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# A NOTE ON FRACTIONAL KDV HIERARCHIES. II. THE BIHAMILTONIAN APPROACH

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**Abstract**

In this paper we show, in a concrete case, that the fractional KdV hierarchies can be naturally derived in the framework of the theory of bihamiltonian manifolds. A general reduction scheme for such manifolds is applied, and a KP-like form for the evolution equations of the Hamiltonian densities of the hierarchy is exhibited.

## 1. INTRODUCTION

After the discovery [10] of the complete integrability of the Korteweg-de Vries (KdV) equation, many efforts have been made in order to find other nonlinear partial differential equations with the same properties of KdV (see, e.g., [5]). In particular, it was shown by Drinfel'd and Sokolov [9] how to associate with every affine Kac–Moody algebra an integrable hierarchy of KdV-type. More recently [11], it has been shown how to generalize such a construction in order to systematically obtain the so–called generalized DS hierarchies (introduced, in particular cases, in, e.g., [1, 2, 15]). They are labelled by two (coprime) integers $N$ and $l$, and are known to possess a bihamiltonian structure. The "second" Hamiltonian structure gives rise to the $\mathcal{W}_N^{(l)}$ algebras, whose quantum extensions play the role of extended conformal symmetries in 2-dimensional quantum field theories (see, e.g., [3]). Since the conformal weights of the generators of the algebra are in this case rational numbers, the generalized DS hierarchies are also called "fractional KdV (fKdV) hierarchies".

In this paper we want to pursue the study of such hierarchies we started in [6], and frame from the onset this study in the theory of bihamiltonian manifolds. Although



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most of the results we will present hold in the general case, for sake of brevity we will concentrate ourself on the simplest (nontrivial) case, $N = 3$, $l = 2$, which we will refer to as the $\mathrm{fKdV}_3^{(2)}$ hierarchy.

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## 2. THE BIHAMILTONIAN SETTING OF FKDV HIERARCHIES

In this section we will begin to construct the bihamiltonian structure of the $\mathrm{fKdV}_3^{(2)}$ hierarchy, starting from Lie-Poisson structures on loop-algebras. This is done also in [4], by means of a Drinfeld-Sokolov reduction. Our approach is different (although equivalent), since we will make use of a general reduction theorem for bihamiltonian manifolds (enlightened in [7]), based on the Marsden-Ratiu theorem for Poisson manifolds [14].

Let $\mathfrak{g}$ be the simple Lie algebra $\mathfrak{sl}(3)$ and let $\mathfrak{G}$ be the space of $C^\infty$–maps from $S^1$ into $\mathfrak{g}$; let us denote by $x$ the coordinate on $S^1$, and by $S$ a map from $S^1$ into $\mathfrak{g}$. As usual, an element in the tangent space $T_S\mathfrak{G}$ will be denoted by $\dot{S}$. We can think of a covector as a map $V$ from $S^1$ into $\mathfrak{g}$, whose value on the tangent vector $\dot{S}$ is given by

$$\langle V, \dot{S}\rangle = \int_{S^1} \mathrm{tr}(V(x)\,\dot{S}(x))\,dx. \tag{2.1}$$

It is well-known (see, e.g., [12]) that $\mathfrak{G}$ is endowed with two compatible Poisson tensors,

$$(P_0)_S V = [A, V] \tag{2.2}$$
$$(P_1)_S V = V_x + [V, S], \tag{2.3}$$

where $V_x$ denotes the derivative of the loop $V$ with respect to $x$ and $A$ is a fixed element in $\mathfrak{G}$. This means that the composition laws on functions on $\mathfrak{G}$ given by $\{f, g\}_i = \langle df, P_i dg\rangle$, $i = 0, 1$, are Poisson brackets, which are compatible in the sense that the pencil $\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_1 - \lambda\{\cdot,\cdot\}_0$ is still a Poisson bracket for all $\lambda \in \mathbb{C}$. Hence $\mathfrak{G}$ is a *bihamiltonian manifold*. The choice of $A$ leading to the $\mathrm{fKdV}_3^{(2)}$ hierarchy is

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.4}$$

Let us now apply the bihamiltonian reduction process to $\mathfrak{G}$. It starts with the choice of a specific symplectic leaf of the Poisson tensor (2.2). All leaves are affine hyperplanes modeled on the orthogonal space $\mathfrak{G}_A^\perp$ (with respect to the pairing (2.1)) of the isotropy algebra $\mathfrak{G}_A = \{V \in \mathfrak{G} \text{ s.t. } [V, A] = 0\}$ of $A$. We consider the symplectic leaf $\mathcal{S}$ passing through the point

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.5}$$

so that the generic element $S \in \mathcal{S}$ has the form

$$S = \begin{pmatrix} p_0(x) & p_2(x) & 1 \\ q_0(x) & p_1(x) - p_0(x) & -p_2(x) \\ q_2(x) & q_1(x) & -p_1(x) \end{pmatrix}. \tag{2.6}$$

The next step in the reduction process concerns the distribution $E = P_1(\mathrm{Ker} P_0) \cap T\mathcal{S}$, which is integrable on account of the compatibility condition between $P_0$ and $P_1$. The main statement of the bihamiltonian reduction theorem is that the quotient manifold $\mathcal{N} = \mathcal{S}/E$ is still a bihamiltonian manifold. Hence the final task is to compute the canonical projection $\pi : \mathcal{S} \to \mathcal{N}$ and the reduced bihamiltonian structure. In order to determine $\pi$ one should, in general, integrate the distribution $E$, but in our case such a process admits an interesting "shortcut", which we call the *Frobenius technique.*

## 3. THE FROBENIUS TECHNIQUE

A basic feature of the fKdV theories is that the integral leaves of $E$ are orbits of a group action, the group being defined by the distribution $E$ at the special point $B$. Indeed, one can easily prove

**Proposition 3.1.** *The vector space $E_B$ is spanned by the tangent vectors $(P_1)_B(V)$ corresponding to covectors $V$ belonging to $\mathfrak{G}_{AB} := \{V \in \mathfrak{G}_A \mid V_x + [V, B] \in \mathfrak{G}_A^{\perp}\}$. This latter is a subalgebra of the algebra $\mathfrak{N}_-$ of loops with values in the nilpotent subalgebra of strictly lower triangular matrices. The distribution $E$ is spanned by the vector fields $P_1 V$ with $V \in \mathfrak{G}_{AB}$, and its integral leaves are the orbits of the gauge action of $G_{AB} :=$* $\exp \mathfrak{G}_{AB}$ *on $\mathcal{S}$ defined by*

$$S' = TST^{-1} + T_x T^{-1}. \tag{3.1}$$

The idea of the Frobenius technique is to write Equation (3.1) in the equivalent form $(-\partial_x + S' + \lambda A) \circ T = T \circ (-\partial_x + S + \lambda A)$, meaning that the points $S$ and $S'$ belong to the same orbit if and only if the associated matrix first-order differential operators $-\partial_x + (S + \lambda A)$ and $-\partial_x + (S' + \lambda A)$ are conjugated by an element $T \in G_{AB}$. Then we consider the two vectors $\langle g^{(0)}(S)| = \langle 1, 0, 0|$ and $\langle g^{(1)}(S)| = \langle p_2(x), 1, 0|$. They are covariant, in the sense that $\langle g^{(j)}(S')| = \langle g^{(j)}(S)|T$ if $S' = T^{-1}ST - T^{-1}T_x$ with $T \in \mathfrak{G}_{AB}$. Moreover, if we use the recurrence relation

$$\langle g^{(j+2)}(S)| = \partial_x \langle g^{(j)}(S)| + \langle g^{(j)}(S)|(S + \lambda A) \tag{3.2}$$

to define $\langle g^{(j)}(S)|$ for $j \geq 0$, we have

**Proposition 3.2.** *The vectors $\langle g^{(j)}(S)|$, $j \geq 0$, are covariant. The first three of them are independent of $\lambda$ and constitute a basis of $\mathbb{C}^3$, to be referred to as the* Frobenius basis *at $S$. The matrix $G(S)$ whose rows are the elements of the Frobenius basis is lower triangular with 1 on the diagonal.*

If we develop the subsequent two vectors, $\langle g^{(3)}(S)|$ and $\langle g^{(4)}(S)|$, on the Frobenius basis, we obtain a system of two equations, to be called the *characteristic system*:

$$\left\{ \begin{array}{lcl} \langle g^{(3)}(S)| & = & u_0\langle g^{(1)}(S)| + (u_1+\lambda)\langle g^{(0)}(S)| \\ \langle g^{(4)}(S)| & = & -u_0\langle g^{(2)}(S)| + (u_2+\lambda)\langle g^{(1)}(S)| + u_3\langle g^{(0)}(S)| \end{array} \right. \tag{3.3}$$

where

$$\begin{array}{lcl} u_0 & = & -p_0 + p_1 + p_2^2 \\ u_1 & = & q_0 + 2p_0p_2 - p_1p_2 - p_2^3 + p_{2x} \\ u_2 & = & q_1 - p_0p_2 + 2p_1p_2 + p_2^3 + p_{2x} \\ u_3 & = & q_2 + p_{0x} - p_2p_{2x} + q_0p_2 + p_0p_1 - p_2^4 - 2p_1p_2^2 - p_2q_1 + 2p_0p_2^2 \end{array} \tag{3.4}$$

This amounts to considering the matrix $\mathcal{U} = G(S)(S+\lambda A)G(S)^{-1} + G(S)_xG(S)^{-1}$, which turns out to be

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 1 \\ u_1+\lambda & u_0 & 0 \\ u_3 & u_2+\lambda & -u_0 \end{pmatrix}. \tag{3.5}$$

It is not difficult to show that the covariance property of the Frobenius basis entails that the functions $(u_0, u_1, u_2, u_3)$ are a complete set of invariant for the action of $\mathfrak{G}_{AB}$, i.e., they can be used to parametrize the quotient manifold $\mathcal{N}$. Therefore Equations (3.4) give the projection $\pi : \mathcal{S} \to \mathcal{N}$ in terms of the coordinates $u_j, p_i, q_i$. At this point the reduced bihamiltonian structure can be computed according to the scheme described in [7] for the KdV hierarchy. The final result can be found in [6].

## 4. FKDV HIERARCHIES AS BIHAMILTONIAN HIERARCHIES

In this section we will construct a set of commuting flows on the symplectic leaf $\mathcal{S}$, which project on the quotient space $\mathcal{N}$ and give rise to the $\mathrm{fKdV}_3^{(2)}$ hierarchy. Our method can be seen as a variant of the one in [11], but here a systematic use of the bihamiltonian structure and of the Frobenius basis is made.

As explained in, e.g., [13], a simple way to construct commuting flows on a bihamiltonian manifold is to find the Casimir functions of the Poisson pencil $P_\lambda = P_1 - \lambda P_0$. More precisely, we want to find formal series $V(\lambda) = \sum_{k\geq -1} V_k\lambda^{-k}$ of 1–forms (possibly defined only on the symplectic leaf $\mathcal{S}$) which solve the equation

$$V_x + [V, S+\lambda A] = 0, \qquad S \in \mathcal{S}, \tag{4.1}$$

and which are exact when restricted to $T\mathcal{S}$. This can be done by means of a geometric version of the method of *dressing transformations* [16, 9]. The argument rests on the remark that the solutions $V$ of Equation (4.1) must commute with the first–order matrix differential operator $-\partial_x + (S+\lambda A)$. Suppose that this differential operator has a set of "eigenvectors" $|\psi\rangle$, obeying the equation

$$-|\psi\rangle_x + (S+\lambda A)|\psi\rangle = \mu|\psi\rangle \tag{4.2}$$

for some suitable function $\mu$, and suppose that these eigenvectors form a basis in $\mathbb{C}^3$. Then any matrix $V$ having constant (in $x$) eigenvalues and eigenvectors $|\psi\rangle$ solves Equation (4.1). The idea is thus to study the auxiliary linear problem (4.2) to characterize the eigenvectors of $V$. We project this equation on the Frobenius basis at the point $S$, to find

$$-\langle g^{(k)}|\psi_x\rangle + \langle g^{(k)}|S + \lambda A|\psi\rangle = \mu\langle g^{(k)}|\psi\rangle. \tag{4.3}$$

After an integration by parts, and using (3.2), we obtain

$$\langle g^{(k+2)}|\psi\rangle = \langle g^{(k)}|\psi\rangle_x + \mu\langle g^{(k)}|\psi\rangle. \tag{4.4}$$

Then, calling $F^{(k)} := \langle g^{(k)}|\psi\rangle$, we get

$$F^{(k+2)} = \partial_x F^{(k)} + \mu F^{(k)}. \tag{4.5}$$

Furthermore, by imposing the normalization condition $F^{(0)} = \langle g^{(0)}|\psi\rangle = 1$, we finally have that all $F^{(k)}$ are determined from $h := F^{(1)}$ and $k := F^{(2)} = \mu$, by means of the recurrence relation (4.5). From the characteristic system (3.3), still projecting on the Frobenius basis, we have that the functions $F^{(k)}$ must satisfy the following system of equations (to be referred to as the *Riccati system)*:

$$\begin{cases} F^{(3)} &= u_0 F^{(1)} + (u_1 + \lambda)F^{(0)} \\ F^{(4)} &= -u_0 F^{(2)} + (u_2 + \lambda)F^{(1)} + u_3 F^{(0)} \end{cases} \tag{4.6}$$

The unknowns of this system are the generators $h$, $k$ of the recurrence relation (4.5). Putting $\lambda = z^3$, one finds that (4.6) has a solution of the form $h = z + \sum_{i\geq 1} h_i z^{-i}$, $k = z^2 + \sum_{i\geq 1} k_i z^{-i}$, where

$$h_1 = u_0, \quad h_2 = \tfrac{1}{3}(2u_1 - u_2), \quad k_1 = \tfrac{1}{3}(u_1 + u_2), \quad k_2 = \tfrac{1}{3}(u_0^2 + u_3 - u_{0x}'), \tag{4.7}$$

and all coefficients $h_i$, $k_i$ are determined iteratively. From the solutions of the Riccati system we easily obtain the eigenvector $|\psi\rangle$ by means of the definition of the $F^{(k)}$. Since the auxiliary linear problem (4.2) contains $z$ only in the form $\lambda = z^3$, it is clear that other solutions of (4.2) are given by $|\psi_a(z)\rangle = |\psi(\omega^a z)\rangle$ and $\mu_a(z) = \mu(\omega^a z)$, where $\omega = \exp\frac{2\pi i}{3}$, for $a = 1, 2$. Moreover, it is easily shown that $\{|\psi_a\rangle\}_{a=0,1,2}$ is a basis of $\mathbb{C}^3$. By construction, if $\Psi$ is the matrix whose columns are the eigenvectors $|\psi_a\rangle$ and $C$ is a diagonal matrix with constant (in $x$) entries, then $V_C = \Psi C \Psi^{-1}$ solves Equation (4.1). Moreover, one can show that if the entries of $C$ are *constant on* $\mathcal{S}$, then $V_C$ (restricted to $T\mathcal{S}$) is exact, its potential being given by $H_C = \langle M, C\rangle$, where $M = \mathrm{diag}(\mu_0, \mu_1, \mu_2)$. In order to recover the fKdV hierarchy we are interested in, it is convenient to choose $V := V_\Delta = \Psi\Delta\Psi^{-1}$, where $\Delta = \mathrm{diag}(z, \omega z, \omega^2 z)$. Then $V$ turns out to be a Laurent series in $\lambda = z^3$, and we can set

$$\partial_{t_j} S = \partial_x (V^j)_+ + [(V^j)_+, S + \lambda A], \tag{4.8}$$

where $(V^j)_+$ is the part of $V^j$ containing the nonnegative powers of $\lambda$. The vector fields (4.8) are by definition Hamiltonian with respect to the Poisson pencil $P_\lambda$, and can be shown to be bihamiltonian. Then, for general results of the bihamiltonian theory [13], they commute and project on the quotient manifold $\mathcal{N}$.

**Remark 4.1.** The coefficients of the "eigenvalue" $k = \mu_0$ play the role of Hamiltonian densities for the hierarchy (4.8). More precisely, the Hamiltonian vector field associated with $\int_{S^1} 3k_{j+3}\, dx$ by means of the projection of $P_0$ on $\mathcal{N}$ is the projection of the vector field (4.8).

## 5. A KP-LIKE SYSTEM FOR THE FKDV HIERARCHY

In this section we will discuss how the generators $h = \langle g^{(1)}|\psi_0\rangle$ and $k = \langle g^{(2)}|\psi_0\rangle$ of the recurrence relation (4.5) evolve along the flows (4.8). This will allows us to embed the hierarchy of the previous section in a bigger hierarchy, which was called $\mathrm{KP}^{(2)}$ in [6], and is, roughly speaking, a jet-extension of the usual KP hierarchy (see, e.g., [8]). The following result is an adaptation to our setting of Lemma 3.23 of [9].

**Proposition 5.1.** *The evolution of the matrix $\Psi$ along the $j$-th flow of the hierarchy (4.8) is given by*

$$\partial_{t_j}\Psi = (V^j)_+\Psi - \Psi W^{(j)}, \tag{5.1}$$

*where $W^{(j)}$ is a suitable diagonal matrix. Furthermore, the eigenvalue $k = \mu_0$ evolves as*

$$\partial_{t_j} k = \partial_x H^{(j)}, \tag{5.2}$$

*where $H^{(j)} := \langle g^{(0)}|(V^j)_+|\psi_0\rangle$ is the $(1,1)$ entry of $W^{(j)}$.*

Notice that $k$ evolves according to a conservation law because it collects the Hamiltonian densities of a hierarchy of commuting flows. Now we will show that the "currents" $H^{(j)}$ can be computed solely in terms of the generators $h$ and $k$, and that they define the evolution of $h$ too. We begin with

**Proposition 5.2.** *The asymptotic behaviour of $H^{(j)}$ is given by*

$$H^{(j)} = z^j + O(z^{-1}). \tag{5.3}$$

**Proof.** From the definition of the $F^{(k)}$ we have that $|\psi_0\rangle = {}^T|1, z + O(1), z^2 + O(z)\rangle$. Since $V|\psi_0\rangle = z|\psi_0\rangle$ and $(V^j)_- = O(\lambda^{-1})$, then

$$H^{(j)} = \langle g^{(0)}|(V^j)_+|\psi_0\rangle = \langle g^{(0)}|V^j|\psi_0\rangle - \langle g^{(0)}|(V^j)_-|\psi_0\rangle = z^j + O(z^{-1}), \tag{5.4}$$

and we are done. □

To push the analysis further, we need to introduce some more notations. Let $\mathcal{L}$ be the space of truncated Laurent series in $z$ whose coefficients are functions of the space variable $x$. To every pair of Laurent series $h = z + \sum_{i\geq 1} h_i z^{-i}$ and $k = z^2 + \sum_{i\geq 1} k_i z^{-i}$ we can associate the *Faà di Bruno iterates* $\{F^{(j)}\}_{j\geq 0}$, defined by

$$F^{(0)} = 1, \qquad F^{(1)} = h, \qquad F^{(j+2)} = (\partial_x + k)\cdot F^{(j)}, \qquad j \geq 0. \tag{5.5}$$

Then we have a splitting $\mathcal{L} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ is generated (over $C^\infty$-functions independent of $z$) by the Faà di Bruno iterates, and $\mathcal{H}_-$ is generated by $z^j$ with $j \leq -1$.

**Proposition 5.3.** *The Laurent series $H^{(j)}$ belong to $\mathcal{H}_+$.*

**Proof.** The Riccati system (4.6) entails that $\lambda F^{(j)} \in \mathcal{H}_+$ for $j = 0, 1$. By definition of $\mathcal{H}_+$, it follows that $\lambda \cdot \mathcal{H}_+ \subset \mathcal{H}_+$, and, more in general, $\lambda^k \cdot \mathcal{H}_+ \subset \mathcal{H}_+$ for all $k \geq 0$. Therefore any combination of elements in $\mathcal{H}_+$ with coefficients that are *polynomials* in $\lambda$ is in $\mathcal{H}_+$. Now, developing $\langle g^{(0)}|(V^j)_+$ on the Frobenius basis, we obtain

$$\langle g^{(0)}|(V^j)_+ = \sum_{i=0}^{2} a_i^j |\langle g^{(i)}| \tag{5.6}$$

for some polynomials $a_i^j$ in $\lambda$. Hence $H^{(j)} = \sum_{i=0}^2 a_i^j \langle g^{(i)}|\psi_0\rangle = \sum_{i=0}^2 a_i^j F^{(i)} \in \mathcal{H}_+$, and the proof is complete. □

We are left with the evolution of the first generator $h$.

**Proposition 5.4.** *Along the flows (4.8), the Laurent series $h$ evolves as*

$$\partial_{t_j} h = -\pi_-(hH^{(j)}), \tag{5.7}$$

*where $\pi_-$ is the projection on $\mathcal{H}_-$ corresponding to the splitting $\mathcal{L} = \mathcal{H}_+ \oplus \mathcal{H}_-$.*

**Proof.** We develop the time derivative of the element $\langle g^{(1)}|$ of the Frobenius basis on the basis itself, to get

$$\partial_{t_j} \langle g^{(1)}| = \sum_{l=0}^{2} b_{jl} \langle g^{(l)}| \tag{5.8}$$

for some coefficients $b_{jl} \in C^\infty(\mathcal{S}, \mathbb{C})$ (independent of $\lambda$). Moreover, from (5.1) we have that

$$\partial_{t_j} |\psi_0\rangle = (V^j)_+ |\psi_0\rangle - H^{(j)} |\psi_0\rangle. \tag{5.9}$$

Then Equation (5.6) implies

$$\partial_{t_j} h = \partial_{t_j} \langle g^{(1)}|\psi_0\rangle = \sum_{l=0}^{2} b_{jl} \langle g^{(l)}|\psi_0\rangle + \langle g^{(1)}|\partial_{t_j}|\psi_0\rangle = \sum_{l=0}^{2} (b_{jl} + a_l^j) F^{(l)} - hH^{(j)}, \tag{5.10}$$

showing that

$$\partial_{t_j} h + hH^{(j)} \in \mathcal{H}_+. \tag{5.11}$$

Applying $\pi_-$ to both sides of this equation we obtain the thesis. □

The hierarchy defined by (5.2) and (5.7), where no constraint is put on the generators $h$ and $k$, has been studied in [6], where it is called the $\mathrm{KP}^{(2)}$ hierarchy. The constraints given by the Riccati system (4.6) lower the number of free fields to four, and correspond to the reduction giving rise to the $\mathrm{KdV}_3^2$ hierarchy of [6]. Hence we can conclude that the projection on $\mathcal{N}$ of the hierarchy (4.8) coincides with the $\mathrm{KdV}_3^2$ hierarchy.

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