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CONVERGENCE TO BROWNIAN MOTION AND COUPLINGS IN THE D=1 GINZBURG-LANDAU EQUATION WITH NOISE

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Abstract. We consider a Ginzburg–Landau equation in the interval $[-\epsilon^{-\kappa}, \epsilon^{-\kappa}]$, $\epsilon > 0$ and $\kappa \geq 1$, perturbed by a white noise of strength $\sqrt{\epsilon}$ and with Neumann boundary conditions. We prove that if the initial datum is close to an "instanton" then the solution remains close to some instanton for times that may grow as $\epsilon \to 0^+$ as fast as any inverse power of ϵ and as long as the "center of the instanton stays away" from the endpoints of the interval. We prove that the center of the instanton suitably normalized converges to a Brownian motion and that there is an asymptotic coupling of two solutions starting from initial data close to instantons such that in the limit $\epsilon \to 0^+$ the time of the coupling (suitably normalized) converges in law to the first encounter of two Brownians starting from the centers of the instantons approximating the initial data.

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INTRODUCTION.

In a quenching experiment a system is initially in thermodynamic equilibrium with a reservoir which is then cooled down from above to below its critical temperature. This process is usually so fast that the state of the system in the meantime does not change significantly. At the end of the cooling, however, it is no longer in equilibrium with its surroundings, its state is still stationary but unstable. What happens next is called "phase separation". At a very early stage the interaction with the reservoir (usually modelled by a small random perturbation) is dominant because without perturbations the system would stay in its initial stationary state. As soon as the state changes, because of its instability the deterministic driving forces internal to the system take over driving it toward equilibrium. For a mathematical analysis of this part of phase separation, see [8] and references therein. Very often, though, equilibrium is not yet reached at this stage: when the system is spatially extended, in each subregion the state approaches a stable thermodynamic phase, but since several ones are equally accessible (as the temperature is below its critical value) there is no reason why the equilibria of far away regions should coincide. The typical picture is thus a collection of phases with interfaces in between and the next regime of phase separation describes the competition between phases. For non conservative evolutions (that we consider here) the larger clusters grow at the expenses of the smaller ones and typically the interfaces move by mean curvature. Once the clusters become very large (i.e., the curvature very small) this process becomes extremely slow and new effects arise, in particular stochastic forces may again be dominant. As a matter of fact there are examples involving deterministic evolutions where the phase separation even stops and the system fails to reach the true equilibrium, being stuck in some locally stable but spurious equilibrium, see [12]. These effects are more frequent in one dimension and we restrict to this case.

Our model equation is the $d = 1$ Ginzburg-Landau equation with an additive white noise of strength $\sqrt{\epsilon}$, see (1.1) below, where $\epsilon > 0$ is the small parameter of the theory and eventually $\epsilon \to 0^+$. We consider the equation in the interval $\mathcal{T}_{\epsilon,\kappa} = [-\epsilon^{-\kappa}, \epsilon^{-\kappa}]$, $\kappa \geq 1$, imposing Neumann boundary conditions. The potential term in the Ginzburg-Landau equation is a symmetric double well potential $V(m)$, $m \in \mathbb{R}$, with minima at $m = \pm 1$ which represent the equilibrium values of the "order parameter" m. The two equilibrium phases are then the constant functions $m(x) = \pm 1$, $x \in \mathcal{T}_{\epsilon,\kappa}$, and the problem of interface dynamics concerns the evolution of profiles where to the left of some point, say x_0, the state is close to -1 and to the right to $+1$. Fusco and Hale, [13], and Carr and Pego, [6], have shown that for the deterministic Ginzburg-Landau equation the motion is extremely slow. The system in fact relaxes after a short time to an apparently stationary state with the two ± 1 phases connected by an "instanton". This state for our choice of the potential is (very close to) $\bar{m}_{x_0}(x) \doteq \tanh(x - x_0)$ (x_0 is the "center of the instanton"). $\bar{m}_{x_0}$ is not truly stationary, it moves almost without changing shape with speed $\approx e^{-c\ell}$, c a positive "slowly varying" factor, ℓ the distance of the center from the boundary of the region. Thus if ℓ is a fraction of the whole interval, i.e., $\ell \approx \epsilon^{-\kappa}$, the motion is exponentially slow. If we take into account the stochastic term, the picture initially does not change much: except for small random fluctuations we still see, after a short relaxation time, the same profile $\bar{m}_{x_0}$. However on dramatically shorter times (when compared with the deterministic case) the instanton moves. At times $t_\epsilon \doteq t\epsilon^{-a}$, $0 < a < 1/3$,

$t > 0$, the shape is still the same but the center x_0 has changed by the order of $\sqrt{\epsilon t_\epsilon}$ and on this scale it is a Brownian motion, [3]. This result is extended up to times $t_\epsilon \doteq t\epsilon^{-1}$, $t > 0$, when the displacement is finite and the center of the instanton converges as $\epsilon \to 0^+$ to a Brownian motion b_t, as shown in [5] with $\kappa = 1$. The motion is still Brownian also at longer times, $t_\epsilon \doteq t\epsilon^{-1-\gamma}$, $t > 0$, $\gamma > 0$ small enough, as shown by Funaki, [14], in a somewhat different setup. At much longer times the picture may change, for instance the system may pick up some drift, as it happens when the potential $V(s)$ is non-symmetric (still with equally deep minima, as shown by Brassesco and Buttà, [4]). Here however we restrict to the symmetric case proving that no drift will appear even at "very long times". Roughly speaking we prove that the process m_t behaves like $\bar{m}_{x_0+\sqrt{\epsilon}b_t}$ (the sup norm of the difference vanishing as $\epsilon \to 0^+$) where b_t has the same law as the previous Brownian motion. Convergence is proven by scaling space and times with the latter "much shorter" than what it takes to a Brownian motion to reach the boundary of $\mathcal{T}_{\epsilon,\kappa}$. The result is far complete, a question of physical interest left open is a precise estimate of the first time when a single phase takes over. We control in this paper the first time τ when one of the two phases occupies the whole $\mathcal{T}_{\epsilon,\kappa}$ except for a "small interval close to one of the endpoints of $\mathcal{T}_{\epsilon,\kappa}$" of size proportional to ϵ^{-1}. We show that τ has the same law in the limit as $\epsilon \to 0^+$ as a suitable stopping time of b_t (this law can be easily computed). We could prove that also after τ the instanton will move like a Brownian motion, but only till it gets close to the endpoints of $\mathcal{T}_{\epsilon,\kappa}$ by a distance $c \log \epsilon^{-1}$, $c > 0$. Thereafter (for a critical value of c) the Fusco-Hale drift should in fact be dominant with the minority phase shrinking deterministically till extinction. The time length (after τ) when all this happens should be significantly smaller than τ itself, which is thus a good estimate for the time of relaxation to equilibrium. This is not a true equilibrium, though, and the game is not yet over: at much longer times tunnelling phenomena will appear, but this is really beyond the purposes of the present discussion.

The convergence to a one dimensional process simply described by a Brownian motion holds in a much stronger sense than one could suspect from the above presentation. In fact by extending the work of Mueller, [15], (to the present case where the stable point is replaced by a one-dimensional manifold of equilibria, i.e., the translates of the instanton) we construct, in the limit as $\epsilon \to 0^+$, an asymptotic coupling of processes starting from two different initial data, where the time of coupling is the same in law as that of the first encounter of two independent Brownian motions in $d = 1$. They represent the motion of the centers of the instantons associated to the different initial conditions. Namely it suffices that the centers meet for the whole profiles to be matched completely everafter. We refer to the next section for a precise and complete list of results. In Section 2 we prove statements concerning the deterministic equation (without noise). In Section 3 we extend the results to the case with noise. In Section 4 we prove convergence to the Brownian motion and in Section 5 we construct the coupling which proves the loss of memory of the initial datum. A brief outline of the main ideas of the proof is given in Section 3 after the proof of Theorem 1.1.

1. Definitions and main results.

In this paper we study a Ginzburg-Landau one-dimensional equation with additive "small" noise. More precisely, we consider the stochastic partial differential equation

$$\begin{aligned} &\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + [m - m^3] + \sqrt{\epsilon}\alpha, \qquad t \geq 0; \quad x \in \mathcal{T}_{\epsilon,\kappa} \doteq [-\epsilon^{-\kappa}, \epsilon^{-\kappa}] \\ &m(x,0) = m_0(x). \end{aligned} \tag{1.1}$$

We impose Neumann boundary conditions (N.b.c.) at $\mathcal{T}_{\epsilon,\kappa}$. The noise $\alpha = \alpha(x,t)$ is a standard space-time white noise (see [17] for a precise definition), and ϵ is a small parameter that will go to zero. The standard way to give a precise meaning to (1.1) is to consider the corresponding integral equation, in terms of the heat kernels associated with the given boundary conditions. In our case, the construction is as follows. Let $H_t^{(\epsilon)}$ be the Green operator associated to the heat equation on $\mathcal{T}_{\epsilon,\kappa}$ with N.b.c. Let $Z_t(x)$ be the Gaussian process defined by the stochastic integral

$$Z_t(x) \doteq \int_0^t ds \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, \alpha(s,y)\, H_{t-s}^{(\epsilon)}(x-y) \tag{1.2}$$

for $x \in \mathcal{T}_{\epsilon,\kappa}$ and extended to $\mathbb{R}$ by successive reflections around $n\epsilon^{-\kappa}$. Call $C^0(\mathbb{R})$ the set of continuous and bounded functions and

$$C_{\epsilon,\kappa}(\mathbb{R}) = \{f : f \in C^0(\mathbb{R}), f \text{ is invariant by reflections around the points } n\epsilon^{-\kappa}, |n| \geq 1\}$$

Given $m_0 \in C^0(\mathbb{R})$ and $Z_t \in C_{\epsilon,\kappa}(\mathbb{R})$ for each t, and continuous in t, we define $m_t =: T_t(m_0; \sqrt{\epsilon}Z)$, $t \geq 0$, as the solution of the equation

$$m_t = H_t \star m_0 - \int_0^t ds H_{t-s} \star (m_s^3 - m_s) + \sqrt{\epsilon} Z_t \tag{1.3}$$

where $\star$ denotes the convolution of functions on $\mathbb{R}$, $H_t(x)$ is the heat kernel in $\mathbb{R}$:

$$H_t(x) \doteq \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \tag{1.4}$$

We will consider the case when Z_t is given by (1.2), and refer to it as the Ginzburg-Landau process, which is adapted to Z_t. For more details, see [5], where $\kappa = 1$ is considered. In particular, it is proved in [5] that (1.3) has a unique continuous solution. If $m_0 \in C_{\epsilon,\kappa}(\mathbb{R})$ then the process lives in $C_{\epsilon,\kappa}(\mathbb{R})$ at all times. In general one refers to this as the the Ginzburg-Landau process on $\mathcal{T}_{\epsilon,\kappa}$ with Neumann boundary conditions, but, for notational convenience, we have extended the range of admissible initial data to the whole $C^0(\mathbb{R})$.

Since the function $\bar{m}(x) \doteq \tanh x$ verifies the identity

$$\frac{1}{2}\frac{d^2\bar{m}}{dx^2} = V'(\bar{m}); \quad V'(m) \doteq \frac{dV(m)}{dm}, \quad V(m) \doteq -\frac{1}{2}m^2 + \frac{1}{4}m^4, \quad m \in \mathbb{R} \tag{1.5}$$

$\bar{m}$ is a stationary solution of the deterministic Ginzburg-Landau equation on $\mathbb{R}$; i.e., $\bar{m} = T_t(\bar{m}) \doteq T_t(\bar{m}; 0)$, for any $t \geq 0$. The function $\bar{m}(x)$ is what we call "instanton", and, its translate by $x_0 \in \mathbb{R}$,

$$\bar{m}_{x_0}(x) \doteq \bar{m}(x - x_0), \qquad x \in \mathbb{R} \tag{1.6}$$

will be called "the instanton centered at x_0" (thus $\bar{m} \equiv \bar{m}_0$). The manifold

$$\mathcal{M} \doteq \{\bar{m}_{x_0}, x_0 \in \mathbb{R}\} \subset C^0(\mathbb{R}) \tag{1.7}$$

is locally attractive under the deterministic flow $T_t(\cdot)$, see Fife and McLeod, [10]–[11]. More precisely, let $\|\cdot\|$ denote the sup norm in $\mathbb{R}$, and, for $\delta \geq 0$, define

$$\mathcal{M}_\delta \doteq \{m \in C(\mathbb{R}) : \mathrm{dist}(m, \mathcal{M}) \doteq \inf_{x_0 \in \mathbb{R}} \|m - \bar{m}_{x_0}\| \leq \delta\} \tag{1.8}$$

Then, Fife and McLeod have proved that there exists $\delta_0 \geq 0$ and a real valued function $\zeta(m)$ defined on $\mathcal{M}_{\delta_0}$, such that

$$\lim_{t \to +\infty} T_t(m) = \bar{m}_{\zeta(m)}, \qquad \text{for all } m \in \mathcal{M}_{\delta_0}, \tag{1.9}$$

in sup norm and exponentially fast. Thus $\mathcal{M}_{\delta_0}$ is foliated by the submanifolds (transversal to $\mathcal{M}$):

$$\mathcal{V}_{x_0} \doteq \{m \in \mathcal{M}_{\delta_0} : \zeta(m) = x_0\} \tag{1.10}$$

one the space translation of the other. $\zeta(m)$ will be called the "true center" of $m \in \mathcal{M}_{\delta_0}$ (in the proofs it will be convenient to work with an approximate center that we call the "linear center", but the above definition has conceptual advantages).

Given $m \in C^0(\mathbb{R})$ we set $m^{\epsilon,\kappa} \doteq m$ if $m \notin C_{\epsilon,\kappa}(\mathbb{R})$, otherwise

$$m^{\epsilon,\kappa}(x) \doteq \begin{cases} m(x) & \text{for } |x| \leq \epsilon^{-\kappa} \\ m(\pm\epsilon^{-\kappa}) & \text{for } x \geq \epsilon^{-\kappa}, \text{ respectively } x \leq -\epsilon^{-\kappa} \end{cases} \tag{1.11}$$

(Observe that if $m \in C_{\epsilon,\kappa}(\mathbb{R})$ then $m \notin \mathcal{M}_\delta$ (if δ is small enough) no matter how close it is to an instanton when restricted to $\mathcal{T}_{\epsilon,\kappa}$, yet $m^{\epsilon,\kappa}$ may be in $\mathcal{M}_\delta$).

We define

$$\zeta^{\epsilon,\kappa}(m) \doteq \begin{cases} x_0 & \text{if } m^{\epsilon,\kappa} \in \mathcal{V}_{x_0} \\ 2\epsilon^{-\kappa} & \text{otherwise} \end{cases} \tag{1.12}$$

Given m_t a solution of (1.3) we set

$$\zeta_t \doteq \zeta^{\epsilon,\kappa}(m_t) \tag{1.13}$$

and, for any $0 < \ell < 1$ and $h > 0$ we define the stopping time

$$\tau_\epsilon(\kappa, \ell, h) \doteq \epsilon^{-h} \wedge \inf\left\{t \geq 0 : |\zeta^{\epsilon,\kappa}(m_t)| \geq \epsilon^{-\kappa} - \ell\epsilon^{-1}\right\} \tag{1.14}$$

and the stopped process $\{\hat{m}_t\}_{t \geq 0}$,

$$\hat{m}_t \doteq m_{t \wedge \tau_\epsilon}, \quad \hat{\zeta}_t \doteq \zeta_{t \wedge \tau_\epsilon}, \qquad \tau_\epsilon \doteq \tau_\epsilon(\kappa, \ell, h),$$

where $a \wedge b$ stands for the minimum between a and b.

Finally we shall denote by P^ϵ the probability distribution of the various stochastic elements we consider and by E^ϵ the expectation value.

1.1 Theorem.

There is $\delta \in (0, \delta_0]$, δ_0 as in (1.9), and for any $\kappa \geq 1$, $h > 0$, $a \in (0, 1/4)$ and $\ell \in (0,1)$ there are $c > 0$ and $p > 0$ so that the following holds.

Let $m \in C^0(\mathbb{R})$, $|\zeta^{\epsilon,\kappa}(m)| < \epsilon^{-\kappa} - \ell\epsilon^{-1}$, then

$$P^\epsilon\Big((m_t)^{\epsilon,\kappa} \in \mathcal{M}_{2\delta}\ \forall t \leq (\log\epsilon)^2, \quad (m_t)^{\epsilon,\kappa} \in \mathcal{M}_{\epsilon^{1/2-a}}\ \forall t \in [(\log\epsilon)^2, \tau_\epsilon(\kappa,\ell,h)]\Big) \geq 1 - ce^{-\epsilon^{-p}} \tag{1.15}$$

Our next result states that the process $\hat{\zeta}_t$ converges as $\epsilon \to 0^+$ to a Brownian motion with diffusion coefficient $D \doteq 3/4$.

1.2 Theorem. *(Convergence to Brownian motion.)*

Let $\kappa \geq 1$, $h > 0$, $t^\star > 0$, $\ell \in (0,1)$ and for all $\epsilon > 0$ let $m_0 \in \mathcal{M}_\delta$, δ as in Theorem 1.1, such that $|\zeta^{\epsilon,\kappa}(m_0)| \leq \epsilon^{-\kappa} - \ell\epsilon^{-1}$ We also suppose that there are $R_\pm \in [-\infty, +\infty]$ such that

$$\lim_{\epsilon\to 0^+} \epsilon^{h/2}[\pm(\epsilon^{-\kappa} - \ell\epsilon^{-1}) - \zeta^{\epsilon,\kappa}(m_0)] = R_\pm$$

Then $\epsilon^{h/2}\hat{\zeta}_{\epsilon^{-h-1}t}$, as a process on $C([0,t^\star])$, converges to a Brownian motion with diffusion coefficient D, starting from 0 and stopped at $R_\pm$.

Theorem 1.1 thus states that the Ginzburg-Landau process is locally attracted by the manifold $\mathcal{M}$ (when the center is sufficiently far from the endpoints of $\mathcal{T}_{\epsilon,\kappa}$) and Theorem 1.2 that it performs a Brownian motion on $\mathcal{M}$. We also have sharper results, see Proposition 3.5 and Corollary 3.6, on the small deviations of the process transversally to $\mathcal{M}$.

Finally, we construct, in the same probability space a pair m_t and m'_t of Ginzburg–Landau processes starting from two initial data. In such space, we define

$$\sigma = \inf\{t : m_t \equiv m'_t\} \tag{1.16}$$

(where we are considering the processes in $\mathcal{T}_{\epsilon,\kappa}$). As the process, by Theorem 1.2, looks like a Brownian motion on $\mathcal{M}$, a natural guess is that σ has the same (limit) law as the first encounter of two independent Brownians that start from the centers of the initial data.

1.3 Theorem. *(Couplings.)*

Let m_0 and m'_0 be in $C_{\epsilon,\kappa}$ and let both verify the assumptions of Theorem 1.2 with respectively $R_\pm$ and $R'_\pm$. We suppose that

$$\lim_{\epsilon\to 0^+} \epsilon^{h/2}(x_0 - x'_0) =: r^\star$$

Then we can construct m_t and m'_t , Ginzburg Landau processes starting from m_0 and m'_0 respectively so that, if σ is defined by (1.16) , then $\epsilon^{h+1}\sigma$ converges in law to the distribution of the variable S defined as follows. Let b_t and b'_t be independent Brownians with diffusion D starting

from 0 and stopped respectively at $R_\pm$ and $R'_\pm$. Then S is the first time when $b'_t - b_t = r^\star$, provided S occurs before any of the Brownians is stopped, in that case $S = +\infty$.

2. The deterministic flow.

In this section we study the deterministic flow $\{T_t(m)\}_{t\geq 0}$, proving that a neighborhhod of $\mathcal{M}$ is attracted exponentially by $\mathcal{M}$ and it is therefore foliated by the submanifolds $\{\mathcal{V}_{x_0}, x_0 \in \mathbb{R}\}$ defined in (1.10).

If $m \in C^0(\mathbb{R})$ then $m(x,t) \doteq T_t(m)(x)$, $t > 0$, is in $C^2(\mathbb{R})$, is differentiable with respect to t and satisfies the Ginzburg-Landau equation

$$\frac{\partial m}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 m}{\partial x^2}(x,t) + m(x,t) - m(x,t)^3 \tag{2.1}$$

The instantons $\bar{m}_{x_0}$, $x_0 \in \mathbb{R}$, are stationary solutions of (2.1). The linearized flow around $\bar{m}_{x_0}$ is the linear semigroup g_{t,x_0} on $C^0(\mathbb{R})$ whose generator L_{x_0} acts on $C^2(\mathbb{R})$ as

$$L_{x_0} f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) + [1 - 3\bar{m}_{x_0}(x)^2] f(x) \tag{2.2}$$

The operator L_{x_0} has 0 as an eigenvalue (for any $x_0 \in \mathbb{R}$) since $L_{x_0}\bar{m}'_{x_0} = 0$. Denoting by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\mathbb{R})$ we set for any $x_0 \in \mathbb{R}$,

$$\tilde{m}'_{x_0} \doteq \frac{\sqrt{3}}{2}\bar{m}'_{x_0}, \quad \langle \tilde{m}'_{x_0}, \tilde{m}'_{x_0}\rangle = 1 \tag{2.3}$$

which is thus a unitary eigenvector in $L^2(\mathbb{R})$ of L_{x_0}. L_{x_0} has a spectral gap:

2.1 Theorem.

There are α and c positive so that for any $f \in C^0(\mathbb{R})$ and $x_0 \in \mathbb{R}$

$$\|g_{t,x_0}[f - \langle \tilde{m}'_{x_0}, f\rangle \tilde{m}'_{x_0}]\| \leq c e^{-\alpha t}\|f - \langle \tilde{m}'_{x_0}, f\rangle \tilde{m}'_{x_0}\| \tag{2.4}$$

A proof of Theorem 2.1 in a L^2 setting may be found in [10], the proof with sup-norms is similar to that in Section 4 of [7] and it is omitted.

We will exploit Theorem 2.1 by observing that $m(x,t) \doteq T_t(m)(x)$ solves for $t > 0$ the equation:

$$\frac{\partial[m - \bar{m}_{x_0}]}{\partial t}(x,t) = L_{x_0}[m - \bar{m}_{x_0}](x,t) - 3\bar{m}_{x_0}(x)[m - \bar{m}_{x_0}]^2(x,t) - [m - \bar{m}_{x_0}]^3(x,t) \tag{2.5}$$

We next define the "linear center" of a function $f \in C^0(\mathbb{R})$. This notion was introduced in [5], where it was called simply "center".

2.2 Definition.

The point $x_0 \in \mathbb{R}$ is a linear center of $m \in C^0(\mathbb{R})$ if

$$\langle \bar{m}'_{x_0}, [m - \bar{m}_{x_0}] \rangle = 0 \tag{2.6}$$

Existence and uniqueness of the linear center are stated in the next lemma, whose proof, being essentially the same as that of Proposition 3.2 of [5], is omitted:

2.3 Lemma.

There is $\delta_0 > 0$ so that any $m \in \mathcal{M}_{\delta_0}$ has a unique linear center $\xi(m)$. Moreover there is $c_0 > 0$ so that if $m \in C^0(\mathbb{R})$, $y_0 \in \mathbb{R}$ and

$$\|m - \bar{m}_{y_0}\| =: \delta \leq \delta_0 \tag{2.7}$$

then the linear center x_0 of m is such that

$$|x_0 - y_0| \leq c_0 \delta; \qquad \left| x_0 - \left[y_0 - \frac{3}{4} \langle \bar{m}'_{y_0}, (m - \bar{m}_{y_0}) \rangle \right] \right| \leq c_0 \delta^2 \tag{2.8}$$

Let m and $\tilde{m}$ be in $\mathcal{M}_{\delta_0}$, x_0 and $\tilde{x}_0$, their respective linear centers and $\|m - \tilde{m}\| \leq \delta_0$. Then

$$|x_0 - \tilde{x}_0| \leq \frac{c_0}{2} |\langle \bar{m}'_{x_0}, m - \tilde{m} \rangle| \ \leq \ c_0 \|m - \tilde{m}\| \tag{2.9}$$

The next one is the main result in this Section:

2.4 Theorem.

There are $\beta > 0$, $c > 0$ and $\delta_1 \in (0, \delta_0]$, ($\delta_0$ as in Lemma 2.3) so that any $m \in \mathcal{M}_{\delta_1}$ has a true center $\zeta(m)$, a unique linear center $\xi(m)$ and

$$|\xi(m) - \zeta(m)| \leq c \Big[\|m - \bar{m}_{\xi(m)}\| \log\{\|m - \bar{m}_{\xi(m)}\|\} \Big]^2 \tag{2.10}$$

Moreover $T_t(m) \in \mathcal{M}_{\delta_0}$ for all $t \geq 0$ and setting $\xi_t \doteq \xi(T_t(m))$,

$$\|T_t(m) - \bar{m}_{\zeta(m)}\| \leq c \|m - \bar{m}_{\xi(m)}\| e^{-\beta t}, \quad \|T_t(m) - \bar{m}_{\xi_t}\| \leq c \|m - \bar{m}_{\xi(m)}\| e^{-\beta t} \tag{2.11}$$

$$\sup_{t \geq 0} |\xi_t - \xi(m)| \leq c \|m - \bar{m}_{\xi(m)}\|, \quad \sup_{t \geq 0} \|T_t(m) - \bar{m}_{\xi(m)}\| \leq c \|m - \bar{m}_{\xi(m)}\| \tag{2.12}$$

We will prove a preliminary Lemma, Lemma 2.5, and then Theorem 2.4. The bound (2.10) is not optimal, in fact a bound without the logarithmic factor holds true; the right hand side of the first inequality in (2.12) can be replaced by $c\|m - \bar{m}_{\xi(m)}\|^2$. Both refinements are not needed and will not be proved.

2.5 Lemma.

There is $c_1 > 0$ so that for any $\lambda \in (0,1]$, $x_0 \in \mathbb{R}$ and $v \in C(\mathbb{R})$ such that $\|v\| \leq 1$ and $\langle \bar{m}'_{x_0}, v\rangle = 0$ the following holds. Let

$$m \doteq \bar{m}_{x_0} + \lambda v, \qquad t_\lambda \doteq \max\{1, (\log \lambda)^2\} \tag{2.13}$$

and c, α as in Theorem 2.1. Then for all $0 \leq t \leq t_\lambda$

$$\left|T_t(m) - \bar{m}_{x_0}\right| \leq ce^{-\alpha t}\lambda + c_1\lambda^2 t \tag{2.14}$$

Proof.

Let $u_t \doteq T_t(m) - \bar{m}_{x_0}$, $u_0 \doteq \lambda v$. By (2.5) and Theorem 2.1 there is $c_2 > 0$ so that

$$\|u_t\| \leq ce^{-\alpha t}\lambda + c_2 \int_0^t ds \left(\|u_s\|^2 + \|u_s\|^3\right) \tag{2.15}$$

Call T the first time such that $\|u_T\| = 2c\lambda$, and suppose that $T \leq t_\lambda$. Then by (2.15)

$$2c\lambda \leq c\lambda + c_2 t_\lambda \left([2c\lambda]^2 + [2c\lambda]^3\right)$$

which does not hold when λ is small enough, say $\lambda \leq \lambda_0$, $\lambda_0 \in (0,1]$.

But (2.14) follows from (2.15) when $\lambda \in (\lambda_0, 1]$, if we use the maximum principle to bound $\|u_s\|$ for all $s \geq 0$. For $\lambda \leq \lambda_0$, we have seen that $T > t_\lambda$ so $\|u_t\| < 2c\lambda$ for all $t \leq t_\lambda$. With this bound on the right hand side of (2.15) we obtain (2.14), completing the proof of the Lemma. □

Proof of Theorem 2.4.

We use the same notation as in Lemmas 2.5 and 2.3. We take $\delta_1 > 0$ (other requests on δ_1 will be specified later) so that for any $\lambda \in (0, \delta_1]$

$$\sup_{t \leq t_\lambda} [ce^{-\alpha t}\lambda + c_1\lambda^2 t] \leq \delta_0, \qquad \delta_0 \text{ as in Lemma 2.3} \tag{2.16}$$

Let $m \in M_{\delta_1}$ and call $x_0 = \xi(m)$. If $m = \bar{m}_{x_0}$, then the theorem follows since $\bar{m}_{x_0}$ is stationary for T_t. We write

$$m = \bar{m}_{x_0} + \frac{m - \bar{m}_{x_0}}{\|m - \bar{m}_{x_0}\|}\|m - \bar{m}_{x_0}\|.$$

We are then in the setting of Lemma 2.5, with $\lambda = \|m - \bar{m}_{x_0}\|$ and, by Lemma 2.3, (2.14) and (2.16), the linear center $\xi(T_t(m))$ is well defined for $t \leq t_\lambda$, and, calling $m^\star \doteq T_{t_\lambda}(m)$ there is a constant $c_2 > 0$ so that,

$$\left|\xi(m^\star) - x_0\right| \leq c_0\|m^\star - \bar{m}_{x_0}\| \leq c_2[\lambda \log \lambda]^2 \tag{2.17}$$

We will prove that there is a constant $c_3 > 0$ so that

$$\left|\xi(m^\star) - \zeta(m^\star)\right| \leq c_3\|m^\star - \bar{m}_{\xi(m^\star)}\| \tag{2.18}$$

Proof of (2.10).

Since $\zeta(m) = \zeta(T_t(m))$, $\zeta(m) = \zeta(m^\star)$. Then by (2.18) and (2.17)

$$|\zeta(m) - x_0| \leq c_2[\lambda \log \lambda]^2 + c_3\|m^\star - \bar{m}_{\xi(m^\star)}\|$$

Recalling that $\bar{m}' \leq 1$,

$$\begin{aligned}\|m^\star - \bar{m}_{\xi(m^\star)}\| &\leq \|m^\star - \bar{m}_{x_0}\| + \|\bar{m}_{x_0} - \bar{m}_{\xi(m^\star)}\| \leq (1+c_0)\|m^\star - \bar{m}_{x_0}\| \\ &\leq \frac{(1+c_0)c_2}{c_0}[\lambda \log \lambda]^2 \end{aligned} \tag{2.19}$$

The last inequality uses (2.17). Thus, (2.10) is proved (provided (2.18) holds).

Proof of (2.18).

We further specify δ_1 by requiring that for all $\lambda \in (0, \delta_1]$

$$(1+c_0)[ce^{-\alpha t_\lambda}\lambda + c_1\lambda^2 t_\lambda] \leq \delta_1 \tag{2.20}$$

By (2.19) and (2.14), (2.20) implies

$$\|m^\star - \bar{m}_{x^\star}\| \leq \delta_1, \quad \text{where} \quad x^\star \doteq \xi(m^\star) \tag{2.21}$$

thus reconstructing the initial assumption also at time t_λ.

We define τ so that

$$e^{-\alpha\tau}c(1+c_0) = \frac{1}{4} \tag{2.22}$$

We require that δ_1 is such that for all $\lambda \in (0, \delta_1]$

$$(1+c_0)[ce^{-\alpha\tau}\lambda + c_1\lambda^2\tau] \leq \frac{\lambda}{2} \tag{2.23}$$

Let $m^{(0)} \doteq m^\star$, $x^{(0)} \doteq x^\star$, $m^{(1)} \doteq T_\tau(m^{(0)})$, $x^{(1)} \doteq \xi(m^{(1)})$ and

$$\lambda_0 \doteq (1+c_0)[ce^{-\alpha t_\lambda}\lambda + c_1\lambda^2 t_\lambda], \quad \lambda \doteq \|m - \bar{m}_{x_0}\|, \quad \|m^\star - \bar{m}_{x^{(0)}}\| \leq \lambda_0$$

and

$$\lambda_1 \doteq (1+c_0)[ce^{-\alpha\tau}\lambda_0 + c_1\lambda_0^2\tau]$$

By iteration we then define $m^{(n)}$, $x^{(n)}$ and λ_n, $n > 1$. We have from (2.23)

$$\|T_{n\tau}(m^{(0)}) - \bar{m}_{x^{(n)}}\| \leq \lambda_n \leq 2^{-n}\lambda_0$$

and since $|x^{(n)} - x^{(n-1)}| \leq c_0\|T_\tau(m^{(n-1)}) - \bar{m}_{x^{(n-1)}}\|$ by (2.14)

$$|x^{(n)} - x^{(0)}| \leq \sum_{i=1}^{n} \lambda_i \leq 2\lambda_0$$

Since $x^{(n)} \to \zeta(m^{(0)})$ as $n \to +\infty$ this proves (2.18) (recall that $m^\star = m^{(0)}$).

Proof of (2.12).

We have

$$|\xi_t - \xi(m)| \le |\xi_t - \zeta(m)| + |\zeta(m) - \xi(m)| \tag{2.24}$$

The last term is bounded using (2.10). For the other one observe that by (2.9) for any $t \ge 0$

$$\begin{aligned} \|\xi_t - \zeta(m)\| &= \|\xi(\bar{m}_{\xi_t}) - \xi(\bar{m}_{\zeta(m)})\| \\ &\le c_0\|\bar{m}_{\xi_t} - \bar{m}_{\zeta(m)}\| \le c_0\|\bar{m}_{\xi_t} - T_t(m)\| + c_0\|\bar{m}_{\zeta(m)} - T_t(m)\| \end{aligned} \tag{2.25}$$

which by (2.11) proves the first inequality in (2.12). The second one follows from the first one and (2.11). Theorem 2.4 is proved. □

3. The stochastic flow.

In this Section we prove Theorem 1.1 and some of the key estimates that will be used in Sections 4 and 5 to prove the other theorems of Section 1. We start with Proposition 3.1 where we derive the basic bounds on the Gaussian process Z_t. The Proposition is proved in [5] for $\kappa = 1$, its extension to $\kappa > 1$ is not difficult, for completeness we report it in the Appendix.

3.1 Proposition.

Let $\kappa \ge 1$, $\epsilon > 0$ and Z_t as defined in (1.2), *and $T_t(m, \sqrt{\epsilon}Z)$ the solution of* (1.3) *starting from m. Then there are positive constants b_0 and b_1 such that, if we set $t_\epsilon \doteq (\log \epsilon)^2$ and*

$$\mathcal{B}_{p,\epsilon} \doteq \Big\{ \sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}Z_t\| \le \epsilon^{\frac{1}{2}-p} \Big\}, \tag{3.1}$$

then, for all $\epsilon > 0$ and $p > 0$

$$P^\epsilon(\mathcal{B}_{p,\epsilon}) \ge 1 - b_0 e^{-b_1|\log\epsilon|^{-1}\epsilon^{-2p}} \tag{3.2}$$

and for all $m \in C^0(\mathbb{R})$ with $\|m\| \le 1 + 10^{-2}$ and any $S \ge t_\epsilon$,

$$P^\epsilon\Big(\sup_{0\le t\le S} \|T_t(m;\sqrt{\epsilon}Z)\| \le 2,\ \|T_S(m;\sqrt{\epsilon}Z)\| \le 1 + 10^{-2} \Big) \ge 1 - Sb_0 e^{-b_1\epsilon^{-1}} \tag{3.3}$$

Note how (3.3) implies that, for any $N > 0$, the process remains uniformly bounded up to times ϵ^{-N} with probability close to 1 exponentially in ϵ^{-1}.

Let $m \in C^0(\mathbb{R})$, set $m_t \doteq T_t(m;\sqrt{\epsilon}Z)$, $t \ge 0$, then, for any $x_0 \in \mathbb{R}$, $u_t \doteq m_t - \bar{m}_{x_0}$ solves the following integral version of the Ginzburg-Landau stochastic equation (also considered in [5])

$$u_t = g_{t,x_0}u_0 - \int_0^t ds\, g_{t-s,x_0}(3\bar{m}_{x_0}u_s^2 + u_s^3) + \sqrt{\epsilon}\hat{Z}_{t,x_0} \tag{3.4}$$

The operator g_{t,x_0} was defined in the beginning of Section 2 and

$$\hat{Z}_{t,x_0} \doteq Z_t + \int_0^t ds\, g_{t-s,x_0}\,[(3\bar{m}_{x_0}^2 - 1)Z_s] \tag{3.5}$$

$\hat{Z}_{t,x_0}$ is also given by the stochastic integral

$$\hat{Z}_{t,x_0} = \int_0^t ds \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, \alpha(s,y), \tag{3.6}$$

where

$$g^{(\epsilon)}_{t-s,x_0}(x,y) \doteq \sum_{j\in\mathbb{Z}} \Big(g_{t-s,x_0}(x, y+4j\epsilon^{-\kappa}) + g_{t-s,x_0}(x, 4j\epsilon^{-\kappa} + 2\epsilon^{-\kappa} - y)\Big), \tag{3.7}$$

and $g_{t,x_0}(\cdot,\cdot)$ stands for the kernel corresponding to the operator g_{t,x_0}. An estimate analogous to that given by the previous Proposition follows for $\hat{Z}_{t,x_0}$:

3.2 Proposition.

Let $\hat{Z}_{t,x_0}$ as above. Then there are positive constants b_0 and b_1 such that, if we set $t_\epsilon \doteq (\log\epsilon)^2$ and

$$\mathcal{B}_{p,\epsilon,x_0} \doteq \Big\{ \sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}\hat{Z}_{t,x_0}\| \le \epsilon^{\frac{1}{2}-p} \Big\}$$

then

$$P^\epsilon\Big(\mathcal{B}_{p,\epsilon,x_0}\Big) \ge 1 - b_0 e^{-b_1\epsilon^{-p}} \tag{3.8}$$

Proof.

It follows easily from (3.2), (3.5) and (A.32).

Recall from Lemma 2.3 that $\xi(m)$ is the linear center of m (see (2.6)), and that there is δ_0 so that if $m \in \mathcal{M}_{\delta_0}$ then $\xi(m)$ is uniquely defined. In analogy with (1.12), given any $\kappa \ge 1$, we set

$$\xi^{\epsilon,\kappa}(m) \doteq \begin{cases} \xi(m^{\epsilon,\kappa}) & \text{if } m^{\epsilon,\kappa} \in \mathcal{M}_{\delta_0} \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

and, given any $\ell \in (0,1)$ and $\delta \in (0,\delta_0]$,

$$M^\epsilon_{\kappa,\ell,\delta} \doteq \{ m \in C^0(\mathbb{R}) : m^{\epsilon,\kappa} \in \mathcal{M}_\delta,\ |\xi^{\epsilon,\kappa}(m)| \le \epsilon^{-\kappa} - \ell\epsilon^{-1} \} \tag{3.10}$$

By (2.10), Theorem 1.1 follows from the analogous statement with the true center replaced by the linear center. Theorem 1.1 will then be a consequence of the following Proposition.

3.3 Proposition.

There is $\delta_2 \in (0,\delta_1]$ (δ_1 as in Theorem 2.4) so that the following holds. Let $a \in (0,1/2)$, $\ell \in (0,1)$, $\kappa \ge 1$, $\delta \in (0,\delta_2]$ and $p \in (0,a/2)$. Then there are positive constants b_0, b_1 and c so that for any ϵ small enough and for any $m \in M^\epsilon_{\kappa,\ell,\delta}$, if $m_t \doteq T_t(m;\sqrt{\epsilon}Z)$, $\ell' = \ell - \epsilon c(\delta \vee \epsilon^{1/2-a})$, and $t_\epsilon = (\log\epsilon)^2$,

$$P^\epsilon\Big(m_t \in M^\epsilon_{\kappa,\ell',c\delta} \text{ for all } t \le t_\epsilon,\ m_{t_\epsilon} \in M^\epsilon_{\kappa,\ell',\epsilon^{1/2-a}}\Big) \ge 1 - b_0 e^{-b_1\epsilon^{-p}} \tag{3.11}$$

Proof.

Let m be as in the statement and consider first the case $m \in \mathcal{M}_{\delta_1}$. We study $m_t \doteq T_t(m;\sqrt{\epsilon}Z)$ as a perturbation of $m^0_t \doteq T_t(m)$. Let

$$x_0 = \xi(m), \quad u^0_t \doteq m^0_t - \bar{m}_{x_0}, \quad D_t \doteq \|m_t - m^0_t\|, \quad \xi_t \doteq \xi(m_t), \quad \xi^0_t \doteq \xi(m^0_t) \tag{3.12}$$

By Theorem 2.4 there is a constant $c = c(\delta_1)$ so that for all $t \geq 0$

$$\|m_t^0 - \bar{m}_{\xi_t^0}\| \leq ce^{-\beta t}, \quad |\xi_t^0 - x_0| \leq c \tag{3.13}$$

Next, we write the integral equation for $m_t - m_t^0$ in terms of the operator g_{t,x_0}. Recalling that $V'(m_t) - V'(m_t^0) = (m_t - m_t^0)[3(m_t^0)^2 - 1 + 3\bar{m}_{x_0}^2 - 3\bar{m}_{x_0}^2] + 3m_t^0(m_t - m_t^0)^2 + (m_t - m_t^0)^3$, we obtain

$$\begin{aligned} m_t - m_t^0 = &- \int_0^t ds\, g_{t-s,x_0}[(m_t - m_t^0)3(m_t^0 - \bar{m}_{x_0})(m_t^0 + \bar{m}_{x_0}) + 3m_t^0(m_t - m_t^0)^2 + (m_t - m_t^0)^3] \\ &+ \epsilon^{1/2}\hat{Z}_{t,x_0}, \end{aligned}$$

where $\hat{Z}_{t,x_0}$ was defined in (3.5). By (2.12), (3.12) and (A.34), we get that in $\mathcal{B}_{p,\epsilon,x_0}$ (which, by (3.8), has probability greater than $1 - b_0 e^{-b_1\epsilon^{-p}}$),

$$D_t \leq c_3 \int_0^t ds\, \{\|u_0^0\| D_s + D_s^2 + D_s^3\} + \epsilon^{1/2-p} \tag{3.14}$$

for some positive constant c_3 Then, by Gronwall's Lemma applied in this set, there is $\tau > 0$ so that, calling $t^\star \doteq \tau|\log\epsilon|$,

$$\sup_{t \leq t^\star} D_t \leq \epsilon^{1/2-2p} \tag{3.15}$$

By (3.12) there is $b = b(\tau)$, $b \in (0, 1/2 - 2p)$, so that calling $y_0 \doteq \xi(m_{t^\star}^0)$,

$$\|m_{t^\star}^0 - \bar{m}_{y_0}\| \leq \epsilon^b \tag{3.16}$$

Then

$$\|m_{t^\star} - \bar{m}_{y_0}\| \leq \epsilon^{1/2-2p} + \epsilon^b \leq 2\epsilon^b \tag{3.17}$$

Hence for $\epsilon > 0$ small enough the linear center $x^\star \doteq \xi(m_{t^\star})$ of $m_{t^\star}$ exists and by Lemma 2.3 $|x^\star - y_0| \leq c_0 2\epsilon^b$. Then

$$|x^\star - x_0| \leq |x^\star - y_0| + |y_0 - x_0| \leq c_0(1 + 2\epsilon^b) \leq: c' \tag{3.18}$$

Thus by (3.17) and (3.18) $m_{t^\star} \in M_{\kappa,\ell_1,\delta}^\epsilon$, $\ell_1 = \ell - \epsilon c'$ and $\delta \doteq 2\epsilon^b$.

Moreover, recalling that $\bar{m}' \leq 1$,

$$\begin{aligned} \|m_{t^\star} - \bar{m}_{x^\star}\| &\leq \|m_{t^\star} - \bar{m}_{y_0}\| + \|\bar{m}_{y_0} - \bar{m}_{x^\star}\| \leq (1 + c_0)\|m_{t^\star} - \bar{m}_{y_0}\| \\ &\leq (1 + c_0)2\epsilon^b \end{aligned} \tag{3.19}$$

We next consider $m_t \doteq T_t(m_{t^\star}; \sqrt{\epsilon}Z)$. We call $m_t^0 \doteq T_t(m_{t^\star})$ and $D_t \doteq \|m_t - m_t^0\|$. Again in $\mathcal{B}_{p,\epsilon,x_0}$, (3.15) holds in this setup, but now, by (3.19), $\|u_0^0\| \leq (1 + c_0)2\epsilon^b$. Hence, by Gronwall's Lemma we get this time that there is $c > 0$ so that $D_t \leq c\epsilon^{1/2-p}$ for all $t \leq t_\epsilon$.

By the same argument used above we then complete the proof of the Proposition under the additional assumption that $m \in \mathcal{M}_{\delta_1}$ and consider next the case $m \in M_{\kappa,\ell,\delta}^\epsilon \cap C_{\epsilon,\kappa}(\mathbb{R})$. Then

$m^{\epsilon,\kappa} \in \mathcal{M}_\delta$ and $|\xi^{\epsilon,\kappa}(m)| \le \epsilon^{-\kappa} - \ell\epsilon^{-1}$ and hence (3.11) holds for $T_t(m^{\epsilon,\kappa}; \sqrt{\epsilon}Z)$. Let $\mathcal{B}_{p,\epsilon}$ be the set in (3.1). Then there are constants c' and V positive so that setting $L_\epsilon \doteq \epsilon^{-\kappa} - Vt_\epsilon$

$$\sup_{0\le t\le t_\epsilon} \sup_{|x|\le L_\epsilon} |T_t(m; \sqrt{\epsilon}Z) - T_t(m^{\epsilon,\kappa}; \sqrt{\epsilon}Z)| \le \hat{c}e^{-t_\epsilon} \tag{3.20}$$

(3.20) is proved in Proposition 5.3 of [5] (Barrier Lemma) for $\kappa = 1$, but the proof is also valid for $\kappa > 1$. Let next $x \in [L_\epsilon, \epsilon^{-\kappa}]$ (the proof for $x \in [-\epsilon^{-\kappa}, -L_\epsilon]$, is similar). By assumption there is a constant $c \in (0,1)$ so that

$$\sup_{|x-\epsilon^{-\kappa}|\le 2Vt_\epsilon} |m(x) - 1| \le c \tag{3.21}$$

Recalling that $m \in C_{\epsilon,\kappa}(\mathbb{R})$, we define $\hat{m} \in C(\mathbb{R})$ as

$$\hat{m}(x) \doteq \begin{cases} m(x) & \text{if } |x - \epsilon^{-\kappa}| \le 2Vt_\epsilon \\ m(\epsilon^{-\kappa} - 2Vt_\epsilon) & \text{if } x \le \epsilon^{-\kappa} - 2Vt_\epsilon \\ m(\epsilon^{-\kappa} + 2Vt_\epsilon) & \text{if } x \ge \epsilon^{-\kappa} + 2Vt_\epsilon \end{cases} \tag{3.22}$$

Using again the Barrier Lemma there is $c > 0$ so that in $\mathcal{B}_{p,\epsilon}$

$$\sup_{0\le t\le t_\epsilon} \sup_{L_\epsilon < x \le \epsilon^{-\kappa}} |T_t(m; \sqrt{\epsilon}Z) - T_t(\hat{m}; \sqrt{\epsilon}Z)| \le ce^{-t_\epsilon} \tag{3.23}$$

Since $m(\cdot) \equiv 1$ is stable (see Lemma A.2 in the Appendix) there is a constant $c > 0$ so that in $\mathcal{B}_{p,\epsilon}$ for any $t \in [0, t_\epsilon]$

$$\|T_t(\hat{m}; \sqrt{\epsilon}Z) - 1\| \le c[e^{-t} + \epsilon^{1/2-p}] \tag{3.24}$$

By (3.20), (3.23) and (3.24) there is $c > 0$ so that $\left(T_t(m; \sqrt{\epsilon}Z)\right)^{\epsilon,\kappa}$ is in $\mathcal{M}_{c\delta}$ for all $t \le t_\epsilon$ and in $\mathcal{M}_{c\epsilon^{1/2-a}}$ at time t_ϵ.

It remains to control the position of the center. From (3.20), (3.11) and (3.24), there is a constant $c > 0$ so that

$$\|\left(T_t(m; \sqrt{\epsilon}Z)\right)^{\epsilon,\kappa} - T_t(m^{\epsilon,\kappa}; \sqrt{\epsilon}Z)\| \le c\delta \tag{3.25}$$

By choosing $\delta_2 > 0$ so small that $c\delta_2 \le \delta_0$, δ_0 as in Lemma 2.3 we conclude that

$$\left|\xi\Big(\left(T_t(m; \sqrt{\epsilon}Z)\right)^{\epsilon,\kappa}\Big) - \xi\Big(T_t(m^{\epsilon,\kappa}; \sqrt{\epsilon}Z)\Big)\right| \le c_0 c\delta \tag{3.26}$$

Since $|\xi\big(T_t(m^{\epsilon,\kappa}; \sqrt{\epsilon}Z)\big)| \le \epsilon^{-\kappa} - \epsilon^{-1}\ell'$ similar conclusion (with ℓ' replaced by $\ell' - \epsilon(c_0 c\delta)$) holds for $\xi\big(\left(T_t(m; \sqrt{\epsilon}Z)\right)^{\epsilon,\kappa}\big)$, and Proposition 3.3 is proved. □

Proof of Theorem 1.1.

By iterating (3.11) and recalling that by (2.10) the true and the linear centers are close we conclude the proof of Theorem 1.1. □

Main ideas of the proofs.

The proof of Theorem 1.1 is a (simple) perturbative argument that relates $T_t(m; \sqrt{\epsilon}Z)$ and $T_t(m)$; convergence to a Brownian motion is a quite different game, its proof much more delicate.

When times are scaled as $\epsilon^{-1}\tau$, τ in a compact of $\mathbb{R}$, the argument used in [5] applies. It is based on the bound

$$\|T_{t_\epsilon}(m;\sqrt{\epsilon}\underline{Z}) - T_{t_\epsilon}(\bar{m}_{\xi(m)};\sqrt{\epsilon}Z)\| \leq c\epsilon^{1-2a} \tag{3.27}$$

which holds if $m \in M^{\epsilon}_{\kappa,\ell,\epsilon^{1/2-a}}$ and $c > 0$ a suitable constant. The bound follows easily from the integral representation (3.4) using the estimates of Section 2. The idea then is to split the time into intervals of length t_ϵ (a different value of t_ϵ is actually used in [5]) and to replace at the beginning of each interval the true process by that starting from the instanton with the same linear center. One can see that if a is small then the sum of all the errors, bounded using (3.27) vanishes when $\epsilon \to 0^+$ so that we can consider a process that at each interval starts from an instanton. Then except for the (negligible) influence of the boundaries, the increments of the linear center are mutually independent and convergence to a Brownian motion is easily proved.

If times are proportional to ϵ^{-h}, h large, the sum of the errors is no longer negligible and this approach fails even though the bound (3.27) is optimal. The way out, at least that is what we do, exploits the fact that there is a much better bound that goes even like $c_n\epsilon^n$ (for any given n) provided we construct the two processes not as simply as when taking just the same noise in the whole interval $[0, t_\epsilon]$. But the most important point is that we only compare the processes modulo translations. We can then conclude that after a time delay t_ϵ the two processes, suitably shifted, are in law very close to each other. The successive increments of the linear centers are independent of the shift (except for the influence of the boundary, controlled by using the Barrier Lemma) so that they are in law very close to each other.

The crucial bound involving $c_n\epsilon^n$ is proved below, see Proposition 3.5 and Corollary 3.6, its application to the convergence to a Brownian motion in Section 4.

To investigate the process modulo translation, i.e. its transversal deviations from $\mathcal{M}$ neglecting the localization along $\mathcal{M}$, it is convenient to introduce a function $D_\epsilon(m, m^\star)$ which plays the role of a distance between m and $m^\star$, but it is not a distance.

3.4 Definition.

Let $f \in C(\mathbb{R})$, $\epsilon > 0$, x and $y \in \mathbb{R}$, we set

$$\|f\|_\epsilon \doteq \sup_{|x|\leq\epsilon^{-1/10}} |f(x)|; \quad \tau_y f(x) \doteq f(x+y) \quad \|f\|_{\epsilon,x} \doteq \|\tau_x f\|_\epsilon \tag{3.28}$$

For m and $m^\star$ in $C(\mathbb{R})$, we then define $D_\epsilon(m, m^\star)$ by

$$D_\epsilon(m, m^\star) \doteq \begin{cases} \inf_{r\in\mathbb{R}} \{|r| \vee \|\tau_{x_0} m - \tau_{x_0^\star + r} m^\star\|_\epsilon\} & \text{if } m^{\epsilon,\kappa}, (m^\star)^{\epsilon,\kappa} \in \mathcal{M}_{\epsilon^{1/2-a}} \\ 1 & \text{if not} \end{cases} \tag{3.29}$$

where $x_0 \doteq \xi^{\epsilon,\kappa}(m)$, $x_0^\star \doteq \xi^{\epsilon,\kappa}(m^\star)$ (see (3.9) for notation) and $a \vee b$ stands for the maximum between a and b.

In general $D_\epsilon(m, m^\star) \neq D_\epsilon(m^\star, m)$. By its definition for any $\delta \in (0,1)$, $D_\epsilon(m, m^\star) \leq \delta$ if and only if $m^{\epsilon,\kappa}, (m^\star)^{\epsilon,\kappa} \in \mathcal{M}_{\epsilon^{1/2-a}}$ and there is η such that $|\eta - (x_0^\star - x_0)| \leq \delta$ and $\|m - \tau_\eta m^\star\|_{\epsilon,x_0} \leq \delta$.

We next prove a contraction property of the evolution with respect to D_ϵ.

3.5 Proposition.

Let $\kappa \geq 1$, $\ell \in (0,1)$, $a \in (0,1/2)$, $n \geq 1$, $b > 0$ and $\gamma \in (0,1/2-a)$. Then there are $p > 0$ and $c > 0$ such that for all $0 < \epsilon < 1$ the following holds. For all pairs m and $m^\star \in C(\mathbb{R})$ so that $D_\epsilon(m,m^\star) \leq \epsilon^b$, we can construct the processes m_t and $m_t^\star$, solutions of (1.1) with initial data m and $m^\star$ respectively in the same probability space and such that, if we set $t_\epsilon = (\log \epsilon)^2$, then

$$P^\epsilon\Big(D_\epsilon(m_{t_\epsilon/n}, m^\star_{t_\epsilon/n}) \leq \epsilon^{b+\gamma}\Big) \geq 1 - ce^{-\epsilon^{-p}} \tag{3.30}$$

Proof.

Notation: for simplicity we consider the case $n = 1$. Let $x_0 \doteq \xi^{\epsilon,\kappa}(m)$, $x_0^\star \doteq \xi^{\epsilon,\kappa}(m^\star)$, $\Delta \doteq x_0^\star - x_0$, $\lambda \doteq \epsilon^{1/2-a}$, $\delta \doteq \epsilon^b$,

$$v(x) \doteq [m(x) - \bar{m}_{x_0}(x)], \qquad u(x) \doteq [m^\star(x) - \bar{m}_{x_0^\star}(x)] \tag{3.31}$$

By assumption there is η, $|\eta - \Delta| \leq \delta$, so that (recall that $\bar{m}' \leq 1$)

$$\|v - \tau_\eta u\|_{\epsilon,x_0} \leq 2\delta \tag{3.32}$$

We divide the proof into several steps.

STEP 1. Construction of the coupling.

Starting from a white noise process α, consider the processes Z_t and $\hat{Z}_{t,x_0}$ as defined in (1.2) and (3.5) (or (3.6)) respectively.

Next, take a second noise $\bar{\alpha}$ independent of α and set

$$\begin{aligned} Z_t^\star(x) \doteq & \int_0^t ds \int dz\, \mathbf{1}_{\{|z+\Delta-x_0^\star| \leq 4\epsilon^{-1/10};\, z+\Delta \in \mathcal{T}_{\epsilon,\kappa}\}}\, H^{(\epsilon)}_{t-s}(x, z+\Delta)\, \alpha(s,z) \\ & + \int_0^t ds \int dy\, \mathbf{1}_{\{|y-x_0^\star| > 4\epsilon^{-1/10};\, y \in \mathcal{T}_{\epsilon,\kappa}\}}\, H^{(\epsilon)}_{t-s}(x,y)\, \bar{\alpha}(s,y) \end{aligned} \tag{3.33}$$

($\mathbf{1}_A$ denotes the characteristic function of the set A). It is easy to check (by comparing covariances) that the processes $Z_t^\star$ and Z_t have the same law. Using them, we construct the Ginzburg-Landau processes by setting

$$m_t \doteq T_t(m; \sqrt{\epsilon} Z), \qquad m_t^\star \doteq T_t(m^\star; \sqrt{\epsilon} Z^\star). \tag{3.34}$$

Define

$$v_t \doteq m_t - \bar{m}_{x_0}, \quad v_0 \doteq v, \qquad u_t \doteq m_t^\star - \bar{m}_{x_0^\star}, \quad u_0 \doteq u \tag{3.35}$$

We also call $v_t^{(\epsilon,\kappa)} \doteq (m_t)^{\epsilon,\kappa} - \bar{m}_{x_0}$ and $u_t^{(\epsilon,\kappa)} \doteq (m_t^\star)^{\epsilon,\kappa} - \bar{m}_{x_0^\star}$.

STEP 2. The good sets.

Let $p \in (0, a/2)$, $\epsilon > 0$, $c > 0$ and

$$\mathcal{B}^{(1)}_{\epsilon,p} \doteq \Big\{ \sup_{0 \leq t \leq t_\epsilon} \{\|\hat{Z}_{t,x_0}\| + \|\hat{Z}^\star_{t,\hat{x}_0^\star}\|\} \leq \epsilon^{-2p},\ \sup_{0 \leq t \leq t_\epsilon} \{\|v_t^{(\epsilon,\kappa)}\| + \|u_t^{(\epsilon,\kappa)}\|\} \leq c\epsilon^{1/2-a} \Big\} \tag{3.36}$$

$$\mathcal{B}_\epsilon^{(2)} \doteq \Big\{ \sup_{0\le t\le t_\epsilon} \|Z_{t,x_0} - \tau_\Delta Z^\star_{t,x_0^\star}\|_{x_0,\epsilon} < e^{-\epsilon^{-1/100}} \Big\}, \tag{3.37}$$

$$\mathcal{B}_\epsilon^{(3)} \doteq \Big\{ \sup_{0\le t\le t_\epsilon} \|\hat{Z}_{t,x_0} - \tau_\Delta \hat{Z}^\star_{t,x_0^\star}\|_{x_0,\epsilon} < e^{-\epsilon^{-1/100}} \Big\} \tag{3.38}$$

where $\hat{Z}_{t,x_0}$ is given by (3.5), and $\hat{Z}^\star_{t,x_0^\star}$ is given by (3.5), with $Z_t^\star$ (given by (3.33)) in the place of Z_t and $x_0^\star$ in the place of x_0 . That is,

$$\hat{Z}^\star_{t,x_0^\star} \doteq Z_t^\star + \int_0^t ds g_{t-s,x_0^\star}[(3\bar{m}^2_{x_0^\star} - 1)Z_s^\star] \tag{3.39}$$

We will prove that there is $c > 0$ so that

$$P^\epsilon(\hat{\mathcal{B}}_{\epsilon,p}) \ge 1 - ce^{-\epsilon^{-p}}, \qquad \hat{\mathcal{B}}_{\epsilon,p} \doteq \mathcal{B}^{(1)}_{\epsilon,p} \cap \mathcal{B}^{(2)}_\epsilon \cap \mathcal{B}^{(3)}_\epsilon \tag{3.40}$$

Since $D_\epsilon(m, m^\star) \le \epsilon^b < 1$, both m and $m^\star$ are in $\mathcal{M}_{\epsilon^{1/2-a}}$. Then by (3.8) and the proof of Proposition 3.3 there is a constant c so that

$$P^\epsilon(\mathcal{B}^{(1)}_{\epsilon,p}) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.41}$$

A similar bound holds for the probability of $\mathcal{B}_\epsilon^{(i)}$, $i = 2, 3$. (See Lemma A.5 of the Appendix and recall that $p < 1/2$).

STEP 3. Bounds close to the center.

Let $\gamma \in (0, 1/2 - a)$ and $b > 0$, $\bar{\gamma} \in (\gamma, 1/2 - a)$ and M a positive integer such that $b + \bar{\gamma} < M(1/2 - a)$. It follows that there exists ϵ_0 so that for $\epsilon < \epsilon_0$

$$[t_\epsilon \epsilon^{1/2-a}]^M \le \delta\epsilon^{\bar{\gamma}} \tag{3.42}$$

We will prove that there are c and p positive so that for $\epsilon < \epsilon_0$, in the set $\hat{\mathcal{B}}_{\epsilon,p}$,

$$\sup_{|x-x_0|\le\epsilon^{-1/10}-M\epsilon^{-1/20}} |v_{t_\epsilon}(x) - \tau_\Delta u_{t_\epsilon}(x)| \le c\delta\epsilon^{\bar{\gamma}} \tag{3.43}$$

Proof of (3.43). We set

$$d_t(x) \doteq v_t(x) - \tau_\Delta u_t(x) \tag{3.44}$$

For x, y and h in $\mathbb{R}$ and $t > 0$

$$g_{t,x_0+h}(x+h, y+h) = g_{t,x_0}(x,y), \qquad \bar{m}_{x_0}(x-h) = \bar{m}_{x_0+h}(x), \tag{3.45}$$

hence, for any function $f \in C^0(\mathbb{R})$,

$$\int dy\, g_{t,x_0^\star}(x+\Delta, y)\, \bar{m}_{x_0^\star}(y)\, f(y) = \int dy\, g_{t,x_0}(x,y)\, \bar{m}_{x_0}(y)\, \tau_\Delta f(y) \tag{3.46}$$

Writing the integral equation (3.4) for v_t and u_t (the former with g_{t,x_0}, the latter with $g_{t,x_0^\star}$) we get

$$d_t = [g_{t,x_0} v - \tau_\Delta g_{t,x_0^\star} u] + \int_0^t ds\, (g_{t-s,x_0} A_s d_s) + \sqrt{\epsilon}[\hat{Z}_{t,x_0} - \tau_\Delta \hat{Z}^\star_{t,x_0^\star}] \tag{3.47}$$

where

$$A_s \doteq -3\bar{m}_{x_0}[v_s + \tau_\Delta u_s] - [v_s^2 + (\tau_\Delta u_s)^2 + v_s \tau_\Delta u_s]$$

For any $k = 0, \ldots, M$ we define

$$B_k \doteq \{y : |y - x_0| \leq \epsilon^{-1/10} - k\epsilon^{-1/20}\} \tag{3.48}$$

$$D_{k,t} \doteq \sup_{x \in B_k} |d_t(x)| \tag{3.49}$$

We next prove that there are $c_1 > 0$, $c_2 > 0$ and for any n there is $c_n' > 0$ so that in $\hat{\mathcal{B}}_{\epsilon,p}$

$$D_{k,t} \leq c_1 \delta \left[1 + \frac{\epsilon^{1/2-a}}{\sqrt{t}}\right] + c_2 \int_0^t ds\, \epsilon^{1/2-a}\, D_{k-1,s} + c_n' \epsilon^n \tag{3.50}$$

To prove (3.50) we first notice that by (3.36) and (3.40) there is $c > 0$ so that in $\hat{\mathcal{B}}_{\epsilon,p}$

$$\sup_{t \leq t_\epsilon} \|A_t\|_{\epsilon,x_0} \leq c\epsilon^{1/2-a} \tag{3.51}$$

Furthermore from (3.38) and (3.40) it follows that for any n there is c_n so that

$$\sqrt{\epsilon}\|\hat{Z}_{t,x_0} - \tau_\Delta \hat{Z}^\star_{t,x_0^\star}\|_{x_0,\epsilon} \leq c_n \epsilon^n \tag{3.52}$$

By (3.45),

$$(g_{t,x_0}v)(x) - (\tau_\Delta g_{t,x_0^\star}u)(x) = \left[g_{t,x_0}(v - \tau_\eta u)\right](x) + \int dy\, \tau_\eta u(y)\left[g_{t,x_0}(x,y) - g_{t,x_0}(x, y - \Delta + \eta)\right] \tag{3.53}$$

We then use (3.32), that $\|u\| \leq 2\epsilon^{1/2-a}$ and (A.33) to conclude that for any $k = 0, .., M$

$$\sup_{x \in B_k} \left|g_{t,x_0}v(x) - \tau_\Delta g_{t,x_0^\star}u(x)\right| \leq c_1 \delta \left[1 + \frac{\epsilon^{1/2-a}}{\sqrt{t}}\right] \tag{3.54}$$

Furthermore by Theorem 2.1 and the Appendix, see (A.32) and (A.34), for any n there is $c_n > 0$ so that for all $k = 0, .., M$

$$\sup_{t \geq 0} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} dy\, g_{t,x_0}(x,y) \leq c_0, \qquad \sup_{t > 0} \sup_{x \in B_k} \int_{\mathbb{R} \setminus B_{k-1}} dy\, g_{t,x_0}(x,y) \leq c_n \epsilon^n, \tag{3.55}$$

so (3.50) follows from (3.47), (3.51), (3.52), (3.54) and (3.55).

By Theorem 2.1

$$\|g_{t_\epsilon,x_0}v\| \leq ce^{-\alpha t_\epsilon}, \qquad \|g_{t_\epsilon,x_0^\star}u\| \leq ce^{-\alpha t_\epsilon} \tag{3.56}$$

By (3.47), (3.56) and (3.51) we have that, in the set $\hat{\mathcal{B}}_{\epsilon,p}$

$$D_{M,t_\epsilon} \leq 2ce^{-\alpha t_\epsilon} + c_2 \int_0^{t_\epsilon} ds\, \epsilon^{1/2-a} D_{M-1,s} + c_n \epsilon^n$$

Therefore by iterating (3.50) we get

$$
\begin{aligned}
D_{M,t_\epsilon} &\le 2ce^{-\alpha t_\epsilon} \\
&+ \sum_{k=1}^{M} [c_2\epsilon^{1/2-a}]^k \int_0^{t_\epsilon} ds_1 \cdots \int_0^{s_{k-1}} ds_k c_1 \delta \left(1 + \frac{\epsilon^{1/2-a}}{\sqrt{s_k}}\right) \\
&+ [c_2\epsilon^{1/2-a}]^{M-1} \int_0^{t_\epsilon} ds_1 \cdots \int_0^{s_{M-1}} ds_M D_{0,s_M} + c_n\epsilon^n
\end{aligned}
$$

By the choice of M and since $D_{0,s_M} \le \epsilon^{1/2-a}$ we conclude the proof of Step 3.

STEP 4. Bounds away from the center.

We prove that there are c, p and ϵ_0 positive so that if $\epsilon < \epsilon_0$ and the processes m_t and $m_t^\star$ are in $\hat{\mathcal{B}}_{\epsilon,p}$, then

$$
\sup_{\epsilon^{-1/10} - M\epsilon^{-1/20} \le |x-x_0| \le 2\epsilon^{-1/10}} |v_t(x) - \tau_\Delta u_t(x)| \le ce^{-t}\lambda, \qquad \text{for all } t \le t_\epsilon \tag{3.57}
$$

We set

$$
r_{\epsilon,M} = \epsilon^{-1/10} - M\epsilon^{-1/20}
$$

and consider the case $x - x_0 \in [r_{\epsilon,M}, 2\epsilon^{-1/10}]$. The analysis of the other interval involved in the sup in (3.57) is similar and omitted. Given $V > 0$ we define

$$
m^+(x) = \begin{cases} m(x) & \text{for } x - x_0 \in \left(r_{\epsilon,M} - Vt_\epsilon, 2\epsilon^{-1/10} + Vt_\epsilon\right) \\ 1 & \text{for } x - x_0 \in \left\{\left(-\infty, r_{\epsilon,M} - 2Vt_\epsilon\right) \cup \left(2\epsilon^{-1/10} + 2Vt_\epsilon, +\infty\right)\right\} \end{cases} \tag{3.58}
$$

and complete the definition of m^+ in the missing intervals by linear interpolation. $\tilde{m}^+$ is defined similarly with m replaced by $m^\star$ and x_0 by $x_0^\star$.

We set

$$
m_t^+ = T_t(m^+, \sqrt{\epsilon}Z), \qquad \tilde{m}_t^+ = T_t(\tilde{m}^+, \sqrt{\epsilon}Z^\star)
$$

We choose V in (3.58) as the parameter entering in the Barrier Lemma (Proposition 5.3 of [5]). Then there is $c > 0$ so that in $\hat{\mathcal{B}}_{\epsilon,p}$ for all $t \le t_\epsilon$,

$$
\sup_{x - x_0 \in [r_{\epsilon,M}, 2\epsilon^{-1/10}]} |m_t(x) - m_t^+(x)| \le ce^{-t} \tag{3.59}
$$

The same bound holds for $m_t^\star(x) - \tilde{m}_t^+(x)$ when x_0 is replaced by $x_0^\star$. We define

$$
v_t^+ = m_t^+ - 1, \qquad u_t^+ = \tilde{m}_t^+ - 1 \tag{3.60}
$$

It is not difficult to prove the following a-priori bound for v_t^+ and u_t^+: for $p < a$ there is $c > 0$ so that

$$
\|v_t^+\| \le 2\lambda c, \qquad \|u_t^+\| \le 2\lambda c, \qquad \text{for all } t \le t_\epsilon \tag{3.61}
$$

We consider next the versions of v_t^+ and u_t^+ given by the corresponding solutions of the equations:

$$v_t^+ = e^{-2t} H_t \star v_0^+ + \int_0^t ds\, e^{-2(t-s)} H_{t-s} \star \big(-3(v_s^+)^2 - (v_s^+)^3 \big) + \sqrt{\epsilon} V_t$$

$$u_t^+ = e^{-2t} H_t \star u_0^+ + \int_0^t ds e^{-2(t-s)} H_{t-s} \star \big(-3(u_s^+)^2 - (u_s^+)^3 \big) + \sqrt{\epsilon} V_t^\star$$

where

$$V_t(x) \doteq \int_0^t ds \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, e^{-2(t-s)} H_{t-s}^{(\epsilon)}(x,y)\, \alpha(s,y)$$

$$V_t^\star(x) \doteq \int_0^t ds \int dz\, \mathbf{1}_{\{|z+\Delta-x_0^\star| \le 4\epsilon^{-1/10};\, z+\Delta \in \mathcal{T}_{\epsilon,\kappa}\}}\, e^{-(t-s)}\, H_{t-s}^{(\epsilon)}(x, z+\Delta)\, \alpha(s,z)$$
$$+ \int_0^t ds \int dy\, \mathbf{1}_{\{|y-x_0^\star| > 4\epsilon^{-1/10};\, y \in \mathcal{T}_{\epsilon,\kappa}\}}\, e^{-(t-s)}\, H_{t-s}^{(\epsilon)}(x,y)\, \bar{\alpha}(s,y) \tag{3.62}$$

We call

$$d_t^+ \doteq \sup_{x-x_0 \in (r_{\epsilon,M}, 2\epsilon^{-1/10})} (v_t^+ - \tau_\Delta u_t^+)$$

From (3.61), we get that there is a constant $c' > 0$ such that

$$d_t^+ \le e^{-2t} 2\lambda c + c'\lambda \int_0^t ds\, e^{-2(t-s)} d_s^+ + \sqrt{\epsilon} \|V_t - \tau_\Delta V_t^\star\|_{\epsilon,x_0} \tag{3.63}$$

But $\|V_t - \tau_\Delta V_t^\star\|_{\epsilon,x_0} \le \|Z_t - \tau_\Delta Z_t^\star\|_{\epsilon,x_0}$, and, by (3.37), a bound like (3.52) holds also for $\|Z_t - \tau_\Delta Z_t^\star\|_{\epsilon,x_0}$. Then by (3.63) there is $c > 0$ so that

$$d_t^+ \le c e^{-2t} \lambda, \qquad \text{for all } t \le t_\epsilon \tag{3.64}$$

Since

$$v_t - \tau_\Delta u_t = (v_t^+ - \tau_\Delta u_t^+) + (m_t - m_t^+) - \tau_\Delta (m_t^\star - \tilde{m}_t^+)$$

from (3.64) and (3.59), we get that

$$\sup_{x-x_0 \in (r_{\epsilon,M}, 2\epsilon^{-1/10})} |v_t(x) - \tau_\Delta u_t(x)| \le c\big(e^{-2t}\lambda + e^{-t}\big), \qquad \text{for all } t \le t_\epsilon \tag{3.65}$$

STEP 5. Conclusion.

By (3.43) and (3.57) there are c and ϵ_0 so that if $\epsilon < \epsilon_0$ and the processes m_t and $m_t^\star \in \hat{\mathcal{B}}_{\epsilon,p}$, then

$$\sup_{|x-x_0| \le 2\epsilon^{-1/10}} |m_{t_\epsilon}(x) - \tau_\Delta m_{t_\epsilon}^\star(x)| = \sup_{|x-x_0| \le 2\epsilon^{-1/10}} |v_{t_\epsilon}(x) - \tau_\Delta u_{t_\epsilon}(x)| \le \delta \epsilon^{\tilde{\gamma}} \tag{3.66}$$

We then observe that from Lemma 2.3 it follows that in $\hat{\mathcal{B}}_{\epsilon,p}$,

$$|\xi^{\epsilon,\kappa}(m_{t_\epsilon}) - x_0| \le c\epsilon^{1/2-a}$$

so that from (3.66) it follows that

$$\|m_{t_\epsilon} - \tau_\Delta m_{t_\epsilon}^\star\|_{\epsilon,\xi(m_{t_\epsilon}^{\epsilon,\kappa})} \le \delta \epsilon^{\tilde{\gamma}} \tag{3.67}$$

By Lemma 2.3

$$|\xi^{\epsilon,\kappa}(m^\star_{t_\epsilon}) - \xi^{\epsilon,\kappa}(m_{t_\epsilon}) - \Delta| \le c_0 |\langle \bar{m}'_{\xi^{\epsilon,\kappa}(m_{t_\epsilon})}, (m_{t_\epsilon})^{\epsilon,\kappa} - \tau_\Delta (m^\star_{t_\epsilon})^{\epsilon,\kappa}\rangle| \tag{3.68}$$

By (3.67) and the exponential decay at infinity of $\bar{m}'_{\xi(m_{t_\epsilon})}$ there is $c > 0$ so that

$$|\xi^{\epsilon,\kappa}(m^\star_{t_\epsilon}) - \xi^{\epsilon,\kappa}(m_{t_\epsilon}) - \Delta| \le c\delta\epsilon^{\bar{\gamma}} \tag{3.69}$$

Since $\bar{\gamma} > \gamma$, we have thus proven that in $\hat{\mathcal{B}}_{\epsilon,p}$, with p small enough,

$$\|m_{t_\epsilon} - \tau_\Delta m^\star_{t_\epsilon}\|_{\epsilon,\xi^{\epsilon,\kappa}(m_{t_\epsilon})} \le \delta\epsilon^\gamma, \qquad |\xi^{\epsilon,\kappa}(m^\star_{t_\epsilon}) - \xi^{\epsilon,\kappa}(m_{t_\epsilon}) - \Delta| \le \delta\epsilon^\gamma \tag{3.70}$$

Proposition 3.5 is proved. □

As a corollary of the previous Proposition we have a fast decay trasversally to $\mathcal{M}$, as we are going to see. We define

$$\psi : C(\mathbb{R}) \longrightarrow C(\mathbb{R}) \qquad \text{such that} \qquad \psi(m)(x) \doteq m^{\epsilon,\kappa}(x + \xi^{\epsilon,\kappa}(m)) \tag{3.71}$$

Note that we omit in the expression of ψ the explicit dependence on ϵ and κ.

3.6 Corollary.

Let $\kappa \ge 1$, $\ell \in (0,1)$, $a \in (0,1/2)$, $m \in M^\epsilon_{\kappa,\ell,\epsilon^{1/2-a}}$. We call

$$\psi \doteq \psi(T_{t_\epsilon}(m;\sqrt{\epsilon}Z)), \qquad \psi^\star \doteq \psi(T_{t_\epsilon}(m^\star;\sqrt{\epsilon}Z))$$

Then for any positive integer N we can construct $T_t(m;\sqrt{\epsilon}Z)$ and $T_t(m^\star;\sqrt{\epsilon}Z)$ in the same probability space and such that there are $c > 0$, and $p > 0$ so that

$$P^\epsilon(\|\psi - \psi^\star\|_\epsilon \le \epsilon^N) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.72}$$

Proof.

Let $\gamma \in (0, 1/2 - a)$, $n > 1$, $m_t \doteq T_t(m, \sqrt{\epsilon}Z)$ and $m^\star_t \doteq T_t(m^\star, \sqrt{\epsilon}Z)$ constructed as in Proposition 3.5. By iterating (3.30) n times we get (for a constant c possibly different from that in (3.30))

$$P^\epsilon(D_\epsilon(m_{t_\epsilon}, m^\star_{t_\epsilon}) \le \epsilon^{1/2-a}\epsilon^{n\gamma}) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.73}$$

Then, see right after Definition 3.4, there is η such that (we bound $\epsilon^{1/2-a}$ with 1)

$$P^\epsilon\Big(|\eta - \Delta| \le \epsilon^{n\gamma}, \quad \|m_{t_\epsilon} - \tau_\eta m^\star_{t_\epsilon}\|_{\epsilon,x_0} \le \epsilon^{n\gamma}\Big) \ge 1 - ce^{-\epsilon^{-p}} \tag{3.74}$$

where $\Delta = x^\star_0 - x_0$, with $x_0 = \xi(m^{\epsilon,\kappa}_{t_\epsilon})$ and $x^\star_0 = \xi((m^\star_{t_\epsilon})^{\epsilon,\kappa})$. We note that

$$\|\psi - \psi^\star\|_\epsilon = \|m_{t_\epsilon} - \tau_\Delta m^\star_{t_\epsilon}\|_{\epsilon,x_0} \le \|m_{t_\epsilon} - \tau_\eta m^\star_{t_\epsilon}\|_{\epsilon,x_0} + \|\tau_{\eta-\Delta} m^\star_{t_\epsilon} - m^\star_{t_\epsilon}\|_{\epsilon,x^\star_0} \tag{3.75}$$

In the Appendix, see Lemma A.3, it is proved the following property of the Ginzburg-Landau process. For any $\delta > 0$, $\alpha \in (0, 1/2)$ there is a constant $c > 0$ so that for any ϵ small enough

$$P^\epsilon\left(\sup_{|x|,|y|\le\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\le 1}} \frac{|m_{t_\epsilon}(x) - m_{t_\epsilon}(y)|}{|x-y|^\alpha} > \delta\right) \le e^{-c\delta^2} \tag{3.76}$$

Applying the previous inequality to $m_t^\star$, we obtain

$$P^\epsilon\left(\sup_{|x|,|y|\le\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\le|\eta-\Delta|}} \frac{|m_{t_\epsilon}^\star(x) - m_{t_\epsilon}^\star(y)|}{|x-y|^{1/4}} > 1\right) \le$$

$$P^\epsilon\left(\sup_{|x|,|y|\le\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\le 1}} \frac{|m_{t_\epsilon}^\star(x) - m_{t_\epsilon}^\star(y)|}{|x-y|^{3/8}} > |\eta-\Delta|^{-1/8}\right) \le e^{-c|\eta-\Delta|^{-1/4}} \tag{3.77}$$

By (3.74), (3.75) and (3.77), and choosing n such that $\epsilon^{n\gamma/4} < \epsilon^N$ we then derive (3.72). Corollary 3.6 is proved. □

Corollary 3.6 proves that two processes that start from different data become almost equal modulo translations, with probability going to 1 as $\epsilon \to 0^+$ as fast as in (3.72). We improve this result in the next Proposition in the sense that we have a similar statement without translations. The price we pay is twofold. The processes must start from data with linear centers close to each other and, more importantly, the rate of convergence is not as fast as before. In particular it is not fast enough for what needed in Section 4, where we prove Theorem 1.2. In that case we use Corollary 3.6 as we can reduce the analysis to events invariant under translations. This is no longer possible when proving Theorem 1.3, where however we need only convergence in probability, without bounds on the rate of convergence: for that Proposition 3.7 below will suffice.

3.7 Proposition.

Let $a \in (0, 1/2)$, $b \ge 1 - a$, $\gamma \in (0, 1/2 - a)$, m and $\tilde m$ both in $M^\epsilon_{\kappa,\ell,\epsilon^{1/2-a}}$ and

$$\|m - \tilde m\| \le \epsilon^b, \qquad |\xi^{\epsilon,\kappa}(m) - \xi^{\epsilon,\kappa}(\tilde m)| \le \epsilon^b \tag{3.78}$$

Then we can construct the Ginzburg-Landau processes $\{m_t\}_{t\ge0}$ and $\{\tilde m_t\}_{t\ge0}$ starting respectively from m and $\tilde m$ in the same probability space and so that

$$\lim_{\epsilon\to0^+} P^\epsilon\Big(\|(m_{t_\epsilon})^{\epsilon,\kappa} - (\tilde m_{t_\epsilon})^{\epsilon,\kappa}\| \le \epsilon^{b+\gamma}, \quad |\xi^{\epsilon,\kappa}(m_{t_\epsilon}) - \xi^{\epsilon,\kappa}(\tilde m_{t_\epsilon})| \le \epsilon^{b+\gamma}\Big) = 1 \tag{3.79}$$

Proof.

We set $\delta \doteq \epsilon^b$ and $\lambda \doteq \epsilon^{1/2-a}$.

We consider first the case when m and $\tilde m$ are not in $C_{\epsilon,\kappa}(\mathbb{R})$. Let $x_0 \doteq \xi(m)$, $\tilde x_0 \doteq \xi(\tilde m)$ and $t_\epsilon^\star \doteq \epsilon^q t_\epsilon$, $q > 0$ will be specified later. Let $\{e_j^{(\epsilon)}\}_{j\ge1}$ be an ortonormal basis of $L^2(\mathcal{T}_{\epsilon,\kappa})$, such that $e_1^{(\epsilon)} = \sqrt{D_\epsilon}\tilde m'_{x_0}$ on $\mathcal{T}_{\epsilon,\kappa}$, D_ϵ the normalization constant, ($D_\epsilon \to D = 3/4$ as $\epsilon \to 0^+$). Set

$$\langle f, g\rangle_{\epsilon,\kappa} \doteq \int_{\mathcal{T}_{\epsilon,\kappa}} dx\, f(x)g(x), \qquad f, g \in L^2(\mathcal{T}_{\epsilon,\kappa}). \tag{3.80}$$

Let $\{b_j(t)\}_{j\geq 0}$ be a family of standard independent brownian motions, and consider the Gaussian process

$$\hat{Z}^{(1)}_{t,x_0}(x) \doteq \int_0^t db_1(s) \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, e^{(\epsilon)}_1(y) + R_t(x) \tag{3.81}$$

where

$$R_t(x) \doteq \sum_{j\geq 2} \int_0^t db_j(s) \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, e^{(\epsilon)}_j(y), \tag{3.82}$$

and $g^{(\epsilon)}_{t-s,x_0}$ was defined in (3.7) . By comparing covariances, it is easy to check that $\hat{Z}^{(1)}_{t,x_0}$ has the same law of the process $\hat{Z}_{t,x_0}$ defined in (3.6). We will construct another Gaussian process $\hat{Z}^{(2)}_{t,x_0}$ with the same law. Consider, for y_0 that will be conveniently chosen later, the process

$$\tilde{b}_1(t) \doteq \begin{cases} b_0(t) & \text{for any } t \in [0,\tau] \\ b_1(t) - y_0 & \text{for any } t \geq \tau \end{cases} \tag{3.83}$$

where

$$\tau \doteq \inf\{\, t \geq 0 \,:\, [b_1(t) - b_0(t)] = y_0 \,\} \tag{3.84}$$

The process $\tilde{b}_1(t)$ is a Brownian motion, independent of $\{b_j(t)\}_{j\geq 2}$. Finally, let

$$\hat{Z}^{(2)}_{t,x_0}(x) \doteq \int_0^t d\tilde{b}_1(s) \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, e^{(\epsilon)}_1(y) + R_t(x). \tag{3.85}$$

Write the integral equations (3.4) for m_t and $\tilde{m}_t$ as in the statement of the Proposition, using the Gaussian processes $\hat{Z}^{(1)}_{t,x_0}$ and $\hat{Z}^{(2)}_{t,x_0}$ respectively. Then,

$$\begin{aligned} m_\tau - \tilde{m}_\tau = g_{\tau,x_0}(m - \tilde{m}) &- \int_0^\tau ds\, g_{\tau-s,x_0}\big(A_{x_0}[v_s] - A_{x_0}[u_s]\big) \\ &+ \sqrt{\epsilon} \int_0^\tau d(b_1 - b_0)(s) \int_{\mathcal{T}_{\epsilon,\kappa}} dy\, g^{(\epsilon)}_{t-s,x_0}(x,y)\, e^{(\epsilon)}_1(y), \end{aligned} \tag{3.86}$$

where $A_{x_0}[f] \doteq 3\bar{m}_{x_0} f^2 + f^3$, $v_t \doteq m_t - \bar{m}_{x_0}$, $u_t \doteq \tilde{m}_t - \bar{m}_{x_0}$. We multiply both sides by $\bar{m}'_{x_0}$ and integrate over $\mathbb{R}$. We get, in $\{\tau \leq t^*_\epsilon\}$:

$$\begin{aligned} \langle \bar{m}'_{x_0}, (m_\tau - \tilde{m}_\tau)\rangle = -\langle \bar{m}'_{x_0}, \tilde{m}\rangle &- \int_0^\tau ds\, \langle \bar{m}'_{x_0}, (A_{x_0}[v_s] - A_{x_0}[u_s])\rangle \\ &+ \sqrt{\epsilon} \int_0^\tau d(b_1 - b_0)(s)\, \langle a_\epsilon, e^{(\epsilon)}_1\rangle_{\epsilon,\kappa}, \end{aligned} \tag{3.87}$$

where

$$a_\epsilon(y) \doteq \langle g^{(\epsilon)}_{\tau-s,x_0}(\cdot,y), \bar{m}'_{x_0}\rangle = \sum_{j\in\mathbb{Z}} \Big(\bar{m}'_{x_0}(x, y + 4j\epsilon^{-\kappa}) + \bar{m}'_{x_0}(x, 4j\epsilon^{-\kappa} + 2\epsilon^{-\kappa} - y)\Big)$$

Choosing now $y_0 = \langle \bar{m}'_{x_0}, \tilde{m}\rangle(\sqrt{\epsilon}\langle a_\epsilon, e^{(\epsilon)}_1\rangle_{\epsilon,\kappa})^{-1}$, from the definition of τ we then get

$$\langle \bar{m}'_{x_0}, m_\tau - \tilde{m}_\tau\rangle = \int_0^\tau ds\, \langle \bar{m}'_{x_0}, A_{x_0}[v_s] - A_{x_0}[u_s]\rangle \tag{3.88}$$

By standard results on Brownian motions and since $\lim_{\epsilon\to 0+}\langle a_\epsilon, e_1^{(\epsilon)}\rangle_{\epsilon,\kappa} = \sqrt{D}$, there is $c > 0$ so that

$$P^\epsilon\big(\tau \le t_\epsilon^\star\big) \ge 1 - c\frac{|\langle \bar m'_{x_0}\tilde m\rangle|}{\sqrt{\epsilon t_\epsilon^\star}} \ge 1 - c\epsilon^{1-a-1/2-q/2}$$

By choosing $q < 1 - 2a$,

$$\lim_{\epsilon\to 0+} P^\epsilon\big(\tau \le t_\epsilon^\star\big) = 1 \tag{3.89}$$

The set

$$\mathcal{G} \doteq \Big\{\sup_{t\le t_\epsilon}\Big(\|v_t\| + \|u_t\|\Big) \le 2\delta\Big\}$$

has, by Propositions 3.3, probability that goes to 1 as $\epsilon \to 0^+$. There is $c_2 > 0$ so that in $\mathcal{G}$

$$\sup_{0\le t\le t_\epsilon^\star} \|m_t - \tilde m_t\| \le c_2\delta \tag{3.90}$$

Let $\bar\gamma \in (\gamma, 1/2 - a)$, then in $\{\tau \le t_\epsilon^\star\}\cap\mathcal{G}$ and for all $\epsilon > 0$ small enough

$$|\langle \bar m'_{x_0}, m_\tau - \tilde m_\tau\rangle| \le \delta\epsilon^{\bar\gamma} \tag{3.91}$$

By Lemma 2.3 there is $c_3 > 0$ so that in the same set

$$|\xi(m_\tau) - \xi(\tilde m_\tau)| \le c_3\delta\epsilon^{\bar\gamma} \tag{3.92}$$

Finally, using the integral equation (3.4) in the time interval $[\tau, t_\epsilon^\star]$, by (3.90) and (3.91), there is $c_4 > 0$ so that in $\{\tau \le t_\epsilon^\star\}\cap\mathcal{G}$

$$|\xi(m_{t_\epsilon^\star}) - \xi(\tilde m_{t_\epsilon^\star})| \le c_4\delta\epsilon^{\bar\gamma} \tag{3.93}$$

We next consider the time interval $[t_\epsilon^\star, t_\epsilon]$. Let $x_\epsilon^\star \doteq \xi(m_{t_\epsilon^\star})$. We set, for any $t \in [0, \bar t_\epsilon]$, $\bar t_\epsilon \doteq (1-\epsilon^q)t_\epsilon$,

$$v_t^\star \doteq m_{t_\epsilon^\star + t} - \bar m_{x_\epsilon^\star}, \qquad u_t^\star \doteq \tilde m_{t_\epsilon^\star + t} - \bar m_{x_\epsilon^\star} \tag{3.94}$$

We write (3.4) for v_t and u_t relatively to $\bar m_{x_\epsilon^\star}$. Setting $\Delta_\star = x_\epsilon^\star - \xi(\tilde m_{t_\epsilon^\star})$,

$$\begin{aligned} g_{\bar t_\epsilon, x_\epsilon^\star}(v_0^\star - u_0^\star) = g_{\bar t_\epsilon, x_\epsilon^\star}(m_{t_\epsilon^\star} - \tilde m_{t_\epsilon^\star}) = g_{\bar t_\epsilon, x_\epsilon^\star}(m_{t_\epsilon^\star} - \tau_{\Delta_\star}\tilde m_{t_\epsilon^\star}) + \\ \int dy\,\big[g_{\bar t_\epsilon, x_\epsilon^\star}(x, y - \Delta_\star) - g_{\bar t_\epsilon, x_\epsilon^\star}(x,y)\big]\tilde m_{t_\epsilon^\star}(y) \end{aligned} \tag{3.95}$$

Then Theorem 2.1, (3.93) and (A.33) imply that there are c_5 and c_6 so that

$$\lim_{\epsilon\to 0+} P^\epsilon\Big(\|g_{\bar t_\epsilon, x_\epsilon^\star}(v_0^\star - u_0^\star)\| \le c_5 e^{-\alpha\bar t_\epsilon} + c_6\delta\epsilon^{\bar\gamma}\Big) = 1 \tag{3.96}$$

Using (3.96) and the integral equations for $v_t^\star$ and $u_t^\star$, we get, for some constant c_7,

$$\lim_{\epsilon\to 0+} P^\epsilon\Big(\|v_{\bar t_\epsilon}^\star - u_{\bar t_\epsilon}^\star\| \le c_7\delta\epsilon^{\bar\gamma}\Big) = 1 \tag{3.97}$$

By (3.97), using Lemma 2.3, we have also, for some constant c_8,

$$\lim_{\epsilon\to 0+} P^\epsilon\Big(|\xi(m_{t_\epsilon}) - \xi(\tilde m_{t_\epsilon})| \le c_8\delta\epsilon^{\bar\gamma}\Big) = 1 \tag{3.98}$$

and so, since $\bar{\gamma} > \gamma$, by (3.97) and (3.98) we finally obtain

$$\lim_{\epsilon \to 0+} P^{\epsilon}\Big(\|m_{t_\epsilon} - \tilde{m}_{t_\epsilon}\| \le \delta\epsilon^{\gamma}, \quad |\xi(m_{t_\epsilon}) - \xi(\tilde{m}_{t_\epsilon})| \le \delta\epsilon^{\gamma}\Big) = 1 \tag{3.99}$$

This proves the Proposition for m, $\tilde{m} \notin C_{\epsilon,\kappa}(\mathbb{R})$. By using conveniently the Barrier Lemma, one easily extends the result to the general case. We omit the details. □

When proving Theorem 1.3 we will consider the case when at some time T we have two data, m_T and $\tilde{m}_T$, both in $C_{\epsilon,\kappa}(\mathbb{R})$ and in $M^{\epsilon}_{\kappa,\ell,\epsilon^{1/2-a}}$, and such that $\xi^{\epsilon,\kappa}(m_T) = \xi^{\epsilon,\kappa}(\tilde{m}_T)$, see the end of Section 5. This case is not directly covered by Proposition 3.7, but we will see in the following Lemma that if we construct with the same noise the two processes then they will verify with large probability the conditions of Proposition 3.7 after a time t_ϵ.

3.8 Lemma.

Let m and $\tilde{m}$ both in $M^{\epsilon}_{\kappa,\ell,\epsilon^{1/2-a}}$, $a \in (0,1/2)$, with $x_0 \doteq \xi^{\epsilon,\kappa}(m) = \xi^{\epsilon,\kappa}(\tilde{m})$ and let $p \in (0,a)$. Then for any $\omega \in (2a, 1/2)$ and any ϵ small enough, in $\mathcal{B}_{p,\epsilon,x_0}$ the following estimate holds:

$$\|T_{t_\epsilon}(m, \sqrt{\epsilon}Z) - T_{t_\epsilon}(\tilde{m}, \sqrt{\epsilon}Z)\| \le \epsilon^{1-\omega}$$

Proof.

Writing the integral equation (3.4) around $\bar{m}_{x_0}$ for the two processes one obtains an equation for the difference like (3.47) but with the same center and noises. Then the estimate follows easily. □

We conclude the section with two lemmas consequence of general properties of the Ginzburg-Landau process. For any $a \in (0,1/2)$ and any $\phi \in \mathcal{M}_{\epsilon^{1/2-a}}$, let us denote by E_ϕ the expectation with respect to the Ginzburg–Landau process starting from ϕ. We indicate with m_t the coordinate map on $C(\mathbb{R}_+; C(\mathbb{R}))$ and let $\xi_t = \xi^{\epsilon,\kappa}(m_t)$.

3.9 Lemma.

For any $t, s \ge 0$,

$$E^{\epsilon}_{\bar{m}}[E^{\epsilon}_{\psi(m_s)}[\xi_t]] = 0 \tag{3.100}$$

Proof.

Consider the symmetry trasformation $\mathcal{R} : C(\mathbb{R}) \to C(\mathbb{R})$ defined by $(\mathcal{R}\phi)(x) \doteq -\phi(-x)$. For any $m \in \mathcal{M}_{\epsilon^{1/2-a}}$, $t \ge 0$, $T_t(m, \sqrt{\epsilon}\underline{Z}) = \mathcal{R}T_t(\mathcal{R}m, \sqrt{\epsilon}\mathcal{R}\underline{Z})$. Since $\bar{m} = \mathcal{R}\bar{m}$, for any bounded functional F on $C(\mathbb{R}_+; C(\mathbb{R}))$, $E_{\bar{m}}[F] = E_{\bar{m}}[\mathcal{R}F]$. In particular the law of $\psi(m_s)$ is symmetric with respect to $\mathcal{R}$ so that

$$E^{\epsilon}_{\bar{m}}[E^{\epsilon}_{\psi(m_s)}[\xi_t]] = \frac{1}{2}E^{\epsilon}_{\bar{m}}\left[E^{\epsilon}_{\psi(m_s)}[\xi_t] + E^{\epsilon}_{\mathcal{R}\psi(m_s)}[\xi_t]\right] \tag{3.101}$$

On the other hand, by symmetry, if $\xi(m) = 0$ then $E_m[\xi_t] = -E_{\mathcal{R}m}[\xi_t]$. The Lemma is proved. □

3.10 Lemma.

Let $a \in (0,1/4)$ and $m \in \mathcal{M}_{\epsilon^{1/2-a}}$ such that $\xi(m) = 0$. Then

$$\lim_{\epsilon \to 0^+} E^\epsilon_m[(\epsilon t_\epsilon)^{-1} \xi^2_{t_\epsilon}] = D, \qquad D = 3/4 \tag{3.102}$$

Proof.

Given $p \in (0,a)$ let $\mathcal{D}_p$ be the nice set where $\sup_{0 \le t \le t_\epsilon} \|\sqrt{\epsilon}\hat{Z}_{t,0}\| \le \epsilon^{1/2-p}$. By Lemma 2.3, for small ϵ, in this set $\bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}$ has a unique linear center ξ^Z_t and it holds

$$|\xi^Z_{t_\epsilon} - \langle \bar{m}', \bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle| \le c_0 |\langle \bar{m}', \bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle|^2 \tag{3.103}$$

Moreover, in $\mathcal{D}_p$,

$$|\langle \bar{m}', \bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle| \le \|\sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\| \le \epsilon^{1/2-p} \tag{3.104}$$

On the other hand, looking at the integral version of the Ginzburg-Landau equation, since m is ortogonal to $\bar{m}'$, one easily obtains that, in $\mathcal{D}_p$,

$$\|m_{t_\epsilon} - (\bar{m} + \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0})\| \le t_\epsilon \epsilon^{1/2-a} \tag{3.105}$$

and then, by Lemma 2.3,

$$|\xi_{t_\epsilon} - \xi^Z_{t_\epsilon}| \le c_0 t_\epsilon \epsilon^{1-2a} \tag{3.106}$$

In the proof of Theorem 4.2 we will use the following conseguence of (3.106); in $\mathcal{D}_p$ one has

$$|\xi_{t_\epsilon} - \bar{\xi}_{t_\epsilon}| \le c_0 t_\epsilon \epsilon^{1-2a} \tag{3.107}$$

where $\bar{\xi}_{t_\epsilon} = \xi(T_{t_\epsilon}(\bar{m}, \sqrt{\epsilon}\underline{Z}))$.

Since $P^\epsilon(\mathcal{D}_p)$ converge to 1 as $\epsilon \to 0^+$ faster than any power of ϵ, from (3.103), (3.104) and (3.106),

$$\lim_{\epsilon \to 0^+} E^\epsilon_m[(\epsilon t_\epsilon)^{-1} \xi^2_{t_\epsilon}] = \lim_{\epsilon \to 0^+} E^\epsilon_m[(\epsilon t_\epsilon)^{-1} |\langle \bar{m}', \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle|^2] \tag{3.108}$$

But one easily compute

$$E^\epsilon_m[(\epsilon t_\epsilon)^{-1} |\langle \bar{m}', \sqrt{\epsilon}\hat{Z}_{t_\epsilon,0}\rangle|^2] = |\langle \bar{m}', e^{(\epsilon)}_1\rangle|^2 \tag{3.109}$$

where $e^{(\epsilon)}_1$ is defined in the proof of Proposition 3.7 (here is considered as an element of $C_{\epsilon,\kappa}(\mathbb{R})$). The Lemma follows then from (3.108) and (3.109) since $\lim_{\epsilon \to 0^+} \langle \bar{m}', e^{(\epsilon)}_1\rangle = \sqrt{D}$. □

4. Convergence to a Brownian motion.

In this Section we prove that the linear center $\xi(m_t)$, suitably normalized, converges to a Brownian motion.

Let $a \in (0,1/2)$, $\kappa \ge 1$, $\epsilon > 0$, and

$$\mathcal{X} \doteq \left\{\psi \in C(\mathbb{R}) : \xi^{\epsilon,\kappa}(\psi) = 0\right\}; \qquad \mathcal{X}_{\epsilon,a} \doteq \mathcal{X} \cap \mathcal{M}_{\epsilon^{1/2-a}} \tag{4.1}$$

As in the previous section, set $t_\epsilon = (\log \epsilon)^2$. We consider an auxiliary Markov chain $(x_n, \psi_n)_{n\in\mathbb{N}}$ with state space $\mathbb{R} \times \mathcal{X}$. We denote by $\mathbb{P}^\epsilon_{(x_n,\psi_n)}$ its transition probabilities, given by:

$$\mathbb{P}^\epsilon_{(x_n,\psi_n)}\Big((x_{n+1},\psi_{n+1}) = (x_n,\psi_n)\Big) = 1 \quad \text{if } \psi_n \notin \mathcal{X}_{\epsilon,a}$$

If instead $\psi_n \in \mathcal{X}_{\epsilon,a}$ we define, for any given B and A Borel sets in $\mathbb{R}$ and $C^0(\mathbb{R})$ respectively

$$\mathbb{P}^\epsilon_{(x_n,\psi_n)}\Big((x_{n+1},\psi_{n+1}) \in B \times A\Big) = P^\epsilon\Big(x_n + \theta \in B, \tau_\theta m_{t_\epsilon} \in A\Big)$$

where

$$m_t \doteq T_t(\psi_n; \sqrt{\epsilon}\underline{Z}), \quad \theta \doteq \xi^{\epsilon,\kappa}(m_{t_\epsilon})$$

We next introduce some stopping times: for $r > 0$ we set

$$t^0(r) \doteq \inf\{n \in \mathbb{N} : |x_n| \geq r\} \tag{4.2}$$

and for $\epsilon > 0$ and $a \in (0, 1/2)$

$$t_{\epsilon,a} \doteq \inf\{n \in \mathbb{N} : \psi_n \notin \mathcal{X}_{\epsilon,a}\} \tag{4.3}$$

Finally, suppose we construct the Markov chain and the Ginzburg-Landau process in the same space, and consider the stopping time $s_{\epsilon,\kappa}(\zeta)$, $\epsilon > 0$, $\kappa \geq 1$, $\zeta > 0$,

$$s_{\epsilon,\kappa}(\zeta) \doteq \inf\Big\{n \in \mathbb{N} : \big|x_n - \xi^{\epsilon,\kappa}(m_{nt_\epsilon})\big| + \|\psi_n(x) - \tau_{x_n} m_{nt_\epsilon}\|_\epsilon \geq \zeta\Big\} \tag{4.4}$$

The seminorm $\|\cdot\|_\epsilon$ is defined in (3.28). In the above definitions the stopping times are set equal to $+\infty$ if the sets on the right hand side are empty.

In the next Proposition we indeed construct the original Ginzburg-Landau process and the auxiliary Markov chain in the same probability space, and prove lower bounds on $s_{\epsilon,\kappa}(\zeta)$ thus showing that they are close.

4.1 Proposition.

Let $\ell \in (0,1)$, $a \in (0,1/2)$, $\kappa \geq 1$, $h > 0$, $q > 0$. Then there is $c > 0$ so that the following holds. Let $\epsilon > 0$, $m \in C^0(\mathbb{R})$ with $m^{\epsilon,\kappa} \in \mathcal{M}_{\epsilon^{1/2-a}}$,

$$x_0 \doteq \xi(m^{\epsilon,\kappa}), \quad \psi_0(x) \doteq m^{\epsilon,\kappa}(x + x_0), \quad |x_0| < \epsilon^{-\kappa} - \ell\epsilon^{-1} \tag{4.5}$$

Then we can construct the Ginzburg-Landau process m_t (that starts from m) and the Markov chain (that starts from (x_0, ψ_0)) in the same probability space so that

$$P^\epsilon\Big(s_{\epsilon,\kappa}(\epsilon^q) \geq t^0\big(\epsilon^{-\kappa} - \ell\epsilon^{-1}\big) \wedge \epsilon^{-h}\ ;\ t_{\epsilon,a} \geq \epsilon^{-h}\Big) \geq 1 - c\epsilon^q \tag{4.6}$$

Proof.

The proof follows by applying iteratively Corollary 3.6 and Proposition 3.3. □

We next study the Markov chain (x_n, ψ_n) and prove convergence to a Brownian motion. We set $z_0 \doteq 0$ and for $n \geq 1$

$$z_n \doteq \frac{x_n - x_{n-1}}{\sqrt{\epsilon t_\epsilon}} \tag{4.7}$$

We then define

$$Z_n \doteq \sum_{i=0}^{n} z_i = \frac{x_n - x_0}{\sqrt{\epsilon t_\epsilon}} \tag{4.8}$$

and given N (eventually we let $N = N(\epsilon)$ and $N(\epsilon) \to +\infty$)

$$X(t) \doteq \frac{1}{\sqrt{N}} Z_n \qquad t = n/N \tag{4.9}$$

We finally extend $X(t)$ to $t \in \mathbb{R}_+$ by linear interpolation.

4.2 Theorem.

Let $h > 0$, $x_0 \in \mathbb{R}$, $a \in (0, 1/4)$ and $\psi_0 \in \mathcal{X}_{\epsilon,a}$. Let $\mathbb{P}^\epsilon$ be the law on $C(\mathbb{R}_+)$ of the process $X(t)$ induced via (4.9) with $N \doteq [\epsilon^{-h}]$ by the Markov chain that starts from (x_0, ψ_0). Then $\mathbb{P}^\epsilon$ converges weakly on the compacts to P as $\epsilon \to 0^+$, where P is the law of a Brownian motion starting from 0 with diffusion equal to 3/4.

Proof.

Without loss of generality we may restrict to $t \in [0, 1]$. Tightness on $C([0, 1])$ follows from the existence of $c > 0$ for which

$$\mathbb{E}^\epsilon\Big(\sup_{n \leq N} Z_n^2\Big) \leq cN \tag{4.10}$$

$$\mathbb{E}^\epsilon\Big([Z_{n_3} - Z_{n_2}]^2 [Z_{n_2} - Z_{n_1}]^2\Big) \leq c(n_3 - n_1)^2, \qquad \text{for all } 1 \leq n_1 < n_2 < n_3 \leq N \tag{4.11}$$

see [2].

We first prove (4.10) and (4.11), then a martingale relation for the limit laws that will identify the law P of the theorem.

We call $\mathcal{F}_n$, $n \in \mathbb{N}$, the σ-algebra generated by the coordinates (x_i, ψ_i), $0 \leq i \leq n$, of the Markov chain and denote by $\mathbb{E}_n^\epsilon$, $n \in \mathbb{N}$, the conditional expectation given $\mathcal{F}_n$ (sometimes we write more explicitly $\mathbb{E}^\epsilon_{(x_n, \psi_n)}$). We set

$$\gamma_{1,n} \doteq \mathbb{E}_n^\epsilon(z_{n+1}), \quad \gamma_{1,n}^\star \doteq \mathbb{E}_n^\epsilon(\gamma_{1,n+1}) \quad \text{for } n \geq 0 \text{ and } \gamma_{1,-1}^\star \doteq \gamma_{1,0} \tag{4.12}$$

We then have for $n \geq 1$

$$Z_n = \Gamma_{1,n-2}^\star + M_{n-1}^\star + M_n \tag{4.13}$$

where

$$\Gamma_{1,n}^\star \doteq \sum_{i=-1}^{n} \gamma_{1,i}^\star, \quad n \geq -1 \tag{4.14}$$

$$M_n^\star \doteq \sum_{i=1}^{n} [\gamma_{1,i} - \gamma_{1,i-1}^\star], \quad n \geq 1, \quad M_0^\star \doteq 0 \tag{4.15}$$

$$M_n \doteq \sum_{i=1}^{n} [z_i - \gamma_{1,i-1}], \quad n \geq 1, \quad M_0 \doteq 0 \tag{4.16}$$

Observe that $M_n^\star$ and M_n are $\mathcal{F}_n$-martingales. The usual semimartingale representation for Z_n in terms of the compensators $\gamma_{1,i}$ and M_n is not useful in the present context: the time delay in the definition of the compensators $\gamma_{1,i}^\star$ allows in fact to exploit the relaxation properties of the Ginzburg-Landau process stated in Corollary 3.6.

The semimartingale representation of M_n^2 is

$$M_n^2 = \Gamma_{2,n-1} + N_n, \quad n \geq 1 \tag{4.17}$$

$$\Gamma_{2,n} \doteq \sum_{i=0}^{n} \gamma_{2,i}, \quad \gamma_{2,n} \doteq \mathbb{E}_n^\epsilon\Big((z_{n+1} - \gamma_{1,n})^2\Big), \quad n \geq 0 \tag{4.18}$$

where we set $N_0 = 0$ and for $n \geq 1$

$$N_n \doteq 2 \sum_{1 \leq j < i \leq n} [z_i - \gamma_{1,i-1}][z_j - \gamma_{1,j-1}] + \sum_{i=1}^{n} [(z_i - \gamma_{1,i-1})^2 - \gamma_{2,i-1}] \tag{4.19}$$

is a $\mathcal{F}_n$ martingale. For $(M_n^\star)^2$ we have

$$(M_n^\star)^2 = \Gamma_{2,n-1}^\star + N_n' + N_n'' \tag{4.20}$$

$$\Gamma_{2,n}^\star \doteq \sum_{i=0}^{n} \gamma_{2,i}^\star, \quad \gamma_{2,n}^\star \doteq \mathbb{E}_n^\epsilon\Big((\gamma_{1,n+1} - \gamma_{1,n}^\star)^2\Big), \quad n \geq 0 \tag{4.21}$$

$$N_n' \doteq \sum_{i=0}^{n-1} \Big[(\gamma_{1,i+1} - \gamma_{1,i}^\star)^2 - \gamma_{2,i}^\star\Big], \quad n \geq 1, \quad N_0' = 0 \tag{4.22}$$

$$N_n'' \doteq 2 \sum_{1 \leq j < i \leq n} [\gamma_{1,i} - \gamma_{1,i-1}^\star][\gamma_{1,j} - \gamma_{1,j-1}^\star], \quad n > 1, \quad N_1'' = N_0'' = 0 \tag{4.23}$$

where N_n' and N_n'' are $\mathcal{F}_n$-martingales.

Proof of tightness. By (4.13)

$$\frac{1}{4}\mathbb{E}^\epsilon\Big(\sup_{n \leq N} Z_n^2\Big) \leq \mathbb{E}^\epsilon\Big(\sup_{n \leq N}(M_{n-1}^\star)^2\Big) + \mathbb{E}^\epsilon\Big(\sup_{n \leq N}(M_n)^2\Big) + 2N^2 \sup_{n \leq N} \mathbb{E}^\epsilon\Big((\gamma_{1,n}^\star)^2\Big)$$

and using Doob's inequality

$$\leq \sup_{n \leq N} \mathbb{E}^\epsilon\Big(2N^2(\gamma_{1,n}^\star)^2 + 4N\gamma_{2,n} + 4N\gamma_{2,n}^\star\Big) \tag{4.24}$$

We set $N(\epsilon) \doteq [\epsilon^{-h}]$, then we obtain

$$\mathbb{E}^\epsilon\Big(\sup_{n \leq N(\epsilon)} Z_n^2\Big) \leq 16N(\epsilon) \sup_{n \leq N(\epsilon)} \mathbb{E}^\epsilon\Big(N(\epsilon)(\gamma_{1,n}^\star)^2 + \gamma_{2,n} + \gamma_{2,n}^\star\Big) \tag{4.25}$$

Since the chain is stopped once it is not in $\mathcal{X}_{\epsilon,a}$ it follows that if $\psi_n \notin \mathcal{X}_{\epsilon,a}$ then $\gamma_{1,n}^\star = \gamma_{2,n} = \gamma_{2,n}^\star = 0$ so that (4.10) holds if there is $c > 0$ so that

$$\sup_{\psi \in \mathcal{X}_{\epsilon,a}} \mathbb{E}_{(0,\psi)}^\epsilon\Big(N(\epsilon)^2(\gamma_{1,0}^\star)^2 + \gamma_{2,0} + \gamma_{2,0}^\star\Big) \leq c \tag{4.26}$$

We will prove (4.26) later.

To prove (4.11) we use the same argument after conditioning on $\mathcal{F}_{n_2}$. We call (x_{n_2}, ψ_{n_2}) the state of the chain at time n_2, $n \doteq n_3 - n_2$ and $\bar{n} \doteq n_2 - n_1$. Then

$$\begin{aligned}
&\mathbb{E}^\epsilon\Big([Z_{n_3} - Z_{n_2}]^2[Z_{n_2} - Z_{n_1}]^2\Big) \\
&\qquad \le \sup_{(x_{n_2},\psi_{n_2})} \mathbb{E}^\epsilon_{n_2}\Big([Z_{n_3} - Z_{n_2}]^2\Big)\mathbb{E}^\epsilon\Big([Z_{n_2} - Z_{n_1}]^2\Big) \\
&\qquad \le \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(Z_n^2\Big) \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(Z_{\bar{n}}^2\Big)
\end{aligned} \tag{4.27}$$

and using (4.26) we have, for a suitable constant $c' > 0$,

$$\begin{aligned}
&\sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(Z_n^2\Big) \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(Z_{\bar{n}}^2\Big) \\
&\le 16^2 n\bar{n} \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(n(\gamma^\star_{1,0})^2 + \gamma_{2,0} + \gamma^\star_{2,0}\Big) \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi)}\Big(\bar{n}(\gamma^\star_{1,0})^2 + \gamma_{2,0} + \gamma^\star_{2,0}\Big) \\
&\le c'(n_3 - n_1)^2
\end{aligned} \tag{4.28}$$

having used (4.26). (4.11) and tightness are proved provided that (4.26) holds.

Let $\mathbb{P}$ be a limit law on $C([0,1])$ of $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$. By Levy's characterization theorem it will be sufficient to prove that the coordinate process $X(t)$ in $C([0,1])$ is a square integrable $\mathbb{P}$-martingale and that $X(t)^2 - 3t/4$ is also a $\mathbb{P}$-martingale.

By (4.13) and (4.24)–(4.26)

$$\lim_{\epsilon\to 0+} \frac{1}{N(\epsilon)} \sup_{\psi_0\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi_0)}\Big(\sup_{n\le N(\epsilon)} (Z_n - M_n)^2\Big) = 0 \tag{4.29}$$

which proves that $X(t)$ is a $\mathbb{P}$-martingale.

By (4.17)

$$\Lambda_n \doteq \frac{1}{N(\epsilon)}\Big(Z_n^2 - \Gamma_{2,n-1}\Big) \tag{4.30}$$

differs from a $\mathbb{P}^\epsilon$-martingale by the term $[Z_n^2 - M_n^2]$ which by (4.29) vanishes in L^1 as $\epsilon \to 0^+$. Thus the proof that $X(t)^2 - 3t/4$ is a $\mathbb{P}$-martingale follows from

$$\lim_{\epsilon\to 0+} \sup_{\psi_0\in\mathcal{X}_{\epsilon,a}} \mathbb{E}^\epsilon_{(0,\psi_0)}\Big(\sup_{n\le N(\epsilon)} \frac{1}{N(\epsilon)}\Big|\Gamma_{2,n-1} - \frac{4}{3}\Big|\Big) = 0 \tag{4.31}$$

which, by Proposition 4.1, is implied by

$$\lim_{\epsilon\to 0+} \sup_{\psi\in\mathcal{X}_{\epsilon,a}} \Big|\gamma_{2,0}(\psi) - \frac{4}{3}\Big| = 0 \tag{4.32}$$

Proof of (4.26) *and* (4.32). Let $\psi \in \mathcal{X}_{\epsilon,a}$, $\phi \doteq T_{t_\epsilon}(\psi; \sqrt{\epsilon}\underline{Z})$, $\chi \doteq \xi^{\epsilon,\kappa}(\phi)$ and $\psi_{t_\epsilon}(x) \doteq \phi(x + \chi)$. By (4.12)

$$\gamma^\star_{1,0}(\psi) = E^\epsilon_\psi\Big(\gamma_{1,0}(\psi_{t_\epsilon})\Big), \quad \gamma_{1,0}(\phi) = E^\epsilon_\phi\Big(\xi^{\epsilon,\kappa}(\phi_{t_\epsilon})\Big) \tag{4.33}$$

By Lemma 3.9 $\gamma^\star_{1,0}(\bar m) = 0$ and $\gamma^\star_{1,0}(\psi) = \gamma^\star_{1,0}(\psi) - \gamma^\star_{1,0}(\bar m)$. We apply Corollary 3.6 so that we can construct the processes starting from ψ and $\bar m$ in the same probability space in such a way that for any $q > 0$ there is $c > 0$ so that $\|\psi_{t_\epsilon} - \bar\psi_{t_\epsilon}\|_\epsilon \le \epsilon^q$ with probability larger than $1 - c\epsilon^q$. ($\bar\psi_{t_\epsilon}$ denotes the value of ψ_{t_ϵ} when $\psi = \bar m$). Thus

$$|\gamma^\star_{1,0}(\psi)| \le c\epsilon^q + \sup_{\phi,\bar\phi\in\mathcal{X}_{\epsilon,a}} \sup_{\|\phi-\bar\phi\|_\epsilon\le\epsilon^q} \left| E^\epsilon_\phi(\xi^{\epsilon,\kappa}(\phi_{t_\epsilon})) - E^\epsilon_{\bar\phi}(\xi^{\epsilon,\kappa}(\bar\phi_{t_\epsilon}))\right| \tag{4.34}$$

By (3.70) with $\Delta = 0$ and $\delta = \epsilon^q$ we get $|\gamma^\star_{1,0}(\psi)| \le c\epsilon^q$ (for a suitable constant $c > 0$). (4.26) is proved.

By symmetry $\gamma_{1,0}(\bar m) = 0$, so that by (3.107)

$$|\gamma_{1,0}(\psi)| \le ct_\epsilon\epsilon^{1/2-2a} \tag{4.35}$$

that vanishes as $\epsilon \to 0^+$ by the assumption $a < 1/4$. Finally, by (4.18)

$$\gamma_{2,0}(\psi) = \frac{1}{\epsilon t_\epsilon} E^\epsilon_\psi(\xi^{\epsilon,\kappa}(\psi_{t_\epsilon})^2) - \gamma_{1,0}(\psi)^2$$

By the previous bound the last term vanishes as $\epsilon \to 0^+$ (uniformly on $\mathcal{X}_{\epsilon,a}$) while the first term on the right hand side converges to 3/4 by Lemma 3.10. We have thus proved (4.32) and it only remains to prove that $\gamma^\star_{2,0}(\psi)$ is uniformly bounded on $\mathcal{X}_{\epsilon,a}$ which follows from (4.21), (4.34) and (4.35).

The proof of the Theorem is complete. □

We next relate the convergence results proved for the auxiliary Markov chain to the Ginzburg-Landau process. We use the same notation as in Theorem 4.2 and Proposition 4.1. We fix the initial position x_0 in the Markov chain so that $\epsilon^{h/2}x_0 \doteq r_0$ (which is independent of ϵ) with $|r_0| < \epsilon^{-\kappa+h/2}$. We consider the Markov and the Ginzburg-Landau process whose initial state is related to that of the Markov chain as in Proposition 4.1. We call $T_M(r)$, $r \in \mathbb{R}$, the subfix M standing for Markov, the first time when the coordinate process $X(t)$ (that we here suppose starting from r_0) reaches r. The analogous variable in the Ginzburg-Landau process is denoted by $T_{\rm GL}(r)$. Let $\ell^\star \in (0,1)$ and let $\ell \in (0,\ell^\star)$, call $r^\star_\epsilon \doteq \epsilon^{h/2}[\epsilon^{-\kappa} - \ell^\star\epsilon^{-1}]$, so that (at least for $\epsilon > 0$ small enough) $|r_0| < r^\star_\epsilon$. Then, by Proposition 4.1 with ℓ as above, for any $q > 0$ there is $c > 0$ so that

$$P^\epsilon\Big(T_M(r^\star_\epsilon - \epsilon^q) \le T_{\rm GL}(r^\star_\epsilon) \le T_M(r^\star_\epsilon + \epsilon^q)\Big) \ge 1 - c\epsilon^q \tag{4.36}$$

Similar statement holds for $-r^\star_\epsilon$.

For any r the law of $T_M(r)$ converges as $\epsilon \to 0^+$ to the law of the stopping time at $\pm r$ for the limit brownian motion b_t (starting from r_0) because the stopping time for the limit process is almost surely continuous, see [2]. Moreover the probability of $|T_M(r \pm \delta) - T_M(r)| > \zeta\ \delta$ and ζ positive, vanishes as $\delta \to 0^+$, hence by (4.36) the law of the stopping time at $\epsilon^{h/2}[\epsilon^{-\kappa} - \ell^\star\epsilon^{-1}]$ in the Ginzburg Landau process converges to the law of the stopping time for the limit Brownian motion at

$$\lim_{\epsilon\to 0^+}\{\epsilon^{h/2}[\epsilon^{-\kappa} - \ell\epsilon^{-1}] - r_0\}$$

This together with Theorem 4.2 proves Theorem 1.2.

5. Asymptotic Coupling.

In this section we prove Theorem 1.3. By Theorem 1.2 and Proposition 3.7 we only need, as we will explain later, the following theorem.

5.1 Theorem.

Let m, $m^\star \in C_{\epsilon,\kappa}(\mathbb{R})$ (eventually depending on ϵ) such that $\|m\|$, $\|m^\star\| \leq 3/2$ and $\|m - m^\star\| \leq \epsilon^{2+\kappa}$. Then, we can construct a pair of Ginzburg–Landau processes m_t and $m_t^\star$, starting from m and $m^\star$ respectively, in the same probability space, and so that, if

$$\eta \doteq \inf\{t \geq 0 : \|m_t - m_t^\star\| = 0\}$$

(η is defined to be infinity if the set above is empty), for any $\alpha < 1$

$$\lim_{\epsilon \to 0^+} P^\epsilon\big(\eta > \epsilon^\alpha\big) = 0 \tag{5.1}$$

Proof.

The proof uses the coupling and the ideas introduced in [15] to prove Theorem 1, but since V' is not monotonic, an extra argument is needed to conclude (5.1). Recall that in fact, we do not prove, as in [15] that η is finite with probability 1.

Consider the pair $(m_t, m_t^\star)$ introduced in [15], which satisfies

$$\begin{aligned}
\frac{\partial m_t}{\partial t} &= \frac{1}{2}\frac{\partial^2 m_t}{\partial x^2} - V'(m_t) + \sqrt{\epsilon}\alpha_1 \\
\frac{\partial m_t^\star}{\partial t} &= \frac{1}{2}\frac{\partial^2 m_t^\star}{\partial x^2} - V'(m_t^\star) + \sqrt{\epsilon}[1 - (|m_t - m_t^\star| \wedge 1)^{1/2}\alpha_1 + (|m_t - m_t^\star| \wedge 1)^{1/2}\alpha_2]
\end{aligned} \tag{5.2}$$

for α_1 and α_2 two independent space-time white noises and with initial conditions

$$m_0 = m, \qquad m_0^\star = m^\star \tag{5.3}$$

Consider the case $m \geq m^\star$. The general case follows from this one as in [15]. If we write the equation for the difference $m_t - m_t^\star$ and approximate the coefficients of the noise by Lipschitz functions as in [15], we can conclude, from theorem 2.3 of [16], that $m_t \geq m_t^\star \quad \forall x \in \mathbb{R}, t \geq 0$. Call $\mathcal{F}_t$ the filtration generated by α_1 and α_2 up to time t. Next, integrate (5.2) from 0 to t and over $[-\epsilon^{-\kappa}, \epsilon^{-\kappa}]$. Call

$$U(t) \doteq \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx\, \big(m_t(x) - m_t^\star(x)\big) \tag{5.4}$$

Proceeding as in [15], we obtain for U the equation

$$U(t) = U(0) + \int_0^t ds\, U(s) - \int_0^t ds \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx\, \big(m_s(x)^3 - m_s^\star(x)^3\big) + M(t) \tag{5.5}$$

where M_t is a martingale with respect to $\mathcal{F}_t$, with compensator

$$\langle M\rangle(t) = 2\epsilon \int_0^t ds \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx \, \frac{|m_s(x) - m_s^\star(x)| \wedge 1}{1 + (1 - |m_s(x) - m_s^\star(x)| \wedge 1)^{1/2}} \tag{5.6}$$

Since

$$\frac{d\langle M\rangle(t)}{dt} \geq \epsilon \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx \, \left(|m_t(x) - m_t^\star(x)| \wedge 1\right) = \epsilon \int_{-\epsilon^{-\kappa}}^{\epsilon^{-\kappa}} dx \, \frac{|m_t(x) - m_t^\star(x)|}{|m_t(x) - m_t^\star(x)| \vee 1} \tag{5.7}$$

we have that

$$\frac{d\langle M\rangle(t)}{dt} = U(t)D(t),$$

for some adapted process $D(t)$ satisfying

$$D(t) \geq \frac{\epsilon}{\|m_t - m_t^\star\| \vee 1} \tag{5.8}$$

Take

$$\varphi(t) = \int_0^t ds \, D(s) \tag{5.9}$$

It is not difficult to see that Lemma 3.3 of [15] also holds in our case and, for each fixed $\epsilon, \varphi(\infty) = \infty$. Then, we can define the time changed process

$$X(t) = U\left(\varphi^{-1}(t)\right) \tag{5.10}$$

which satisfies

$$X(t) = U(0) + \int_0^t ds \, \frac{X(s)}{\varphi'(\varphi^{-1}(s))} + \int_0^t ds \, C(s) + \int_0^t dB(s) \, X^{1/2}(s)$$

for some Brownian motion $B(s)$ and nonpositive adapted process $C(s)$. Applying Ito's formula with the function $f(x) = 2x^{1/2}$, we have that, as long as $X(t) \geq 0$,

$$Y(t) \doteq 2\sqrt{X(t)} \tag{5.11}$$

satisfies

$$Y(t) = 2\sqrt{U(0)} + \int_0^t ds \left(\frac{2C(s)}{Y(s)} - \frac{1}{2Y(s)} + \frac{Y(s)}{2\varphi'(\varphi^{-1}(s))} \right) + B(t) \tag{5.12}$$

Now, let us prove (5.1). From (5.11) and the definition of the time change, for any positive y

$$P^\epsilon(\eta > y) \leq P^\epsilon(\gamma > \varphi(y)) \tag{5.13}$$

where

$$\gamma \doteq \inf\{t \geq 0 : Y(t) = 0\}$$

Now, take $\alpha < 1$ as in the statement. From (5.13) we can write, for any given positive a,

$$P^\epsilon(\eta > \epsilon^\alpha) \leq P^\epsilon(\gamma > \varphi(\epsilon^\alpha), \varphi(\epsilon^\alpha) > a) + P^\epsilon(\gamma > \varphi(\epsilon^\alpha), \varphi(\epsilon^\alpha) \leq a) \leq$$
$$P^\epsilon(\gamma > a) + P^\epsilon(\varphi(\epsilon^\alpha) \leq a)$$

Using (5.8) and the a-priori bound on the sup-norm of m_t and $m_t^\star$ (see Proposition 3.1), if we take $a = \epsilon^\beta$, for any $\beta > 1+\alpha$, it is not difficult to prove that this last probability goes to zero as $\epsilon \to 0$, and so, to prove (5.1) we only need to show that

$$\lim_{\epsilon \to 0+} P^\epsilon\big(\gamma \le \epsilon^\beta\big) = 1 \tag{5.14}$$

for some β as above. Recall equation (5.11) for Y and consider

$$\tau \doteq \inf\{t : B(t) + 2\sqrt{U(0)} = 0 \text{ or } B(t) + 2\sqrt{U(0)} = \epsilon|\log\epsilon|\}$$

and the set S

$$S \doteq \{\varphi'(\varphi^{-1}(s)) \ge \epsilon \quad \forall\, s \le \epsilon^\beta\}$$

Call

$$E \doteq S \cap \{B(\tau) + 2\sqrt{U(0)} = 0\} \cap \{\tau < \epsilon^\beta\}$$

We shall prove that for all ϵ small enough

$$E \subset \{\gamma \le \epsilon^\beta\} \tag{5.15}$$

Define the stopping time

$$t_0 = \inf\left\{ s \,:\, \frac{1}{2Y(s)} \le \frac{Y(s)}{2\varphi'(\varphi^{-1}(s))} \right\}$$

Take $\omega \in E$ and suppose by contradiction that, for this ω, $Y(t) > 0$ for all $t \le \tau$. Then, equation (5.12) holds for $Y(t)$ for any $t \le \tau$, and so, for the ω we are considering,

$$Y(t) \le B(t) + 2\sqrt{U(0)} \quad \forall\, t \le t_0 \wedge \tau \tag{5.16}$$

If $\tau \le t_0$ the evaluation of the previous expression at τ yields $Y(\tau) \le B(\tau) + 2\sqrt{U(0)} = 0$, which is a contradiction. Then $\tau > t_0$ and since $\omega \in E \supset S$,

$$Y(t_0) = \frac{\varphi'(\varphi^{-1}(t_0))}{Y(t_0)} \ge \frac{\epsilon}{Y(t_0)}$$

which implies

$$Y(t_0) \ge \sqrt{\epsilon}$$

and this contradicts (5.16) for small ϵ for the definition of τ, which finishes the proof of (5.15). To conclude, we only have to show that we can take $\beta > 1+\alpha$ such that $P^\epsilon(E) \to 1$ as $\epsilon \to 0$. But, if we recall that $U(0) \le 2\epsilon^2$, we obtain

$$P^\epsilon\big(B(\tau) + 2\sqrt{U(0)} = 0\big) = \frac{\epsilon|\log\epsilon| - 2\sqrt{U(0)}}{\epsilon|\log\epsilon|} \ge \frac{\epsilon|\log\epsilon| - 2\sqrt{2}\epsilon}{\epsilon|\log\epsilon|} \to 1 \quad \text{as } \epsilon \to 0$$

Also, taking $\beta < 2$, it follows that

$$\lim_{\epsilon \to 0+} P^\epsilon\big(\tau \le \epsilon^\beta\big) = 1$$

To finish, let us prove

$$\lim_{\epsilon\to 0+} P^\epsilon(S) = 1 \tag{5.17}$$

First recall that by Proposition 3.1 for any $\beta < 2$

$$\lim_{\epsilon\to 0+} P^\epsilon\left(\frac{1}{\|m_s - m_s^\star\| \vee 1} > \frac{1}{4} \quad \forall s < \epsilon^\beta\right) = 1$$

so, from (5.8) and (5.9),

$$\lim_{\epsilon\to 0+} P^\epsilon\left(\varphi(s) > \frac{\epsilon}{4} \quad \forall s < \epsilon^\beta\right) = 1$$

and then

$$\lim_{\epsilon\to 0+} P^\epsilon\left(\varphi^{-1}(s) \le \frac{4s}{\epsilon} \quad \forall s < \epsilon^\beta\right) = 1 \tag{5.18}$$

Recalling $\varphi' = D$, from (5.8) and the Proposition 3.1

$$\lim_{\epsilon\to 0+} P^\epsilon\left(\varphi'(t) > \frac{\epsilon}{4} \quad \forall t < 4\epsilon^{\beta-1}\right) = 1 \tag{5.19}$$

for any $\beta \ge 0$. Finally, (5.18) and (5.19) imply (5.17), and the Theorem is proved. □

Proof of Theorem 1.3.

The coupling is constructed as follows. The two processes m_t and m'_t are independent of each other till the first time T_1 when $\xi^{\epsilon,\kappa}(m_{T_1}) = \xi^{\epsilon,\kappa}(m'_{T_1})$. Let $a \in (0, 1/4)$, then with probability going to 1 a $\epsilon \to 0^+$, both $(m_t)^{\epsilon,\kappa}$ and $(m'_t)^{\epsilon,\kappa}$ are in $\mathcal{M}_{\epsilon^{1/2-a}}$, we can thus suppose that such a condition is verified. By Lemma 3.8 at time $T = T_1 + t_\epsilon$ with large probability we are in the hypothesis of Proposition 3.7. We construct the processes in the time interval $[T, T + t_\epsilon]$ using Proposition 3.7 with $b = 1 - \omega$, $\omega \in (2a, 1/2)$, so that (3.79) is verified and we can suppose that the processes at time $T + t_\epsilon$ are in the set which appears on its left hand side. We can thus apply again Proposition 3.7 with $b = 1 - \omega + \bar\gamma$ and iterate this procedure $N > 1$ times. Then calling $S \doteq T + Nt_\epsilon$:

$$\|(m_S)^{\epsilon,\kappa} - (m'_S)^{\epsilon,\kappa}\| \le \epsilon^{1/2-a+N\bar\gamma}$$

with probability going to 1 as $\epsilon \to 0^+$. Since by assumption m and m' are both in $C_{\epsilon,\kappa}(\mathbb{R})$ the above holds as well for the sup norm (without the cutoff (ϵ, κ)).

We can then apply Theorem 5.1 to conclude that if N is large enough there is a coupling before $T + (N+1)t_\epsilon$ with probability going to 1 as $\epsilon \to 0^+$. Thus the time of coupling differes from the time of first encounter of the linear centers by a term bounded by $(N+1)|\log\epsilon|^2$. The law of first encounter of the linear centers converges to that of the brownians limit of the linear centers by Theorem 4.2. As the difference between true and linear centers vanishes in the limit, see the proof of Theorem 1.2 at the end of Section 4 we then obtain the proof of Theorem 1.3. □

A.1 Lemma.

Let $Z_t(x)$ be the process defined in Section 1. There are positive constants b_0 and b_1 so that for all $p > 0$,

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}Z_t\| > \epsilon^{\frac{1}{2}-p}\Big) \le b_0 e^{-b_1|\log\epsilon|^{-1}\epsilon^{-2p}} \tag{A.1}$$

Proof.

The process Z_t is a Gaussian centered process, with bounded and continuous paths a.e. Define

$$D_\epsilon \doteq \{(x,t) : x \in \mathcal{T}_{\epsilon,\kappa}; 0 \le t \le t_\epsilon\} \qquad |||Z||| \doteq \sup_{(x,t)\in D_\epsilon} Z_t(x), \qquad \sigma_\epsilon^2 \doteq \sup_{(x,t)\in D_\epsilon} E^\epsilon[Z_t(x)^2] \tag{A.2}$$

Using the explicit form of the covariance of Z_t, see for instance [18], it is not difficult to prove that there exists a constant C_1, independent of ϵ such that

$$E^\epsilon[Z_t(x)^2] \le C_1 t\epsilon^\kappa + C_1\sqrt{t}, \tag{A.3}$$

what yields

$$\sigma_\epsilon^2 \le C\sqrt{t_\epsilon} \tag{A.4}$$

for some C independent of ϵ. Then, we can apply the following inequality, that follows from a symmetry argument from Theorem 2.1 of [1]: for any $\lambda > E^\epsilon|||Z|||$:

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon} \|Z_t\| > \lambda\Big) \le 4\exp\Big[-\frac{(\lambda - E^\epsilon|||Z|||)^2}{2\sigma_\epsilon^2}\Big] \tag{A.5}$$

To give an upper bound to $E^\epsilon|||Z|||$, we use Corollary 4.15 of [1]: there exists a universal constant K such that

$$E^\epsilon|||Z||| \le K\int_0^\infty dr\sqrt{\log N_\epsilon(r)}, \tag{A.6}$$

where $N_\epsilon(r)$ is the minimal number of balls of radius r nedeed to cover D_ϵ, with respect to the metric

$$d((x,t),(y,s)) \doteq \sqrt{E^\epsilon\big[\big(Z_t(x) - Z_s(y)\big)^2\big]} \tag{A.7}$$

It can be proved that there are positive constants k_1 and k_2 (independent of ϵ, t, s, x and y) such that, for any $x, y \in \mathbb{R}$ and $t, s \in \mathbb{R}_+$:

$$E^\epsilon\big[\big(Z_t(x) - Z_t(y)\big)^2\big] \le k_1|x-y| \tag{A.8}$$

$$E^\epsilon\big[\big(Z_t(x) - Z_s(x)\big)^2\big] \le k_2\sqrt{|t-s|} \tag{A.9}$$

(see for example Proposition 4.2 in [18]). From (A.8) and (A.9) it is easy to check that there is a constant c such that

$$N_\epsilon(r) \le \max\{1, c|\log\epsilon|^2\epsilon^{-\kappa}r^{-3}\} \tag{A.10}$$

By (A.6) and (A.10) it follows that there is a constant K' such that

$$E^\epsilon|||Z||| \le K'\log\epsilon^{-(\kappa+1)} \tag{A.11}$$

Using (A.4) and (A.11), from inequality (A.6) with $\lambda = \epsilon^{-p}$ we finally obtain

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon} \|Z_t\| > \epsilon^{-p}\Big) \le 4\exp\Big[-\frac{(\epsilon^{-p} - K'\log\epsilon^{-(\kappa+1)}}{2C|\log\epsilon|}\Big] \tag{A.12}$$

The bound (A.12), which is valid for ϵ small enough, implies the estimate (A.1) for some constants b_0 and b_1. The estimate can be then extended to any $\epsilon \in (0,1]$ simply by modifing conveniently the values of b_0 and b_1. The Lemma is so proved. □

A.2 Lemma.

Let $m \in C^0(\mathbb{R})$, $\|m\| \le 1 + 10^{-2}$. Then there are constants c_0 and c_1 so that

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon} \|T_t(m;\sqrt{\epsilon}Z)\| \le 2,\ \|T_{t_\epsilon}(m;\sqrt{\epsilon}Z)\| \le 1 + 10^{-2}\Big) \ge 1 - c_0 e^{-c_1\epsilon^{-1}} \tag{A.13}$$

Proof.

A Comparison Theorem holds for the stochastic Ginzburg-Landau equation, see Proposition 5.1 in [5]. So, if $m \in C(\mathbb{R};[-1-10^{-2}, 1+10^{-2}])$, for any $t \ge 0$ it holds

$$m_t^- \le T_t\big(m;\sqrt{\epsilon}Z\big) \le m_t^+ \qquad P^\epsilon \text{ - a.s.} \tag{A.14}$$

where $m_t^\pm \doteq T_t\big(\pm(1+10^{-2});\sqrt{\epsilon}Z\big)$. It is then sufficient to prove (A.13) with $T_t(m;\sqrt{\epsilon}Z)$ replaced by $m_t^\pm$. We define $u_\pm(x,t) \doteq m_t^\pm(x) \mp 1$. Then $u_\pm(x,t)$ solve the equations

$$\frac{\partial u_\pm}{\partial t} - \frac{1}{2}\frac{\partial^2 u_\pm}{\partial x^2} + 2u_\pm = \mp 3u_\pm^2 - u_\pm^3 + \sqrt{\epsilon}\alpha \tag{A.15}$$

that is, the integral equations

$$u_\pm(t) = e^{-2t}H_t \star u_\pm(0) + \int_0^t ds e^{-2(t-s)}H_{t-s} \star [\mp 3u_\pm^2 - u_\pm^3](s) + \sqrt{\epsilon}V_t \tag{A.16}$$

where $e^{-2t}H_t \star u_\pm(0) = \pm e^{-2t}[1+10^{-2}]$ and

$$V_t(x) \doteq \int ds dy\, \alpha(s,y)\mathbf{1}_{0\le s\le t}e^{-2(t-s)}H_{t-s}(y-x) \tag{A.17}$$

(note that $e^{-2t}H_t(x-y)$ is the Green function for the operator $\partial_t - (1/2)\partial_x^2 + 2Id$). By arguing as in the proof of Lemma A.1, it is easy to prove that for any $b > 0$ there are constants h_0 and h_1 such that

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}V_t\| > b\Big) \le h_0 e^{-h_1\epsilon^{-1}} \tag{A.18}$$

Let $T \doteq \sup\{t \ge 0 : \|u_\pm(\cdot,t)\| < 2(10^{-2} + b)\}$. We will prove that there exists b and ϵ_0 such that for all $\epsilon \le \epsilon_0$

$$P^\epsilon\Big(\sup_{0\le t\le t_\epsilon} \|m_t^\pm\| \le 2,\ \|m_{t_\epsilon}^\pm\| \le 1 + 10^{-2}\Big) \ge P^\epsilon\Big(T \ge t_\epsilon,\ \sup_{0\le t\le t_\epsilon} \|\sqrt{\epsilon}V_t\| \le b\Big) \tag{A.19}$$

and

$$P^\epsilon\Big(T \geq t_\epsilon, \sup_{0\leq t\leq t_\epsilon} \|\sqrt{\epsilon}V_t\| \leq b\Big) = P^\epsilon\Big(\sup_{0\leq t\leq t_\epsilon} \|\sqrt{\epsilon}V_t\| \leq b\Big) \tag{A.20}$$

Clearly, by definition of T and $u_\pm$, $\sup_{0\leq t\leq t_\epsilon} \|m_t^\pm\| \leq 2$ is implied by $T \geq t_\epsilon$ if b is small enough. Moreover, in the set

$$\{T \geq t_\epsilon, \sup_{0\leq t\leq t_\epsilon} \|\sqrt{\epsilon}V_t\| \leq b\},$$

by equation (A.16), one has the estimate

$$\|u_\pm(\cdot, t_\epsilon)\| \leq e^{-2t_\epsilon}10^{-2} + 12(10^{-2}+b)^2 + 8(10^{-2}+b)^3 \tag{A.21}$$

Now there exists $\epsilon_0 > 0$ and $b_0 > 0$ such that, for any $\epsilon \leq \epsilon_0$ and $b \leq b_0$, (A.21) implies $\|u_\pm(\cdot, t_\epsilon)\| \leq 10^{-2}$ and hence $\|m_{t_\epsilon}^\pm\| \leq 1 + 10^{-2}$. (A.19) is then proven. To prove (A.20) we note that if $\sup_{0\leq t\leq t_\epsilon} \|\sqrt{\epsilon}V_t\| \leq b$ and $T \leq t_\epsilon$ then

$$2(10^{-2}+b) = \|u_\pm(\cdot, T)\| \leq 10^{-2} + b + 12(10^{-2}+b)^2 + 8(10^{-2}+b)^3 \tag{A.22}$$

which gives a contradiction if, for example, $b \leq 10^{-2}$. Then, for $b = b_0 \wedge 10^{-2}$, both (A.19) and (A.20) holds. The estimate (A.13) follows then from (A.18) for $\epsilon \leq \epsilon_0$ (with $c_0 = h_0$ and $c_1 = h_1$) and it extends to any ϵ simply by modifing the values of the constants c_0 and c_1. □

A.3 Lemma.

Let $m \in C^0(\mathbb{R})$, $m_t \doteq T_t(m, \sqrt{\epsilon}Z)$. For any $\delta > 0$, $\alpha \in (0, 1/2)$ there is a constant $c > 0$ so that for all ϵ small enough

$$P^\epsilon\left(\sup_{|x|,|y|\leq\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\leq 1}} \frac{|m_{t_\epsilon}(x) - m_{t_\epsilon}(y)|}{|x-y|^\alpha} > \delta\right) \leq e^{-c\delta^2} \tag{A.23}$$

Proof.

We first prove an analogous estimate for the Gaussian process $\sqrt{\epsilon}Z$. We use Theorem 2.1 of [1] applied to the gaussian process

$$G_\epsilon(x,y) \doteq \sqrt{\epsilon}\,\frac{Z_{t_\epsilon}(x) - Z_{t_\epsilon}(y)}{|x-y|^\alpha} \tag{A.24}$$

By arguing as in the proof of Lemma A.1 we have, for $\delta > E^\epsilon[\hat{G}_\epsilon]$,

$$P^\epsilon\Big(\sup_{|x|,|y|\leq\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\leq 1}} |G_\epsilon(x,y)| > \delta\Big) \leq 4\exp\left[-\frac{(\delta - E^\epsilon[\hat{G}_\epsilon])^2}{2\sigma_\epsilon^2}\right] \tag{A.25}$$

where

$$\hat{G}_\epsilon \doteq \sup_{|x|,|y|\leq\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\leq 1}} G_\epsilon(x,y), \qquad \sigma_\epsilon^2 \doteq \sup_{|x|,|y|\leq\epsilon^{-\kappa}} \sup_{\substack{x\neq y\\ |x-y|\leq 1}} E^\epsilon[G_\epsilon(x,y)^2] \tag{A.26}$$

Using Corollary 4.15 of [1] as in Lemma A.1 one easily proves that $E^\epsilon[\hat{G}_\epsilon] \le c_1\sqrt{\epsilon}$ for some $c_1 > 0$. To estimate σ_ϵ^2 we note that

$$E^\epsilon[G_\epsilon(x,y)^2] = \frac{\sqrt{\epsilon}}{|x-y|^{2\alpha}} \int_0^{t_\epsilon} ds \, \frac{1}{\sqrt{2\pi s}} \left(1 - e^{-(x-y)^2/2s}\right) \tag{A.27}$$

and, for some constant $c_3 > 0$ depending on α,

$$\left|1 - e^{-(x-y)^2/2s}\right| \le c_3 \frac{|x-y|^{2\alpha}}{s^\alpha} \tag{A.28}$$

From (A.27) and (A.28) we obtain $\sigma_\epsilon^2 \le c_4\sqrt{\epsilon}\, t_\epsilon^{1/2-\alpha}$ for some constant $c_4 > 0$. Then by (A.25) we recover an estimate like (A.23) for the noise $\sqrt{\epsilon}Z$. To prove (A.23) we use the integral form (1.3) of the Ginzburg–Landau equation and write

$$\begin{aligned} m_{t_\epsilon}(x) - m_{t_\epsilon}(y) &= H_1 \star m_{t_\epsilon - 1}(x) - H_1 \star m_{t_\epsilon - 1}(y) + \\ &\int_0^1 ds \, [H_{1-s}(x-\cdot) - H_{1-s}(y-\cdot)] \star [m_{t_\epsilon - 1 + s} - m^3_{t_\epsilon - 1 + s}] + G_\epsilon(x,y)|x-y|^\alpha \end{aligned} \tag{A.29}$$

We consider the intersection of the sets where m_{t_ϵ} is bounded by 2 and where the noise satisfies the bound like (A.23) with δ' to be fixed. In this set we can estimate

$$|m_{t_\epsilon}(x) - m_{t_\epsilon}(y)| \le 2|x-y| + c_5|x-y| + \delta'|x-y|^\alpha \tag{A.30}$$

for any $x \ne y$ such that $|x|, |y| \le \epsilon^{-\kappa}$ and $|x-y| \le 1$. Choosing δ' small enough (A.23) follows easily. □

A.4 Lemma.

Let $x_0 \in \mathbb{R}$ and let $g_{t,x_0}(x,y)$ be the foundamental solution of the equation $\partial_t u = L_{x_0} u$. Then the following holds.

$$g_{t,x_0}(x,y) \ge 0 \qquad \text{for any } x, y \in \mathbb{R} \tag{A.31}$$

$$\sup_{t \ge 0} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} dy \, g_{t,x_0}(x,y) \le c_0, \qquad c_0 > 0 \tag{A.32}$$

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} dy \, |g_{t,x_0}(x,y+d) - g_{t,x_0}(x,y)| \le c_1 \frac{d}{\sqrt{t}} \qquad c_1 > 0 \tag{A.33}$$

$$\sup_{t \le t_\epsilon} \sup_{B_2} \int_{\mathbb{R} \setminus B_1} dy \, g_{t,x_0}(x,y) \le c e^{2t_\epsilon} e^{-(r_1 - r_2)^2}, \qquad B_i \doteq \{x \in \mathbb{R} : |x - x_0| \le r_i\}, \; r_1 > r_2 \tag{A.34}$$

Proof.

From (3.45) we can restrict ourselves to the case $x_0 = 0$.

From the Feynman–Kac formula (A.31) follows. (A.32) follows from Theorem 2.1. To prove (A.33) we use the following integral equation:

$$g_t(x,y) = H_t(x,y) + \int_0^t \int dz \, H_{t-s}(x-z)[1 - 3\bar{m}^2(z)] g_s(z,y) \tag{A.35}$$

We define

$$f_t(d) = \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} dy\, |g_t(x,y) - g_t(x,y+d)|$$

From (A.35) we then have

$$f_t(d) \le c_1 \frac{d}{\sqrt{t}} + c_2 \int_0^t f_s(d) \tag{A.36}$$

From (A.36) the inequality (A.33) follows immediately for $t \in (0,1]$. For $t > 1$ we use the semigroup property getting that $f_t(d) \le c_0 f_1(d)$ with c_0 as in (A.32). (A.34) is based on the estimate

$$g_t(x,y) \le \frac{e^{2t}}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

which follows easily from the Feynman–Kac formula. □

A.5 Lemma.

Consider the set $\mathcal{B}_\epsilon^{(i)}$, $i = 2, 3$, defined in (3.37) and (3.38). Then, there are c_0 and c_1 positive constants such that

$$P^\epsilon\Big(\mathcal{B}_\epsilon^{(i)}\Big) \ge 1 - c_0 e^{-c_1 \epsilon^{-1/2}}, \qquad i = 2,3 \tag{A.37}$$

Proof.

Recall equations (3.5) for $\hat{Z}_t$, and (3.39) for $\hat{Z}_t^\star$, in terms of Z_t and $Z_t^\star$. Using (3.46), we obtain

$$\begin{aligned}
\hat{Z}_{t,x_0} - \tau_\Delta \hat{Z}^\star_{t,x_0^\star} &= Z_t - \tau_\Delta Z_t^\star + \int_0^t ds\, g_{t-s,x_0}(3\bar{m}^2_{x_0^\star} - 1)(Z_{t,x_0} - \tau_\Delta Z^\star_{t,x_0^\star}) \\
&= Z_t - \tau_\Delta Z_t^\star + \int_0^t ds\, g_{t-s,x_0} \mathbf{1}_{|y-x_0|\le 2\epsilon^{-1/10}}(3\bar{m}^2_{x_0^\star} - 1)(Z_{t,x_0} - \tau_\Delta Z^\star_{t,x_0^\star}) \\
&+ \int_0^t ds\, g_{t-s,x_0} \mathbf{1}_{|y-x_0|>2\epsilon^{-1/10}}(3\bar{m}^2_{x_0^\star} - 1)(Z_{t,x_0} - \tau_\Delta Z^\star_{t,x_0^\star}) \\
&\doteq Z_t - \tau_\Delta Z_t^\star + A_1(x,t) + A_2(x,t).
\end{aligned} \tag{A.38}$$

We will prove below that

$$P^\epsilon\Big(\sup_{\substack{0\le t\le t_\epsilon \\ |x-x_0|\le 2\epsilon^{-1/10}}} |Z_t - \tau_\Delta Z_t^\star| > e^{-\epsilon^{-1/50}} \Big) \le c_0 e^{-c_1\epsilon^{-1/2}} \tag{A.39}$$

The bound (A.37) for $i = 2$ follows immdiately from this inequality. Moreover, from (A.39) and (A.36), we can estimate

$$P^\epsilon\Big(\sup_{t\le t_\epsilon} \|A_1(x,t)\|_{x_0,\epsilon} > \frac{e^{-1/100}}{3} \Big) \le P^\epsilon\Big(\sup_{\substack{0\le t\le t_\epsilon \\ |x-x_0|\le 2\epsilon^{-1/10}}} |Z_t - \tau_\Delta Z_t^\star| > \frac{K e^{-\epsilon^{-1/100}}}{t_\epsilon} \Big) \le c_0 e^{-c_1\epsilon^{-1/2}} \tag{A.40}$$

Also, for A_2, we obtain

$$P^\epsilon\Big(\sup_{t\le t_\epsilon} \|A_2(x,t)\|_{x_0,\epsilon} > \frac{e^{-1/100}}{3} \Big) \le P^\epsilon\Big(\sup_{\substack{0\le t\le t_\epsilon \\ x\in \bar{T}_{\epsilon,\kappa}}} |Z_t - \tau_\Delta Z_t^\star| > \epsilon^{-1/2} \Big) \le b_0 e^{-b_1\epsilon^{-1/2}}$$

Then, let us prove (A.39), to conclude the proof of the Lemma. Recall that

$$
\begin{aligned}
&|Z_t - \tau_\Delta Z_t^\star| \leq \\
&\left| \int_0^t ds \int dy \left(\mathbf{1}_{\{y \in \mathcal{T}_{\epsilon,\kappa}\}} H_{t-s}^{(\epsilon)}(x,y) - \mathbf{1}_{\{|y-x_0| \leq 4\epsilon^{-1/10}, y+\Delta \in \mathcal{T}_{\epsilon,\kappa}\}} H_{t-s}^{(\epsilon)}(x+\Delta, y+\Delta) \right) \alpha(y,s) \right| \\
&+ \left| \int_0^t ds \int dy \, \mathbf{1}_{\{|y-x_0^\star| > 4\epsilon^{-1/10}; y \in \mathcal{T}_{\epsilon,\kappa}\}} \, H_{t-s}^{(\epsilon)}(x+\Delta, y) \, \bar{\alpha}(y,s) \right| \\
&\doteq |I_1(x,t)| + |I_2(x,t)|
\end{aligned}
$$

Both I_1 and I_2 are centered Gaussian processes, for which estimates like (A.8) and (A.9) are valid. Moreover, recalling that

$$
H_t^{(\epsilon)} = \sum_{j \in \mathbb{Z}} \left(H_t(x, y + 4j\epsilon^{-\kappa}) + H_t(x, 4j\epsilon^{-\kappa} + 2\epsilon^{-\kappa} - y) \right),
$$

it is not difficult to prove that

$$
(\sigma_i)^2 \doteq \sup_{\substack{0 \leq t \leq t_\epsilon \\ |x-x_0| \leq 2\epsilon^{-1/10}}} E\left(I_i(x,t)^2 \right) \leq e^{-c\epsilon^{-1/16}}
$$

Then, proceeding as in the proof of Lemma A.1 , (A.39) follows. □

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