, the Jacobi forms will be given by a generating function which has some notably resemblances with the generating function of the invariant polynomials.

First of all, in the algebra of Jacobi forms there can be defined two natural operators, one linear $\mathfrak{D} : J_{\bullet,\bullet} \to J_{\bullet,\bullet}$ and the other bilinear $\mathfrak{M} : J_{\bullet,\bullet} \otimes J_{\bullet,\bullet} \to J_{\bullet,\bullet}$. Before we need some definitions

Definition 1.6 *The* **intersection form** *is the covariant second–rank tensor $\mathfrak{I}$ obtained by inversion of the contravariant tensor $\mathfrak{I}^*$ given by*

$$\mathfrak{I}^* := -\partial_\tau \otimes \partial_u - \partial_u \otimes \partial_\tau + < \partial_{\mathbf{x}} \underset{,}{\otimes} \partial_{\mathbf{x}} >^* \Rightarrow \mathfrak{I} := -du \otimes d\tau - d\tau \otimes du + < d\mathbf{x} \underset{,}{\otimes} d\mathbf{x} > ;$$

its associated Laplace–Beltrami operator is denoted by

$$\Delta := -2\partial_\tau \, \partial_u + \Delta_{\mathbf{x}} \ ,$$

where $\Delta_{\mathbf{x}}$ is the Laplacian associated to the positive definite metric $<,>$ (proportional to the Killing form of $\mathfrak{h}$).

The intersection form $\mathfrak{I}$ is conformally invariant under the Jacobi group $\mathbb{J}$, namely

$$\gamma_* \mathfrak{I}^* = (c\tau + d)^2 \mathfrak{I}^* \ .$$

The laplacian Δ is not conformally invariant (contrary to what stated in [KP84] on page 190) but enjoys the property shown in next lemma.

Lemma 1.1 Let $\varphi(u, \mathbf{x}, \tau)$ be any (holomorphic) function: we have

$$\Delta\left[\tau^{\frac{-l}{2}}\varphi\left(u-\frac{\|\mathbf{x}\|^2}{2\tau}, \frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right)\right] = \tau^{-2-\frac{l}{2}}(\Delta\varphi)\left(u-\frac{\|\mathbf{x}\|^2}{2\tau}, \frac{\mathbf{x}}{\tau}, -\frac{1}{\tau}\right) .$$

Proof. It is a straightforward computation. □

This lemma can be used to define an operator on sections of the line bundles $\mathcal{L}_{k,m}$; we do this hereafter.

Definition 1.7 *Let $\varphi \in J_{-k,m}$ and $\psi \in J_{-h,n}$; then we define*

$$\begin{aligned}\mathfrak{D}(\varphi) &:= \boldsymbol{\eta}^{-2k-l}\Delta(\boldsymbol{\eta}^{2k+l}\varphi) \in J_{-k+2,m} \ ;\\ \mathfrak{M}(\varphi,\psi) &:= \boldsymbol{\eta}^{-2k-2h}\mathfrak{I}^*\left(d\left(\boldsymbol{\eta}^{2k}\varphi\right), d\left(\boldsymbol{\eta}^{2h}\psi\right)\right) \in J_{-k-h+2,m+n} \ .\end{aligned}$$

We will call the matrix elements $\mathfrak{M}(\varphi_i, \varphi_j)$, for two arbitrary generators (whose existence and properties are stated in Thm. 1.2), the **intersection elements**.

In this way we have defined two $\mathbb{C}$–linear maps $\mathfrak{D}, \mathfrak{M}$ of bi–graded modules which are both of bi–grading $(2, 0)$, i.e.

$$\begin{aligned}\mathfrak{D} &: J_{k,m} \mapsto J_{k+2,m}\\ \mathfrak{M} &: J_{k,m} \otimes_{\mathbb{C}} J_{h,n} \mapsto J_{k+h+2,m+n} \ .\end{aligned}$$

We have still to check that the definition is well posed, namely that we have not added any singularity;

Proposition 1.3 For any $\varphi \in J_{k,m}$, $\psi \in J_{h,n}$, we have $\mathfrak{D}(\varphi) \in J_{k+2,m}$ and $\mathfrak{M}(\varphi, \psi) \in J_{h+k+2,m+n}$.

Proof. We must check that the results of the application of the two operators are still invariant Jacobi forms; invariance is obvious and follows from invariance of the intersection form an of the laplacian. Also the bigradings of the resulting functions are obvious and follow from Lemma 1.1.
We must check that $\mathfrak{D}(\varphi)$ and $\mathfrak{M}(\varphi, \psi)$ are still locally bounded functions of $\mathbf{x}$ as $\Im(\tau) \to +\infty$; now the Dedekind's $\boldsymbol{\eta}$ has neither zeroes nor poles in $\mathcal{H}$ and vanishes as $q^{1/24}$ at $q = 0$ (where $q = e^{2i\pi\tau}$). Hence, $\Delta\left(\boldsymbol{\eta}^{2k}\varphi\right)$ is holomorphic and goes like $q^{\frac{k}{12}}$ as $\tau \to i\infty$, so that finally $\boldsymbol{\eta}^{-2k}\Delta\left(\boldsymbol{\eta}^{2k}\varphi\right)$ is well defined and holomorphic also as $\tau \to i\infty$: this proves that $\mathfrak{D}(\varphi)$ is still locally bounded. A similar reasoning also holds for $\mathfrak{M}$.
In passing we notice that $\mathfrak{D}(\varphi) \notin \mathbf{M}_\bullet[\varphi]$, namely it is never proportional to itself for the reason that $\mathbf{M}_2 = \{0\}$; for the same reason $\mathfrak{M}(\varphi, \psi) \notin \mathbf{M}_\bullet[\varphi\psi]$. □

The function τ and the generators of the algebra $J_{\bullet,\bullet}$, being algebraically independent, form a set of coordinates $\{\varphi_{-1} := \tau, \varphi_1, ..., \varphi_{l+1}\}$ in a neighbourhood of the generic point of the Tits cone Ω; they play a similar role to that of invariant polynomials of Coxeter groups. In the case of Coxeter invariants, the contravariant intersection form can be expressed in the local coordinates given by the invariant polynomials and we want to perform a similar computation

in the context of Jacobi groups. To this end, the operator $\mathfrak{M}$ and the intersection elements will be very important; we stress that the intersection elements are not the entries of the contravariant intersection form in the coordinates $\{\varphi_\mu\}_{\mu=-1,1,\ldots,l+1}$ nonetheless they are useful to compute them as we show now. It follows from the definition that

$$\mathfrak{I}^* \left(d(\boldsymbol{\eta}^{2k_i}\varphi_i), d(\boldsymbol{\eta}^{2k_j}\varphi_j)\right) = \boldsymbol{\eta}^{2k_j+2k_i}\mathfrak{M}(\varphi_i, \varphi_j) \ .$$

This formula shows that the intersection elements can be used to compute the intersection form.

It appears convenient to introduce a special notation which will shorten formulae; indeed, from the above formula we see that it is of advantage to consider the functions $\boldsymbol{\eta}^{2k_i}\varphi_i, \ i = 1..l{+}1$ as coordinates (indeed, they are still algebraically independent). Therefore we introduce a **hat operator** which transforms a Jacobi form of weight k into a Jacobi form of weight zero but with some multiplier system under the metaplectic group; it will be defined by

Definition 1.8 *For any Jacobi form $\varphi \in J_{k,m}$, its* **hat–form** *is defined by $\widehat{\varphi} := \boldsymbol{\eta}^{-2k}\varphi$.*

It is clear that the hatted–forms still are section of a line–bundle (since $\boldsymbol{\eta}$ never vanishes on $\mathcal{H}$) which now has *constant* transition functions.
Moreover the hat–operator preserves the index, and it satisfies the identity, $\widehat{\varphi\psi} = \widehat{\varphi}\,\widehat{\psi}$, for any $\psi, \varphi \in J_{\bullet,\bullet}$.
With this notation we can write

$$\boldsymbol{\eta}^4\widehat{\mathfrak{M}(\varphi, \psi)} = \mathfrak{I}^* \left(d(\widehat{\varphi}), d(\widehat{\psi})\right) \ .$$

This means that when we have computed the intersection elements for a set of generators, by putting hats over every Jacobi forms (including the modular forms), we obtain the *tensor* $\mathfrak{I}^*$ expressed in the coordinates $\widehat{\varphi}_j$, which are locally well–defined on the orbit space; this will shorten much of the computations we are dealing with.

Remark 1.5 The explicit form of the operator $\mathfrak{D}$ is rather simple: for any $\varphi \in J_{-k,m}$ we find

$$\mathfrak{D}\varphi = -4i\pi m(2k+l)g_1\varphi - 4i\pi m\partial_\tau\varphi + \Delta_{\mathbf{x}}\varphi \ .$$

We observe that if the lowest–degree homogeneus polynomial in the Taylor expansion (w.r.t. $\mathbf{x}$) of φ is $P(\mathbf{x})$ (possibly depending on τ), then the Taylor expansion of $\mathfrak{D}(\varphi)$ begins with $\Delta_{\mathbf{x}}(P)$.

Analogously we find for $\varphi \in J_{-k,m}$ and $\psi \in J_{-f,n}$ that

$$\mathfrak{M}(\varphi, \psi) = -2i\pi m\varphi\,\partial_\tau\psi - 2i\pi f\psi\,\partial_\tau\varphi - 4i\pi(k\,m + f\,n)g_1\varphi\,\psi + < d_{\mathbf{x}}\varphi, d_{\mathbf{x}}\psi >^* \ .$$

Again, the leading term in the Taylor expansion w.r.t. $\mathbf{x}$ is simply $< d_{\mathbf{x}}P, d_{\mathbf{x}}Q >$, if P, Q are the leading terms for φ, ψ.

Now suppose that we have the generator of $J_{\bullet,\bullet}$ of the minimal weight (the *lightest*), $\varphi_l \in J_{-k_l,m}$ whose existence is stated in Thm. 1.2: it follows from the above that $\mathfrak{D}\varphi_\tau \in J_{-k_l+2,m}$ *if not zero*; in this latter case one can prove

Proposition 1.4 [Prop. 13.3 in [Ka]] The kernel of the laplacian Δ on the space $\widetilde{Th_m}$, for $m \geq 0$ is spanned over $\mathbb{C}$ by the theta functions $\Theta_{\mathbf{p},m}$, where $\mathbf{p} \in \Lambda \bmod mP$.

Therefore if $\mathfrak{D}\varphi_j = 0$ then $\eta^{2k_j+l}\varphi_j$ is a linear combination with constant coefficients of the theta functions; this could occur because the function $\eta^{2k_j+l}\varphi_l$ transforms under the inversion $\tau \mapsto -\frac{1}{\tau}$ as a weight $l/2$ form (with some multiplier system χ which doesn't matter in this context), and all also the $\mathbb{C}$–linear combinations of theta functions transform as a (multiplet) of *weight* $l/2$) Jacobi forms (for details see [KP84]).

Since the leading order in the Taylor expansion w.r.t. $\mathbf{x}$ of a generator (actually, of any Jacobi form) is an invariant polynomial, then a sufficient condition for $\mathfrak{D}\varphi_l$ not to vanish is that its leading term $P_{k_l}(\mathbf{x})$ is not a harmonic polynomial. In the analogous context of Coxeter groups, as one can check directly, for the classical root systems (namely the series A_l, B_l, C_l, D_l), all polynomials obtained by recursive application of the invariant Laplacian $\Delta_{\mathbf{x}}$ are algebraically independent as these examples show (we will use this information later).

Example 1.2 For A_l (see [Bo] planche I), we realize the Cartan subalgebra in $\mathbb{C}^{l+1}$ with coordinates $z_1, .., z_l$ such that $\sum_{j=1}^{l+1} z_j = 0$. The coordinates $\mathbf{x}$ are chosen as $z_1 = x_1,\ z_2 = x_2 - x_1 \ldots z_{l+1} = -x_l$, or the coefficients of the vector in the root basis. The W-invariant polynomial of maximal degree $l+1$ can be choosen as

$$P_{l+1}(\mathbf{x}) := \left[\prod_{j=1}^{l+1} z_j\right]_{|\sum z_j = 0} .$$

Applying $\Delta_{\mathbf{x}}$ iteratively we obtain all the polynomials whose degree has the same parity

$$P_{l+1-2k}(\mathbf{x}) := (\Delta_{\mathbf{x}})^k P_{l+1}(\mathbf{x}) .$$

To obtain the remaining we notice that, applying the operator $\sum_{j=1}^{l+1} \frac{\partial}{\partial z_j}$ we get the invariant polynomial

$$P_l(\mathbf{x}) := \left[\sum_{j=1}^{l+1} \prod_{k \neq j} z_k\right]_{|\sum z_j = 0} ,$$

from which we recover all the remaining applying $\Delta_{\mathbf{x}}$. We remark that an alternative definition of these polynomials which is completely equivalent (up to multiplicative constants) is the following, using a generating polynomial in an auxiliary indeterminate λ, as in

$$\mathcal{P}_{\mathbf{x}}(\lambda) := \prod_{j=1}^{l+1} (\lambda + z_j) = \lambda^{l+1} + p_2(\mathbf{x})\lambda^{l-1} + .. + p_l(\mathbf{x})\lambda + p_{l+1}(\mathbf{x}) .$$

It is useful for what is coming to rewrite the same polynomial in an equivalent form which better generalizes to the case of Jacobi forms.
To do this let us remark that the Weyl group of A_l is represented as the permutation group of $l+1$ elements acting on $\mathbb{C}^{l+1}$ with coordinates $z_1, .., z_{l+1}$, restricted to $\Sigma := \{\sum z_j = 0\}$. Now there exist a unique (up to scalar) holomorphic vector field in $\mathbb{C}^{l+1}$ which is orthogonal to Σ and Weyl invariant, namely

$$\mathbf{Z} := \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j} .$$

This means that it is a derivation on the algebra of W invariant extended polynomials (namely all symmetric polynomials in the indeterminates $z_1..z_{l+1}$). We can recover the invariant polynomials for A_l as

$$\mathcal{P}_{\mathbf{x}}(\lambda) := \left[e^{\lambda \mathbf{z}} \cdot \left(\prod_{j=1}^{l+1} z_k \right) \right]_{|\sum x_j = 0}$$

Notice that

$$\frac{\partial^2}{\partial \lambda^2} \mathcal{P}_{\mathbf{x}}(\lambda) = -(l+1) \Delta_{\mathbf{x}} \mathcal{P}_{\mathbf{x}}(\lambda) \ , \tag{5}$$

which follows from the fact that $\Delta_{\mathbf{z}} = \Delta_x + (l+1)\partial_\lambda^2$: this will hold true, *mutatis mutandis*, also for the Jacobi forms.
As for B_l, C_l ([Bo], planches II,III), they are realized in $\mathbb{C}^l$ and the highest degree invariant polynomial can be chosen as

$$P_{2l}(\mathbf{x}) := \prod_{j=1}^{l} (x_j)^2 \ ,$$

and the Laplacian is just the usual one in $\mathbb{C}^l$. Then it is an easy computation to show that the invariant polynomials

$$P_{2l-2k}(\mathbf{x}) := 2^{-k} (\Delta_x)^k P_{2l}(\mathbf{x})$$

are non–zero and algebraically independent. Notice that B_l, C_l have the same invariant polynomials because their Weyl groups are isomorphic and their action actually coincides in this realization; in fact [Bo] both Weyl groups act by permutation of the x_i and independent change of signs and are of order $2^l l!$.
Again we introduce the generating polynomial in two forms as we did for A_l

$$\mathcal{P}_{\mathbf{x}}(\lambda) := \prod_{k=1}^{l} (\lambda + x_i^2) = \lambda^l + P_2(\mathbf{x}) \lambda^{l-1} + ... + P_{2l-2}(\mathbf{x}) \lambda + P_{2l}(\mathbf{x})$$

$$\mathcal{P}_{\mathbf{x}}(\lambda) \equiv e^{\frac{\lambda}{2} \Delta_{\mathbf{x}}} \cdot \left(\prod_{j=1}^{l} (x_j)^2 \right) \equiv \left(\lambda - \sum_{j=1}^{l} \frac{\partial}{\partial (x_j^2)} \right)^l \cdot \left(\prod_{j=1}^{l} (x_j)^2 \right) \ . \tag{6}$$

As for D_l we have exactly the same construction as before but starting with the highest polynomial

$$P_{2l-2}(\mathbf{x}) := \sum_{j=1}^{l} \prod_{k \neq j} (x_k)^2$$

and adding the middle degree $\widetilde{P}_l = \prod_{j=1}^{l} x_j$, which is invariant because the Weyl group of D_l acts by permutations of the x_j and by change of an even number of signs (and hence has order $2^{l-1} l!$). Notice that $\Delta_{\mathbf{x}} \widetilde{P}_l = 0$, namely it is a harmonic polynomial.

In a completely similar manner we can build the Jacobi forms starting from the lightest generator ($\leftrightarrow$ highest degree polynomial) as we see hereafter.

1.4 The root system of type A_l: fundamental Jacobi forms.

In the case of A_l, it follows from Thm. 1.2 that the fundamental Jacobi forms $\varphi_0, \varphi_2, .., \varphi_{l+1}$ belong to the spaces (we labelled the forms according to minus their weight) $J_{-l-1+k,1}$ for $k = l+1, l-1, ..0$.
If we realize the Cartan subalgebra $\mathfrak{h}$ of sl_{l+1} as in [Bo], planche I, then Wirthmüller [Wi92] found that the lightest generator (which corresponds to a maximal degree generating polynomial in the setting of the corresponding Coxeter group) is given

$$\varphi_{l+1}(u, \mathbf{x}, \tau) := e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j, \tau)_{|\sum z_i = 0} \quad \in J_{-l-1,1} \ ,$$

where the function α is defined by

$$\alpha(v, \tau) := \frac{\theta_1(v, \tau)}{\theta_1'(0, \tau)} = -\frac{1}{2\pi} \frac{\theta_1(v, \tau)}{\eta(\tau)^3} = v + O(v^2) \ ,$$

θ_1 being the Jacobi theta function. It enjoys the following property (from the properties of θ_1, see e.g. [Ba])

$$\begin{aligned}
&\partial_\tau \alpha(x, \tau) = \frac{1}{4i\pi} \partial_x^2 \alpha(x, \tau) - 3g_1 \alpha(x, \tau) \\
&\alpha(-v, \tau) = -\alpha(v, \tau) \ ; \qquad \alpha(v+1, \tau) = -\alpha(v, \tau) \ ; \qquad \alpha(v, \tau+1) = \alpha(v, \tau) \\
&\alpha(v+\tau, \tau) = -e^{-2i\pi v - i\pi\tau} \alpha(v, \tau) \ \Rightarrow \alpha(v + n\tau, \tau) = (-1)^n e^{-2i\pi n v - i\pi\tau n^2} \alpha(v, \tau) \ ; \\
&\alpha\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) = \frac{1}{\tau} e^{i\pi \frac{v^2}{\tau}} \alpha(v, \tau) \ ,
\end{aligned} \tag{7}$$

We can obtain the Jacobi forms of weights $l+1-2k$ by recursive application of $\mathfrak{D}$. To obtain the remaining we define the function

$$\varphi_l(u, \mathbf{x}, \tau) := \sum_{k=1}^{l+1} \alpha'(z_k, \tau) \prod_{j \neq k} \alpha(z_k, \tau) \ .$$

It is a straightforward exercise to show that this is a Jacobi form of weight $-l$.
Therefore we can now build the remaining Jacobi forms for A_l of weight $-l+2k$ by application of $\mathfrak{D}^k$ as in

Proposition 1.5 A system of generators for the algebra $J_{\bullet,\bullet}$ of the Jacobi group $\mathbb{J}(A_l)$ is given by

$$\begin{aligned}
\varphi_{l+1-2k} &:= \mathfrak{D}^k(\varphi_{l+1}) \\
\varphi_{l-2k} &:= \mathfrak{D}^k(\varphi_l) \ ,
\end{aligned}$$

where φ_{l+1} and φ_l have been defined above.
Any other set of generators is a weighted *linear* combination of these with coefficients in $\mathbf{M}_\bullet$.

Proof. Since we already saw that these are Jacobi forms belonging to the spaces $J_{-l-1+2k,1}$ and $J_{-l+2k,1}$ we only have to show that they are algebraically independent: but this is promptly seen by looking at the lowest term in the Taylor expansion w.r.t. $\mathbf{x}$ for

$$\begin{aligned}\varphi_{l+1-2k} &= e^{2i\pi u}\left(\Delta^k\left(P_{l+1}(\mathbf{x})\right) + O(\|\mathbf{x}\|^{l-2k})\right)\\ \varphi_{l-2k} &= e^{2i\pi u}\left(\Delta^k\left(P_l(\mathbf{x})\right) + O(\|\mathbf{x}\|^{l-2k-1})\right)\end{aligned}$$

and this suffices to show their algebraic independence.
As for the second assertion, since any other set of generators must have index $m = 1$ then it must be at most linear in these generators, and if we want that they have definite weights, the combination must be actually a weighted one. □

Remark 1.6 In the setting of Coxeter groups, the set of generators of the algebra of invariant polynomials is uniquely specified up to weighted polynomial transformations; on the other hand, here we can only perform linear transformations with coefficients in the modular forms.

We can describe these forms much more concisely with the aid of a generating function. In order to show how to build it, first of all we consider the enlarged space $(u, \mathbf{z}, \tau) := (u, z_1, ...z_{l+1}, \tau) \in \Omega' := \mathbb{C} \oplus \mathbb{C}^{l+1} \oplus \mathcal{H}$ and the following vector field

$$\mathcal{Z} := \mathbf{Z} - 4i\pi\left(\sum_{j=1}^{l+1} z_j\right) g_1(\tau)\mathbf{E}\ ,$$

where, as before, $\mathbf{Z} := \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j}$.

Lemma 1.2 The generators of the algebra $J_{\bullet,\bullet}$ are given by

$$\varphi_{l+1-k}(u, \mathbf{x}, \tau) := \left[\mathcal{Z}^k\left(e^{2i\pi u}\prod_{j=1}^{l+1}\alpha(z_j)\right)\right]_{|\sum z_j=0}.$$

Followingly, they are the coefficients of the generating function

$$\Phi_{u,\mathbf{x},\tau}(v) := \left[e^v\mathcal{Z}\left(e^{2i\pi u}\prod_{j=1}^{l+1}\alpha(z_j,\tau)\right)\right]_{|\sum z_j=0}.$$

Proof. We extend the action of the Jacobi group $\mathbb{J}$ to the enlarged space in such a way that the complex crystallographic Weyl group acts –exactly as before– by permutation of the coordinates and translation by the root lattice, while the metaplectic group acts by

$$(u, \mathbf{z}, \tau) \mapsto \left(u + \frac{c\|\mathbf{z}\|^2}{2(c\tau+d)}, \frac{\mathbf{z}}{(c\tau+d)}, \frac{a\tau+b}{c\tau+d}\right)\ ,$$

where the Killing form has been extended to the obvious $ds^2 = \sum_{j=1}^{l+1} dz_j{}^2$. We realize that the vector $\mathcal{Z}$ is conformally invariant of weight $+1$ (namely it increases by one the weight of

an extended Jacobi form); this means that if $\varphi(u,\mathbf{z},\tau)$ is an invariant function of the extended Jacobi group, then (setting $p := (l+1)\sum_1^{l+1} z_i = p(\mathbf{z})$)

$$\left(\mathbf{Z} - 4i\pi\frac{p}{l+1}g_1(\tau)\mathbf{E}\right)\left[\varphi\left(u - \frac{\|\mathbf{z}\|^2}{2\tau}, \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right)\right] =$$
$$= -\frac{2i\pi}{l+1}\frac{p}{\tau}(\mathbf{E}\varphi)(*) + \frac{1}{\tau}(\mathbf{Z}\varphi)(*) - 4i\pi\frac{p}{l+1}\left[\frac{\eta'(-1/\tau)}{\eta(-1/\tau)}\frac{1}{\tau^2} - \frac{1}{2\tau}\right](\mathbf{E}\varphi)(*) =$$
$$= \frac{1}{\tau}\left[\left(\mathbf{Z} - \frac{4i\pi}{l+1}p\left(\frac{\mathbf{z}}{\tau}\right)\frac{\eta'(-1/\tau)}{\eta(-1/\tau)}\mathbf{E}\right)\varphi\right](*) .$$

One may ask why we put a term proportional to $p = (l+1)\sum_1^{l+1} z_i$ since in the end we want to restrict to $p = 0$: the reason is that when we apply more than once the operator $\mathcal{Z}$ and do not include the term we get a non $\mathbb{J}$–covariant result as we see in this

Counterexample. Consider the push–forward under the map $\mathfrak{S}$ of the second iterate of $\mathbf{Z}$:

$$\mathbf{Z}\left(\mathbf{Z}\left(\varphi\left(u - \frac{\|\mathbf{z}\|^2}{\tau}, \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right)\right)\right) = \mathbf{Z}\left(-\frac{1}{\tau}v(l+1)(\partial_u\varphi)(*) + \frac{1}{\tau}(\mathbf{Z}\varphi)(*)\right) =$$
$$-\frac{l+1}{\tau}\partial_u\varphi(*) + \frac{v^2(l+1)^2}{\tau^2}\partial_u^2\varphi(*) - 2\frac{p(l+1)}{\tau}(\mathbf{Z}\varphi)(*) + \frac{1}{\tau^2}(\mathbf{Z}^2\varphi)(*) ,$$

where $*$ stands for the point $\left(u - \frac{\|\mathbf{z}\|^2}{\tau}, \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right)$, and we see that even restricting on $p = 0$ we get

$$\mathbf{Z}\left(\mathbf{Z}\left(\varphi\left(*\right)\right)\right) = +\frac{1}{\tau^2}(\mathbf{Z}^2\varphi)(*) - 2i\pi\frac{l+1}{\tau}\mathbf{E}\varphi(*) .$$

We see that the vector $\mathcal{Z}$ commutes with $\mathbf{E}$, hence preserves the index; we have a natural Jacobi form on the enlarged space which is simply $e^{2i\pi u}\prod_{j=1}^{l+1}\alpha(z_j)$. Since the conformal weight of $\mathcal{Z}$ is $+1$ we can write alternatively the fundamental Jacobi forms by means of the simpler formula

$$\varphi_{l+1-k}(u,\mathbf{x},\tau) := \left[\mathcal{Z}^k\left(e^{2i\pi u}\prod_{j=1}^{l+1}\alpha(z_j)\right)\right]_{|\sum z_j=0} .$$

We should check that with this definition they are not algebraically dependent, but again this is obtained by looking at the lowest order in the Taylor expansion w.r.t. $\mathbf{x}$. This proves that the functions defined in the Lemma are a set of generators.

Next, in analogy with the case of the Weyl–invariant polynomials, it is useful to introduce the generating function, which obviously has the form

$$\Phi_{u,\mathbf{x},\tau}(v) := \left[e^{v\mathcal{Z}}\prod_{j=1}^{l+1}\alpha(z_j,\tau)\right]_{|\sum z_i=0} = \varphi_{l+1} + v\varphi_l + v^2\varphi_{l-1} + \ldots + v^{l+1}\varphi_0 + \ldots ;$$

by $e^{v\mathcal{Z}}$ we mean the flow generated by $\mathbb{Z}$ on the extended cone Ω'. □

We remark that the series does not stop, but, by virtue of the structure of the algebra of Jacobi forms, the higher terms are polynomial combinations of these with coefficients in $\mathbf{M}_\bullet$. We will resum the series afterwards, while now it is useful to write down explicitly the generating function; to do this we must integrate the flow of $\mathcal{Z}$ on Ω'. We could perform this

straightforward computation in a direct way, but it is interesting to point out that $\mathcal{Z}$ is a covariantly flat vector for a suitable *flat* metric on $T(\Omega')$ (the tangent bundle). This metric extends the intersection form of $T(\Omega)$ under the natural embedding $\Omega \hookrightarrow \Omega'$ and it is worked out in the following section. After finding the flat coordinates of this extended intersection form, we will have also the integration of the flow of $\mathcal{Z}$ by shifting the coordinate along $\mathcal{Z}$. The extended intersection form will be used later to compute the intersection elements.

Extension of the intersection form .
We are to build a metric on $T(\Omega')$ which extends the intersection form on $T\Omega$; we ask the following conditions on the extended intersection form:

i) the extended metric must be *flat*;

ii) it must coincide with the previous one when restricted on the hypersurface $\sum_{j=1}^{l+1} z_j = 0$.

To this end we introduce the coordinates on $\mathbb{C}^{l+1}$ as

$$\begin{cases} z_1 = x_1 + p \\ z_2 = x_2 - x_1 + p \\ \dots. \\ z_{l+1} = -x_l + p \end{cases}$$

In a concise form we have $\mathbf{z} = \sum_{j=1}^{l} x_i \alpha_i^\vee + p[1, 1..., 1]$: in the following we will often write $z_i(\mathbf{x}, p)$ or $z_i(\mathbf{x}) := z_i(\mathbf{x}, 0)$.
In these coordinates the flat metric $\sum^{l+1} dz_i^2$ becomes

$$dl^2 =< d\mathbf{x}, d\mathbf{x} > +(l+1)dp^2 \ .$$

The most easy form for the extended metric suitable for our purposes is

$$\widetilde{\mathfrak{J}} = -du \otimes d\tau - d\tau \otimes du + dl^2 + pB(\tau)(dp \otimes d\tau + d\tau \otimes dp) + p^2 B'(\tau) d\tau^2$$

One can check directly that the curvature vanishes for any choice of the function $B(\tau)$ (whose explicit form will be fixed to our convenience later), but it is sufficient to introduce the new coordinate

$$s := u - \frac{1}{2} p^2 B(\tau)$$

and the metric becomes

$$\widetilde{\mathfrak{J}} = -ds \otimes d\tau - d\tau \otimes ds + dl^2 \ .$$

If we choose

$$B(\tau) := -\frac{2}{l+1} g_1(\tau) \ ,$$

we obtain that the previously introduced vector field $\mathcal{Z}$ in the coordinates $(s, \mathbf{x}, p, \tau)$ now reads

$$\mathcal{Z} = \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j} - \frac{2}{l+1} g_1(\tau) p \partial_u = \partial_p \ .$$

The vector field $\mathcal{T} := \partial_\tau$ of these coordinates reads, in the old ones $u, \mathbf{z}, \tau$ as

$$\mathcal{T} = -\frac{p^2}{l+1}\left(g_1(\tau)\right)' \partial_u + \partial_\tau \ , \tag{8}$$

and the extended intersection form $\widetilde{\mathfrak{J}}^*$ reads

$$\widetilde{\mathfrak{J}}^* = -\mathcal{T} \otimes \mathcal{Z} - \mathcal{Z} \otimes \mathcal{T} + \sum_{j=1}^{l+1} \frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial z_j} \ . \tag{9}$$

Now we can integrate easily the flow generated by $\mathcal{Z}$ simply shifting p *with constant* s; after these computation we can rewrite the generating function of the Jacobi forms as in

Proposition 1.6 The generating function can be written as

$$\Phi_{u,\mathbf{x},\tau}(v) := \left[e^{v\mathcal{Z}} \prod_{j=1}^{l+1} \alpha(z_j, \tau) \right]_{|\sum z_i = 0} = e^{2i\pi\left(u-(l+1)v^2 g_1(\tau)\right)} \prod_{j=1}^{l+1} \alpha\left(z_j(\mathbf{x}, v), \tau\right) \ .$$

It follows from the transformation rules of α that the generating function enjows the following properties under the Jacobi group:

$$\Phi\left(u - \frac{\|\mathbf{x}\|^2}{2\tau}, \frac{\mathbf{x}}{\tau}, v, -\frac{1}{\tau}\right) = \tau^{-l-1}\Phi(u, \mathbf{x}, \tau v, \tau)$$

$$\Phi(u, \mathbf{x}, v, \tau + 1) = \Phi(u, \mathbf{x}, v, \tau)$$

$$\Phi\left(u + < \mu, \mathbf{x} > + \frac{\tau}{2}\|\mu\|^2, \mathbf{x} + \tau\mu + \mathbf{t}, v, \tau\right) = \Phi(u, \mathbf{x}, v, \tau)$$

$$\Phi(u, \mathbf{x}, v + 1, \tau) = (-1)^{l+1}\Phi(u, \mathbf{x}, v, \tau)$$

$$\Phi(u, \mathbf{x}, v + \tau, \tau) = \left(-e^{-2i\pi v - i\pi\tau}\right)^{l+1} \Phi(u, \mathbf{x}, v, \tau) \ .$$

Remark 1.7 The function $v^{-l-1}\Phi_{u,\mathbf{x},\tau}(v)$ is an invariant function of index 1 on the total space of the line–bundle as in remark (1.2); notice that it has a pole of order $l+1$ in v as a consequence of the fact that the spectrum of the weighting operator $\mathcal{K}$ on the subspace $\mathbb{J}_{\bullet,1}$ is $-l-1, -l, ..., -2, 0, 1, ...$ It will be useful later to consider the function $\lambda(v) := \sigma^{-l-1}(v)\Phi_{u,\mathbf{x},\tau}(v) = \alpha^{-l-1}(v)e^{2i\pi u}\prod_{j=1}^{l+1}\alpha\left(z_j(\mathbf{x}, v)\right)$, which has the same singular tail (in v) but is also invariant under the complex lattice $\mathbb{Z} + \tau\mathbb{Z}$.

1.4.1 The generating function for A_l: resummation of the series.

As we saw, the first $l+1$ coefficients in the taylor expansion w.r.t. v of the extended Jacobi form

$$\Phi(u, \mathbf{x}, v, \tau) := \Phi_{u,\mathbf{x},\tau}(v) = \left[e^{v\mathcal{Z}} \left(e^{2i\pi u} \prod_{j=1}^{l+1} \alpha(z_j, \tau) \right) \right]_{|\sum z_j = 0} = e^{2i\pi u - 2i\pi v^2 (l+1)\frac{\eta'}{\eta}} \prod_{k=1}^{l+1} \alpha\left(z_i(\mathbf{x}) + v, \tau\right)$$

provide us with the desired fundamental Jacobi forms. Unlikely to the case of Coxeter invariants, this generating function is not just a polynomial in the auxiliary variable (in this case v) but a full series. It is therefore useful to resummate this series in order also to analyze its coefficients.

We realize that the generating function is a classical theta function belonging to Th_1^W for the lattice of A_l depending on the parameters v, τ. Since we know from Thm. 1.2 and 1.1, that the Jacobi forms span Th_1^W we can write the following equality

$$\Phi(s, \mathbf{x}, v, \tau) = e^{-2i\pi(l+1)v^2 g_1(\tau)} \sum_{k=0}^{l+1} C_k(v, \tau)\varphi_{l+1-k}(s, \mathbf{x}, \tau)$$

and we must find the coefficients $C_k(v, \tau)$. If we analyze the modular properties of Φ, we promptly find that the coefficients $C_k(v, \tau)$ are Jacobi forms of weight $-k$ for the lattice $\mathbb{C} \bmod 2(\mathbb{Z} + \tau\mathbb{Z})$, namely

$$\begin{aligned} C_k(v+1, \tau) &= (-1)^{l+1} C_k(v, \tau); \\ C_k(v+\tau, \tau) &= (-1)^{l+1} e^{-2i\pi v - i\pi(l+1)\tau} C_k(v, \tau); \\ C_k\left(\frac{v}{\tau}, -\frac{1}{\tau}\right) &= \tau^{-k} e^{i\pi(l+1)\frac{v^2}{\tau}} C_k(v, \tau) \ . \end{aligned}$$

The formula which describes concisely the generating function is contained in the

Theorem 1.5 Up to normalization, the generators of the algebra of Jacobi forms of type A_l already defined recursively in Prop. 1.5, are given by the generating function

$$\begin{aligned} \Phi_{u,\mathbf{x},\tau}(v) \ &:= \ e^{2i\pi u - 2i\pi(l+1)v^2 g_1} \prod_{j=1}^{l+1} \alpha(z_i(\mathbf{x}) - v) = \sigma^{l+1}(v) \sum_{j=0}^{l+1} \frac{(-1)^{l-1-k}}{(l-k)!} \wp^{(k-2)}(v)\varphi_k(u, \mathbf{x}, \tau) = \\ &= \frac{-1}{l!} \frac{\det \begin{pmatrix} 1 & \wp(v) & \wp'(v) & \dots & \wp^{(l-1)}(v) \\ 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-1)}(z_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-1)}(z_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-2)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \dots & \wp^{(l-2)}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-2)}(z_l) \end{pmatrix}} \sigma^{l+1}(v) \prod_{j=0}^{l+1} \alpha\left(z_j(\mathbf{x})\right) \ . \end{aligned} \tag{10}$$

Proof. We recall a classical formula which can be found in [Du93] pag. 199. If $\sum_1^{l+1} z_i = 0$ then

$$\frac{\prod_1^{l+1} \alpha\left(z_i - v\right)}{\alpha^{l+1}\left(v\right) \prod_1^{l+1} \alpha(z_j)} = \frac{\prod_1^{l+1} \sigma\left(z_i - v\right)}{\sigma^{l+1}\left(v\right) \prod_1^{l+1} \sigma(z_j)} = \frac{-1}{l!} \frac{\det \begin{pmatrix} 1 & \wp(v) & \wp'(v) & \dots & \wp^{(l-1)}(v) \\ 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-1)}(z_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-1)}(z_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-2)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \dots & \wp^{(l-2)}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-2)}(z_l) \end{pmatrix}} \ .$$

Expanding the determinant w.r.t. the first row of the determinant in the numerator and rearranging terms we promptly get

$$e^{2i\pi u}\prod_{1}^{l+1}\alpha\left(z_i(\mathbf{x})-v\right)=\left[\frac{(-1)^{l-1}}{l!}\alpha^{l+1}(v)\wp^{(l-1)}(v)\right]\varphi_{l+1}(\mathbf{x})+\left[\frac{(-1)^{l-2}}{(l-1)!}\alpha^{l+1}(v)\wp^{(l-2)}(v)\right]\varphi_l(\mathbf{x})+$$
$$+\ldots+\left[\alpha^{l+1}(v)\wp(v)\right]\varphi_2(\mathbf{x})+\alpha^{l+1}(v)\varphi_0(\mathbf{x})=$$
$$:=C_0(v)\varphi_{l+1}(\mathbf{x})+C_1(v)\varphi_l(\mathbf{x})+\ldots+C_{l-1}(v)\varphi_2(\mathbf{x})+C_{l+1}(v)\varphi_0(\mathbf{x}).$$
$$C_k(v):=\frac{(-1)^{l-1-k}}{(l-k)!}\alpha^{l+1}(v)\wp^{(l-1-k)}(v) \tag{11}$$

In this formula the coefficients of the fundamental Jacobi forms are chosen in such a way that the leading term in the v expansion in front of the Jacobi form φ_{l+1-k} is v^k.
In order to be sure that these φ_k are really the Jacobi forms we are looking for, we first notice that they have no singularities because the LHS in (11) is regular and the coefficients C_k are linearly independent.
Moreover since the coefficients C_k do satisfy the transformation rule

$$C_k\left(\frac{v}{\tau},-\frac{1}{\tau}\right)=\tau^{-k}C_k(v,\tau)$$

and the LHS is of weight $-l-1$ under the map $\mathbf{x}\mapsto\frac{\mathbf{x}}{\tau}$, $v\mapsto\frac{v}{\tau}$, $\tau\mapsto-\frac{1}{\tau}$, it follows by counting the weight that φ_k is of weight $-k$ as well. We now have to check that we exactly resummed the series in Prop. 1.6: first of all notice that $\wp(v)=\frac{1}{v^2}+$ regular and hence for $k=0,1,2,...,l-1,l+1$ we find

$$\wp^{(k)}(v)=(-1)^k k!v^{-k-2}+\text{regular}\Downarrow$$
$$e^{-2i\pi(l+1)v^2g_1}C_k(v,\tau)=\left\{v^{-l-1+k}+O(1)\right\}\left(v^{l+1}+O(v^{l+3})\right)=v^k+O(v^{l+1})$$

and therefore

$$\left[\partial_v^j e^{-2i\pi(l+1)p^2g_1}C_k(v,\tau)\right]_{|v=0}=\delta_{jk},\quad k,j=1,2,...,l-1,l+1.$$

This proves that we actually gave the resummation of the previous series and that the above functions φ_k are indeed the searched Jacobi forms. □

Remark 1.8 As in the case of the generating function for the invariant polynomials (eq. 5) we have for the "gauge transformed" function $\mathcal{P}(v):=e^{2i\pi(l+1)v^2g_1}\Phi(v)$ (see next section)

$$\frac{d^2}{dv^2}\mathcal{P}_{u,\mathbf{x},\tau}(v))=-(l+1)\mathfrak{D}_{u,\mathbf{x},\tau}\mathcal{P}_{u,\mathbf{x},\tau}(v)+4i\pi\,(l+1)g_1\mathcal{P}_{u,\mathbf{x},\tau}(v) \tag{12}$$

where we have put a subscript to the operator $\mathfrak{D}$ to stress on which variables it acts. This formula follows from the fact that (recall $\mathbf{z}=\mathbf{z}(\mathbf{x},v)$)

$$\widetilde{\mathfrak{D}}(\mathcal{P}(u,\mathbf{z},\tau):=\eta^{-3l-3}\widetilde{\Delta}\left(\eta^{3l+3}\mathcal{P}(v)\right)=-4i\pi\left(\partial_\tau+(3l+3)g_1+\sum_{j=1}^{l+1}\frac{\partial^2}{\partial z_j^2}\right)e^{2i\pi u}\prod_{k=1}^{l+1}\alpha(z_k)=0\ ,$$

which is a consequence of the properties of $\alpha(z,\tau)$. But now, expressing the Laplacian in the coordinates $\mathbf{x}, v$ we get

$$0 = \widetilde{\mathfrak{D}}(\mathcal{P}(u,\mathbf{z},\tau)) = \left(-4i\pi g_1 + \frac{1}{l+1}\frac{\partial^2}{\partial v^2}\right)\mathcal{P}\left(u,\mathbf{z}(\mathbf{x},v),\tau\right) + \mathfrak{D}_{u,\mathbf{x},\tau}\left(\mathcal{P}\left(u,\mathbf{z}(\mathbf{x},v),\tau\right)\right) \ ,$$

which proves eq. 12. The comparison with the formula 5 has to be made considering the limit (genus 0 limit)

$$\lim_{\epsilon\to 0}\epsilon^{-l+1}(\text{Eq.}12)_{\epsilon u,\epsilon\mathbf{x},\tau}(\epsilon v) = (\text{Eq. } 5) \ .$$

The formula 12 can be written in a very concise form; recalling that $\mathcal{P}(v) = \sum_{j=0}^{l+1} C_{l+1-k}(v,\tau)\varphi_k(u,\mathbf{x},\tau)$, we have

$$\widetilde{\mathfrak{D}}\mathcal{P} = \sum_{j=0}^{l+1}\left[\mathfrak{D}(C_{l+1-k})\,\varphi_k + C_{l+1-k}\,\mathfrak{D}(\varphi_k)\right] \ ,$$

where we have defined

$$\mathfrak{D}(C_k) := -4i\pi\,\eta^{-2k-1}\partial_\tau\left(\eta^{2k+1}C_k\right) + \frac{1}{l+1}\partial_v^2 C_k \ .$$

1.4.2 Computation of the intersection form with the generating function

In this paragraph we compute the generating function of the matrix elements of the intersection form in terms of our fundamental Jacobi forms. This computation is the translation in the present setting of the analog formula by Saito, Yano and Sekiguchi for the polynomial invariants in [SYS80].
To simplify the computational steps we will consider the functions

$$\mathcal{P}(u,z_1...z_{l+1}) := e^{2i\pi u}\prod_1^{l+1}\alpha(z_i) = e^{2i\pi(l+1)v^2 g_1(\tau)}\Phi(v)$$
$$\lambda(v) := \varphi_0 + \wp(v)\varphi_2 - \frac{1}{2}\wp'(v)\varphi_3 + ... + \frac{(-1)^{l-1}}{l!}\wp^{(l-1)}(v)\varphi_{l+1} = \alpha^{-l-1}(v)\mathcal{P}(v) \ , \quad (13)$$

with no particular relation between the z_i's. The first one is clearly related to the previous generating function Φ by a *gauge* transformation

$$\mathcal{P}\left(u,\mathbf{z}(\mathbf{x},v),\tau\right) := e^{2i\pi(l+1)v^2\frac{\eta'}{\eta}}\Phi(u,\mathbf{x},v,\tau) = e^{2i\pi u}\prod_{i=1}^{l+1}\alpha\left(z_i(\mathbf{x},v)\right) \ .$$

We now prove the

Theorem 1.6 The intersection elements $\mathfrak{M}(\varphi_k,\varphi_j)$ are recovered from the generating function

$$\sum_{k,j=0}^{l+1}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\wp^{(k-2)}(v)\wp^{(j-2)}(v')\mathfrak{M}(\varphi_k,\varphi_j) =$$
$$= 2i\pi\left(\mathbb{D}^\bullet\lambda(v)\lambda(v') + \lambda(v)\mathbb{D}^\bullet\lambda(v')\right) - \frac{1}{l+1}\lambda'(v)\lambda'(v') +$$

$$+\frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')}\left(\lambda(v)\frac{d}{dv'}\lambda(v')-\lambda(v')\frac{d}{dv}\lambda(v)\right) , \tag{14}$$

where $\mathbb{D}^{\bullet}_{\tau}\lambda(v) := \sum_{j=0}^{l+1}\frac{(-1)^{j-1}}{(j-1)!}\mathbb{D}(\wp^{(j-2)})(v)\varphi_j$ (see appendix for the definition of $\mathbb{D}$).

In practice, to find the coefficients $\mathfrak{M}(\varphi_k,\varphi_j)$ we have to multiply both sides by $\wp(v)-\wp(v')$ and compare the Laurent expansions w.r.t. v, v' (see example for A_2 later).

The proof of this short formula involves many steps and lemmata.

Lemma 1.3 For the extended intersection operator $\widetilde{\mathfrak{M}}$ we have

$$\begin{aligned} &e^{2i\pi(l+1)(v^2+v'^2)g_1(\tau)}\widetilde{\mathfrak{M}}(\Phi(v),\Phi(v')) = \\ &= 2i\pi(l+1)\frac{\nabla_\tau\alpha(v-v')}{\alpha(v-v')}\mathcal{P}(v)\mathcal{P}(v') + \frac{\alpha'(v-v')}{\alpha(v-v')}\left\{\mathcal{P}(v)\frac{d}{dv'}\mathcal{P}(v') - \mathcal{P}(v')\frac{d}{dv}\mathcal{P}(v)\right\} . \end{aligned} \tag{15}$$

Proof. The extended intersection form is given by (see formula (9))

$$\widetilde{\mathfrak{I}}^* = -\mathcal{T}\otimes\mathcal{Z} - \mathcal{Z}\otimes\mathcal{T} + \sum_{j=1}^{l+1}\frac{\partial}{\partial z_j}\otimes\frac{\partial}{\partial z_j} ,$$

and the extended intersection operator $\widetilde{\mathfrak{M}}$ is defined accordingly as

$$\widetilde{\mathfrak{M}}(\Phi(v),\Phi(v')) = \eta^{-4(l+1)}\mathfrak{I}^*\left(d\left(\eta^{2l+2}\Phi(v)\right), d\left(\eta^{2l+2}\Phi(v')\right)\right) .$$

Since the "gauge–transformed" vectors $\mathcal{Z}$ and $\mathcal{T}$ (the latter being defined in (8)) are

$$\begin{aligned} e^{+2i\pi(l+1)v^2g_1}\mathcal{Z}e^{-2i\pi(l+1)v^2g_1} &= \sum_{1}^{l+1}\frac{\partial}{\partial z_i} = \partial_v \\ e^{+2i\pi(l+1)v^2g_1}\mathcal{T}e^{-2i\pi(l+1)v^2\frac{\eta'}{\eta}} &= \partial_\tau \end{aligned}$$

we have that for the function $\mathcal{P}(v) := \mathcal{P}\left(u, \mathbf{z}(\mathbf{x}, p+v), \tau\right) := \mathcal{P}_{u,\mathbf{z},\tau}(v)$ (where we consider the z_j unconstrained, and v a parameter of deformation) the *gauge transformed* intersection element reads

$$e^{2i\pi(l+1)(v^2+v'^2)g_1(\tau)}\widetilde{\mathfrak{M}}\left(\Phi(v),\Phi(v')\right) = -\partial_u\mathcal{P}(v)\nabla_\tau\mathcal{P}(v') - \partial_u\mathcal{P}(v')\nabla_\tau\mathcal{P}(v) + \sum_{j=1}^{l+1}\frac{\partial\mathcal{P}(v)}{\partial z_j}\frac{\partial\mathcal{P}(v')}{\partial z_j} .$$

We now compute

$$\begin{aligned} \text{lhs of (15)} &= -2i\pi\eta^{-2l-2}\left[\mathcal{P}(v)\partial_\tau\left(\eta^{2l+2}\mathcal{P}(v')\right)+\mathcal{P}(v')\partial_\tau\left(\eta^{2l+2}\mathcal{P}(v)\right)\right]+\sum_{1}^{l+1}\frac{\partial}{\partial z_i}\mathcal{P}(v)\frac{\partial}{\partial z_i}\mathcal{P}(v') = \\ &= -\frac{1}{2}\sum_{1}^{l+1}\left(\frac{\alpha''(z_i)}{\alpha(z_i)}+\frac{\alpha''(w_i)}{\alpha(w_i)}-2\frac{\alpha'(z_i)\alpha'(w_i)}{\alpha(z_i)\alpha(w_i)}\right)\mathcal{P}(v)\mathcal{P}(v') + 4i\pi(l+1)g_1\mathcal{P}(v)\mathcal{P}(v') , \end{aligned}$$

where $z_i := z_i(\mathbf{x}, p+v)$ and $w_i := z_i(\mathbf{x}, p+v')$. We now must analyze the terms in round brackets: using the formulae in Prop. A.2, the intersection form becomes (recall that $z_i - w_i = v - v'$, $\forall\, i = 1..l+1$)

$$\begin{aligned}\text{lhs of (15)} &= 2i\pi(l+1)\left[\frac{\partial_\tau \alpha\,(v-v')}{\alpha\,(v-v')} + 2g_1\right]\mathcal{P}(v)\mathcal{P}(v') - \frac{\alpha'\,(v-v')}{\alpha\,(v-v')}\left\{\mathcal{P}(v')\frac{d}{dv}\mathcal{P}(v) - \mathcal{P}(v)\frac{d}{dv'}\mathcal{P}(v')\right\} = \\ &= 2i\pi(l+1)\frac{\nabla_\tau \alpha\,(v-v')}{\alpha\,(v-v')}\mathcal{P}(v)\mathcal{P}(v') + \frac{\alpha'\,(v-v')}{\alpha\,(v-v')}\left\{\mathcal{P}(v)\frac{d}{dv'}\mathcal{P}(v') - \mathcal{P}(v')\frac{d}{dv}\mathcal{P}(v)\right\}.\end{aligned}$$

This ends the proof of lemma. □

Although very concise, this formula gives the intersection elements of the extended intersection operator $\widetilde{\mathfrak{M}}$: this is insufficient to our purposes since we want to analyze the intersection operator $\mathfrak{M}$.
In order to find the intersection elements $\mathfrak{M}(\varphi_j, \varphi_k)$, we have the further lemma.

Lemma 1.4 For the intersection elements of $\mathfrak{M}$ we have the formula

$$\begin{aligned}&\sum_{k,j} C_k(v)C_j(v')\mathfrak{M}(\varphi_{l+1-k}, \varphi_{l+1-j}) = \\ &= 2i\pi(l+1)\frac{\nabla_\tau \alpha(v-v')}{\alpha(v-v')}\mathcal{P}(v)\mathcal{P}(v') + \frac{\alpha'(v-v')}{\alpha(v-v')}\left\{\mathcal{P}(v)\frac{d}{dv'}\mathcal{P}(v') - \mathcal{P}(v')\frac{d}{dv}\mathcal{P}(v)\right\} + \\ &- \sum_{k,j}\mathfrak{M}(C_k(v), C_j(v'))\varphi_{l+1-k}\varphi_{l+1-j}\ ,\ ,\end{aligned} \tag{16}$$

where $C_l(v)$ are defined in (11) and we have set for short

$$\mathfrak{M}(C_k(v), C_j(v')) := -2i\pi\left(C_k(v)\boldsymbol{\eta}^{-2j}\partial_\tau\left(\boldsymbol{\eta}^{2j}C_j(v')\right) + C_j(v')\boldsymbol{\eta}^{-2k}\partial_\tau\left(\boldsymbol{\eta}^{2k}C_k(v)\right)\right) + \frac{1}{l+1}C_k'(v)C_j'(v')\,.$$

Proof. From the definitions, by a straightforward rearrangement of terms, we have

$$\begin{aligned}e^{2i\pi(l+1)(v^2+v'^2)g_1}\widetilde{\mathfrak{M}}(\Phi(v), \Phi(v')) &= \sum_{k,j} C_k(v)C_j(v')\mathfrak{M}(\varphi_{l+1-k}, \varphi_{l+1-j}) + \\ &+ \sum_{k,j}\mathfrak{M}(C_k(v), C_j(v'))\varphi_{l+1-k}\varphi_{l+1-j}\ ,\end{aligned} \tag{17}$$

where we have set for short (as in the statement of lemma)

$$\begin{aligned}\mathfrak{M}(C_k(v), C_j(v')) &:= -2i\pi\left(C_k(v)\boldsymbol{\eta}^{-2j}\partial_\tau\left(\boldsymbol{\eta}^{2j}C_j(v')\right) + C_j(v')\boldsymbol{\eta}^{-2k}\partial_\tau\left(\boldsymbol{\eta}^{2k}C_k(v)\right)\right) + \\ &+ \frac{1}{l+1}C_k'(v)C_j'(v')\,.\end{aligned}$$

We can now recast eq. (17) using eq. (15) into the formula in the statement of the lemma, which is now proved. □

Proof of Theorem 1.6 We use the function $\lambda(v)$ defined here below

$$\lambda(v) := \varphi_0 + \wp(v)\varphi_2 - \frac{1}{2}\wp'(v)\varphi_3 + \ldots + \frac{(-1)^{l-1}}{l!}\wp^{(l-1)}(v)\varphi_{l+1} = \alpha^{-l-1}(v)\mathcal{P}(v)\ ,$$

$$\mathcal{P}'(v) = \alpha^{l+1}(v)\left[\lambda'(v) + (l+1)\frac{\alpha'(v)}{\alpha(v)}\lambda(v)\right] .$$

Rewriting the formula in Lemma 1.3 for this $\lambda(v)$, we have

$$\begin{aligned}
&\frac{e^{2i\pi(l+1)(v^2+v'^2)g_1}\widetilde{\mathfrak{M}}(\Phi(v),\Phi(v'))}{\alpha^{l+1}(v)\alpha^{l+1}(v')} = \\
&= (l+1)\underbrace{\left\{2i\pi\frac{\nabla_\tau\alpha(v-v')}{\alpha(v-v')} - \frac{\alpha'(v-v')}{\alpha(v-v')}\left[\frac{\alpha'(v)}{\alpha(v)} - \frac{\alpha'(v')}{\alpha(v')}\right]\right\}}_{:=Q}\lambda(v)\lambda(v') + \\
&+\frac{\alpha'(v-v')}{\alpha(v-v')}\left(\lambda(v)\frac{d}{dv'}\lambda(v') - \lambda(v')\frac{d}{dv}\lambda(v)\right) .
\end{aligned} \tag{18}$$

On the other hand we can compute[2], recalling the definition of the modular connection $\nabla^\bullet_\tau$ and the elliptic connection $\mathbb{D}^\bullet$ defined in Appendix, Prop. 33,

$$\begin{aligned}
&\sum_{k,j}\frac{\mathfrak{M}(C_k(v),C_j(v'))}{\alpha^{l+1}(v)\alpha^{l+1}(v')}\varphi_{l+1-k}\varphi_{l+1-j} = \\
&= -2i\pi\left(\nabla^\bullet_\tau\lambda(v)\lambda(v') + \lambda(v)\nabla^\bullet_\tau\lambda(v')\right) - 2i\pi(l+1)\left[4g_1(\tau) + \frac{\partial_\tau\alpha(v)}{\alpha(v)} + \frac{\partial_\tau\alpha(v')}{\alpha(v')}\right] + \\
&+\frac{1}{l+1}\left[\lambda'(v) + (l+1)\frac{\alpha(v)}{\alpha(v)}\lambda(v)\right]\left[\lambda'(v') + (l+1)\frac{\alpha(v')}{\alpha(v')}\lambda(v')\right] = \\
&= -2i\pi\left(\nabla^\bullet_\tau\lambda(v)\lambda(v') + \lambda(v)\nabla^\bullet_\tau\lambda(v')\right) + \\
&-(l+1)\overbrace{\left[8i\pi g_1(\tau) + 2i\pi\left(\frac{\partial_\tau\alpha(v)}{\alpha(v)} + \frac{\partial_\tau\alpha(v')}{\alpha(v')}\right) - \frac{\alpha'(v')\alpha'(v)}{\alpha(v')\alpha(v)}\right]}^{=-Q \text{ by means of eq. (34)}}\lambda(v)\lambda(v') + \\
&+\frac{1}{l+1}\lambda'(v)\lambda'(v') + \frac{\alpha'(v)}{\alpha(v)}\lambda(v)\lambda'(v') + \frac{\alpha'(v')}{\alpha(v')}\lambda(v')\lambda'(v) = \\
&= -2i\pi\left(\nabla^\bullet_\tau\lambda(v)\lambda(v') + \lambda(v)\nabla^\bullet_\tau\lambda(v')\right) + (l+1)Q\lambda(v)\lambda(v') + \frac{1}{l+1}\lambda'(v)\lambda'(v') + \\
&+\frac{\alpha'(v)}{\alpha(v)}\lambda(v)\lambda'(v') + \frac{\alpha'(v')}{\alpha(v')}\lambda(v')\lambda'(v) = \\
&= -2i\pi\left[\lambda(v)\mathbb{D}^\bullet\lambda(v') + \lambda(v')\mathbb{D}^\bullet\lambda(v)\right] + (l+1)Q\,\lambda(v)\lambda(v') + \frac{1}{l+1}\lambda'(v)\lambda'(v') +
\end{aligned}$$

[2]We shall use the notation $\nabla^\bullet_\tau\lambda(v)$ and $\mathbb{D}^\bullet\lambda(v)$ understanding them as

$$\nabla^\bullet_\tau\lambda(v) := \sum_{j=0}^{l+1}\frac{(-1)^{j-1}}{(j-1)!}\eta^{2j}\partial_\tau\left(\eta^{-2j}\wp^{(j-2)}(v)\right)\varphi_j ;$$

$$\mathbb{D}^\bullet_\tau\lambda(v) := \sum_{j=0}^{l+1}\frac{(-1)^{j-1}}{(j-1)!}\mathbb{D}(\wp^{(j-2)})(v)\varphi_j ,$$

namely the modular and elliptic connections are supposed to operate only on the functions depending on v.

$$+\left[\frac{\alpha'(v)}{\alpha(v)}-\frac{\alpha'(v')}{\alpha(v')}\right]\left(\lambda(v)\frac{d}{dv'}\lambda(v')-\lambda(v')\frac{d}{dv}\lambda(v)\right) . \tag{19}$$

Therefore we finally find (for compactness we define $\wp^{(-2)}(v) := 1$ and $\wp^{(-1)}(v) := 0$)

$$\sum_{k,j=0}^{l+1}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\wp^{(k-2)}(v)\wp^{(j-2)}(v')\mathfrak{M}(\varphi_k,\varphi_j) = (18)-(19) =$$

$$= 2i\pi\left(\mathbb{D}^{\bullet}\lambda(v)\lambda(v')+\lambda(v)\mathbb{D}^{\bullet}\lambda(v')\right)+$$

$$+\left[\underbrace{\frac{\alpha'(v-v')}{\alpha(v-v')}-\frac{\alpha'(v)}{\alpha(v)}+\frac{\alpha'(v')}{\alpha(v')}}_{=\zeta(v-v')+\zeta(v')-\zeta(v)}\right]\left(\lambda(v)\frac{d}{dv'}\lambda(v')-\lambda(v')\frac{d}{dv}\lambda(v)\right)-\frac{1}{l+1}\lambda'(v)\lambda'(v') \tag{20}$$

From the classical pseudo–addition formula [Ba]

$$\zeta(v-v')+\zeta(v')-\zeta(v) = \frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')} ,$$

we can finally prove the theorem. □

Notice the resemblance of eq. 15 with the formula worked out in [SYS80] by Saito, Yano and Sekiguchi in the case of the finite Weyl group A_l: in that case they had $\mathcal{P}(v) = \prod_1^{l+1}(z_i - v)$ and the formula was

$$\sum_{i=1}^{l+1}\frac{\partial}{\partial z_i}\mathcal{P}(v)\frac{\partial}{\partial z_i}\mathcal{P}(v') = \frac{1}{v'-v}\left\{\mathcal{P}(v')\frac{d}{dv}\mathcal{P}(v)-\mathcal{P}(v)\frac{d}{dv'}\mathcal{P}(v')\right\} . \tag{21}$$

We can realize that eq. 15 is an elliptic deformation of eq. 21.
Indeed, under a suitable limit the former formula goes into the latter

$$\lim_{\epsilon\to 0}\epsilon^{-2l}\left[e^{2i\pi(l+1)(v^2+v'^2)g_1}\widetilde{\mathfrak{M}}\left(\Phi(v),\Phi(v')\right)\right]_{(u,\mathbf{z},v,v')\mapsto(\epsilon u,\epsilon\mathbf{z},\epsilon v,\epsilon v')} = \text{RHS of eq. 21} .$$

Example 1.3 The case of $\mathbf{A_2}$ In this example we compute explicitly the intersection form in the coordinates $\varphi_{-1} = \tau$, $\widehat{\varphi_0} = \varphi_0$, $\widehat{\varphi_2} = \eta^4\varphi_2$, $\widehat{\varphi_3} = \eta^6\varphi_3$; this result will be useful in the computations for G_2.
Using the generating function in Thm. 1.5 one can compute explicitly the Jacobi forms to be (recall that we have $z_1 = x_1;\ z_2 = x_2 - x_1;\ z_3 = -x_2$)

$$\varphi_3(u,\mathbf{x}) = -2e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3);$$
$$\varphi_2(u,\mathbf{x}) = e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)} = -2e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\left[\zeta(z_1)+\zeta(z_2)+\zeta(z_3)\right]$$
$$\varphi_0(u,\mathbf{x}) = e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\frac{\wp(z_1)\wp'(z_2)-\wp'(z_1)\wp(z_2)}{\wp(z_2)-\wp(z_1)} = e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\cdot$$
$$\cdot\left\{\frac{1}{3}\left[\wp'(z_1)+\wp'(z_2)+\wp'(z_3)\right]+\frac{2}{3}\left[\zeta(z_1)+\zeta(z_2)+\zeta(z_3)\right]\left(\wp(z_1)+\wp(z_2)+\wp(z_3)\right)\right\} ,$$

where we have used some classical formulae in dealing with the $\wp$ functions [WW].
We now compute the elements $\mathfrak{M}_{i,j} := \mathfrak{M}(\varphi_i, \varphi_j)$ since we are going to use them for G_2 later. Using formula (14) and setting $\varphi_{-1} := \tau$ we find

$$\mathfrak{M}(\varphi_i, \varphi_j) = \begin{pmatrix} 0 & -2i\pi\varphi_0 & -2i\pi\varphi_2 & -2i\pi\varphi_3 \\ -2i\pi\varphi_0 & \frac{1}{24}g_2{}^2\varphi_3{}^2 - \frac{1}{6}g_2\varphi_0\varphi_2 - \frac{1}{2}g_3\varphi_2{}^2 & \frac{3}{4}g_3\varphi_3{}^2 - \frac{1}{3}g_2\varphi_2{}^2 & -\frac{5}{12}g_2\varphi_2\varphi_3 \\ -2i\pi\varphi_2 & \frac{3}{4}g_3\varphi_3{}^2 - \frac{1}{3}g_2\varphi_2{}^2 & \frac{1}{2}g_2(\varphi_3)^2 - 2\varphi_0\varphi_2 & -3\varphi_3\varphi_0 \\ -2i\pi\varphi_3 & -\frac{5}{12}g_2\varphi_3\varphi_2 & -3\varphi_3\varphi_0 & \frac{2}{3}(\varphi_2)^2 \end{pmatrix} \tag{22}$$

1.5 Jacobi forms of type G_2. Saito's flat invariants.

The Cartan subalgebra for G_2 is realized in [Bo], planche IX as the subspace of $\mathbb{C}^3$ such that $z_1 + z_2 + z_3 = 0$; the root lattice is the same as the one of A_2 and the Weyl group is the dihedral group of order 12. It can be seen that $W(G_2)/W(A_2) \simeq \mathbb{Z}_2$ and it is generated by the involution

$$\mathfrak{S} : (z_1, z_2, z_3) \mapsto (-z_3, -z_2, -z_1) \ .$$

The Jacobi forms we have to build are

$$\varphi_0 \in J_{0,1}; \qquad \varphi_2 \in J_{-2,1}; \qquad \varphi_6 \in J_{-6,2} \ .$$

They can be built directly as follows:

1. As for φ_6 we can take

$$\varphi_6(u, \mathbf{x}, \tau) := 2e^{4i\pi u} \prod_{j=1}^{3} \alpha^2(z_j, \tau) = \frac{1}{2}\left(\varphi_3^{(A_2)}\right)^2 \quad \in J_{-6,2}$$

   namely te square of the lightest Jacobi form for A_2, which is clearly invariant under the involution $\mathfrak{S}$ defined above.

2. As for φ_2, φ_0 one can check that the same Jacobi forms which work for A_2 do work for this case (we only have to check invariance under the extra involution $\mathfrak{S}$) in fact we have for $z_1 + z_2 + z_3 = 0$,

$$\begin{aligned} \varphi_2(u, \mathbf{x}) &= e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} = \\ &= -2e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\left[\zeta(z_1) + \zeta(z_2) + \zeta(z_3)\right] \\ \varphi_0(u, \mathbf{x}) &= e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\frac{\wp(z_1)\wp'(z_2) - \wp'(z_1)\wp(z_2)}{\wp(z_2) - \wp(z_1)} = \\ &= e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\left\{\frac{1}{3}\left[\wp'(z_1) + \wp'(z_2) + \wp'(z_3)\right] + \right. \\ &\quad \left. + \frac{2}{3}\left[\zeta(z_1) + \zeta(z_2) + \zeta(z_3)\right]\left(\wp(z_1) + \wp(z_2) + \wp(z_3)\right)\right\} \ , \end{aligned}$$

   and since both are product of two antiinvariant functions, they are invariant under $\mathfrak{S}$.

This identification allows us to compute the Jacobi invariant intersection form $\mathfrak{M}$ since it is the same as for A_2 being the Weyl invariant inner product the same $[(dz_1)^2+(dz_2)^2+(dz_3)^2]_{|z_1+z_2+z_3=0}$ in both cases. Therefore we can easily compute the intersection elements $\mathfrak{M}(\varphi_j,\varphi_k)$ out of those computed for A_2 in eq. (22) and find the following expression for the intersection form in the coordinates $\tau =: \widehat{\varphi_{-1}}, \widehat{\varphi_0}, \widehat{\varphi_2}, \widehat{\varphi_6}$ (where the hat on the Jacobi forms are given in Def. 1.8)

$$\mathfrak{I}^*(d(\widehat{\varphi_i}), d(\widehat{\varphi_j})) =$$
$$= \eta^4 \begin{pmatrix} 0 & -2i\pi\widehat{\varphi_0}\eta^{-4} & -2i\pi\widehat{\varphi_2}\eta^{-4} & -4i\pi\widehat{\varphi_6}\eta^{-4} \\ -2i\pi\widehat{\varphi_0}\eta^{-4} & \frac{1}{12}\widehat{g_2}^2\widehat{\varphi_6}-\frac{1}{6}\widehat{g_2}\widehat{\varphi_0}\widehat{\varphi_2}-\frac{1}{2}\widehat{g_3}\widehat{\varphi_2}^2 & \frac{3}{2}\widehat{g_3}\widehat{\varphi_6}-\frac{1}{3}\widehat{g_2}\widehat{\varphi_2}^2 & -\frac{5}{6}\widehat{g_2}\widehat{\varphi_6}\widehat{\varphi_2} \\ -2i\pi\widehat{\varphi_2}\eta^{-4} & \frac{3}{2}\widehat{g_3}\widehat{\varphi_6}-\frac{1}{3}\widehat{g_2}\widehat{\varphi_2}^2 & \widehat{g_2}\widehat{\varphi_6}-2\widehat{\varphi_0}\widehat{\varphi_2} & -6\widehat{\varphi_6}\widehat{\varphi_0} \\ -4i\pi\widehat{\varphi_6}\eta^{-4} & -\frac{5}{6}\widehat{g_2}\widehat{\varphi_6}\widehat{\varphi_2} & -6\widehat{\varphi_6}\widehat{\varphi_0} & \frac{4}{3}\widehat{\varphi_6}\widehat{\varphi_2}^2 \end{pmatrix} \tag{23}$$

Notice that the matrix is linear in the generator φ_6: this is obvious when one counts the indices of the matrix elements and it is connected with the fact that G_2 is a "codimension one" case in the sense of [Sa90].
The matrix $\frac{\partial}{\partial\widehat{\varphi_6}}\mathfrak{I}^*$ (which is the tensor J^* in Saito's notation) reads

$$\frac{\partial}{\partial\widehat{\varphi_6}}\mathfrak{I}^* = \eta^4 \begin{pmatrix} 0 & 0 & 0 & -4i\pi\eta^{-4} \\ 0 & \frac{1}{12}\widehat{g_2}^2 & \frac{3}{2}\widehat{g_3} & -\frac{5}{6}\widehat{g_2}\widehat{\varphi_2} \\ 0 & \frac{3}{2}\widehat{g_3} & \widehat{g_2} & -6\widehat{\varphi_0} \\ -4i\pi\eta^{-4} & -\frac{5}{6}\widehat{g_2}\widehat{\varphi_2} & -6\widehat{\varphi_0} & \frac{4}{3}\widehat{\varphi_2}^2 \end{pmatrix} .$$

It follows from Saito's paper [Sa90] that this tensor defines a contravariant metric whose covariant form is flat; one could now try and look for the flat coordinates of this metric (the "flat theta invariants").
We are in a position to give the flat coordinates associated to this second flat metric.
In order to find them we must integrate the geodesic equations; it is a rather muscolar though non completely straightforward exercise for which it is very helpful the paper [Sat93]. Hereafter we report only the result.

Proposition 1.7 The flat coordinates of Saito's metric are τ, t_1, t_2, t_6, given in the following system

$$\begin{aligned} \widehat{\varphi_0} &= (2\pi)^2 2^{-2/3}\frac{i\pi}{\eta^2}\Big(F_1'(\tau)t_1 + F_2'(\tau)t_2\Big) \\ \widehat{\varphi_2} &= (2\pi)^2 2^{-2/3}\,\eta^2\Big(F_1(\tau)t_1 + F_2(\tau)t_2\Big) \\ \widehat{\varphi_6} &= t_6 - 4i\pi g_1 t_1\, t_2 + \frac{3^2}{2^{11}\pi^{12}}\widehat{g_3}\widehat{\varphi_0}^2 - \frac{1}{2^{11}\pi^{12}}\widehat{g_2}^2\widehat{\varphi_0}\widehat{\varphi_2} + \frac{3}{2^{13}\pi^{12}}\widehat{g_2}\widehat{g_3}\widehat{\varphi_2}^2 , \end{aligned}$$

where the two functions F_1, F_2 are given, in terms of the modular invariant

$$z(\tau) := \frac{1}{2}\left[\frac{\sqrt{-27}}{(2\pi)^6}\widehat{g_3} + 1\right] \quad ; \qquad \frac{dz}{d\tau} = \frac{\sqrt{-27}}{2(2\pi)^6}\frac{\eta^4}{6i\pi}\widehat{g_2}^2 ,$$

by,

$$F_1(\tau) = F_1(z(\tau)) = z(\tau)^{\frac{1}{3}} \; ; \qquad F_2(\tau) = F_2(z(\tau)) = (z(\tau) - 1)^{\frac{1}{3}} \; .$$

In these coordinates Saito's metric becomes,

$$J^*(dt_i, dt_j) = \begin{pmatrix} 0 & 0 & 0 & -4i\pi \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ -4i\pi & 0 & 0 & 0 \end{pmatrix} . \tag{24}$$

The flat coordinates show up a nontrivial monodromy around $z(\tau) = 0, 1$; these points both correspond to the elliptic curves where $g_2 = 0$, namely the symmetric tori.

1.6 The root system of type B_l: Jacobi forms

The Jacobi forms for $\mathbb{J}(B_l)$ can be constructed in a similar but easier way as those for $\mathbb{J}(A_l)$.

Similar computations leading to the generating function for A_l, took us to the following

Theorem 1.7 The generating function

$$\begin{aligned}
\mathcal{P}(v) &:= e^{2i\pi u} \prod_{i=1}^{l} \alpha(v - x_i)\alpha(v + x_i) := \\
&= \frac{\wp^{(2l-2)}(v)}{(2l-1)!}\alpha^{2l}(v)\varphi_{2l}(\mathbf{x}) + \frac{\wp^{(2l-4)}(v)}{(2l-3)!}\alpha^{2l}(v)\varphi_{2l-2}(\mathbf{x}) + \ldots + \alpha^{2l}(v)\varphi_0(\mathbf{x}) = \\
&= \frac{\det \begin{pmatrix} 1 & \ldots & \wp^{(2l-2)}(v) \\ \vdots & \vdots & \vdots \\ 1 & \ldots & \wp^{(2l-2)}(x_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \ldots & \wp^{(2l-4)}(x_1) \\ \vdots & \vdots & \vdots \\ 1 & \ldots & \wp^{(2l-4)}(x_l) \end{pmatrix}} \alpha^{2l}(v) \prod_{j=1}^{l} \alpha^2(x_j) = \\
&= \varphi_{2l}(\mathbf{x}) + v^2\varphi_{2l-2}(\mathbf{x}) + \ldots + v^{2l}\varphi_0(\mathbf{x}) + O(v^{2l+2}) \; ,
\end{aligned} \tag{25}$$

gives a basis of generators for the algebra of Jacobi forms of type B_l.

Any other set of generators is a weighted *linear* combination of these with coefficients in $\mathbf{M}_\bullet$.

Proof. We must show first of all, that the functions defined by the formula have all the properties of smoothness. This follows from the fact that the ratio of determinants is a holomorphic function of all its variables without poles.
In order to show the properties of invariance, consider now the elliptic function

$$F(v) := \det \begin{pmatrix} 1 & \wp(v) & \wp''(v) & \ldots & \wp^{(2l-2)}(v) \\ 1 & \wp(x_1) & \wp''(x_1) & \ldots & \wp^{(2l-2)}(x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(x_l) & \wp''(x_l) & \ldots & \wp^{(2l-2)}(x_l) \end{pmatrix} .$$

Since only even derivatives of the $\wp$ functions are involved, this is clearly invariant under the change of sign of any between $v, x_1, ..x_l$. As a function of v it has a pole at $v = 0$ of order $2l$ and hence it has as many zeroes; it is clear that these are situated at $\pm x_1, ..., \pm x_l$. The same holds clearly for the variables x_i therefore if we express it by means of the function σ [WW], considering the antisymmetry we must have

$$F(v) \propto \frac{\prod_{i=1}^{l} \sigma(v - x_i)\sigma(v + x_i) \prod_{i<j} \sigma(x_i - x_j)\sigma(x_i + x_j)}{\sigma^{2l}(v) \prod_{i=1}^{l} \sigma^{2l}(x_i)} .$$

Therefore we can compute

$$\frac{\det \begin{pmatrix} 1 & \wp(v) & \wp''(v) & \dots & \wp^{(2l-2)}(v) \\ 1 & \wp(x_1) & \wp''(x_1) & \dots & \wp^{(2l-2)}(x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(x_l) & \wp''(x_l) & \dots & \wp^{(2l-2)}(x_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(x_1) & \wp''(x_1) & \dots & \wp^{(2l-4)}(x_1) \\ 1 & \wp(x_2) & \wp''(x_2) & \dots & \wp^{(2l-4)}(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(x_l) & \wp''(x_l) & \dots & \wp^{(2l-4)}(x_l) \end{pmatrix}} = (2l-1)! \frac{\prod_{i=1}^{l} \sigma(v - x_i)\sigma(v + x_i)}{\sigma^{2l}(v) \prod_{i=1}^{l} \sigma^2(x_i)}$$

Expanding the LHS w.r.t. the first row of the determinant in the numerator and multiplying both sides by
$e^{2i\pi u + 2i\pi \frac{\eta'}{\eta}(\|\mathbf{x}\|^2 + 2lv^2)} \sigma^{2l}(v) \prod_1^l \sigma^2(x_i)$ we obtain exactly the formula (25).

As for the second statement, the proof is exactly as in the case A_l. □

Notice that, up to a normalization, all the Jacobi forms can be recovered by applying the operator $\mathfrak{D}$ to the lightest one

$$\varphi_{2l}(u, \mathbf{x}, \tau) = \prod_{j=1}^{l} (x_j)^2 + O(\|\mathbf{x}\|^{2l+1})$$

and hence for any $b = 1..l$ the Jacobi forms $\varphi_{l-b} := \mathfrak{D}^b(\varphi_l)$ do not vanish identically because their leading term is $\Delta_{\mathbf{x}}^b \left(\prod_{j=1}^{l} (x_j)^2 \right) \not\equiv 0$. This gives the whole basis of fundamental invariant Jacobi forms; moreover (as in the A_l case) any other basis is obtained from this one by a weighted linear transformation with coefficients in $\mathbf{M}_\bullet$.

Proposition 1.8 The formula

$$\mathcal{P}(v) = e^{2i\pi u} \alpha^{2l}(v) \left[\prod_{j=1}^{l} \left(\wp(v) - \wp(x_j) \right) \right] \prod_{k=1}^{l} \alpha^2(x_k) = \alpha^{2l}(v) \sum_{j=0}^{l} (\wp(v))^j \psi_{2j}(\mathbf{x}) ,$$

defines a basis of Jacobi forms which is equivalent to that in formula (25).

Proof. It is clear that, as a function of v, this is the same generating function because it has the same poles and zeroes, but the fundamental Jacobi forms ψ_{2j} are weighted linear combination

of those defined previously; indeed, expanding the product we obtain the Jacobi forms as the coefficients in front of powers of $\wp(v)$ while in the former formula they were the coefficients in front of even derivatives of $\wp$ (recall that even derivatives of $\wp$ can be expressed as polynomials in $\wp$). $\square$

Although this is a more compact formula, the previous one is more effective in actual computations of the intersection elements; indeed in the next paragraph the intersection elements will be computed only for the first basis of Jacobi forms. However this formula for the generating function is the closest to the classical one (6) as one can compute that

$$\mathcal{P}(v) = \exp\left(v^2\mathfrak{D}\right)\prod_{j=1}^{l}\left[e^{2i\pi u}\alpha^2(x_j)\right] \ ,$$

which follows from the formula (coming from the case A_1)

$$\left(-4i\pi\nabla_\tau + \frac{1}{2}\frac{d^2}{dx^2}\right)\alpha^2(x) = \wp(x)\alpha^2(x) \ .$$

Remark 1.9 Although the Weyl group of C_l is the same as the one of B_l, the *affine Weyl groups* (namely the complex crystallographic lattices) are different: indeed the root lattice of B_l is simply $\mathbb{Z}^l$ while the one of C_l is $(z_1,..,z_l) \in \mathbb{Z}^l$ such that $\sum z_j \in 2\mathbb{Z}$. There follows in particular that the normalization of the invariant Killing form to 2 for the shortest roots, is different: in this realization of the Cartan subalgebra (which follows the corresponding planches in [Bo]) the Killing metrics are $ds^2 := 2\sum {dx_j}^2$ for B_l and $\frac{1}{2}ds^2 = \sum_j {dx_j}^2$ for C_l.

A consequence of this fact is that if $\varphi(u,\mathbf{x},\tau) \in J_{k,m}^{(B_l)}$ then $\varphi(2u,\mathbf{x},\tau)$ belongs to $J_{k,2m}^{(C_l)}$. This amounts to saying that the algebra $J_{\bullet,\bullet}^{(B_l)}$ of Jacobi form for B_l is isomorphic to a subalgebra of $J_{\bullet,\bullet}^{(C_l)}$ and the isomorphism doubles the index.

Now, from [Wi92], we know that the generators belong to the spaces

$$\varphi_0 \in J_{0,1};\ \varphi_2 \in J_{-2,1};\ \varphi_4 \in J_{-4,1};\ \varphi_{2k} \in J_{-2k,2}\ ,\quad k = 3...l.$$

The isomorphism which injects $J_{\bullet,\bullet}^{(B_L)} \hookrightarrow J_{\bullet,\bullet}^{(C_L)}$ allows us to identify the $l-2$ generators of index 2 for the Jacobi algebra of type C_l as the images of the corresponding generators of B_l.

1.6.1 Computation of the intersection form with the generating function: B_l

As we did in the A_l case, we can now exploit the generating function to compute the intersection elements $\mathfrak{M}_{2j,2k} := \mathfrak{M}(\varphi_{2j},\varphi_{2k})$; again the computational details are quite cumbersome but the result is a remarkably simple formula which –again– is in deep analogy with the corresponding formula for the invariant polynomials in [SYS80]. The result will be a generating function in two variables for the elements $\mathfrak{M}_{2j,2k}$ and is contained in Thm. 1.8.

The intersection form for the Jacobi group of type B_l reads

$$\mathfrak{I} := -du\otimes d\tau - d\tau\otimes du + 2\sum_{j=1}^{l}{dx_j}^2\ ,$$

where (following [Bo] planche II), we have realized the CSA of B_l as the vectors $\mathbf{x} := (x_1, ..., x_l) \in \mathbb{C}^l$; notice the normalization of the Weyl invariant inner product.

Theorem 1.8 The intersection elements $\mathfrak{M}(\varphi_{2i}, \varphi_{2j})$ are recovered from the generating function

$$\sum_{j,k=0}^{l} \frac{\wp^{(2k)}(v')}{(2k-1)!} \frac{\wp^{(2j)}(v)}{(2j-1)!} \mathfrak{M}(\varphi_{2j}, \varphi_{2k}) = 2i\pi \left[\lambda(v)\mathbb{D}^{\bullet}\lambda(v') + \lambda(v')\mathbb{D}^{\bullet}\lambda(v)\right] + \\ + \frac{1}{2\left(\wp(v) - \wp(v')\right)} \left\{ \wp'(v')\lambda(v)\frac{d}{dv'}\lambda(v') - \wp'(v)\lambda(v')\frac{d}{dv}\lambda(v) \right\} . \tag{26}$$

Proof. We compute

$$\mathfrak{M}\left(\mathcal{P}(v), \mathcal{P}(v')\right) := \boldsymbol{\eta}^{-8l}\mathfrak{I}^{*}\left(d\left(\boldsymbol{\eta}^{4l}\mathcal{P}(v)\right), d\left(\boldsymbol{\eta}^{4l}\mathcal{P}(v')\right)\right) = \\ = -2i\pi\boldsymbol{\eta}^{4l}\left[\mathcal{P}(v)\partial_{\tau}\left(\boldsymbol{\eta}^{4l}\mathcal{P}(v')\right) + \mathcal{P}(v')\partial_{\tau}\left(\boldsymbol{\eta}^{4l}\mathcal{P}(v)\right)\right] + \frac{1}{2}\sum_{1}^{l}\frac{\partial}{\partial x_i}\mathcal{P}(v)\frac{\partial}{\partial x_i}\mathcal{P}(v') = \\ = -\frac{1}{2}\sum_{1}^{l}\left(\frac{\alpha''(v-x_i)}{\alpha(v-x_i)} + \frac{\alpha''(v+x_i)}{\alpha(v+x_i)} + \frac{\alpha''(v'-x_i)}{\alpha(v'-x_i)} + \frac{\alpha''(v'+x_i)}{\alpha(v'+x_i)} - \frac{\alpha'(v-x_i)\alpha'(v'-x_i)}{\alpha(v-x_i)\alpha(v'-x_i)}\right. \\ \left. - \frac{\alpha'(v+x_i)\alpha'(v'+x_i)}{\alpha(v+x_i)\alpha(v'+x_i)} - \frac{\alpha'(v-x_i)\alpha'(v'+x_i)}{\alpha(v-x_i)\alpha(v'+x_i)} - \frac{\alpha'(v+x_i)\alpha'(v'-x_i)}{\alpha(v+x_i)\alpha(v'-x_i)}\right)\mathcal{P}(v)\mathcal{P}(v') + \\ + 8i\pi l g_1 \mathcal{P}(v)\mathcal{P}(v') = \\ = \sum_{i=1}^{l}\left[2i\pi\frac{\partial_{\tau}\alpha(v-v')}{\alpha(v-v')} + 2i\pi\frac{\partial_{\tau}\alpha(v+v')}{\alpha(v+v')} + \right. \\ - \frac{1}{2}\left(\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')}\right)\left(\frac{\alpha'(v-x_i)}{\alpha(v-x_i)} + \frac{\alpha(v+x_i)}{\alpha(v+x_i)}\right) + \\ \left. + \frac{1}{2}\left(\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')}\right)\left(\frac{\alpha'(v'-x_i)}{\alpha(v'-x_i)} + \frac{\alpha(v'+x_i)}{\alpha(v'+x_i)}\right)\right]\mathcal{P}(v)\mathcal{P}(v') + 8i\pi l g_1 \mathcal{P}(v)\mathcal{P}(v') = \\ = 2i\pi l\left[\frac{\nabla_{\tau}\alpha(v-v')}{\alpha(v-v')} + \frac{\nabla_{\tau}\alpha(v+v')}{\alpha(v+v')}\right]\mathcal{P}(v)\mathcal{P}(v') + \\ + \frac{1}{2}\left(\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')}\right)\mathcal{P}(v)\frac{d}{dv'}\mathcal{P}(v') - \frac{1}{2}\left(\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')}\right)\mathcal{P}(v')\frac{d}{dv}\mathcal{P}(v) \tag{27}$$

Remark 1.10 Notice again the resemblance of eq. 27 with Saito's formula [SYS80] in the case of the finite Weyl group B_l (here $\mathcal{P}(v) = \prod_1^l \left((x_i)^2 - v^2\right)$)

$$\sum_{i=1}^{l+1}\frac{\partial}{\partial x_i}\mathcal{P}(v)\frac{\partial}{\partial x_i}\mathcal{P}(v') = \frac{2}{(v')^2 - (v)^2}\left\{\mathcal{P}(v')v^2\frac{d}{dv}\mathcal{P}(v) - \mathcal{P}(v)(v')^2\frac{d}{dv'}\mathcal{P}(v')\right\} . \tag{28}$$

We can actually obtain Saito's formula with the limit

$$\lim_{\epsilon\to 0}\epsilon^{-4l+2}\left[\mathfrak{M}\left(\mathcal{P}_{\mathbf{x}}(v), \mathcal{P}_{\mathbf{x}}(v')\right)\right]_{u,\mathbf{x},v,v'\mapsto\epsilon u,\epsilon\mathbf{x},\epsilon v,\epsilon v'} = \text{ RHS of eq. 28 .}$$

Again, we can recast equation (27) into a more useful form using the function

$$\lambda(v) := \alpha^{-2l}(v)\mathcal{P}(v) = \sum_{j=0}^{l} \frac{\wp^{(2j-2)}(v)}{(2j-1)!}\varphi_{2j} =: \alpha^{-2l}(v)\sum_{j=0}^{l} C_j(v)\varphi_{2(l-j)} \ .$$

First of all, since $\alpha^{-2l}(v)\nabla_\tau^\bullet \alpha^{2l}(v) = \nabla_\tau^\bullet + 2l\frac{\nabla_\tau \alpha(v)}{\alpha(v)}$, then a straightforward computation gives

$$\begin{aligned}
&\mathfrak{M}(\mathcal{P}(v),\mathcal{P}(v')) - \sum_{j,k=0}^{l} C_k(v')C_j(v)\mathfrak{M}(\varphi_{2l-2j},\varphi_{2l-2k}) = \\
&= -2i\pi \sum_{k,j} \left(\nabla_\tau C_k(v')C_j(v) + \nabla_\tau C_j(v)C_k(v')\right)\varphi_{2l-2k}\varphi_{2l-2j} = \\
&= -2i\pi\alpha^{2l}(v')\alpha^{2l}(v)\left\{[\lambda(v')\nabla_\tau^\bullet\lambda(v) + \lambda(v)\nabla_\tau^\bullet\lambda(v')] + 2l\left(\frac{\nabla_\tau\alpha(v')}{\alpha(v')} + \frac{\nabla_\tau\alpha(v)}{\alpha(v)}\right)\lambda(v')\lambda(v)\right\} \ .
\end{aligned}$$

Using now formula (27) for $\mathfrak{M}(\mathcal{P}(v),\mathcal{P}(v'))$, we can finally compute

$$\begin{aligned}
&\sum_{j,k} \frac{\wp^{(2j)}(v)}{(2j-1)!}\frac{\wp^{(2k)}(v')}{(2k-1)!}\mathfrak{M}(\varphi_{2k},\varphi_{2j}) = \\
&= 2i\pi\left[\lambda(v)\nabla_\tau^\bullet\lambda(v') + \lambda(v')\nabla_\tau^\bullet\lambda(v)\right] + \\
&+2i\pi\, l\left[2\frac{\nabla_\tau\alpha(v')}{\alpha(v')} + 2\frac{\nabla_\tau\alpha(v)}{\alpha(v)} + \frac{\nabla_\tau\alpha(v-v')}{\alpha(v-v')} + \frac{\nabla_\tau\alpha(v+v')}{\alpha(v+v')}\right]\lambda(v)\lambda(v') + \\
&+l\left\{\frac{\alpha'(v-v')}{\alpha(v-v')}\left[\frac{\alpha'(v')}{\alpha(v')} - \frac{\alpha'(v)}{\alpha(v)}\right] - \frac{\alpha'(v+v')}{\alpha(v+v')}\left[\frac{\alpha'(v')}{\alpha(v')} + \frac{\alpha'(v)}{\alpha(v)}\right]\right\}\lambda(v)\lambda(v') + \\
&\quad +\frac{1}{2}\left[\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')}\right]\lambda(v)\frac{d}{dv'}\lambda(v') - \frac{1}{2}\left[\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')}\right]\lambda(v')\frac{d}{dv}\lambda(v) = \\
&= 2i\pi\left[\lambda(v)\nabla_\tau^\bullet\lambda(v') + \lambda(v')\nabla_\tau^\bullet\lambda(v)\right] + \\
&\quad +\frac{1}{2}\left[\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')}\right]\lambda(v)\frac{d}{dv'}\lambda(v') - \frac{1}{2}\left[\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')}\right]\lambda(v')\frac{d}{dv}\lambda(v) = \\
&= 2i\pi\left[\lambda(v)\mathbb{D}^\bullet\lambda(v') + \lambda(v')\mathbb{D}^\bullet\lambda(v)\right] + \frac{1}{2}\left[\frac{\alpha'(v-v')}{\alpha(v-v')} - \frac{\alpha'(v+v')}{\alpha(v+v')} + 2\frac{\alpha'(v')}{\alpha(v')}\right]\lambda(v)\frac{d}{dv'}\lambda(v') + \\
&\quad -\frac{1}{2}\left[\frac{\alpha'(v-v')}{\alpha(v-v')} + \frac{\alpha'(v+v')}{\alpha(v+v')} - 2\frac{\alpha'(v)}{\alpha(v)}\right]\lambda(v')\frac{d}{dv}\lambda(v) = \\
&= 2i\pi\left[\lambda(v)\mathbb{D}^\bullet\lambda(v') + \lambda(v')\mathbb{D}^\bullet\lambda(v)\right] + \frac{1}{2}\left[\zeta(v-v') - \zeta(v+v') + 2\zeta(v')\right]\lambda(v)\frac{d}{dv'}\lambda(v') + \\
&\quad -\frac{1}{2}\left[\zeta(v-v') + \zeta(v+v') - 2\zeta(v)\right]\lambda(v')\frac{d}{dv}\lambda(v) \ .
\end{aligned}$$

To write this in the final form we use the classical formula (see [WW] pag.458, example 18)

$$-\zeta(w-u) + \zeta(w-v) + \zeta(v) - \zeta(u) = \frac{1}{2}\left\{\frac{\wp'(u)+\wp'(w)}{\wp(u)-\wp(w)} - \frac{\wp'(v)+\wp'(w)}{\wp(v)-\wp(w)}\right\} \ ,$$

which allows us to complete the proof by substition. □

2 Relation between Jacobi groups and Frobenius structures on Hurwitz spaces

In this second part we will identify the Jacobi group of type A_l with the monodromy of a suitable Frobenius structure; the identification will be based on the explicit formula for the generating function of the Jacobi forms.

In order to be self–contained as far as it is possible, we recall the due definitions of the objects we are going to use, namely Frobenius manifolds on one hand, Hurwitz spaces on the other and explain how it is possible to give Hurwitz space a structure of Frobenius manifold, following [Du93]. The main result of this part can be expressed as follows:
the quotient space $\Omega/\mathbb{J}(\mathcal{A}_l)$ is naturally isomorphic to the moduli space of elliptic functions of degree $l+1$ with only one pole.

This identification will appear explicitly and allows us to build a structure of Frobenius manifold over a suitable covering of this space; this covering branches around a divisor in $\Omega/\mathbb{J}$ defined by the zero locus of the lightest Jacobi form.

Before entering the detail we give an account of the necessary mathematical objects.

2.1 Frobenius manifolds

We recall the basic definitions and properties of a Frobenius manifold.

Definition 2.1 *A* **Frobenius algebra** $\mathcal{A}$ *is a unital, commutative, associative* ($\mathbb{C}$) *algebra endowed with a* invariant *nondegenerate bilinear pairing* $\eta(\circ,\circ) : \mathcal{A}\otimes\mathcal{A} \to \mathbb{C}$, *in the sense that*

$$\eta(A\cdot B, C) = \eta(A, B\cdot C) \qquad \forall A, B, C \in \mathcal{A}\ .$$

It follows that $\eta(,)$ is symmetric, for $\eta(A,B) = \eta(\mathbf{1}, A\cdot B) = \eta(\mathbf{1}, B\cdot A) = \eta(B,A)$.
The notion of Frobenius manifold is now the following

Definition 2.2 *A* **Frobenius manifold** M *is a smooth manifold which is endowed with a structure of Frobenius algebra in the tangent space at each point (and henceforth a nondegenerate symmetric tensor* $\eta(,)$ *of type* $(0,2)$ *and a* $(0,3)$ *(symmetric) tensor of the structure constants* $g(X,Y,Z) := \eta(X, Y\cdot Z)$*) and the following properties hold:*

1. *the Levi–Civita connection defined by the metric* $\eta(\circ,\circ)$ *is flat;*
2. *the unit vector field* $\mathbf{1}$ *is parallel, namely* $\nabla_X \mathbf{1} = 0$*;*
3. *the* $(0,4)$ *tensor of its (covariant) derivatives* $(\nabla_X g)(Y,Z,W)$ *is completely symmetric;*
4. *there exist a vector field* $\mathbf{E}$ *(the* Euler vector*) which is covariantly linear* $(\nabla\nabla)_{X,Y}\mathbf{E} = 0$, $\forall X, Y \in \Gamma(TM)$, *and*

   (a) $[\mathbf{E}, \mathbf{1}] = -\mathbf{1}$*;*

(b) $(\mathcal{L}_{\mathbf{E}}\eta)(X,Y) := \mathbf{E}\eta(X,Y) - \eta([\mathbf{E},X],Y) - \eta(X,[\mathbf{E},Y]) = (2-d)\eta(X,Y)$;

(c) $(\mathcal{L}_{\mathbf{E}}g)(X,Y,Z) = \mathbf{E}(g(X,Y,Z)) - g([\mathbf{E},X],Y,Z) - g(X,[\mathbf{E},Y],Z) - g(X,Y,[\mathbf{E},Z]) = (3-d)g(X,Y,Z)$;

namely the $\mathbf{E}$ *generates conformal rescalings of the metric and of the Frobenius structure.*

5. *The* $(4,0)$*-tensor* (∇g) *is totally symmetric, or equivalently*

$$(\nabla g)(X,Y,Z,W) := (\nabla_W g)(X,Y,Z) = (\nabla_X g)(W,Y,Z) \ .$$

Observe that since $\mathbf{E}$ is a conformal Killing vector field it must satisfy also $div(\mathbf{E}) = -\frac{(2-d)}{2n}$; moreover it follows from the above axioms that

$$[\mathbf{E}, X\cdot Y] - [\mathbf{E},X]\cdot Y - X\cdot[\mathbf{E},Y] = X\cdot Y \ . \tag{29}$$

On the spectrum of $\mathbf{E}$ we have

Lemma 2.1 If the grading operator $Q := \nabla\mathbf{E}$ is diagonalizable, then the Euler vector can be represented by

$$\mathbf{E} = \sum_1^n ((1-q_i)t_i + r_i)\,\partial_i \tag{30}$$

for suitable constants q_i, r_i and suitable flat coordinates t_i.

Notice that, up to a translation in the flat coordinates, we can then recast the Euler vector in the form

$$\mathbf{E} = \sum_1^n (1-q_i)t_i\partial_i + \sum_{i|q_i=1} r_i\partial_i \ .$$

Remark 2.1 The flat coordinates t_i's which diagonalize the grading operator $\nabla\mathbf{E}$ are unique up to linear transformations which do not mix coordinates with different scaling dimension.
Moreover notice that $\mathbf{1}$ is an isotropic vector for η except in the case $d=0$; indeed $(2-d)\eta(\mathbf{1},\mathbf{1}) = (\mathcal{L}_{\mathbf{E}}\eta)(\mathbf{1},\mathbf{1}) = \mathbf{E}\eta(\mathbf{1},\mathbf{1}) - 2\eta(\mathcal{L}_{\mathbf{E}}\mathbf{1},\mathbf{1}) = 2\eta(\mathbf{1},\mathbf{1})$.

We now define the scaling exponents as

Definition 2.3 *A function* $\varphi : M \to \mathbb{C}$ *is said to be quasi–homogeneous of* **scaling exponent** d_φ *if it is an eigenfunction of the Euler vector,*

$$\mathbf{E}(\varphi) = d_\varphi\varphi \ .$$

This means that the coordinate functions t_i defined before are quasihomogeneous with scaling dimensions $d_i = (1-q_i)$.
We now give the

Proposition 2.1 The structure constants tensor $g(X,Y,Z)$ is the third covariant derivative of a locally well defined function F (called **Free energy**),

$$g(X,Y,Z) = (\nabla\nabla\nabla F)_{XYZ} \ . \tag{31}$$

Moreover this function is almost-quasihomogeneous of degree $d_F = 3 - d$, namely quasihomogeneous up to a function in the kernel of $\nabla\nabla\nabla$ (i.e. a function which is at most quadratic in local flat coordinates)

Proof. Since ∇g is completely symmetric, the proof of existence of the local function F follows easily in flat coordinates.
From $(\mathcal{L}_{\mathbf{E}} g) = (3-d)g$ it follows (using $\mathcal{L}_{\mathbf{E}} \circ \nabla = \nabla \circ \mathcal{L}_{\mathbf{E}}$) that

$$\nabla\nabla\nabla\left((3-d)F - \mathbf{E}(F)\right) = 0 \ .$$

The proof is thus complete. □

2.1.1 Intersection form

Since the invariant metric η gives a isomorphism between the tangent and cotangent bundle, we can define a Frobenius structure on the cotangent bundle as well, which we will indicate again as $\omega \cdot \alpha$, for $\omega, \alpha \in \Gamma(T^*M)$.

Definition 2.4 *The* **intersection form** *is the bilinear pairing in T^*M defined by*

$$(\omega, \alpha)^* := \underbrace{(\omega \cdot \alpha)}_{\in\Gamma(T^*M)}(\mathbf{E}) \ .$$

One can prove that it is almost everywhere nondegenerate, hence it defines a new "metric" (denoted by $\mathfrak{I}(,)$ as its associated (2,0) tensor) on the tangent bundle: one can show that [Du93]

Proposition 2.2 The metric $\mathfrak{I}(,)$ is flat and moreover $\forall\lambda \in \mathbb{C}$, the contravariant metric $G^*(\lambda) := \eta^*(,) + \lambda\mathfrak{I}^*(,)$ is flat as well and the contravariant connection $\widetilde{\nabla}_{(\lambda)}$ on forms is given by $\widetilde{\nabla}_{(\lambda)} := \widetilde{\nabla}^{(\eta)} + \lambda\widetilde{\nabla}^{(\mathfrak{I})}$, where $\widetilde{\nabla}^{(\eta)}$ and $\widetilde{\nabla}^{(\mathfrak{I})}$ denote the contravariant connections (acting on one–forms) of the metrics η and $\mathfrak{I}$ respectively[3] The family of metrics $G^*(\lambda)$ as λ varies, is called a *flat pencil* of metrics.

[3]By **contravariant connection on one–forms** of a metric g over a manifold M we mean the map

$$\begin{aligned} \nabla : \quad & T^*M \otimes \Gamma(T^*M) \to \Gamma(T^*M) \\ & \xi \otimes \alpha \to \nabla_{\xi^\sharp}\alpha \ , \end{aligned}$$

where ∇ is the Levi–Civita connection and $\xi^\sharp$ is the vector associated to ξ by means of the isomorphism given by the metric.

Properties and relations. There are some differential relation between the two metrics, as it is shown hereafter; these relation allow to reconstruct the invariant metric η and the free energy from the knowledge of the intersection form and the unit vector field as we will see.

Lemma 2.2 We have $\mathcal{L}_{\mathbf{1}} g = \nabla_{\mathbf{1}} g = 0$.

Proof. For any $X, Y, Z \in \Gamma(TM)$ and from the definitions (recalling that $\nabla \mathbf{1} = 0$) we find

$$\begin{aligned}(\mathcal{L}_{\mathbf{1}} g)(X,Y,Z) &= \mathbf{1}\left(g(X,Y,Z)\right) - g([\mathbf{1},X],Y,Z) - g(X,[\mathbf{1},Y],Z) - g(X,Y,[\mathbf{1},Z]) = \\ &= (\nabla_{\mathbf{1}} g)(X,Y,Z) = (\nabla_Z g)(X,Y,\mathbf{1}) = (\nabla_Z \eta)(X,Y) = 0 \ .\end{aligned}$$

□

Hence we find

Lemma 2.3 We have, for any $x, y \in \Gamma(T^*M)$,

$$\eta^*(x,y) = (\mathcal{L}_{\mathbf{1}} \mathfrak{I}^*)(x,y) \ .$$

Proof. Let X, Y denote the dual vectors to x, y (which are covectors) (explicitly $x(\bullet) =: \eta(X, \bullet)$), we find (recall $\mathfrak{I}^*(x,y) := g(X,Y,\mathbf{E})$)

$$\begin{aligned}(\mathcal{L}_{\mathbf{1}} \mathfrak{I}^*)(x,y) &= \mathbf{1}\left(g(X,Y,\mathbf{E})\right) - g(\overbrace{[\mathbf{1},X]}^{=\nabla_{\mathbf{1}} X},Y,\mathbf{E}) - g(X,\overbrace{=[\mathbf{1},Y]}^{\nabla_{\mathbf{1}} Y},\mathbf{E}) = \\ &\underbrace{(\nabla_{\mathbf{1}} g)}_{=0}(X,Y,\mathbf{E}) + g(X,Y,\underbrace{\nabla_{\mathbf{1}}\mathbf{E}}_{=\mathbf{1}}) = \eta(X,Y) =: \eta^*(x,y) \ .\end{aligned}$$

□

Lemma 2.4 The functions $G^{ij} := \mathfrak{I}^*(dt_i, dt_j)$ are homogeneous of degree $d_{ij} = (1+d) - q_i - q_j$.

Proof. Recalling that $[\mathbf{E}, \frac{\partial}{\partial t_i}] = (q_i - 1)\frac{\partial}{\partial t_i}$ we find

$$\begin{aligned}\mathbf{E}\left((dt_i, dt_j)^*\right) &= \mathbf{E}\left(g\left((dt_i)^\#, (dt_j)^\#, \mathbf{E}\right)\right) = (\mathcal{L}_{\mathbf{E}} g)\left((dt_i)^\#, (dt_j)^\#, E\right) + \\ &+ g\left([\mathbf{E}, (dt_i)^\#], (dt_j)^\#, \mathbf{E}\right) + g\left((dt_i)^\#, [\mathbf{E}, (dt_j)^\#], \mathbf{E}\right) = (**) \ .\end{aligned}$$

We now recall that (from $\mathcal{L}_{\mathbf{E}}(\eta^{-1}) = (d-2)\eta^{-1}$) we have, for an arbitrary $\omega \in \Gamma(T^*M)$,

$$[\mathbf{E}, \omega^\#] = (d-2)\omega^\# + (\mathcal{L}_{\mathbf{E}}(\omega))^\# \ ,$$

and hence $[\mathbf{E}, (dt_i)^\#] = (d - 2 + 1 - q_i)(dt_i)^\#$. Therefore we can complete the chain of equation $(**)$ as follows

$$(**) = [3 - d + (d - 1 - q_i) + (d - 1 - q_j)]\, \mathfrak{I}^*(dt_i, dt_j) = (1 + d - q_i - q_j)\mathfrak{I}^*(dt_i, dt_j) \ .$$

□

Lemma 2.5 We have

$$\mathfrak{I}^*(dt_i, dt_j) = (1 + d - q_i - q_j)\nabla_{(dt_i)^\#}\nabla_{(dt_j)^\#} F \ .$$

Proof. Recalling that

$$\mathfrak{I}^*(dt_i, dt_j) = (\nabla\nabla\nabla F)_{(dt_i)^\#(dt_j)^\#\mathbf{E}}$$

and that $[\mathcal{L}_{\mathbf{E}}, \nabla] = 0$ we find (for brevity we denote $(dt_i)^\#$, $(dt_j)^\#$ by I, J respectively)

$$(\nabla\nabla\nabla F)_{\mathbf{E}IJ} = \nabla_{\mathbf{E}}\nabla_I\nabla JF = \mathbf{E}(\nabla_I\nabla_J F) = \mathcal{L}_{\mathbf{E}}(\nabla_I\nabla_J F) + \nabla_{[\mathbf{E},I]}\nabla_J F + \nabla_I\nabla_{[\mathbf{E},J]}F =$$
$$= \underbrace{\mathcal{L}_{\mathbf{E}}(\nabla_I\nabla_J F)}_{=(3-d)\nabla_I\nabla_J F} + (d-1-q_i+d-1-q_j)\nabla_I\nabla_J F = (1+d-q_i-q_j)\nabla_{(dt_i)^\#}\nabla_{(dt_j)^\#}F \ .$$

□

2.1.2 Reconstruction

Let us suppose we are given a Frobenius manifold M and we know only the scaling dimensions $d, q_1, ..., q_n$, the Euler vector field $\mathbf{E}$, the unit vector field $\mathbf{1}$ and the intersection form $\mathfrak{I}(,)$; then, from the previous lemmas we have as a corollary that we can uniquely reconstruct the full Frobenius structure by setting

$$\eta^{-1} := \mathcal{L}_{\mathbf{1}}(\mathfrak{I}^*)$$

and finding the flat coordinates of η as homogeneous functions and then find the structure constants by imposing that

$$(1+d-q_i-q_j)H^F\big((dt_i)^\#, (dt_j)^\#\big) = \mathfrak{I}^*(dt_i, dt_j) \ . \tag{32}$$

Of course this procedure goes through if $q_i + q_j \neq d+1$, $\forall i, j = 1..n$, otherwise there may be some obstruction or ambiguity in the construction of the free energy F; this is the only effective way to find the free energy in many actual examples as we shall do for G_2 later.

2.1.3 Monodromy group of a Frobenius manifold

Both metrics $\eta(X,Y)$ and $\mathfrak{I}(X,Y)$ are flat and it is natural to study their mutual relations. Since $\mathfrak{I}$ comes from the inversion of *almost everywhere* nondegenerate bilinear pairing of the cotangent bundle, it is defined almost everywhere, namely outside the locus Δ where the determinant of $\mathfrak{I}^* : T^*M \to TM$ vanishes.
Now the flat complex manifold, $(M/\Delta, \mathfrak{I})$ is not simply connected and hence we have a nontrivial holonomy group at any point, which is a discrete subgroup of $O(n, \mathbb{C})$.
To be more specific let $y_1, ..., y_n$ be the flat normal coordinates at $p_0 \in M$ (fixed and outside Δ) of the intersection form $\mathfrak{I}$, and express them as functions of the flat coordinates $t_1, ..., t_n$ of the invariant metric η; since $\mathfrak{I}$ degenerates on Δ, its Christoffel symbols (which enter in the equation defining the flat coordinates y_i) have singularities on Δ. As a result, the germs of functions $y_i(t_1, ..., t_n)$ will be in general multivalued for loops around the discriminant. This implies that the result of a non-contractible loop γ around the discriminant will be a linear affine transformation of the y_i's whose linear part is clearly an $\mathfrak{I}$–orthogonal transformation: it is also clear that the correspondence which associates to each noncontractible loop $\gamma \in \pi_1(p_0; M/\Delta)$ this linear affine transformation of the coordinates y_j is a group homomorphism. This map

$$\mathbf{M} \ : \pi_1(p_0; M/\Delta) \ \longrightarrow Aff(\mathfrak{I}, n)$$

$$\gamma \qquad \mapsto \mathbf{M}_\gamma \ ,$$

is a group homomorphism from the fundamental group of M/Δ onto a (discrete) subgroup of the affine transformations of the functions y_i's: the image under $\mathbf{M}$ of the fundamental group $\pi_1(p_0; M/\Delta)$ is called the **monodromy group of the Frobenius manifold**.

Viceversa one could ask to solve the inverse problem, namely that of finding a Frobenius manifold whose monodromy group is a given discrete group of affine transformations preserving a (nondegenerate) bilinear form $\mathfrak{I}$: this is actually the problem that motivated this study.

2.2 Hurwitz spaces and Frobenius structures

In this section we rephrase the contents of [Du93], Lecture 5, and adapt the notation to the present purposes.

Hurwitz spaces are moduli spaces of certain meromorphic functions on algebraic smooth curves of fixed genus. Hereafter we give a short account of their structure and definitions and later we explain how there can be defined Frobenius structures.

Let C be a compact Riemann surface of genus g and let $\lambda : C \to \mathbb{C}P^1$ of degree N; we moreover fix the type of ramification over the point at infinity $\infty \in \mathbb{C}P^1$ assuming that $\lambda^{-1}(\infty) = \{\infty_0, \infty_1, ..., \infty_m \in C\}$ and that the respective degrees at these points be $n_0 + 1, n_1 + 1, ..., n_m + 1$.
These data (namely the genus g, the number of sheets and the ramification at infinity) fix uniquely the total number of ramification points in C, say $\{P_1, .., P_n, \infty_0, .., \infty_m\}$; by means of the Riemann–Roch theorem we find (notice that we must have $N = m + 1 + n_0 + ... + n_m$)

$$\overbrace{deg(K_C)}^{=2g-2} = N \overbrace{deg(K_{\mathbb{C}P^1})}^{=-2} + \overbrace{deg(B)}^{=n+n_0+...+n_m} \quad \Rightarrow n = 2g + n_0 + .. + n_m + 2m \ .$$

Therefore the smooth (i.e. non-orbifold) part of the moduli space $M_{g,m;\{n_j\}}$ of these data has dimension n.
As parameters for the point in this space we can take $(u_1, ..., u_n) := (\lambda(P_1), ..., \lambda(P_n)) \in (\mathbb{C}P^1)^{\times n}$; Summarizing

Definition 2.5 *The* **Hurwitz space** $M_{g,m;n_0,..,n_m}$ *is the moduli space of curves C of genus g endowed with a N branched covering λ of $\mathbb{C}P^1$, $\lambda : C \to \mathbb{C}P^1$ with $m+1$ branching points over $\infty \in \mathbb{C}P^1$ of branching degree $n_\nu + 1$, $\nu = 0..m$.*

In this kind of spaces the usual notion of equivalence involves also the quotient by the diffeomorphism group of the target space (here $\mathbb{C}P^1$) [Na84], but for the present purposes we will consider the (trivial) principal bundle with structure group the diffeomorphisms of $\mathbb{C}P^1$ which fix one point (the infinity); this simply means that in the kind of spaces we are considering the notion of equivalence involves only the diffeomorphism group of C and moreover the affine group $\mathbb{C}^* \rtimes \mathbb{C}$ acts on the functions λ as $a\lambda + b$, for $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. For the sake of clarity we specify the relevant notion of equivalence:

Definition 2.6 *Two pairs (C, λ) and $(\widetilde{C}, \widetilde{\lambda})$ are* **Hurwitz–equivalent** *if there exists an analytic isomorphisms $\vartheta : C \to \widetilde{C}$ such that $\lambda \circ \vartheta = \widetilde{\lambda}$.*

In the following we will use a covering of the Hurwitz spaces which we now describe.
First of all notice that, since we are actually considering curves of genus g with $m+1$ marked points, for $2g+m+1 \geq 3$ the curve C is stable, that is the group of automorphisms is discrete; for $g \geq 1$ the covering of their moduli space is accomplished considering a symplectic basis of cycles in the homology of the curve[4].
We have also to consider the points ∞_ν, $\nu = 0..m$: therefore, for each of them, ∞_ν we fix a local uniformizing function, namely a function w_ν such that ${w_\nu}^{n_\nu+1} = \lambda$ in a neighbourhood of ∞_ν. Summarizing

Definition 2.7 *The covering space $\widehat{M}_{g,m;n_0...n_m}$ is defined as the sets of genus g curves with $m+1$ marked points endowed with a symplectic basis in the homology and a N-sheeted covering λ of $\mathbb{C}P^1$ with a fixation of local uniformizing coordinates $\{w_\nu\}$,*

$$\widehat{M}_{g,m;n_0...n_m} := \{(C, \lambda; w_0, .., w_m; \{a_1..a_g, b_1.., b_g\})\} \ .$$

2.2.1 Frobenius structures on $\widehat{\mathbf{M}}_{\mathbf{g,m;n_0...n_m}}$

Over the space $M_{g,m;n_0...n_m}$ we consider the coordinates $u_1, ..., u_n$ as spanning a semisimple commutative, associative, graded and unital algebra in the tangent space, with $\{u_i = \lambda(P_i) \mid d\lambda(P_i)\}$ as local coordinates. Explicitly the algebra structure is as follows (setting $\partial_i := \frac{\partial}{\partial u_i}$)

1. Multiplication: $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$;
2. Unity: $\mathbf{1} := \sum_1^n \partial_i$;
3. Euler field $\mathbf{E} := \sum_1^n u_i\partial_i$.

In order to obtain a Frobenius structure we must define an invariant inner product such that the resulting metric is flat and satisfies all the axioms of Frobenius manifold.
For any one form $\Omega \in \Gamma(T^*M)$ we can define an invariant inner product as

$$< X, Y >_\Omega := \Omega(X \cdot Y)$$

which is nondegenerate provided that Ω nowhere vanishes. There is a natural way to build one forms on the Hurwitz space starting from a representative as follows. Let Q be a quadratic differential on C (i.e. $Q \in \Gamma(T^*C \otimes T^*C)$) and set

$$\Omega_Q := \sum_1^n du_i \, \underset{P_i}{res} \frac{Q}{d\lambda}$$

Notice that this definition of the one form Ω_Q is independent of the representative and also invariant under the structure group (here simply the affine transformations of the plane): in fact, if $\vartheta : \widetilde{C} \to C$ and $\widetilde{\lambda} = a\lambda + b$, then $\widetilde{u}_i := a\,\lambda \circ \vartheta + b = a\,u_i + b$, $\widetilde{Q} = \vartheta^* Q$, therefore

$$\sum_1^n d\widetilde{u}_i \, \underset{P_i}{res} \frac{\widetilde{Q}}{d\widetilde{\lambda}} = \sum_1^n du_i \, \underset{P_i}{res} \frac{Q}{d\lambda} \ ,$$

[4] We recall that this amounts to choosing $2g$ cycles $\{a_1, .., a_g, b_1, .., b_g\}$ which have intersection number $a_i \cdot a_j = b_i \cdot b_j = 0$ and $a_i \cdot b_j = \delta_{ij}$.

and hence the definition is well posed.
It is clear that if the differential Q is **dλ–divisible**, namely if it has the form $\phi \otimes d\lambda$ and ϕ is holomorphic on the zeroes of $d\lambda$, then the corresponding differential on the Hurwitz space is zero; this allows for an enlargement of the class of quadratic differential that we can use. In fact we can consider the larger class of quadratic differentials on the universal covering of the curve C which have the property that their continuation along a closed curve $\gamma \subset C$ is changed by a $d\lambda$–divisible differential, namely

$$Q \underset{\gamma}{\mapsto} Q + q_g \otimes d\lambda \ .$$

One can consider quadratic forms Q which are squares of a differential ϕ of certain type, namely $Q = \phi \otimes \phi$: these differentials ϕ are called **primary differentials**. The types that lead to Frobenius structures are listed below

1. An Abelian differential of the second kind[5] with poles only at the poles of λ, $\infty_0, ..., \infty_m$ with orders less than n_ν (in such a way that $\frac{\phi\otimes\phi}{d\lambda}$ has no residues there) and such that

$$\oint_{a_j} \phi = 0, \qquad i = 1..g \ .$$

   (Such differentials are said *normalized*)

2. An Abelian differential of the second kind of the form

$$\phi = \sum_{k=1}^{m} D_k \theta_k$$

   where θ_k are normalized Abelian differentials of the second kind with only one pole at ∞_k

$$\theta_k = -d\lambda + \text{ regular terms near } \infty_k \ .$$

3. An Abelian differential of the third kind

$$\phi = \sum_{k=1}^{m} D_k \rho_k$$

   where ρ_k are normalized Abelian differentials of the third kind and simple poles at ∞_k and ∞_0 of residues $+1$ and -1 respectively.

4. A multivalued normalized differential (namely a differential on the universal cover of C) with increment along one of the b–cycle, say the j-th, of the form

$$\phi \underset{b_i}{\mapsto} \phi + \delta_{ij} d\lambda \ .$$

[5]On a curve C the Abelian differentials (namely the analytic one forms) are said to be of the **first kind** if it is holomorphic everywhere, of the **second kind** if it has only poles without residues and of the **third kind** of there are poles with residues.

5. Any differential of the first kind, which can be written for convenience as

$$\phi = \sum_{j=1}^{g} C_j \omega_j$$

where the holomorphic differentials ω_j are the Poincare' duals to the b–cycles, namely $\oint_{a_k} \omega_j = \delta_{jk}$.

In [Du93] it was showed that any quadratic form $Q = \phi \otimes \phi$ where ϕ is one of the above primary differentials gives rise to a Frobenius structure, namely to a flat invariant metric η along the lines we drew before, namely

$$\eta\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = \sum_{i=1}^{n} \underset{P_i}{res} \frac{\phi \otimes \phi}{d\lambda} du_i \left(\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j}\right) = \sum_{i=1}^{n} \underset{P_i}{res} \frac{\phi \otimes \phi}{d\lambda} .$$

In order to build the **superpotential** of this Frobenius structure we take λ considered as a function on the universal covering $\widetilde{C}$ (namely a multivalued function) by considering it as depending on the multivalued coordinate

$$v(P) := \int_{\infty_0}^{P} \phi ,$$

where the principal value prescription (if necessary) is understood in this integral. The necessity to consider λ as a function of v is that we need to make differentiations of λ along the moduli space and it must be clear which is the (local) coordinate w.r.t. make the differentiation. We cite the Theorem to be found in [Du93] and which we will use later.

Theorem 2.1 For any primary differential ϕ the corresponding invariant metric η_ϕ along with the canonical multiplication rules in the coordinates u_i, endows the Hurwitz space $\hat{M}_{g,m;n_0..n_m}$ with a structure of Frobenius manifold.

The flat coordinates of the invariant Frobenius metric η are the $n = 2g + 2m + n_0 + ... + n_m$ coordinates

$$\begin{aligned}
t_{\nu;a} &:= \underset{\infty_\nu}{res} (w_\nu)^a \, v \, d\lambda; \qquad \nu = 0..m, \ a = 1..n_\nu; \\
v_\nu &:= \int_{\infty_0}^{\infty_\nu} dv = \int_{\infty_0}^{\infty_\nu} \phi; \qquad V_\nu := -\underset{\infty_\nu}{res} \, \lambda \, dv = -\underset{\infty_\nu}{res} \, \lambda \, \phi; \qquad \nu = 1..m; \\
B_j &:= \oint_{b_j} dv \ ; \qquad C_j := \oint_{a_j} \lambda \, dv; \qquad j = 1..g,
\end{aligned}$$

and the nonzero entries of η in these coordinates read

$$\begin{aligned}
\eta_{t_{\nu;a}, t_{\mu;b}} &= \frac{1}{n_\nu + 1} \delta_{\mu\nu} \delta_{a+b, n_\nu + 1} \\
\eta_{v_\nu, V_\mu} &= \frac{1}{n_\nu + 1} \delta_{\mu\nu} \\
\eta_{B_j, C_k} &= \frac{1}{2i\pi} \delta_{jk} \ .
\end{aligned}$$

In order to identify the remaining objects in the framework of Frobenius manifolds, we notice that

Proposition 2.3 [Remark 5.3 in [Du93]] The intersection form associated to the Frobenius structure specified in Theorem 2.1 is given by

$$\mathfrak{I} = \sum_{d\lambda(P_i)=0} (du_i)^2 \underset{P_i}{res} \left(\frac{dv \otimes dv}{\lambda\, d\lambda} \right) .$$

The flat coordinates of this metric are the described by the same formulae as above with the substitution $\lambda \mapsto \log(\lambda)$ and with the $t_{\nu;a}$ changed into the

$$z_i = v(Q_i) \ , \qquad \text{s.t. } \lambda(Q_i) = 0 \ .$$

Remark 2.2 If we have any other set of Hurwitz moduli and we consider derivations w.r.t. these ones, we find

$$\mathfrak{I}(\partial, \partial') = \sum \underset{d\lambda=0}{res} \frac{\partial\lambda\, dv \otimes \partial'\lambda\, dv}{\lambda\, d\lambda} = \sum \underset{d\lambda=0}{res} \frac{\partial\left(\log(\lambda)\right) dv \otimes \partial'\left(\log(\lambda)\right) dv}{d\left(\log(\lambda)\right)} .$$

Proposition 2.4 Scaling dimensions The scaling dimensions of the flat invariants of η are given by

$$\begin{array}{ll} \mathbf{E}(t_{\nu;a}) = \dfrac{n_\nu + 1 - a}{n_\nu + 1} t_{\nu;a} \ , & \nu = 0, ..., m; \ a = 1, ..., n_\nu \ ; \\ \mathbf{E}(v_\nu) = 0 \ , \ \ \mathbf{E}(V_\nu) = V_\nu \ ; & \nu = 1, ..., m \ ; \\ \mathbf{E}(B_j) = 0 \ , \ \ \mathbf{E}(C_j) = C_j \ ; & j = 1, ..., g \ . \end{array}$$

Proof. The proof is immediate considering the expressions of the invariant and noticing that the Euler vector field is a rescaling in the variable λ, leaving unchanged the primary differential dv. □

In the following section we apply this general theorems to the case of genus 1.

2.3 The space $M_{1,0;l}$ as orbit space of $\mathbb{J}(A_l)$

In this section we build a Frobenius structure over the Hurwitz space $M_{1,0;l}$ (or better its covering $\widehat{M}_{1,0;l}$); now C is a torus and the superpotential λ has only one pole ∞_0 of degree $l+1$. Applying the general theory we shall identify the flat coordinates of the intersection form and find the monodromy group of the resulting Frobenius manifold: it will turn out that this is the Jacobi group of type A_l, $\mathbb{J}(A_l)$ and that the moduli of the superpotential -when expressed as functions of the flat coordinates of the intersection form-, are the generators of the algebra of Jacobi forms $J_{\bullet,\bullet}$ (see formula 11).

We call τ the modular parameter of the torus C and we think of it as $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. It is useful to use the Weierstrass uniformization, realizing the torus $C(\tau)$ as the affine curve[6]

$$\begin{aligned} &Y^2 = 4X^3 - g_2(\tau)X - g_3(\tau) \\ &X = \wp(v)\ ; \qquad Y = \wp'(v)\ . \end{aligned}$$

The point ∞_0 can be chosen (modulo the diffeomorphisms of the torus) as the point $v=0$ of $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Since λ must be a meromorphic elliptic function with a pole of order $l+1$ at $v = 0 \bmod \mathbb{Z}+\tau\mathbb{Z}$, it follows that the most generic form is (using a convenient normalization of the moduli)

$$\begin{aligned} \lambda(v) &= \sum_{j=0}^{l+1} \frac{(-1)^{l+1-j}}{(l-j)!}\wp^{(l-1-j)}(v)\varphi_{l+1-j} = \frac{1}{v^{l+1}}\varphi_{l+1} + \frac{1}{v^l}\varphi_l + \frac{1}{v^{l-1}}\varphi_{l-1} + \ldots + \frac{1}{v^2}\varphi_2 + O(1) \\ \partial_k\lambda &:= \frac{\partial\lambda}{\partial\varphi_k} = \frac{(-1)^k}{(k-1)!}\wp^{(k-2)}(v)\ , \end{aligned}$$

where we numbered the coefficients φ_k according to the order of the pole and we have set, for notational brevity, $\wp^{(-1)}(v) \equiv 0$, $\wp^{(-2)}(v) \equiv 1$.

From the modular properties of $\wp$ it follows that the parameters φ_k transform as modular forms of weight $-k$ under $SL(2,\mathbb{Z})$: indeed by assumption the superpotential has to be independent of the isomorphism class of the torus namely,

$$\lambda(v|\tau) = \lambda\left(\frac{v}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right)\ , \qquad \forall\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z}) \ \Rightarrow \varphi_k \underset{\tau\mapsto\tau'}{\mapsto} (c\tau+d)^{-k}\varphi_k\ .$$

We observe that the moduli φ_j's are equivariant and we anticipate here that they will be identified with the fundamental Jacobi forms. At this stage they are simply some parameters which play the same role as the Jacobi forms in formula (13), but the identification will be complete in Thm. 2.2. In view of this identification we shall occasionally call them "Jacobi forms" by an abuse of language.

The primary differential we will use is simply the holomorphic differential $\phi = dv$ and hence λ will be the superpotential of our Frobenius manifold as a function of the multivalued coordinate v on the torus (hence as a function on the Jacobian of the torus).
Let us analyze the structure:

1. The parameters $\varphi_0, \varphi_2, \ldots, \varphi_{l+1}$ and the modular parameter $\tau =: \varphi_{-1}$ are local coordinates

[6] In this section we use a slightly different definition for the Weierstrass functions, namely

$$\begin{aligned} \sigma(v|\tau) &:= e^{-2i\pi v^2 g_1(\tau)}\frac{\theta_1(v,\tau)}{\theta_1'(0,\tau)} = e^{-2i\pi v^2 g_1(\tau)}\alpha(v|\tau) \\ \zeta(v|\tau) &:= \frac{d}{dv}\log\left(\sigma(v|\tau)\right) \\ \wp(v|\tau) &:= -\frac{d}{dv}\zeta(v|\tau) = \frac{1}{v^2} + \sum_{m^2+n^2\neq 0}\left(\frac{1}{(v+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right) \end{aligned}$$

of the Frobenius manifold $M := M_{1,0;l}$ and the invariant metric $\eta(,)$ is given by [7]

$$\eta\left(\frac{\partial}{\partial\varphi_i},\frac{\partial}{\partial\varphi_j}\right) = \sum_{\substack{\lambda'(v_n)=0\\ i,j=-1,0,2,...,l+1}} \frac{\partial u_n}{\partial\varphi_i}\frac{\partial u_n}{\partial\varphi_j}\,\underset{v=v_n}{res}\,\frac{dv}{\lambda'(v)} = \sum \underset{\lambda'=0}{res}\,\frac{\boldsymbol{\partial}_i\lambda(v)\boldsymbol{\partial}_j\lambda(v)\,dv}{\lambda'(v)}\ ,$$

For $i,j = 0..l+1$ the functions $\frac{\wp^{(i-2)}(v)\wp^{(j-2)}(v)}{\lambda'(v)}$ are elliptic functions (recall that we have set $\wp^{(-2)} \equiv 1$; $\wp^{(-1)} \equiv 0$) and hence the sum of all residues in a fundamental mesh is zero: therefore we can compute the residues at the points defined by $\lambda' = 0$ by computing the residue at $v = 0$ with opposite sign, (we suppress the v dependence to shorten the formulae)

$$\eta\Big(\boldsymbol{\partial}_i,\boldsymbol{\partial}_j\Big) = -\underset{v=0}{res}\,\frac{\boldsymbol{\partial}_i\lambda\,\boldsymbol{\partial}_j\lambda\,dv}{\lambda'}\ .$$

A different problem is to compute the matrix elements where $i = -1$ ($\varphi_{-1} := \tau$) for in this case the function $\partial_\tau\lambda$ *is not* an elliptic function, but we can compute the residues for

$$\mathbb{D}^\bullet\lambda(v) \equiv \nabla^\bullet_\tau\lambda(v) - \frac{1}{2i\pi}\frac{\alpha'(v)}{\alpha(v)}\lambda'(v) = \partial_\tau\lambda(v) - \frac{\boldsymbol{\eta}'}{\boldsymbol{\eta}}\sum_{j=0}^{l+1} 2j\,\wp^{(j-2)}(v)\varphi_j - \frac{1}{2i\pi}\frac{\alpha'(v)}{\alpha(v)}\lambda'(v) =$$

$$=: \partial_\tau\lambda(v) + \boldsymbol{\Sigma}(v) + \gamma(v)\lambda'(v)$$

(which *is elliptic*: notice that $\boldsymbol{\Sigma}$ is elliptic). Hence if $F(v)$ is any elliptic function with a pole *only* at the origin, we find

$$\sum \underset{\lambda'=0}{res}\,\frac{\partial_\tau\lambda\,F\,dv}{\lambda'} = \sum \underset{\lambda'=0}{res}\left[\frac{\Big(\mathbb{D}^\bullet\lambda - \boldsymbol{\Sigma}\Big)\,F\,dv}{\lambda'}\right] =$$

$$= -\underset{v=0}{res}\,\frac{\Big(\mathbb{D}^\bullet\lambda - \boldsymbol{\Sigma}\Big)\,F\,dv}{\lambda'} = -\underset{v=0}{res}\left[\left(\frac{\partial_\tau\lambda\,F\,dv}{\lambda'(v)}\right) + \gamma\,F\,dv\right]\ .$$

The computational advantage is that we have converted a sum over residues which we do not know where are situated, into a single evaluation at a fixed pole. Along these lines we can compute

$$\eta\Big(\boldsymbol{\partial}_{-1},\boldsymbol{\partial}_k\Big) = -\underset{v=0}{res}\left[\left(\frac{\partial_\tau\lambda\,\boldsymbol{\partial}_k\lambda\,dv}{\lambda'}\right) + \gamma\boldsymbol{\partial}_k\lambda\,dv\right] =$$

[7] The formula follows from the Jacobian of the parameters u_i in terms of the moduli φ_k as follows: the defining equation of the u's are

$$\begin{cases} u_i(\underline{\varphi}) := \lambda(P_i;\underline{\varphi}) \\ \lambda'(P_i;\underline{\varphi}) = 0 \end{cases}$$

and upon differentiation w.r.t. φ_k we get

$$\begin{cases} \frac{\partial u_i(\underline{\varphi})}{\partial\varphi_k} = \frac{\partial\lambda}{\partial\varphi_k}(P_i;\underline{\varphi}) + \lambda'(P_i;\underline{\varphi})\frac{\partial P_i}{\partial\varphi_k} = \frac{\partial\lambda}{\partial\varphi_k}(P_i;\underline{\varphi}) \\ d\lambda(P_i;\underline{\varphi}) = 0 \end{cases}\ .$$

$$= \underset{v=0}{res} \left[\frac{\alpha'(v)}{2i\pi\alpha(v)} \boldsymbol{\partial}_k \lambda(v)\, dv \right] ,$$

where, in the last equality, the term with $\left(\partial_\tau \lambda(v) \boldsymbol{\partial}_k \lambda(v)\right)/\lambda'(v) \propto \left(\partial_\tau \lambda(v) \wp^{(k-2)}(v)\right)/\lambda'(v)$ has disappeared because it is regular at $v = 0$ for $k = 2..l+1$.

In order to compute $\eta\left(\boldsymbol{\partial}_{-1}, \boldsymbol{\partial}_{-1}\right)$ we use a similar trick

$$\sum \underset{\lambda'=0}{res} \frac{(\partial_\tau \lambda)^2}{\lambda'} = \sum \underset{\lambda'=0}{res} \left\{ \frac{\left(\mathbb{D}^\bullet \lambda - \boldsymbol{\Sigma}\right)\left(\mathbb{D}^\bullet \lambda - \boldsymbol{\Sigma}\right) dv}{\lambda'} \right\} =$$

$$= -\underset{v=0}{res} \left\{ \frac{\left(\partial_\tau \lambda + \gamma\lambda'\right)\left(\partial_\tau \lambda + \gamma\lambda'\right) dv}{\lambda'} \right\} = -\underset{v=0}{res} \left\{ \frac{(\partial_\tau \lambda)^2\, dv}{\lambda'} + 2\gamma \partial_\tau \lambda\, dv + \gamma^2\, \lambda'\, dv \right\} =$$

$$= -\underset{v=0}{res} \left\{ \left(2\partial_\tau \lambda + \gamma\lambda'\right) \gamma\, dv \right\} .$$

2. The multiplication is defined as

$$\eta\left(\boldsymbol{\partial}_i, \boldsymbol{\partial}_j \cdot \boldsymbol{\partial}_k\right) = \sum \underset{\lambda'=0}{res} \frac{\boldsymbol{\partial}_i \lambda\, \boldsymbol{\partial}_j \lambda\, \boldsymbol{\partial}_k \lambda\, dv}{\lambda'} .$$

Again, if all indices are nonnegative we can evaluate the residue at zero changing sign; as for the remaining cases we get, after computations similar to those of before,

$$\eta\left(\boldsymbol{\partial}_i, \boldsymbol{\partial}_j \cdot \boldsymbol{\partial}_k\right) = -\underset{v=0}{res} \left(\frac{\boldsymbol{\partial}_i \lambda\, \boldsymbol{\partial}_k \lambda\, \boldsymbol{\partial}_k \lambda\, dv}{\lambda'} \right)$$

$$\eta\left(\boldsymbol{\partial}_{-1}, \boldsymbol{\partial}_i \cdot \boldsymbol{\partial}_j\right) = -\underset{v=0}{res} \left\{ \frac{\partial_\tau \lambda}{\lambda'} \boldsymbol{\partial}_i \lambda\, \boldsymbol{\partial}_j \lambda\, dv + \gamma\, \boldsymbol{\partial}_i \lambda\, \boldsymbol{\partial}_j \lambda\, dv \right\}$$

$$\eta\left(\boldsymbol{\partial}_{-1}, \boldsymbol{\partial}_{-1} \cdot \boldsymbol{\partial}_i\right) = -\underset{v=0}{res} \left\{ \left(2\partial_\tau \lambda + \gamma\, \lambda'\right) \boldsymbol{\partial}_i \lambda\, dv \right\}$$

$$\eta\left(\boldsymbol{\partial}_{-1}, \boldsymbol{\partial}_{-1} \cdot \boldsymbol{\partial}_{-1}\right) = -\underset{v=0}{res} \left\{ 3\gamma\, (\partial_\tau \lambda)^2 dv + 3\gamma^2\, \partial_\tau \lambda\, \lambda'\, dv + \gamma^3\, (\lambda')^2\, dv \right\} .$$

Applying Theorem 2.1 we find

Proposition 2.5 The flat coordinates of the invariant Frobenius metric η are (the principal values prescriptions are understood in the integrations)

$$\begin{aligned} t_0 &:= \oint_b dv = \int_0^\tau dv = \tau \\ t_1 &:= \oint_a \lambda(v) dv = \int_0^1 \lambda(v) dv = \varphi_0 + \left[\sum_{k=0}^{l-1} \frac{(-1)^{l+1-k}}{(l-k)!} \zeta^{(l-1-k)}(v) \varphi_{l+1-k} \right]_{0+\epsilon}^{1+\epsilon} = \\ &= \varphi_0 + 4i\pi g_1 \varphi_2; \end{aligned}$$

$$t_a \;:=\; \underset{v=0}{res}\left(v\lambda^{-\frac{a-1}{l+1}}(v)d\lambda(v)\right); \qquad a=2..l+1;$$

$$\eta(\partial_t,\partial_t)=\begin{pmatrix} 0 & \frac{1}{2i\pi} & 0 & 0\cdots 0 & 0 \\ \frac{1}{2i\pi} & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 0 & 0 & 0\cdots 0 & \frac{1}{l+1} \\ \vdots & \vdots & \vdots & \iddots & 0 \\ 0 & \cdots & \frac{1}{l+1} & 0\cdots 0 & 0 \end{pmatrix}$$

While the flat (local) coordinates of the intersection form are the zeroes of $\lambda(v)$, τ and $u := \frac{1}{2i\pi}\oint_a \log(\lambda)dv$.

Notice that the second Jacobi form φ_2 is always a quadratic polynomial in the flat coordinates $t_2,...,t_{l+1}$ and the lightest Jacobi form φ_{l+1} is a power of the coordinate t_{l+1} as we now prove.

Corollary 2.1 The second Jacobi form φ_2 satisfies

$$\varphi_2=\frac{1}{2}\sum_{i,j\neq 1,0}\eta_{t_i,t_j}t_it_j=\frac{1}{2(l+1)}\sum_{j=2}^{l+1}t_j\,t_{l+3-j}\ ,$$

while for the lightest one φ_{l+1} we find

$$\varphi_{l+1}=(-l-1)^{-(l+1)}(t_{l+1})^{l+1}\ .$$

Proof. The second statement is easily proven by computing the residue

$$t_{l+1}=\underset{v=0}{res}\left(v\lambda^{-\frac{l}{l+1}}d\lambda\right)=\underset{v=0}{res}\left(v\left(\frac{\varphi_{l+1}}{v^{l+1}}+...\right)^{-\frac{l}{l+1}}\left((-1)^{l+1}(l+1)\frac{\varphi_{l+1}}{v^{l+2}}+....\right)\right)=$$
$$=\underset{v=0}{res}\left((-1)^{l+1}(l+1)\varphi_{l+1}{}^{\frac{1}{l+1}}\frac{1}{v}+O(1)\right)\ .$$

As for the first statement, we introduce the local coordinate $z=(\lambda)^{-\frac{1}{l+1}}$ (choosing one branch of the root), and hence find ($\lambda=\frac{1}{z^{l+1}}$)

$$t_j=\underset{v=0}{res}\left(v\lambda^{\frac{j-1}{l+1}}d\lambda\right)=-(l+1)\underset{z=0}{res}\left(v\,z^{j-l-3}\,dz\right)\ , \qquad j=2,...,l+1\ ,$$

where we implicitly solve the equation $\lambda(v)=\frac{1}{z^{l+1}}$ for v as a function of z; in other words the flat coordinates $t_2,..,t_{l+1}$ are the coefficients in the expansion of v

$$v(z)=\frac{-1}{l+1}\left(t_{l+1}\,z+t_l\,z^2+...+t_2z^l+O(z^{l+2})\right)\ ,$$

in which the term in z^{l+1} vanishes because $\underset{v=0}{res}(vd\lambda)=0$. We now compute

$$\frac{1}{l+1}\sum_{j=2}^{l-1}t_j\,t_{l+3-j}=\underset{z=0}{res}\frac{(l+1)v^2(z)dz}{z^{l+1}}=-\underset{v=0}{res}\,v^2d\lambda=2\varphi_2\ ,$$

and this ends the proof. □

From this corollary it follows that

$$t_1 = \varphi_0 + 2i\pi \frac{\eta'}{\eta} \sum_{i,j\neq 0,1} \eta_{t_i,t_j} t_i\, t_j \ .$$

In order to complete the description of the Frobenius structure it is important to express the two vector fields $\mathbf{1}$ and $\mathbf{E}$ in these coordinates.
Recall that the unity vector field $\mathbf{1}$ in the coordinates $u_i \equiv \lambda(P_i)$ where P_i are the critical points, read $\mathbf{1} = \sum_{i=1}^{l+2} \frac{\partial}{\partial u_i}$, while the Euler vector field is given by $\sum_{i=1}^{l+2} u_i \frac{\partial}{\partial u_i}$. We now prove that

Proposition 2.6 The unity vector field $\mathbf{1}$ and the Euler vector field $\mathbf{E}$ in the coordinates $t_0, ..., t_{l+1}$ read,

$$\mathbf{1} = \frac{\partial}{\partial t_1} \ ; \qquad \mathbf{E} = \sum_{k=1}^{l+1} \frac{l+2-k}{l+1} t_k \frac{\partial}{\partial t_k} \ .$$

Proof. The effect of these two vectors on λ are of shifting or dilating it, hence they read $\mathbf{1} = \frac{\partial}{\partial \lambda}$ and $\mathbf{E} = \lambda \frac{\partial}{\partial \lambda}$; now, from Corollary 2.1 it follows that the vector $\frac{\partial}{\partial t_1}$ exactly shifts λ, while the expression for the Euler vector field comes from the degrees of the flat coordinates as stated in Proposition 2.4. □

Since the superpotential is invariant under the group $SL(2,\mathbb{Z})$, it follows easily that

Proposition 2.7 Under a transformation of the modular group with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, the flat coordinates transform as follows : they are invariant under the map corresponding to $\tau \mapsto \tau + 1$ and under the inversion $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we find

$$\begin{aligned} t_0 &\mapsto t_0' = -\frac{1}{t_0} \\ t_1 &\mapsto t_1' = t_1 + i\pi \frac{1}{t_0} \sum_{j=2}^{l+1} t_j\, t_{l+3-j} \\ t_j &\mapsto t_j' = \frac{1}{t_0} t_j \ , \qquad j = 2...l+1 \end{aligned}$$

namely, the flat coordinates $t_2, ..., t_{l+1}$ behave like modular forms of weight -1.

Proof. The transformation rule for $t_0 = \tau$ is obvious since it is the modular parameter of the torus; the one for t_1 follows immediately once expressing it in terms of the Jacobi form φ_0 and φ_2 and from the modular properties of the Dedekind η–function. As for the transformation rule for the remaining, this is a consequence of their definition as residues w.r.t. a weight -1 form which is $v\, \lambda^{\varkappa}\, d\lambda$ (recall that under the inversion the coordinate v on the universal covering of the torus is mapped to $\frac{v}{\tau}$). □

Twisted Frobenius structure

The Frobenius structure which has been constructed is well defined on $M_{1,0;l}$ only locally, because the diffeomorphism group of the curve C ($SL(2,\mathbb{Z})$ in this case, full modular group) acts in a nontrivial way by means of a *symmetry* (see [Du93] for details on symmetries of Frobenius structures). This picture can be interpreted in two ways; we can consider that the invariant metric η takes values in a suitable line bundle L over $M = M_{1,0;l}$; namely we consider the Frobenius structure to be well–defined on $L \otimes TM$ rather that on TM itself.
Equivalently we can consider the covering space $\widehat{M} = \widehat{M}_{1,0;l}$ and then we get a *bona fide* equivariant Frobenius structure on $T\widehat{M}$.

Notice that the coordinates t_j's have to be thought of as coordinates on $\widehat{M}$ in view of the branching around the surface $\varphi_{l+1} = 0$: in fact in the definition of $\widehat{M}$, the fixation of the branching of the root around the infinities makes t_{l+1} a one valued function.

2.4 Free energy

To complete this study we must give the structure constants of the bundle of Frobenius algebras on the tangent bundle of the manifold; in principle this could be done by changing coordinates from the φ_j's to the t_j's, since the structure constants form a tensor, but it is more useful and satisfactory to express the free energy in terms of the flat coordinates.
To achieve this goal we compute free energy F by means of the particular bilinear pairing of forms which is described in Appendix A.5; it is proven it [Du93] that the **free energy** for the Frobenius structure we have build is given by

$$F := -\frac{1}{2} < v\, d\lambda, v\, d\lambda > \ .$$

where we use, in the notation of the cited appendix, $\Phi = \Psi = v d\lambda$. In order to find all the due constants we first notice that no logarithmic polydromy is present in this differential; hence we are to find the coefficients c_k ($\equiv c_{\Phi,k}$ in the notation of the appendix): we already know the first l, which are (up to a constant) the flat coordinates $t_2, ..., t_l$. In order to find the remaining we have to write the expansion of the differential around the infinity point ∞_0 in terms of the local coordinate $z := (\lambda)^{-\frac{1}{l+1}}$ and therefore we must expand v as a function of z by means of the inversion formula

$$\lambda(v) = \frac{1}{z^{l+1}} \Rightarrow v = v(z) = -\frac{1}{l+1}\left(t_{l+1}z + t_l z^2 + ... + t_2 z^l + c_0 z^{l+2} + ... + c_{l-1} z^{2l+1} + O\left(z^{2l+2}\right)\right) ,$$

where we have numbered the coefficients in a convenient way for the following application.
Plugging into the formula we find

$$\begin{aligned} v\, d\lambda &= v(z) d\left(z^{-l-1}\right) = -(l+1)\frac{v(z)}{z^{l+2}} dz = \\ &= \left[t_{l+1}z^{-l-1} + t_l z^{-l} + ... + t_2 z^{-2} + c_0 + z c_1 + ... + c_{l-1} z^{l-1} + O\left(z^l\right)\right] dz , \end{aligned}$$

where we notice that $c_{-1} = 0$ because the differential $v\, d\lambda$ has no residue. As for the polydromy we compute

$$v d\lambda \underset{v \mapsto v+a}{\longrightarrow} v d\lambda - d\lambda \ \Rightarrow \ A(\lambda) \equiv \lambda$$

$$vd\lambda \underset{v\mapsto v+b}{\longrightarrow} vd\lambda - \tau d\lambda \;\Rightarrow\; B(\lambda) \equiv \tau\lambda \;.$$

Finally we can write the pairing

$$< vd\lambda, vd\lambda >= -\sum_{k=0}^{l-1} c_k \frac{1}{k+1} c_{-k-2} + \frac{1}{2i\pi}\oint_a vd\lambda \oint_b vd\lambda - \frac{\tau}{2i\pi}\oint_a \lambda\, v\, d\lambda + \frac{1}{2i\pi}\oint_b \lambda\, v\, d\lambda \;.$$

To compute the periods in the formula we recall that we have to realize the cycles a, b as paths with base–point a zero of λ; in the specific we can think of them as the segments $a \equiv [x, x+1]$ and $b \equiv [x, x+\tau]$ in the complex v plane (which realizes the universal covering of the torus; here x is a zero of λ in a fundamental mesh) and hence compute

$$\begin{aligned}
&\oint_{a(x)} v\, d\lambda = -\oint_a \lambda\, dv = -t_1 \;,\; \oint_a \lambda\, dv = \varphi_0 + 4i\pi g_1\varphi_2 =: t_1 \\
&\oint_{b(x)} v\, d\lambda = -\oint_b \lambda\, dv = -t_0 t_1 - 2i\pi\varphi_2 \;, \\
&\oint_b \lambda\, dv = \tau\varphi_0 + 4i\pi\tau g_1\varphi_2 + 2i\pi\varphi_2 = \tau t_1 + 2i\pi\varphi_2 =: t_0 t_1 + 2i\pi\varphi_2 \\
&\left(\oint_{a(x)} v\, d\lambda\right)\left(\oint_{b(x)} v\, d\lambda\right) = t_0 (t_1)^2 + 2i\pi t_1\varphi_2 \;.
\end{aligned}$$

The other periods give

$$\begin{aligned}
&\oint_{a(c)} B(\lambda)\, v\, d\lambda = \tau \oint_{a(c)} v\lambda d\lambda = -\frac{\tau}{2}\oint_a \lambda^2\, dv \\
&\oint_{b(c)} A(\lambda)\, v\, d\lambda = \oint_{b(c)} v\lambda d\lambda = -\frac{1}{2}\oint_b \lambda^2\, dv \\
&\oint_{b(c)} A(\lambda)\, v\, d\lambda - \oint_{a(c)} B(\lambda)\, v\, d\lambda = \frac{\tau}{2}\oint_a \lambda^2\, dv - \frac{1}{2}\oint_b \lambda^2\, dv = \\
&\qquad = -\frac{1}{2}\left(\oint_b - \tau\oint_a\right)\lambda^2(v) dv = -i\pi \operatorname*{res}_{v=0}\left(v\,\lambda^2(v)\, dv\right)
\end{aligned}$$

where we notice that the differential $\lambda^2\, dv$ has only one pole at $v = 0$ and no residue[8]: we have then used formula (35) so that finally (recall that $\tau = t_0$)

Proposition 2.8 The *Free energy* of the Frobenius structure associated to the primary differential dv is

$$\begin{aligned}
&F := -\frac{1}{2} < v\, d\lambda, v\, d\lambda >= \\
&= \frac{1}{2}\sum_{k=0}^{l-1} c_{-2-k}\frac{1}{k+1}c_k - \frac{1}{4i\pi}\left[\int_{b(x)} v\, d\lambda \int_{a(x)} v\, d\lambda - \tau\int_{a(x)} v\,\lambda d\lambda + \int_{b(x)} v\,\lambda d\lambda\right] =
\end{aligned}$$

[8]This follows from the fact that λ^2 is an elliptic function with only one pole, and from the general fact that the sum of residues in a fundamental mesh of an elliptic function is zero

$$= \frac{1}{2}\sum_{k=0}^{l-1} c_k \frac{1}{k+1} c_{-k-2} - \frac{1}{4i\pi} t_0(t_1)^2 - \frac{1}{2} t_1 \varphi_2 + \frac{1}{4} \underset{v=0}{res} \left(v\, \lambda^2(v)\, dv \right) =$$
$$= \frac{1}{2}\sum_{k=0}^{l-1} c_k \frac{1}{k+1} t_{k+2} - \frac{1}{4i\pi} t_0(t_1)^2 - \frac{1}{2} t_1 \varphi_2 + \frac{1}{4} \underset{v=0}{res} \left(v\, \lambda^2(v)\, dv \right) .$$

We will give an explicit example in the case of A_2 later.

2.5 Flat coordinates of the intersection form

We now analyze the structure of the flat coordinates of the intersection form. This enables to identify this Frobenius manifold as a suitable covering of the orbit space of the Jacobi group of type A_l.
In particular it will become clear in which sense the moduli $\varphi_k 4$ are Jacobi forms; the point is that as functions of the flat coordinates of the intersection form, they are exactly the previously studied generators of the algebra $J_{\bullet,\bullet}$.

From Thm. 2.1 we know that the flat coordinates of the intersection form are the functions $v(Q_i)$ with Q_i a zero of the superpotential λ, and $\oint_b dv = \tau$ and $u = \frac{1}{2i\pi}\oint_a \log(\lambda) dv$.

To begin with, the number of zeroes of $\lambda(v)$ is $l+1$ because λ is an elliptic function with a pole of order $l+1$ at the origin; they are linearly related since the divisor of zeroes must be congruent to the divisor of poles. Therefore we have $\sum_1^{l+1} z_i = 0 \bmod (\mathbb{Z} + \tau\mathbb{Z})$.
We already know how to express the parameters $\varphi_0, \varphi_2, ..., \varphi_{l+1}$ in terms of the zeroes of $\lambda(v)$; this follows from the explicit construction of the Jacobi forms for A_l, which is accomplished in eq. (11). This clearly leaves an arbitrariness, since the knowledge of the divisor of zeroes fixes a function modulo multiplication by a nonzero number; this multiplicative coefficient will be denoted by $e^{2i\pi s}$.
The explicit formula which expresses the superpotential as a function of its zeroes $z_1 ... z_{l+1}$ (s.t. $\sum_1^{l+1} z_i = 0$), of τ and s is thus

$$\lambda(v) = e^{2i\pi s + 2i\pi g_1 \|\mathbf{z}\|^2} \frac{\prod_1^{l+1} \sigma(z_i - v)}{\sigma^{l+1}(v)} =$$
$$= \frac{-1}{l!} \underbrace{\left(e^{2i\pi s} \prod_{j=1}^{l+1} \alpha(z_j) \right)}_{=\varphi_{l+1}} \frac{\det \begin{pmatrix} 1 & \wp(v) & \wp'(v) & \dots & \wp^{(l-1)}(v) \\ 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-1)}(z_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-1)}(z_l) \end{pmatrix}}{\det \begin{pmatrix} 1 & \wp(z_1) & \wp'(z_1) & \dots & \wp^{(l-2)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \dots & \wp^{(l-2)}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(z_l) & \wp'(z_l) & \dots & \wp^{(l-2)}(z_l) \end{pmatrix}}$$

In order to identify completely the flat coordinates of the intersection form with the coordinates we used in constructing the Jacobi forms of A_l in formula (11) we have to compute explicitly the flat coordinate u; to this end we give the

Lemma 2.6 The flat coordinate $u = \frac{1}{2i\pi}\oint_a dv \log(\lambda)$ equals exactly s.

Proof. We have

$$u := \frac{1}{2i\pi}\oint_a dv\,\log(\lambda) = \frac{1}{2i\pi}\int_0^1 dv\,\log(\lambda) = s + g_1\|\mathbf{z}\|^2 + \frac{1}{2i\pi}\int_0^1 dv\,\log\left(\frac{\prod_{j=1}^{l+1}\sigma(v-z_j)}{\sigma^{l+1}(v)}\right) .$$

Considering this function depending on z_i we have

$$\frac{\partial}{\partial z_k}\int_0^1 dv\left[\sum_{j=1}^{l+1}\log\left(\frac{\sigma(v-z_i)}{\sigma(v)}\right)\right] = \int_0^1 dv\,\zeta(v-z_k) =$$
$$= \log\left(\frac{\sigma(1-z_k)}{\sigma(z_k)}\right) = \left(-4i\pi\frac{\eta'}{\eta}z_k\right) = -2i\pi g_1\frac{\partial}{\partial z_k}(\|\mathbf{z}\|^2) .$$

Since for $\mathbf{z}=0$ the integral is obviously zero we have finally

$$\int_0^1 dv\left[\sum_{j=1}^{l+1}\log\left(\frac{\sigma(v-z_i)}{\sigma(v)}\right)\right] = -2i\pi\frac{\eta'}{\eta}\|\mathbf{z}\|^2 ,$$

and therefore $s \equiv u$. □

From this explicit formula we recognize that the Hurwitz moduli $\varphi_{-1}, \varphi_0, \varphi_2, ..., \varphi_{l+1}$ –as functions of the flat coordinates of the intersection form, $\tau, u, x_1, ..., x_l$– are exactly the invariant Jacobi forms constructed in 11 for the Jacobi group $\mathbb{J}(A_l)$; we have thus proven

Theorem 2.2 The Hurwitz moduli $\varphi_{-1} = \tau$, $\varphi_0, \varphi_2, ..., \varphi_{l+1}$ as functions of the flat coordinates of the intersection form $u, z_1, ..., z_{l+1}, \tau$ (where $\sum z_i = 0$), are the Jacobi forms for the Jacobi group of type A_l.
Moreover the intersection form in Proposition 2.3 coincides with the intersection form in Definition 1.6.

2.6 Monodromy

By means of the identification in Thm. 2.2 between the moduli $(\varphi_0, \varphi_2, ..., \varphi_{l+1}, \tau)$ on the space $\hat{M}_{1,0;l}$ and the Jacobi forms for the Jacobi group associated to A_l, we have therefore constructed a Frobenius structure on a suitable covering of the orbit space $\frac{\Omega}{\mathbb{J}} := \frac{\mathbb{C}\oplus\mathbb{C}^l\oplus\mathcal{H}}{\mathbb{J}}$. From the above formulae expressing the flat coordinates in terms of the Jacobi forms it is clear that the multivaluedness of this covering comes uniquely from the $t_l = -(l+1)(\varphi_{l+1})^{\frac{1}{l+1}}$ coordinate; looking at the explicit form (defining $x_{l+1} := 0 =: x_0$)

$$\varphi_{l+1}(x_1, .., x_l) = \prod_{j=1}^{l+1}\alpha(x_i - x_{i-1})$$

we realize that the zeroes are situated at the walls (recall that $\alpha_k^\vee$ are the coroots spanning over $\mathbb{Z}$ the lattice Λ of A_l in $\mathbb{C}^{l+1}$)

$$\alpha_j^\vee(\mathbf{x}) = 0 \bmod \mathbb{Z} + \tau\mathbb{Z} .$$

namely at the walls of the alcove for the complex crystallographic group

$$\mathcal{A} := \{\mathbf{x} \in \mathfrak{h} \mid \alpha_j^\vee(\mathbf{x}) \in \mathcal{A}_0\}$$

where $\mathcal{A}_0$ is the fundamental mesh of $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.
In other words for fixed τ the quotient of $\mathfrak{h} \simeq \mathbb{C}^l$ by the complex crystallographic group is a torus $\mathbb{T}^l$ which is simply the product of identical tori of modular parameter τ; our Frobenius structure lives on a $l+1$ sheeted covering of this torus with branching divisor $Y := \{\varphi_{l+1} = 0\}$.

Example 2.1 The case $\mathbf{A_1}$ This example was worked out explicitly also in [Du93] but it is useful to use the present formalism. The superpotential is

$$\lambda(v) := \wp(v)\varphi_2 + \varphi_0 \ ,$$

and the Jacobi forms read

$$\varphi_2(u, x, \tau) = -e^{2i\pi u}\alpha^2(x) \ ; \qquad \varphi_0(u, x, \tau) = -e^{2i\pi u}\alpha^2(x)\wp(x) \ .$$

The flat coordinates of the invariant metric η are

$$t_0 := \tau \ ; \qquad t_1 := \varphi_0 + 4i\pi g_1\varphi_2 \ ; \qquad t_2 := -2\sqrt{\varphi_2} = -2\alpha(x)$$
$$\eta := \begin{pmatrix} 0 & \frac{1}{2i\pi} & 0 \\ \frac{1}{2i\pi} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} .$$

The free energy is found to be

$$F^{(A_1)} := \frac{1}{4i\pi}t_0(t_1)^2 + \frac{1}{4}t_1(t_2)^2 - \frac{i\pi}{8}g_1(t_0)(t_2)^4$$

Notice that these flat coordinates live on a double covering of the quotient space $\mathbb{C}\oplus\mathbb{C}\oplus\mathcal{H}/\mathbb{J}$, inasmuch the coordinate t_2 changes sign under the action of the Weyl group (in this case the Weyl group is simply $\mathbb{Z}_2$ acting as $x \mapsto -x$); this comes from the fact that t_2 is a square-root of the truly invariant lightest Jacobi form φ_2, and this is what happens in the general case.

Example 2.2 The case $\mathbf{A_2}$ We have the superpotential

$$\lambda(v) := -\frac{1}{2}\wp'(v)\varphi_3 + \wp(v)\varphi_2 + \varphi_0$$

where the numbers $\varphi_3, \varphi_2, \varphi_0$ in terms of the zeroes of the superpotential read (we set $z_1 = x_1, z_2 = x_2 - x_1, z_3 = -x_2$)

$$\begin{aligned}
\varphi_3(u, \mathbf{x}) &= -2e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3); \\
\varphi_2(u, \mathbf{x}) &= e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\frac{\wp'(x_1) - \wp'(x_2 - x_1)}{\wp(x_1) - \wp(x_2 - x_1)}; \\
\varphi_0(u, \mathbf{x}) &= e^{2i\pi u}\alpha(z_1)\alpha(z_2)\alpha(z_3)\frac{\wp(x_1)\wp'(x_2 - x_1) - \wp'(x_1)\wp(x_2 - x_1)}{\wp(x_2 - x_1) - \wp(x_1)} \ .
\end{aligned}$$

The flat coordinates of η and its entries are

$$t_0 = \tau; \qquad t_2 = -\frac{\varphi_2}{(\varphi_3)^{\frac{1}{3}}}; \qquad t_3 = -3(\varphi_3)^{\frac{1}{3}};$$

$$t_1 = \oint_a \lambda\, dv = \varphi_0 + 4i\pi g_1 \varphi_2 = \varphi_0 + \frac{4i\pi}{3} g_1 t_2 t_3 = \varphi_0 + \frac{2i\pi}{3}\frac{\eta'}{\eta}\sum_{j=2}^{3} t_j\, t_{l+1-j}\ ;$$

$$\varphi_0 = t_1 - \frac{4}{3} i\pi g_1 t_2 t_3$$

$$\varphi_2 = \frac{1}{3} t_2 t_3; \qquad \varphi_3 = -\frac{1}{27}(t_3)^3\ ;$$

$$\eta_{t_i,t_j} = \begin{pmatrix} 0 & \frac{1}{2i\pi} & 0 & 0 \\ \frac{1}{2i\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}$$

To compute the free energy we have to find the coefficients $c_{-3}, c_{-2}, c_{-1}, c_0, c_1$ in the expansion

$$\lambda(v) = z^{-3} \Rightarrow v(z) = c_{-3}z + c_{-2}z^2 + c_{-1}z^3 + c_0 z^4 + c_1 z^5 + O(z^6)$$

and then plug into the general formula; the result of this rather long but strighforward computation is

$$c_{-3} = -\frac{1}{3} t_3\ ; \qquad c_{-2} = -\frac{1}{3} t_2\ ; \qquad c_{-1} = 0$$

$$c_0 = -\frac{1}{9}\, t_3\, t_1 + \frac{1}{9}\frac{{t_2}^3}{{t_3}^2} + \frac{2}{27}\, i\pi g_1\, {t_3}^2\, t_2$$

$$c_1 = -\frac{2}{9}\, t_2\, t_1 - \frac{1}{9}\frac{{t_2}^4}{{t_3}^3} + \frac{4}{27}\, i\pi g_1\ t_3\, {t_2}^2 + \frac{1}{14580}\, {t_3}^5\, g_2(\tau)$$

and the free energy reads

$$F^{(A_2)} := \frac{1}{4i\pi} t_0\, {t_1}^2 + \frac{1}{3}\, t_2\, t_3\, t_1 - \frac{1}{12}\frac{t_2^4}{{t_3}^2} - \frac{2i\pi}{9} {t_3}^2 {t_2}^2 g_1(t_0) - \frac{1}{2^4\, 3^6\, 5}\, {t_3}^6\, g_2(t_0)\ .$$

These computations can be handled algorithmically by a computer; just for fun we list the next free energies of the series A_l.

$$\begin{aligned} F^{(A_3)} := {} & \frac{1}{4}\frac{i\, t_0\, {t_1}^2}{\pi} + \frac{1}{4}\, t_4\, t_1\, t_2 + \frac{1}{8}\, {t_3}^2\, t_1 + \frac{1}{6}\frac{{t_2}^3}{t_4} - \frac{1}{2}\frac{{t_3}^2\, {t_2}^2}{{t_4}^2} - \frac{1}{8}\, i\, {t_4}^2\, \pi\, g_1(t_0)\, {t_2}^2 + \frac{1}{4}\frac{t_2\, {t_3}^4}{{t_4}^3} \\ & - \frac{1}{8}\, i\, t_4\, \pi\, g_1(t_0)\, {t_3}^2\, t_2 + \frac{1}{61440}\, {t_4}^5\, g_2(t_0)\, t_2 - \frac{1}{24}\frac{{t_3}^6}{{t_4}^4} - \frac{1}{32}\, i\, \pi\, g_1(t_0)\, {t_3}^4 \\ & - \frac{1}{24576}\, {t_3}^2\, {t_4}^4\, g_2(t_0) + \frac{1}{5505024}\, g_3(t_0)\, {t_4}^8 \end{aligned}$$

$$\begin{aligned} F^{(A_4)} := {} & -\frac{1}{4}\frac{i\, t_0\, {t_1}^2}{\pi} + \frac{1}{5}\, t_5\, t_1\, t_2 + \frac{1}{5}\, t_4\, t_3\, t_1 + \frac{1}{2}\frac{{t_2}^2\, t_3}{t_5} - \frac{1}{2}\frac{{t_2}^2\, {t_4}^2}{{t_5}^2} - \frac{2}{25}\, i\, {t_5}^2\, \pi\, g_1(t_0)\, {t_2}^2 \\ & - \frac{t_4\, t_2\, {t_3}^2}{{t_5}^2} + \frac{t_3\, t_2\, {t_4}^3}{{t_5}^3} - \frac{4}{25}\, i\, t_5\, \pi\, g_1(t_0)\, t_3\, t_2\, t_4 - \frac{1}{5}\frac{t_2\, {t_4}^5}{{t_5}^4} + \frac{1}{37500}\, {t_5}^4\, g_2(t_0)\, t_2\, t_4 - \frac{1}{8}\frac{{t_3}^4}{{t_5}^2} \\ & + \frac{2}{3}\frac{{t_4}^2\, {t_3}^3}{{t_5}^3} - \frac{3}{4}\frac{{t_3}^2\, {t_4}^4}{{t_5}^4} - \frac{2}{25}\, i\, \pi\, g_1(t_0)\, {t_3}^2\, {t_4}^2 - \frac{1}{50000}\, {t_5}^4\, g_2(t_0)\, {t_3}^2 + \frac{3}{10}\frac{t_3\, {t_4}^6}{{t_5}^5} \\ & - \frac{1}{75000}\, {t_5}^3\, g_2(t_0)\, t_3\, {t_4}^2 - \frac{1}{4375000}\, {t_5}^7\, g_3(t_0)\, t_3 - \frac{1}{24}\frac{{t_4}^8}{{t_5}^6} - \frac{1}{50000}\, g_2(t_0)\, {t_5}^2\, {t_4}^4 \\ & + \frac{1}{2^3 5^7 3}\, g_3(t_0)\, {t_5}^6\, {t_4}^2 - \frac{1}{2^7 5^{11} 3}\, g_2(t_0)^2\, {t_5}^{10} \end{aligned}$$

2.7 The Frobenius structures of the orbit space of $\mathbb{J}(G_2)$

The exceptional root system G_2 displays an interesting and unexpected double Frobenius structure in the sense we are going to explain. We have seen in section 1.5 that the Jacobi algebra $J^{(G_2)}_{\bullet,\bullet}$ is naturally embedded as a subalgebra of $J^{(A_2)}_{\bullet,\bullet}$; hence, the flat structure on $Spec(J^{(A_2)}_{\bullet,\bullet})$ can be interpreted as a flat structure also on the (suitable covering of the) orbit space of the Jacobi group $\mathbb{J}(G_2)$. Indeed, recall that the Jacobi forms of G_2 are given by

$$\varphi_0^{(G_2)} = \varphi_0^{(A_2)}; \qquad \varphi_2^{(G_2)} = \varphi_2^{(A_2)}; \qquad \varphi_6^{(G_2)} = \frac{1}{2}\left(\varphi_3^{(A_2)}\right)^2$$

and therefore the flat coordinates associated to the invariant metric $\eta^*(,) = \mathcal{L}_{\frac{\partial}{\partial\varphi_0}}\mathfrak{I}^*(,)$ are exactly the same as those computed in example 2.2, namely

$$\varphi_0^{(G_2)} = t_1 - \frac{4}{3}i\pi g_1(\tau)t_2t_3; \qquad \varphi_2^{(G_2)} = \frac{1}{3}t_2t_3; \qquad \varphi_6^{(G_2)} = \frac{1}{2\,27^2}(t_3)^6\ ;$$

therefore we have a Frobenius structure inherited from the one of A_2, where the free energy is given by $F^{(G_2)} = F^{(A_2)} := \frac{1}{4i\pi}t_0\,{t_1}^2 + \frac{1}{3}\,t_2\,t_3\,t_1 - \frac{1}{12}\frac{t_2^4}{{t_3}^2} - \frac{2i\pi}{9}{t_3}^2{t_2}^2 g_1(t_0) - \frac{1}{2^4\,3^6\,5}\,{t_3}^6\,g_2(t_0)$.

On the other hand G_2 falls also into the category of "codimension one cases" [Sa90], and hence it has another natural flat structure which we have computed in Prop. 1.7. An easy computation of indices shows that Saito's tensor $J^* := \mathcal{L}_{\frac{\partial}{\partial\widehat{\varphi_6}}}\mathfrak{I}^*$ form a linear pencil of flat metric with the intersection form, $\mathfrak{I}^* + \lambda J^*$. As a consequence there exists a unique associated Frobenius structure whose free energy can be recovered by application of the formulae described in section 2.1.2. In order to find it, we only have to find the scaling dimensions q_j and the Euler vector. Now, since the unit vector must have scaling dimension -1, and is given by $\mathbf{1} := \frac{\partial}{\partial\widehat{\varphi_6}}$, then we must take as Euler vector

$$E := \frac{1}{2}\mathbf{E} = \frac{1}{4i\pi}\partial_u = t_6\frac{\partial}{\partial t_6} + \frac{t_1}{2}\frac{\partial}{\partial t_1} + \frac{t_2}{2}\frac{\partial}{\partial t_2} = \varphi_6\frac{\partial}{\partial\varphi_6} + \frac{\varphi_0}{2}\frac{\partial}{\partial\varphi_0} + \frac{\varphi_2}{2}\frac{\partial}{\partial\varphi_2}\ ;$$

from this expression we read off the the scaling degrees, $q_\tau = 1$, $q_1 = q_2 = \frac{1}{2}$, $q_6 = 0$ (namely, $d_\tau = 0, d_1 = d_2 = \frac{1}{2}, d_6 = 1$) and the scaling of the metric of equation (24), namely $\mathcal{L}_{\mathbf{1}}J = J$. The free energy of this Frobenius–Saito structure is found to be

$$\begin{aligned} F^{(G_2)}_{(Saito)} \ := \ & \frac{1}{8i\pi}\,\tau\,{t_6}^2 + \frac{1}{2}\,t_6t_1t_2 + \frac{i\pi^2\sqrt{3}}{36}\,(2z)^{1/3}\left({t_1}^4 - 2\,{t_2}^3t_1\right)\boldsymbol{\eta}^4 + \\ & -i\pi g_1(t_0){t_2}^2{t_1}^2 + \frac{i\pi^2\sqrt{3}}{36}\,(2(z-1))^{1/3}\,\left({t_2}^4 - 2\,t_2{t_1}^3\right)\boldsymbol{\eta}^4\ , \end{aligned}$$

where we recall the definition $z(\tau) := \frac{1}{2}\left[\frac{\sqrt{-27}}{(2\pi)^6}\widehat{g_3}(\tau) + 1\right]$. Again this expression follows from a straightforward but quite long computation which we spare. Notice that this free energy is well–behaved in t_1, t_2, t_6 and has singularities only in τ coming from the Dedekind's eta function and from the branching around the divisor $z(\tau) = 0$ and $z(\tau) = 1$, which both correspond to the $\mathbb{Z}_3$–symmetric torus with $g_2(\tau) = 0$.

Summarizing, we have two Frobenius structures in which the Euler and unit vectors are

1. $\mathbf{E} = \varphi_0\partial_{\varphi_0} + \varphi_2\partial_{\varphi_2} + 2\varphi_6\partial_{\varphi_6}$ and $\mathbf{1} = \partial_{\varphi_0}$ for the Frobenius structure inherited from the one of A_2;

2. $\mathbf{E} = \frac{1}{2}\varphi_0\partial_{\varphi_0} + \frac{1}{2}\varphi_2\partial_{\varphi_2} + \varphi_6\partial_{\varphi_6}$ and $\mathbf{1} = \partial_{\varphi_6}$ for the Frobenius structure associated with Saito's flat structure.

The situation is that we have a two–dimensional linear pencil of flat metrics; by this we mean that the (contravariant) metric

$$G^* := \mathfrak{I}^* + a\,\eta^* + b\,J^* \ ,$$

is flat for any choice of the constants $a, b \in \mathbb{C}$ and the (contravariant) Levi–Civita connection is the same linear combination of the corresponding connections.

The fact that $\mathfrak{I}^*$ is linear in φ_0 is not predictable from the counting of bi–grades; in fact one could expect *a priori* the occurrence of a term proportional to ${\varphi_0}^2$ in $\mathfrak{M}(\varphi_0, \varphi_2) \in J_{0,2}^{(G_2)}$, which is not the case –as a matter of fact– as we saw in the explicit computation.
In the picture of [Sa90] the flat structure we have constructed in the case of A_l, and –as a by-product – in the case of G_2 is unexpected; it essentially comes from the additional structure provided by the modular properties of the Jacobi forms, which are "unseen" in the framework of the theta invariants studied in loc. cit.

There are hints at the possibility that a similar double Frobenius structure could be found also in the other exceptional cases F_4, E_6, E_7 (the latter two corresponding, in Saito's notation, to the "simple–elliptic cases" $\widetilde{E_6}, \widetilde{E_7}$), but this goes far beyond the scope of the present paper; we will study these structures in further publications.

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A Formulae

In these appendix we list and derive all formulae involving the elliptic functions that are used in the previous sections, and we explain the pairing of forms used in the context of Hurwitz spaces.

A.1 Normalized elliptic functions

In the text we have always used the normalized elliptic functions.
In order to match with the more common formulae we give here the table of conversion.

Here *normalized* means that the periods are 1 and τ while usually the half periods are ω_1, ω_2, where $\tau = \frac{\omega_2}{\omega_1}$. We define the Weierstrass eta functions by means of the quasi–periodicity of the unnormalized Weierstrass zeta function

$$\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1$$

$$\begin{aligned}
&\zeta(z+2\omega_2) = \zeta(z) + 2\eta_2 \\
&\eta_1 := \zeta(\omega_1)\ ; \qquad \eta_2 := \zeta(\omega_2) \\
&\eta_1\,\omega_2 - \eta_2\omega_1 = \frac{i\pi}{2}\ .
\end{aligned}$$

If $\boldsymbol{\eta}(\tau)$ is the Dedekind's eta function we have the following relation

$$\begin{aligned}
2\omega_1\eta_1 &= -2i\pi g_1(\tau) \\
2\omega_1\eta_2 &= -2i\pi\,\tau\,g_1(\tau) - i\pi\ ,
\end{aligned}$$

and hence the quasi–periodicity of the normalized zeta function (here –and only here– we put a subscript $_N$ to mean the normalized function)

$$\begin{aligned}
&\zeta_N(v) := 2\omega_1\zeta(z) \\
&\zeta_N(v+1) = \zeta_N(v) - 4i\pi g_1(\tau) \\
&\zeta_N(v+\tau) = \zeta_N(v) - 4i\pi\,\tau\,g_1(\tau) - 2i\pi\ .
\end{aligned}$$

In the previous sections we have always used normalized elliptic functions.

A.2 The elliptic connection $\mathbb{D}^\bullet$

Proposition A.1 Let $F(v|\tau)$ be an elliptic function of weight k, namely

$$F\left(\frac{v}{\tau}\middle| -\frac{1}{\tau}\right) = \tau^k F(v|\tau)\ , \tag{33}$$

then the function

$$(\mathbb{D}^{(k)}F)(v|\tau) := \nabla_\tau F(v|\tau) - \frac{\alpha'(v,\tau)}{2i\pi\alpha(v,\tau)}F'(v|\tau) := \boldsymbol{\eta}^{2k}\partial_\tau\left(\boldsymbol{\eta}^{-2k}F(v|\tau)\right) - \frac{\alpha'(v,\tau)}{2i\pi\alpha(v,\tau)}F'(v|\tau)$$

is an elliptic function of weight $k+2$.

Proof. Differentiating both members of eq. (33) w.r.t. τ we get

$$(\partial_\tau F)\left(\frac{v}{\tau}\middle| -\frac{1}{\tau}\right) = \tau^{k+2}\partial_\tau F(v|\tau) + k\tau^{k+1}F(v|\tau) + \tau^{k+3}vF'(v|\tau);$$

$$(\partial_\tau F)(v+\tau|\tau) = \partial_\tau F(v|\tau) - F'(v|\tau); \qquad (\partial_\tau F)(v+1|\tau) = \partial_\tau F(v|\tau)\ ,$$

and in particular we find

$$\begin{aligned}
&(\nabla_\tau F)\left(\frac{v}{\tau}\middle| -\frac{1}{\tau}\right) = \tau^{k+2}\nabla_\tau F(v|\tau) + \tau^{k+3}\,vF'(v|\tau)\ ; \\
&(\nabla_\tau F)(v+\tau|\tau) = \nabla_\tau F(v|\tau) - F'(v|\tau)\ .
\end{aligned}$$

Recall now that

$$\gamma(v|\tau) := \frac{1}{2i\pi}\frac{\alpha'(v|\tau)}{\alpha(v|\tau)} = 2vg_1(\tau) + \frac{\zeta(v|\tau)}{2i\pi}$$

$$\gamma\left(\frac{v}{\tau}\middle| -\frac{1}{\tau}\right) = \tau\gamma(v|\tau) + v\tau^2; \qquad \gamma(v+\tau) = \gamma(v|\tau) - 1\ ; \qquad \gamma(v+1|\tau) = \gamma(v|\tau)\ .$$

Therefore we finally have

$$\begin{aligned}
&(\mathbb{D}^{(k)}F)\left(\frac{v}{\tau}\middle| -\frac{1}{\tau}\right) = \tau^{k+2}\,\mathbb{D}^{(k)}F(v|\tau)\ ; \qquad (\mathbb{D}^{(k)}F)(v|\tau+1) = \mathbb{D}^{(k)}F(v|\tau)\\
&(\mathbb{D}^{(k)}F)(v+\tau|\tau) = \mathbb{D}^{(k)}F(v|\tau)\ ; \qquad (\mathbb{D}^{(k)}F)(v+1|\tau) = \mathbb{D}^{(k)}F(v|\tau)\ .
\end{aligned}$$

This ends the proof. □

In order to interpret this operator we consider the **universal torus**, namely the fibration over $(\mathbb{C}^2)_+ := \left\{(\omega,\omega')\ s.t.\ \frac{\omega'}{\omega} \in \mathcal{H}\right\}$ whose fiber is the unnormalized torus $\mathbb{C}/(2\omega\mathbb{Z} + 2\omega'\mathbb{Z})$, or better

$$\begin{array}{c} E_\Lambda \\ \downarrow \\ \mathcal{M} := \ \mathcal{L}\ , \end{array}$$

where $\mathcal{L}$ is the set of all lattices $\Lambda := 2\omega\mathbb{Z} + 2\omega'\mathbb{Z}$.
Cursorily speaking we can take $(z;\omega,\omega')$ as local coordinates over $\mathcal{M}$ (which can be seen also as a smooth principal fiber bundle with structure group $S^1 \times S^1$).
The analytic sections of this bundle are the (unnormalized) elliptic functions with periods $2\omega, 2\omega'$.
Over this fiber bundle we have three natural vector fields [Du93]

$$\begin{aligned}
D_1 &:= \omega\frac{\partial}{\partial\omega} + \omega'\frac{\partial}{\partial\omega'} + z\frac{\partial}{\partial z}\\
D_2 &:= \frac{\partial}{\partial z}\\
D_3 &:= \eta_1\frac{\partial}{\partial\omega} + \eta_2\frac{\partial}{\partial\omega'} + \zeta\frac{\partial}{\partial z}
\end{aligned}$$

where the ζ function is the unnormalized one.

Consider now the projection over the **universal elliptic curve** $\mathcal{E}$ defined as

$$\begin{array}{cccc} \Pi: & \mathcal{M} & \longrightarrow & \mathcal{E} \\ & (z;\omega,\omega') & \mapsto & (v|\tau) := \left(\frac{z}{2\omega}\middle|\frac{\omega'}{\omega}\right) \end{array}$$

and the push–forward of the above three vectors. It appears that D_1 is the vertical vector field and Π realizes the universal torus $\mathcal{M}$ as a trivial $\mathbb{C}^*$–principal fibre–bundle over the universal elliptic curve; the other two vectors are horizontal and hence they induce a natural connection over this fibration $\Pi : \mathcal{M} \to \mathcal{E}$.
The sections of the associated line bundle is the sheaf $\mathcal{S}$ of *homogeneous elliptic functions*

$$\begin{aligned}
&f(c\,z;c\omega,c\omega') = c^{-k}f(z;\omega,\omega') \in \mathcal{S}^{\otimes k}\\
&f(z+2m\omega+2n\omega';\omega,\omega') = f(z;\omega,\omega')\ ,
\end{aligned}$$

and the horizontal connection D induced by the vectors D_2, D_3 is

$$D_z = \frac{\partial}{\partial v}$$

$$D_\tau := D_\tau^{(k)} = \eta^{2k}\frac{\partial}{\partial\tau}\eta^{-2k} - \frac{\alpha'(v|\tau)}{2i\pi\alpha(v|\tau)}\frac{\partial}{\partial v} .$$

It follows that the elliptic connection $\mathbb{D}^{(k)}$ is just the horizontal vector D_τ acting on section of the sheaf $\mathcal{S}^{\otimes k}$. Since we also have a natural structure of graded algebra, we see that we can define the connection on the whole algebra by

$$\mathbb{D}^\bullet := \bigoplus_{j=0}^{\infty} \mathbb{D}^{(k)} .$$

A.3 Pseudo addition formulae

Proposition A.2 The following formula holds

$$\left(\frac{\alpha''(x)}{\alpha(x)} + \frac{\alpha''(y)}{\alpha(y)} - 2\frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)}\right) = -4i\pi\frac{\partial_\tau\alpha(x-y)}{\alpha(x-y)} + 2\frac{\alpha'(x-y)}{\alpha(x-y)}\left\{\frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)}\right\} .$$

Proof. Recall first that $\alpha(x) = \frac{1}{2\pi}\eta^{-3}e^{2i\pi\frac{\eta'}{\eta}x^2}\sigma(x)$ where σ is the Weierstrass sigma function. Recall also the standard notation

$$\zeta := \frac{d}{dx}\log(\sigma); \qquad \wp := -\frac{d}{dx}\zeta = \frac{d^2}{dx^2}\log(\sigma) .$$

Thus we can compute[9] (putting $A := 2i\pi\frac{\eta'}{\eta}$)

$$\left(\frac{\alpha''(x)}{\alpha(x)} + \frac{\alpha''(y)}{\alpha(y)} - 2\frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)}\right) =$$
$$= 4A + 4A^2(x-y)^2 + 4A(x-y)\left(\zeta(x) - \zeta(y)\right) + \frac{\sigma''(x)\sigma(y) + \sigma''(y)\sigma(x) - 2\sigma'(x)\sigma'(y)}{\sigma(x)\sigma(y)}$$

and the long fraction involving the sigma functions can be rewritten as

$$\frac{\sigma(y)\sigma''(x) + \sigma(x)\sigma''(y) - 2\sigma'(x)\sigma'(y)}{\sigma(x)\sigma(y)} = -\wp(x) - \wp(y) + (\zeta(x) - \zeta(y))^2 .$$

We now need the following pseudo–addition theorem for the ζ function

Theorem A.1 [see [WW], pag 446] If $x + y + z = 0$ then

$$[\zeta(x) + \zeta(y) + \zeta(z)]^2 = \wp(x) + \wp(y) + \wp(z)$$

[9] We use the straightforward formulae

$$\alpha'(x) = e^{Ax^2}\left(2xA\sigma(x) + \sigma'(x)\right) ; \qquad \alpha''(x) = e^{Ax^2}\left(4x^2A^2\sigma(x) + 4xA\sigma'(x) + 2A\sigma(x) + \sigma''(x)\right) ,$$

with $A := 2i\pi\frac{\eta'}{\eta}$.

Applying this formula with the substitution $y \mapsto -y$ we can write

$$[\zeta(x)-\zeta(y)]^2 - \wp(x) - \wp(y) = \wp(z) - \zeta^2(z) - 2\zeta(z)\,[\zeta(x)-\zeta(y)]$$

and hence, expanding and rearranging terms we obtain

$$\begin{aligned}
&\left(\frac{\alpha''(x)}{\alpha(x)} + \frac{\alpha''(y)}{\alpha(y)} - 2\frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)}\right) = \\
&= 12i\pi g_1(\tau) - \frac{\alpha''(x-y)}{\alpha(x-y)} + 2\frac{\alpha'(x-y)}{\alpha(x-y)}\left\{\frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)}\right\} = \\
&= -4i\pi\frac{\partial_\tau\alpha(x-y)}{\alpha(x-y)} + 2\frac{\alpha'(x-y)}{\alpha(x-y)}\left\{\frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)}\right\}
\end{aligned} \tag{34}$$

This ends the proof. □

Corollary A.1 From the previous it follows

$$4i\pi\left[\frac{\nabla_\tau\alpha(x)}{\alpha(x)} + \frac{\nabla_\tau\alpha(y)}{\alpha(y)}\right] - 2\frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)} = -4i\pi\frac{\nabla_\tau\alpha(x-y)}{\alpha(x-y)} + 2\frac{\alpha'(x-y)}{\alpha(x-y)}\left\{\frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)}\right\}$$

Proof. It follows from the proposition recalling the definition $\nabla_\tau\alpha(x) := \partial_\tau\alpha(x) + 2g_1\alpha(x)$, and the equation

$$\partial_\tau\alpha(x) = \frac{1}{4i\pi}\alpha''(x) - 3g_1(\tau)\alpha(x)\ .$$

□.

A.4 Periods of elliptic functions

Let $\Phi(v)$ be a meromorphic elliptic differential of the second kind (i.e. without residues), namely $\Phi(v) = f(v)\,dv$, with $f(v)$ meromorphic elliptic function.
For the sake of simplicity assume that ϕ has only one pole at, say, $v = 0$; if the principal part is

$$f(v) = \sum_{-N}^{0} C_k v^k + O(v) \qquad C_{-1} = 0\ ,$$

then the periods of the differential ϕ are computed by expressing $f(v)$ in terms of the ζ function

$$f(v) = C_0 - \sum_{k=2}^{N}\frac{(-1)^k}{(k-1)!}C_k\zeta^{(k-1)}(v)\ .$$

Then we promptly find

$$\begin{aligned}
&\oint_a \Phi = \int_c^{c+1} f(v)dv = C_0 + 4i\pi g_1 C_{-2} \implies \underset{v=0}{res}\left[\frac{f(v)dv}{v} + 4i\pi g_1 v\, f(v)dv\right] \\
&\oint_b \Phi = \int_c^{c+\tau} f(v)dv = \tau\, C_0 + 4i\pi\,(\tau\, g_1 + 2i\pi)\, C_{-2} \Rightarrow \underset{v=0}{res}\left[\tau\frac{f(v)dv}{v} + (4i\pi\tau g_1 + 2i\pi)\, v\, f(v)dv\right] \\
&\oint_b \Phi - \tau\oint_a \Phi = 2i\pi\,\underset{v=0}{res}\,(v\, f(v)dv)\ .
\end{aligned} \tag{35}$$

A.5 The pairing of forms.

Let $\widehat{M} := \widehat{M}_{g,m;n_0,...,n_m}$ be the covering of the Hurwitz space $M_{g,m;n_0,...,n_m}$ introduced in Def. 2.7; this means that its points are the set of data $(C, \lambda, \{a_1, ..., a_g; b_1, ..., b_g\}, k_0, k_1, ..., k_m)$ where

i) C is a smooth curve of genus g with $m+1$ marked points $\infty_0, ..., \infty_m$;

ii) λ is a map $\lambda : C \to \mathbb{C}P^1$ with poles at the marked points and of branching degree respectively $n_0, ..., n_m$;

iii) $\{a_1, ..., a_g; b_1, ..., b_g\}$ is a symplectic basis in the homology of the curve C;

iv) $k_0, k_1, ..., k_m$ are some fixed branches of the roots of λ at the marked points (infinities), namely

$$k_\nu := (\lambda)^{\frac{1}{n_\nu+1}} \ ; \qquad \nu = 0, .., m \ .$$

Let $\mathbf{C}$ be the universal covering of the punctured curve $C/\{\infty_0, ..., \infty_m\}$ with canonical projection Π

$$\Pi : \mathbf{C} \longrightarrow C/\{\infty_0, ..., \infty_m\}$$

and let Φ, Ψ be two holomorphic differentials on this covering (namely with poles possibly only at the poles of λ). We moreover assume that their polydromy is fixed and *independent of the moduli $u_1, ..., u_n$ of the Hurwitz space.* Explicitly we work in the following hypotheses: let $\{a_1, ..., a_g; b_1, ..., b_g\}$ be a symplectic basis of cycles on the curve C realized by paths which avoid the infinities, with basepoint $P_0 \in C$ such that $\lambda(P_0) = 0$; let $s_0, ..., s_m$ be pairwise nonintersecting paths joining P_0 to the infinities and $\gamma_0, ..., \gamma_m$ small loops around the infinities in counterclockwise orientation.

Let $\widetilde{C}$ be a simply connected domain in the universal covering $\mathbf{C}$ constructed as follows: we lift the cycle a_1 to $\mathbf{C}$ and hence obtain a segment from Q_0 to Q_1 such that $\Pi(Q_0) = \Pi(Q_1) = P_0$. We then lift the cycles $b_1, a_2, b_2, ..., b_g, a_1^{-1}, b_1^{-1}, ..., b_g^{-1}$ in this order and consequently obtain $4g$ points $Q_0, Q_1, ..., Q_{4g}$ on the covering. Now we lift the paths $s_0, \gamma_0, s_0^{-1}, s_1, \gamma_1, s_1^{-1}, ..., s_m, \gamma_m, s_m^{-1}$ and at the end, by definition of universal covering, we have come back to the initial point Q_0 and hence have a boundary $\partial\widetilde{C}$.

We now make the following assumptions on the differentials Φ, Ψ;

1. **Polydromy** we assume that the two differentials are **$d\lambda$–multivalued**, namely there exist suitable analytic functions $A^j_\Phi, B^j_\Phi, A^j_\Psi, B^j_\Psi$ on the complex plane such that for any cycles a_j, b_j we have the following properties

$$\Phi(P + a_j) = \Phi(P) + dA^j_\Phi(\lambda) = \Phi(P) + \frac{d}{d\lambda}A^j_\Phi(\lambda)d\lambda$$
$$\Phi(P + b_j) = \Phi(P) + dB^j_\Phi(\lambda) = \Phi(P) + \frac{d}{d\lambda}B^j_\Phi(\lambda)d\lambda \ ,$$

and similar formulae for Ψ; we stress that the cycles are to be meant such that their lift to the universal covering lies inside the simply connected domain $\widetilde{C} \subset \mathbf{C}$ defined above. We further ask that the branching around the infinities is at most $d\lambda$–logarithmic, namely

there exist suitable analytic functions $F^{\mu}_{\Phi}(\lambda)$, $F^{\mu}_{\Psi}(\lambda)$, $\mu = 0, 1, ..., m$ such that, near ∞_μ the polydromy is the same of the differential $d\left(F^{\mu}_{\Phi}(\lambda)\log(\lambda)\right)$.
All these polydromy data are assumed to be *independent of the moduli.*

2. **Periods** We assume that the a–periods are independent of the moduli.

3. **Poles** We assume that the singular part (up to the polydromy described above) does not depend on the moduli, namely, near an infinity ∞_μ, in the *fixed local coordinate*[10] $z_\mu := \frac{1}{k_\mu} = \frac{1}{\sqrt[n_\mu+1]{\lambda}}$

$$\Phi(P) = \sum_{-N}^{\infty} c^{\mu}_{\Phi,k} z^k_\mu dz_\mu + d\left(F^{\mu}_{\Phi}(\lambda)\log(\lambda)\right) \ ,$$

and the singular part, namely the numbers $c^{\mu}_{\Phi,-N}, ..., c^{\mu}_{\Phi,-1}$ are all independent on the moduli.

We remark that we have used the fixed branches k_μ of the roots of λ at the infinities ∞_μ, which is a datum of the Hurwitz space.

Definition A.1 *The pairing is defined by means of*

$$< \Phi, \Psi >:= -\sum_{\mu=0}^{m}\left\{\sum_{\mathbf{k}\geq 0} c^{\mu}_{\Phi,-k-2}\frac{c^{\mu}_{\Psi,k}}{k+1} + c^{\mu}_{\Phi,-1} v.p.\int_{P_0}^{\infty_\mu}\Psi + v.p.\int_{P_0}^{\infty_\mu} F_\Phi(\lambda)\Psi\right\} +$$
$$+\frac{1}{2i\pi}\sum_{j=1}^{g}\left\{-\int_{P_0}^{P_0+a_j} B_\Phi(\lambda)\Psi + \int_{P_0}^{P_0+b_j} A_\Phi(\lambda)\Psi + \int_{a_j}\Phi\int_{b_j}\Psi\right\} \ .$$

The expression is rather cumbersome, though it is important to notice that the dependence of the moduli enters only through Ψ.

Consider the tautological fiber bundle over the Hurwitz space such that the fiber is the punctured curve C; onto this fiber bundle we define a lift of the vectors $\frac{\partial}{\partial u_i}$

Definition A.2 *We define the connection on the locally trivial fiber–bundle*

$$\begin{array}{c} \downarrow C \\ M \end{array}$$

defined implicitly by the formula

$$\nabla_i \lambda \equiv 0 \ .$$

This means that all derivatives w.r.t. the moduli have to be done *at λ constant*; this implies that if we realize the curves C_u as locally identical curves, the connection defines some vector fields on this curve by means of the formula (in a local coordinate w on the curve)

$$\frac{\partial w}{\partial u_i} = -\frac{1}{\lambda'(w;u)}\frac{\partial\lambda(w;u)}{\partial u_i} \ .$$

[10]This is to stress that the coefficients are to be defined w.r.t. the Laurent expansion w.r.t. these variables

Notice that if w_i is a local coordinate in the neighborhood of the point P_i we have that

$$\frac{\partial w_j}{\partial u_i} = -\frac{1}{\lambda'(w_j;u)}\frac{\partial \lambda(w_j;u)}{\partial u_i}$$

is a function with a simple pole only if $i = j$ and regular otherwise.

We have the following

Lemma A.1 [Lemma 5.1 in [Du93]] Let $u_i := \lambda(P_i)$, s.t. $d\lambda(P_i) = 0$ be the moduli of the Hurwitz space, then we have $\forall i = 1..n$

$$\frac{\partial}{\partial u_i}(<\Phi,\Psi>) = -\underset{P_i}{res}\frac{\Phi\Psi}{d\lambda}\ .$$

Corollary A.2 The pairing is symmetric up to a constant independent of the moduli.

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