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Generalized Exclusion Statistics in the Kondo Problem

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Abstract

We consider the generalized exclusion statistics in the Kondo problem. The thermodynamic Bethe ansatz equations have been used for a multicomponent system of particles obeying the generalized exclusion principle. We have found a relation between the derivative of the phase shift of the scattering matrix for Fermi particles and for particles characterized by generalized exclusion statistics. We show that the statistical matrix in the Kondo problem has a universal form in high and low temperature limits.

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1. Introduction

Generalized exclusion statistics (GES) [1] has attracted much interest in recent papers [2 – 8] A typical example of the application of generalized exclu-

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sion statistics is the Cologero-Sutherland model [9, 10, 11]. D. Bernard and Y.-S. Wu [12] studied the GES in the thermodynamic Bethe ansatz (TBA) using equivalence of the Bethe ansatz (BA) equations and the ones obtained by the Haldane principle [11]. M. Wadati has shown [13] that a change in the statistics is determined by an appropriate choice of the phase shift of the scattering matrix. The generalization of exclusion statistics for multicomponent systems was provide in the paper [8].

The goal of this paper is to consider the exclusion statistics of excitations in the Kondo problem. To solve this problem we have applied the TBA for a multicomponent system of particles obeying the GES. We have found the universal behavior of the statistical matrix in high- and low-temperature limits. In these limits the statistical matrix is proportional to the Cartan matrix for the A_n-algebra.

The paper is organized as follows. In the second section we provide the main equations of the exclusion statistics theory to complete the discussion. In this section we have also derived the relation between the derivative of phase shift (DPS) of the scattering matrix and the statistical matrix in the framework of the TBA equations for the multicomponent system of particles. The third section deals with the application of this result to the Kondo problem. We show that distribution function and statistical matrix in this problem have the universal form in high and low temperature limits.

2. Exclusion Statistics Equations

A change in the number $D_{\alpha i}$ of the vacant states due to the addition of the number $N_{\beta j}$ of the particles, according to Haldane [1] is defined as

$$\frac{\partial D_{\alpha i}}{\partial N_{\beta j}} = -g_{\alpha i,\beta j}\,. \tag{1}$$

Here $g_{\alpha i,\beta j}$ is the matrix of statistical interaction. The indices α ($\alpha = 1, 2, ...M$) and i correspond to the internal and the dynamical degrees of freedom, respectively. The solution of Eq.(1) has the form:

$$D_{\alpha i} = -\sum_{\beta j} g_{\alpha i,\beta j} N_{\beta j} + D^0_{\alpha i}\,, \tag{2}$$

where $D^0_{\alpha i}$ is the number of vacant states of the αi-th type without particles. The number of holes $D_{\alpha i}$ determines the statistical weights W as follows

$$W = \prod_{\alpha,i} \frac{(N_{\alpha i} - 1 + D_{\alpha i}(\{N_{\beta j}\}) + \sum_{\beta j} g_{\alpha i,\beta j}\delta_{\alpha\beta}\delta_{ij})!}{(N_{\alpha i})!(D_{\alpha i}(\{N_{\beta j}\}) - 1 + \sum_{\beta j} g_{\alpha i,\beta j}\delta_{\alpha\beta}\delta_{ij})!}\,. \tag{3}$$

In the specific cases of $g_{\alpha i,\beta j} = 0$ and $g_{\alpha i,\beta j} = \delta_{\alpha\beta}\delta_{ij}$ Eq.(3) yields the well-known expression for the statistical weights of Bose and Fermi particles.

The distribution function $n_{\alpha i}$ is defined usually [2, 12, 4] in the following way $n_{\alpha i} = N_{\alpha i}/G^0_{\alpha i}$. This is not convenient for systems with internal degrees of freedom. For example, in the hierarchical basis of the states in the fractional quantum Hall effect $n_{\alpha i} = \infty$ if $\alpha = 2, 3, ...$ because $G^0_{\alpha i} = 0$ for spin degrees of freedom. The definition of the distribution function in the form

$$n_{\alpha i} = \frac{N_{\alpha i}}{G_{\alpha i}} \tag{4}$$

with

$$G_{\alpha i} = G^0_{\alpha i} + N_{\alpha i} - \sum_{\beta j} g_{\alpha i,\beta j} N_{\beta j} \tag{5}$$

is more convenient because it takes into account the influence of the statistical interaction on the number of the states. The equilibrium distribution function $n_{\alpha i}$ can be found in this case from the extremum of the grand partition function as a solution of the following equations:

$$\frac{1}{n_{\alpha i}} \prod_{\beta j} [1 - n_{\beta j}]^{g_{\beta j,\alpha i}} = \exp\{(\epsilon_{\alpha i} - \mu_\alpha)/T\}\,. \tag{6}$$

Here $\epsilon_{\alpha i}$ is the energy and μ_α is the chemical potential for the particles of type α. The distribution function $n_{\alpha i}$ determines the free energy

$$F = \sum_\alpha \mu_\alpha N_\alpha - T \sum_{\alpha i} G^0_{\alpha i} \ln \left[\frac{1}{1 + n_{\alpha i}} \right] \tag{7}$$

as well as the value of the entropy $S = \ln W$.

The interection between quasiparticles from the TBA point of view is encoded in the phases $\Theta_{\alpha i,\beta j}$ of the dynamical scattering matrix $S_{\alpha i,\beta j} = -\exp(-i\Theta_{\alpha i,\beta j})$. The statistical properties expressed by the statistical matrix $g_{\alpha i,\beta j}$ depend on the DPS of the scattering matrix [12, 13].

Let us consider the TBA equations for the set of multicomponent particles obeying the GES. Quantizing a gas of such particles on a circle of length L requires that the momentum $k_{\alpha i}$ of the αi-th particle satisfies the following condition:

$$\exp\{ik_\alpha(\theta_i)L\} \sum_{\beta j}^{N} S_{\alpha,\beta}(\theta_i - \theta_j) = 1 \,. \tag{8}$$

The momentum and the energy of the particles are parametrized by the rapidity θ. Going to the limits $L \to \infty$ and $N \to \infty$ with the finite value of N/L and taking the derivative of the log of Eq.(8) yield

$$2\pi q_\alpha(\theta) = \frac{dk_\alpha(\theta)}{d\theta} + \sum_{\beta=1}^{M} \int_{-\infty}^{\infty} \Phi_{\alpha\beta}(\theta - \theta')\rho_\beta(\theta')d\theta' \,, \tag{9}$$

where

$$\Phi_{\alpha\beta}(\theta) = \frac{1}{i}\frac{d}{d\theta} \ln S_{\alpha\beta}(\theta) \tag{10}$$

is DPS. The function $\rho_\beta(\theta)$ in Eq.(9) is the density of the particles of β-type. $q_\alpha(\theta)$ is the density of the states.

The information about the statistical properties of the system is contained in the distribution functions $n_{\alpha i}$ and in the entropy S. In the framework of

the thermodynamic Bethe ansatz [17], we have neither information about the ground state energy, nor about the structure of low-lying excitations with the energy $\epsilon_\alpha^0(\theta)$. They are the objects of the traditional Bethe ansatz approach. The equilibrium state at temperature T is obtained by minimizing the free energy $F = E - TS$, where the energy of the system is

$$E = \sum_{\alpha=1}^{M} \int \epsilon_\alpha^0(\theta)\rho_\alpha(\theta)d\theta\,. \tag{11}$$

The variation of F with respect to ρ_α yields the following equations for the dressed energy $\epsilon_\alpha(\theta)$:

$$\epsilon_\alpha(\theta) = \epsilon_\alpha^0(\theta) + \mu_\alpha - \frac{T}{2\pi}\sum_{\beta=1}^{M}\int \Phi_{\beta\alpha}(\theta-\theta')\ln\left[\frac{1}{1-n_\beta(\theta')}\right]d\theta'\,. \tag{12}$$

Here the function n_α relates to the functions ρ_α and q_α in Eq.(9) as follows: $n_\alpha = \rho_\alpha/q_\alpha$.

Let us assume that the particles are fermions, i.e., $g_{\beta j,\alpha i} = \delta_{\alpha\beta}\delta_{ij}$. We see from Eq.(6) that

$$n_\alpha = \frac{1}{1+\exp\left[(\epsilon_\alpha^f(\theta)-\mu_\alpha)/T\right]}\,. \tag{13}$$

After substituting this expression into Eq.(12), we have the standard TBA equations

$$\epsilon_\alpha^f(\theta) = \epsilon_\alpha^0(\theta) + \mu_\alpha - \frac{T}{2\pi}\sum_{\beta=1}^{M}\int \Phi_{\beta\alpha}^f(\theta-\theta')\ln\left[1+\exp\left(-\frac{\epsilon_\beta^f(\theta')-\mu_\beta}{T}\right)\right]d\theta' \tag{14}$$

for fermions (superscript "f" denotes the Fermi statistics).

Each statistics corresponds to the specific value of DPS $\Phi_{\alpha\beta}(\theta)$ in Eq.(12). The transition to the Fermi statistics leads to the new value $\Phi_{\alpha\beta}^f(\theta)$ of this

function. Assuming that the function $n_\beta(\theta)$ in Eq.(12) coincides with Eq.(13) we can find from Eqs.(12), (14) the relation between DPS $\Phi_{\alpha\beta}(\theta)$ for the arbitrary statistics and for the Fermi statistics. This relation has the form

$$\Phi(\theta-\theta') = \Phi^f(\theta-\theta') - 2\pi\delta(\theta-\theta')[\mathcal{I}-\mathcal{G}]\,, \tag{15}$$

where $\mathcal{I}$ is the unit matrix and

$$\mathcal{G} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & \cdots \\ g_{21} & g_{22} & g_{23} & \cdots \\ g_{31} & g_{32} & g_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{16}$$

is the statistical matrix. It is easy to see from Eq.(15) that for models with the function $\Phi^f_{s\alpha}(\theta-\theta') \sim \delta(\theta-\theta')$ one can find such statistical matrix $\mathcal{G}$ which gives a zero value of the r.h.s. in this equation. In these models the phase of the scattering matrix has the structure of the step function. From the TBA point of view these systems look like a gas of non-interacting particles having the GES. This case is known [12, 6] as the ideal exclusion statistics. In these models the correlations between particles can be transformed to the statistical interaction. The distribution function of excitations in the systems with the ideal exclusion statistics can be obtained from Eq.(6) where the statistical matrix is now $g_{\alpha,\beta} = \delta_{\alpha,\beta} - \frac{1}{2\pi}\Phi^f_{\alpha\beta}$ and the dressed energy of excitations coincides with bare energy ϵ^0_α. Note that the structure of Eq.(6) looks like that for Eq.(15) (after some simple transformation) because the function n_α satisfies Eq.(13).

3. Generalized exclusion statistics in the Kondo problem.

Let us consider the Kondo problem from the GES point of view. Bethe-ansatz equations for the isotropic $s-d$ exchange model [18] in the Kondo

problem are

$$\exp(ik_jL) = \exp\left(\frac{iIS}{2}\right)\prod_{\gamma}^{P}\left(\frac{\lambda_\gamma + i/2}{\lambda_\gamma - i/2}\right), \tag{17}$$

$$\left(\frac{\lambda_\gamma + i/2}{\lambda_\gamma - i/2}\right)^N\left(\frac{\lambda_\gamma + 1/g + iS}{\lambda_\gamma + 1/g - iS}\right) = -\prod_{\beta}^{P}\left(\frac{\lambda_\gamma - \lambda_\beta + i}{\lambda_\gamma - \lambda_\beta - i}\right). \tag{18}$$

These equations solve the problem of diagonalization of the $s-d$ exchange hamiltonian with the impurity spin S and with the coupling constant I. Here the total number of electrons (up-spin electrons) is denoted as $N(P)$. The general solutions of Eqs.(17), (18) have the form of n-strings according to the string hypothesis [14]. The n-string is a set of n solutions given by

$$\lambda_\gamma^{(n,j)} = \lambda_\gamma^n + i\left(\frac{n+1}{2} - j\right), \qquad j = 1, ..., n. \tag{19}$$

Here λ_γ^n is the real number and n is the order of the string. The distribution of the n-type particles (holes) density in the thermodynamic limit is $\rho_n(\lambda)$ ($\rho_n^{(h)}(\lambda)$). Assuming that the particles obey the Fermi statistics, the TBA equations for the function $\epsilon_n(\lambda)$ have the form

$$\begin{aligned}\epsilon_n(\lambda) &= -\frac{2\epsilon_F}{\pi}\tan^{-1}\exp(\pi\lambda)\delta_{n1} + \frac{T}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi(\lambda-\lambda'))} \\ &\left[\ln\left(1 + e^{\epsilon_{n-1}(\lambda')/T}\right) + \ln\left(1 + e^{\epsilon_{n+1}(\lambda')/T}\right)\right]d\lambda'.\end{aligned} \tag{20}$$

Here and below the dressed energy is $\epsilon_n(\lambda) = T\ln[\rho_n^{(h)}(\lambda)/\rho_n(\lambda)]$.

The solutions of these nonlinear integral equations describe the thermodynamic properties of the $s-d$-model. The external magnetic field H of the problem enters in the boundary condition as follows:

$$\lim_{n\to\infty}\frac{\epsilon_n(\lambda)}{n} - H = 0. \tag{21}$$

Note that this boundary condition means the condition of compensation of the internal magnetic field $h = \lim_{n\to\infty}(\epsilon_n / n)$ by external magnetic field H. This situation takes place as well for some phase states in $(2+1)$ D systems.

The spin free energy

$$F^{sp} = -NT\int_{-\infty}^{\infty}\frac{1}{2\cosh(\pi\lambda)}\ln\left(1+e^{\epsilon_1(\lambda)/T}\right)d\lambda$$
$$-T\int_{-\infty}^{\infty}\frac{1}{2\cosh(\pi(\lambda+1/g))}\ln\left(1+e^{\epsilon_{2S}(\lambda)/T}\right)d\lambda \qquad (22)$$

is expressed by the functions $\epsilon_n(\lambda)$. The first term in the r.h.s. of Eq.(22) corresponds to the spin free energy in the absence of impurity. The second term is the impurity contribution to the free energy. We will focus on the universal properties of the solutions of Eqs.(20) in the limits $T\to\infty$ and $T\to 0$.

Comparing BA Eqs.(20) and TBA Eqs.(14) one can see that the relation between them exists if

$$\epsilon_n^f(\lambda) = -\epsilon_n(\lambda)\,. \qquad (23)$$

In other words, the energy of the particles in TBA Eqs.(14) corresponds to the energy of the holes in BA Eqs.(20) of the Kondo problem. The α-type particle in the TBA equations is made corresponding to the n-th string solution of Eqs.(20). By changing the index n by α we can rewrite Eqs.(20) in the new notations as follows

$$\epsilon_\alpha^f(\lambda) = \frac{2\epsilon_F}{\pi}\tan^{-1}\exp(\pi\lambda)\delta_{\alpha 1} - \frac{T}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi(\lambda-\lambda'))}$$
$$\left[\ln\left(1+e^{-\epsilon_{\alpha-1}^f(\lambda')/T}\right)+\ln\left(1+e^{-\epsilon_{\alpha+1}^f(\lambda')/T}\right)\right]d\lambda'\,. \qquad (24)$$

The boundary conditions have now the form $\epsilon_0^f=\infty$ and $\lim_{\alpha\to\infty}\epsilon_\alpha^f/\alpha = -H$.

From Eqs.(24) and Eqs.(14) we have

$$\Phi^f(\lambda-\lambda') = \frac{1}{\cosh(\pi(\lambda-\lambda'))}\mathcal{L} \tag{25}$$

and

$$\epsilon^0_\alpha(\lambda) = \frac{2\epsilon_F}{\pi}\tan^{-1}\exp(\pi\lambda)\delta_{\alpha 1}\,. \tag{26}$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & 1 & & 0 \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{pmatrix}\,. \tag{27}$$

Unlike in the TBA Eqs.(14), we know the function $\epsilon^0_\alpha(\lambda)$ in Eqs.(26) because it was found by the BA method.

Let us assume now that the particles have the GES with any statistical matrix $\mathcal{G}$. In this case the matrix Φ in Eqs.(15) for the Kondo problem is

$$\Phi(\lambda-\lambda') = \frac{1}{\cosh(\pi(\lambda-\lambda'))}\mathcal{L} - 2\pi\delta(\lambda-\lambda')\left[\mathcal{I}-\mathcal{G}\right]\,. \tag{28}$$

It is easy to see from the last expression that in the general case we cannot find a matrix $\mathcal{G}$ to set the matrix Φ to zero. However, it is possible to consider the particular cases of high- and low-temperature limits. The asymptotic behavior of the spin free energy determines the different values of the rapidity range required for the consideration of the high- and low-temperature limits. When $T\to\infty$ (weak coupling limit), the impurity free energy is given by

$$F_{imp} = -\frac{T}{2}\ln\left(1+\exp\left(-\frac{\epsilon^f_{2S}(-\infty)}{T}\right)\right)\,. \tag{29}$$

When $T \to 0$ (strong coupling limit),

$$F_{imp} = -\frac{T}{2} \ln \left(1 + \exp \left(-\frac{\epsilon^f_{2S}(+\infty)}{T} \right) \right) . \tag{30}$$

Therefore, in these temperature limits we have to consider $\epsilon^f_{2S}(\lambda)$ at $|\lambda| \to \infty$. Taking the limit $|\lambda| \to \infty$ in Eqs.(24) we have the equations for $\epsilon^f_\alpha(\pm\infty)$:

$$\epsilon^f_\alpha(\pm\infty) = -\frac{T}{2} \ln \left[\left(1 + \exp \left(-\frac{\epsilon^f_{\alpha-1}(\pm\infty)}{T} \right) \right) \left(1 + \exp \left(-\frac{\epsilon^f_{\alpha+1}(\pm\infty)}{T} \right) \right) \right] . \tag{31}$$

The boundary conditions are: $\lim_{\alpha \to \infty} \epsilon^f_\alpha(\pm\infty)/\alpha = -H$, $\epsilon^f_0(-\infty) = \infty$ and $\epsilon^f_1(+\infty) = \infty$. The solution of these equations is [18]

$$\epsilon^f_\alpha(+\infty) = \epsilon^f_{\alpha+1}(-\infty) = -T \ln \left[\left(\frac{\sinh \left[\frac{H}{2T}(\alpha + 1) \right]}{\sinh(\frac{H}{2T})} \right)^2 - 1 \right] . \tag{32}$$

This result will be the same if we assume that the matrix $\Phi^f(\lambda)$ in Eqs.(24) is proportional to the δ-function, i.e.,

$$\lim_{|\lambda| \to \infty} \Phi^f(\lambda - \lambda') = \pi \delta(\lambda - \lambda') \mathcal{L} . \tag{33}$$

It is clear because the function $\lim_{\lambda \to \infty} 1/[\cosh(\lambda - \lambda')] = 2/[\exp(\infty - \lambda')]$ acts as the δ-function at $\lambda' = \infty$. Therefore, in the GES case the matrix $\Phi(\lambda)$ has the following form

$$\Phi(\lambda - \lambda') = \pi \delta(\lambda - \lambda') \left[\mathcal{L} - 2\mathcal{I} + 2\mathcal{G} \right] \tag{34}$$

for these temperature limits. From the condition $\Phi = 0$ of the ideal statistics we have the following form of the statistical matrix:

$$\mathcal{G} = \frac{1}{2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} . \tag{35}$$

We see that it is proportional to the Cartan matrix for algebra, A_n.

To find the distribution function n_α we need to know the structure of the low-lying excitations. It follows from Eq.(24) that $\epsilon_\alpha(\lambda) = 0$ for all $\alpha \neq 0$ when $\lambda = -\infty$ and for all $\alpha \neq 1$ when $\lambda = +\infty$. The structure of the equations for the distribution function of particles n_α, which can be obtained from Eq.(6) with the statistical matrix (35), resembles that of the equations for $\epsilon_\alpha(|\lambda| = \infty)$ (31):

$$\left(\frac{1}{n_\alpha} - 1\right)^2 = (1 - n_{\alpha+1})(1 - n_{\alpha-1}) . \tag{36}$$

The solution of Eqs.(36) (with the boundary condition $\lim_{\alpha\to\infty}\left(\frac{1}{n_\alpha} - 1\right) \to \exp(-\alpha H/T)$) coincides with the solutions (32) :

$$\left(\frac{1}{n_\alpha} - 1\right) = e^{\epsilon_\alpha^f/T} = \left[\left(\frac{\sinh(\frac{H}{2T}(\alpha+1)}{\sinh(\frac{H}{2T})}\right)^2 - 1\right] \tag{37}$$

at $T \to 0$ and

$$\left(\frac{1}{n_\alpha} - 1\right) = e^{\epsilon_\alpha^f/T} = \left[\left(\frac{\sinh(\frac{H}{2T}\alpha)}{\sinh(\frac{H}{2T})}\right)^2 - 1\right] \tag{38}$$

at $T \to \infty$. The spin free energy does not vary when we change the particle statistics.

The function Φ^f does not change in the TBA equations for multichannel Kondo effect [16, 19, 20]. But the function $\epsilon^0_\alpha(\lambda)$ is changed. It depends on the number of channels, k, as

$$\epsilon^0_\alpha = \delta_{\alpha k} \exp(\pi\lambda) . \tag{39}$$

This leads to the new solutions for $\epsilon^f_n(\lambda)$ in the low-temperature limit ($\lambda \to +\infty$):

$$\epsilon^f_\alpha(+\infty) = \begin{cases} -T \ln \left[\left(\frac{\sin(\pi(\alpha+1)/(k+2))}{\sin(\pi/(k+2))} \right)^2 - 1 \right] & \alpha < k \\ -T \ln \left[\left(\frac{\sinh(\frac{H}{2T}(\alpha+1-k))}{\sinh(\frac{H}{2T})} \right)^2 - 1 \right] & \alpha \geq k \end{cases} . \tag{40}$$

The solutions for the higher-temperature limit ($\lambda \to -\infty$) are not changed. To find the solution (40) using the statistical matrix (35) we have to impose the additional boundary condition $n_\alpha = 1$ at $\alpha = k$.

4. Conclusion

Let us discuss in conclusion what new insights into the thermodynamics of integrable systems (the Kondo problem in particular) are gained by considering these systems in terms of the GES.

Using the GES principle we introduce an additional "parameter", that is, particle statistics determined by the form of the statistical matrix. If we suppose that the statistical matrix is arbitrary, we can write the TBA equations for a system of particles with any statistics. Each form of the statistical matrix in this case has a corresponding distribution- and the DPS-function. Therefore, we can write the TBA equations in a more convenient form. A

successful choice of the statistical matrix may lead to a simpler form of the coupled equations for the dressed energy and for the distribution function. In particular, in the case of ideal statistics one can find a statistical matrix such that the solution of the TBA equations for the dressed energy coincides with the bare energies, and the equations for the distribution function repeat (up to transformation) the TBA equations for Fermi particles.

Let us focus on the features of the Kondo problem which are studied using the GES principle. The consideration of spin excitations as quasiparticles in the Kondo problem leads to the conclusion that spin excitations correspond to the holes in the TBA approach. To find the distribution function in the system of particles with internal degrees of freedom we need additional information besides the statistical matrix. We should know the detailed structures of the ground state [8]. The energy of the ground state shows itself as the boundary conditions for the equations determining the distribution function. The boundary conditions include the external magnetic field which compensates the internal "conformal" magnetic field of the system.

The solution for the distribution function in high and low-temperature regions has the universal form. It is determined by the q-deformed dimension $[\ \alpha\ +\ 1\ ]_q$ of irreducible representations of the quantum group $U_q(sl_2)$. Here $[\ x\ ]_q\ =\ (q^x - q^{-x})/(q - q^{-1})$. In the case of the single channel Kondo problem, the deformation parameter $q\ =\ \exp\ (\ H\ /\ 2\ T\ )$ is real. It depends on the external magnetic field and the temperature. In the multichannel Kondo problem (at $\alpha < k$), the deformation parameter $q = \exp[i\pi/(k+2)]$ being the root of unity, is determined by the number, k of the channels.

In summary, we used the TBA equations for a system of particles with internal degrees of freedom to consider the features of the GES in the Kondo problem. It is shown that the statistical matrix and the distribution function in the Kondo problem have a universal form in high- and low-temperature limits.

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References

[1] F. D. M. Haldane, *Phys. Rev. Lett.* **67**, 937 (1991).

[2] Y.-S. Wu, *Phys. Rev. Lett.* **73**, 922 (1994).

[3] C. Nayak and F. Wilczek, *Phys. Rev. Lett.* **73**, 2740 (1994).

[4] S. B. Isakov, *Phys. Rev. Lett.* **73** 2150 (1994); *Int. J. Mod. Phys.* **A9**, 2563 (1994).

[5] A. K. Pajagopal, *Phys. Rev. Lett.* **74**, 1048 (1995).

[6] M. V. N. Murthy and R. Shankar, *Phys. Rev. Lett.* **72**, 3629 (1994).

[7] A. Protogenov, "*Haldanes statistical interactions and universal properties of anyon systems*", IC/95/26 preprint, March 1995.

[8] T. Fukui and N. Kawakami, *Phys. Rev.* **B51**, 5239 (1995); *J. Phys.* **A28** 6027.

[9] F. Calogero, *J. Math. Phys.* **10**, 2197 (1962).

[10] B. Sutherland, *J. Math. Phys.* **12**, 246 (1971); *Phys.Rev.* **A4**, 2019 (1971).

[11] Z. N. C. Ha, *Phys. Rev. Lett.* **73**, 1574 (1994).

[12] D. Bernard and Y.-S.Wu, in *New developments of integrable systems and long-ranged interaction models,* ed. M. L. Ge and Y.-S. Wu (World Scientific, Singapore, 1995).

[13] M. Wadati, *J. Phys. Soc. Jpn.* **64**, 1552 (1995).

[14] C. N. Yang and C. P. Yang, *J. Math. Phys.* **10**, 1115 (1969).

[15] X. G. Wen and A. Zee, *Phys. Rev.* **B46**, 2299 (1992).

[16] P. Nozières and A. Blandin, *J. Physique* **41**, 193 (1980)

[17] Al. B. Zamolodchikov, *Nucl. Phys.* **342**, 695 (1991).

[18] A. M. Tsvelik and P. B. Wiegmann, *Adv. Phys.* **32**, 453 (1983).

[19] A. M. Tsvelik and P. B. Wiegmann, *J. Phys.* **A17**, 2321 (1983).

[20] A. M. Tsvelik, *J. Phys.* **C18**, 159 (1985).