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Vanishings for the cohomology of certain families of stable rank two vector bundles on P^3

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Introduction

We work over an algebraically closed field of characteristic zero.

One of the motivations of the present paper is provided by the following problem, which, as far as we know, is widely open (see [H2;9.17]):

determine the least integer $m_{c_1}(c_2)$ s.t. $h^1(E(k)) = 0$ if $k \geq m_{c_1}(c_2)$ for every stable, normalized, rank two vector bundle E on $\mathbf{P}^3$.

A similar problem was solved by Gurrola (see [G]) in the context of stable reflexive sheaves. Indeed he was able to show that, given a stable, normalized, rank two reflexive sheaf F with Chern classes c_1, c_2, c_3, there is a function $m(c_1, c_2, c_3)$ s.t. $h^1(F(k)) = 0$ if $k \geq m(c_1, c_2, c_3)$. Moreover that bound turned out to be sharp in the sense that it is realized by reflexive sheaves with maximal c_3. However, in the case $c_3 = 0$, that bound gives a vanishing which should be very far from being sharp for vector bundles. Indeed on one hand the function $m(c_1, c_2, c_3)$ found by Gurrola is quadratic in c_2, on the other hand it was conjectured by Chang ([C]) that it should hold:

$$m_0(c_2) \leq 2c_2 + 1 - (4c_2 + 5)^{\frac{1}{2}}$$

$$m_{-1}(c_2) \leq 2c_2 + \frac{1}{2} - (c_2 + \frac{1}{4})^{\frac{1}{2}}$$

In the paper [E], Ellia was able to improve in the linear part the vanishing by Gurrola but it seems to be very hard to prove the conjecture by Chang or, at least, a vanishing which is linear in c_2. Partial results are available when the spectrum of E has a particular shape or when c_2 is very small. In [H2;9.17] is proved that the conjecture holds for stable vector bundles with $c_1 = 0$, $c_2 = 2m + 1$ and maximum spectrum, in [E] for stable vector bundles with spectra (0^{c_2}), $(-1, 0^{c_2-2}, 1)$ and for stable vector bundles with $c_1 = 0$, $c_2 \leq 5$.

In the present paper we will proceed a little bit in this direction proving the conjecture for vector bundles with $c_1 = 0$ and $sp = (-m, .., -1, 0^2, ..., m)$ and with $c_1 = -1$ and maximum

spectrum. In the first two sections of the paper we will achieve the results, interesting in their own, of the complete characterization of such vector bundles (see theorems 1.1, 2.1). The vanishing will turn out as a corollary (see corollaries 1.4,2.7). Moreover we will show (see section 3) that the vanishing holds also for vector bundles with $c_1 = -1$ and $sp = (-1^a, 0^a)$. Finally we will prove the conjecture for $-1 \leq c_1 \leq 0$, $c_2 \leq 6$.

I am grateful to Ph. Ellia for suggesting me the problem and several improvements of the manuscript.

I) The family of stable rank two vector bundles with spectrum $(-m, .., -1, 0^2, 1, .., m)$

In this section we will prove the following

Theorem 1.1: Let E be a stable vector bundle with spectrum $-m$,$-m+1$,...-1,0^2,1, ... $m-1$,m. Then one of the following must happen:

a) $E(1)$ has a section whose zero locus is a curve Y' which is the disjoint union of a line D with a curve Y which is a double structure on a degree $m+1$ plane cuve and s.t. $\omega_{Y_2} = O_{Y_2}(-2)$.

b) $E(2)$ has a section whose zero locus is a curve Y which is the scheme theoretic union of two skew lines L_1 and L_2 with a curve Y_0 which is a double structure on a degree $m+2$ plane curve and s.t. $\omega_{Y_0} = O_{Y_0}(-\Delta)$, where $\Delta := Y_0 \cap (L_1 \cup L_2)$ and $deg\Delta = 4$;

The proof of the theorem will follow after several lemmas.

Lemma 1.2: Let E be as in 1.1, then E has an instable plane H of order m. Performing the reduction step with respect to H gives us E as an extension

$$0 \longrightarrow E' \longrightarrow E \longrightarrow I_{Z,H}(-m) \longrightarrow 0 \qquad (1.1)$$

where the Chern classes of E' are $(-1, m+2, m^2+2m+2)$.

Proof. From well known properties of the spectrum (see section 3 and references therein) we have $h^1(E(-1-m)) = 1$, $h^1(E(-m)) = 3$ therefore there exists a section $h \in H^0(O(1))$ such that $h \in Ker(H^0(O(1)) \otimes H^1(E(-m-1)) \to H^1(E(-m)))$. If H is the plane defined by $h = 0$ we have

$$0 \longrightarrow E(-m-1) \xrightarrow{.h} E(-m) \longrightarrow E_H(-m) \longrightarrow 0 \qquad (+).$$

The exact cohomology sequence shows that $h^0(E_H(-m)) \neq 0$ and H is instable of order m. The last part of the statement comes from [H2;9.1]. $\diamond$

Lemma 1.3: Let E' be the sheaf coming from the reduction step (1.1), then E' has an instable plane H' of oder m+1. Performing the reduction step with respect to H' we get

$$0 \longrightarrow E''(-1) \longrightarrow E' \longrightarrow I_{Z',H'}(-m-1) \longrightarrow 0 \qquad (1.2).$$

the Chern classes of E'' being $(0, 1, c_3'')$. Moreover the plane H' can be chosen to be the same of H arising in lemma 1.2.

Proof. Dualizing the reduction step sequence (1.1) we get ([H1;9.1])

$$0 \longrightarrow E \xrightarrow{f} E'(1) \longrightarrow O_H(m+1) \longrightarrow 0 \qquad (++)$$

The cohomology long exact sequence shows that $h^2(E'(m-2)) = h^2(E(m-3)) = 1$, $h^2(E'(m-3)) = h^2(E(m-4)) = 3$. Moreover the map f induces a morphism $\hat{f} : H^2_*(E) \to H^2_*(E'(1))$ so the following diagram commutes

$$\begin{array}{ccc} H^2(E(m-4)) & \xrightarrow{.h} & H^2(E(m-3)) \\ \downarrow \hat{f} & & \downarrow \hat{f} \\ H^2(E'(m-3)) & \xrightarrow{.h} & H^2(E'(m-2)) \end{array}$$

Since $h^2(E_H(m-3)) = h^0(E_H(-m)) = 1$ we have from (+) that the map multiplication by h in the top row is zero. Finally because the vertical rows are isomorphisms, we have that the bottom row is also zero and $h^2(E'_H(m-2)) = h^2(E'(m-2)) = 1$. $\diamond$

Proof of the theorem 1.1. Since E is stable, E'' is semi-stable. It follows ([H2;8.2]) that $c_3'' = 2$ or 0 (c_3'' must be even). If $c_3'' = 2$ since $\chi(E'') > 0$ and $h^2(E'') = 0$ [H2;8.2] we get $h^0(E'') \neq 0$.

If $c_3'' = 0$ then E'' is a vector bundle with $c_1 = 0$, $c_2 = 1$; since it is semi-stable, it is in fact stable.

In conclusion there are two possibilities for E'':

a) $c_3'' = 2$, E'' is semi-stable given by

$$0 \to O \to E" \to I_D \to 0$$

where D is a line;

b) $c_3'' = 0$, E'' is a null-correlation bundle.

a) From (1.1,1.2): $H^0(E'') \simeq H^0(E'(1)) \simeq H^0(E(1))$. Hence $E'(1)$ has a section which vanishes on a degree $m+2$ curve X' and $E(1)$ has a section which vanishes on a degree $2m+3$ curve Y' s.t. $\omega_{Y'} = O_{Y'}(-2)$. Moreover, since E'', E' and E are isomorphic outside H and D cannot be contained in H (otherwise $h^0(E'(1)) \geq 2$ but $h^0(E'(1)) = h^0(E'') = 1$), we have $Res_H(X') = D$ and $Res_H(Y') = X'$ where $Res_H(C)$ is the residual scheme of the curve C with respect to H. We conclude that Y' is the schematic union of D and a double structure on a plane curve (contained in H) of degree $m+1$.

b) This time: $H^0(E''(1)) \simeq H^0(E'(2)) \simeq H^0(E(2))$. Arguing as above we get that $E(2)$ has a section vanishing along a curve Y, such that $\omega_Y = O_Y$, $degY = 2m+6$, $Y = Y_0 \cup C$ where $C = L_1 \cup L_2$ is the union of two skew lines and where Y_0 is a double structure on a curve $P \subset H$, ($degP = m+2$). We may assume $L_i \not\subset H$ ($E''(1)$ is globally generated). Moreover since $\omega_Y = O_Y$, L_i intersects Y_0 in a subscheme of length two. This means that L_i is contained in $T_{x_i}(Y_0)$ the Zariski tangent plane to Y_0 at the point $x_i = L_i \cap P$. Set $\Delta := C \cap Y_0$ and $supp\Delta := x_1 \cup x_2$. Let Y_0 be given by the Ferrand

construction

$$0 \longrightarrow I_{Y_0} \longrightarrow I_P \longrightarrow L \longrightarrow 0$$

where L is an invertible sheaf on P. Restricting to P the sequence

$$0 \longrightarrow \omega_{Y_0} \oplus \omega_C \longrightarrow \omega_Y \simeq O_Y \longrightarrow \omega_\Delta \longrightarrow 0 \qquad (*)$$

we find $\omega_{Y_0}|_P \simeq O_P(-x_1 - x_2)$. Since $\omega_{Y_0}|_P \simeq \omega_P \otimes L^{-1}$ [B] we deduce $L \simeq \omega_P(x_1 + x_2)$. In particular $h^1(L) = 0$ which implies that the natural restriction map from $Pic(Y_0)$ to $Pic(P)$ is injective [B]; it follows $\omega_{Y_0} = O_{Y_0}(-\Delta)$.

To conclude the proof we need to show that cases a) and b) are effective. For the case a) it is suffices to take the disjoint union of a line with a suitable double structure on a plane curve of degree $m+1$ (see [HR] 2.10, 2.11).

b) Let $P \subset H$ be a smooth curve of degree $m+2$. Let x_1, x_2 denote two points of P. Let $L := \omega_P(x_1 + x_2)$. Observe that $N_P \otimes L$ is globally generated. Therefore taking a suitable surjection $N_P^* L$ we can construct a double structure, Y_0, on P s.t. $I_{P,Y_0} \simeq L$. We may also assume that the embedded Zariski tangent plane to Y_0 at x_i is different from H (otherwise Y_0 would be a plane curve but this is impossible because $h^1(I_{Y_0}(-1)) = h^0(\omega_P(x_1 + x_2) \otimes O(-1)) \neq 0$). Let L_i be a line intersecting P at x_i and such that: $L_i \not\subset H$, $L_i \subset T_{x_i}Y_0$, $L_1 \cap L_2 = \Phi$. We define Y as the scheme theoretic union of Y_0 and $C := L_1 \cup L_2$.

To construct the desired vector bundle E we have to show

i) ω_Y has a section which generates almost everywhere;

ii) $h^1(I_Y(-m+1)) \neq 0$.

Indeed Y is a locally Cohen Macaulay and generally locally complete intersection curve, thus a section as in i) will yield an extension: $0 \to O \to E(2) \to I_Y(4) \to 0$ with E a rank two reflexive sheaf. Since $p_a(Y) = p_a(Y_0) + 2$ (because $L_i \subset T_{x_i}Y_0$) we easily see that $c_3(E) = 2p_a(Y) - 2 = 0$ and E is a vector bundle with the required Chern classes.

Finally ii) is equivalent to $h^1(E(-m-1)) \neq 0$. This condition implies $sp(E) = (-m, ..., -1, 0^2, 1, ...$

i) From $h^1(L) = h^1(\omega_P(x_1 + x_2)) = 0$ we get the injectivity of the restriction map from $Pic(Y_0)$ to $Pic(P)$. Moreover $\omega_{Y_0}(\Delta) \mid_P \simeq L^{-1} \otimes \omega_P \otimes O(x_1 + x_2) \simeq O_P$ implies $\omega_{Y_0}(\Delta) \simeq O_{Y_0}$ hence a generic section of $\omega_{Y_0}(\Delta) \simeq \omega_Y \mid_{Y_0}$ generates everywhere (on Y_0). Since $\omega_Y \mid_C \simeq \omega_C(\Delta) \simeq O_{L_1} \oplus O_{L_2}$ we see that we can find sections of $\omega_Y \mid_{Y_0}$ and $\omega_Y \mid_C$ which glue to give a section of ω_Y generating almost everywhere (in fact everywhere).

ii) To prove that $h^1(I_Y(-m+1)) \neq 0$ we proceed as follows. Firstly we have $h^1(I_Y(-m+1)) = h^0(O_Y(-m+1)) = h^1(O_Y(m-1))$. Moreover from the sequence $0 \to I_{Y_0,Y} \to O_Y \to O_{Y_0} \to 0$ where $I_{Y_0,Y} \simeq O_{L_1}(-2) \oplus O_{L_2}(-2)$, we see that $h^1(O_Y(m-1)) = h^1(O_{Y_0}(m-1))$. Finally from the cohomology exact sequence associated to $0 \to L(m-1) \to O_{Y_0}(m-1) \to O_P(m-1) \to 0$ and from $h^1(L(m-1)) = h^0(L^*(-m+1) \otimes \omega_P) = 0$ we get $h^1(O_Y(m-1)) = h^1(O_{Y_0}(m-1)) = h^1(O_P(m-1)) = h^0(\omega_P(1-m)) = h^0(O_P) = 1$. $\diamond$

Corollary 1.4: Let E be as in theorem 1.1. Then $h^1(E(p)) = 0$ if $p \geq \frac{3c_2}{2} - 1$

Proof. a) The sequence (1.2) reads:

$$0 \longrightarrow E''(-1) \longrightarrow E' \longrightarrow I_{Z',H}(-m-1) \longrightarrow 0 \qquad (\tilde{.})$$

with $lengthZ' = \frac{c_3(E'')}{2} = 1$. From the defining sequence of E'' we see that $h^1(E''(l)) = 0$ if $l \in \mathbf{Z}$ thus $h^1(E'(k)) = 0$ if $k \geq m+2$. Dualizing $(\tilde{.})$ we have (see the proof of theorem 9.1 of [H2]):

$$0 \longrightarrow E'(1) \longrightarrow E''(1) \longrightarrow I_{W,H}(m+2) \longrightarrow 0 \qquad (\tilde{.})$$

where W is a subscheme of Z (see (1.1)) and $lengthW = lengthZ - 1$. . From the properties of the spectrum [H2;7.1-5] we have ($sp(E') := \{k_i\}$): $h^2(E'(l)) = h^1(\oplus O(k_i + l + 1)) = 0$ if $l \geq c_2(E') - 2 = m$ and from $(\tilde{.})$ we get $h^1(I_{W,H}(r)) = 0$ if $r \geq 2m+1$. This implies $h^1(I_{Z,H}(l)) = 0$ if $l \geq 2m+2$. Indeed from the sequence $0 \to I_{Z,H} \to I_{W,H} \to I_{W,Z} \to 0$ we get $h^1(I_{Z,H}(2m+1)) \leq h^0(I_{W,Z}(2m+1))$ and from $0 \to I_{W,Z} \to O_{Z,H} \to O_{W,H} \to 0$

we find $h^0(I_{W,Z}(2m+1)) = 1$ thus $h^1(I_{Z,H}(2m+1)) \leq 1$. As the Hilbert function of Z is strictly increasing until it attains $lengthZ$, we conclude $h^1(I_{Z,H}(l)) = 0$ if $l \geq 2m+2$. Finally from the sequence (1.1) we find $h^1(E(p)) = 0$ if $p \geq 3m+2 = \frac{3c_2}{2} - 1$.

b) This time the sequence (1.2) reads:

$$0 \longrightarrow E''(-1) \longrightarrow E' \longrightarrow O_H(-m-1) \longrightarrow 0 \qquad (\circ)$$

As E'' is a null-correlation bundle we have $h^1(E''(l)) = 0$ if $l \geq c_2(E'') - 1 = 0$ and $h^1(E'(r)) = 0$ if $r \geq 1$. Dualizing $(\circ)$ we get:

$$0 \longrightarrow E'(1) \longrightarrow E''(1) \longrightarrow I_{Z,H}(m+2) \longrightarrow 0$$

where Z is the same scheme of (1.1) ([H2;9.1]). Arguing as above we find $h^1(E(p)) = 0$ if $p \geq \frac{3c_2}{2} - 2$. $\diamondsuit$

II) The family of stable rank two vector bundles with spectrum $(-m+1, ..., m-2)$

In this section we will be concerned with the family of rank two stable vector bundles on $\mathbf{P}^3$ with $c_1 = -1$, $c_2 = 2m-2$ and the maximum spectrum $(-m+1, ..., m-2)$. As our situation is very similar to that of rank two stable vector bundle with $c_1 = 0$ and maximum spectrum, we will be sketchy and follow sec. 9.3-17 of the paper [H2].

What we are going to prove is the following

Theorem 2.1: For any $m \geq 2$ the stable vector bundles with Chern classes $(-1, 2m-2)$ and maximum spectrum, form an irreducible, non singular, rational family of dimension $3m^2 + m - 1$. Moreover, given any vector bundle E in the family, then the following are true:

1)$h^0(E(1)) = 1$ and if $s \in H^0(E(1))$, then $(s)_0$ describes a degreee $2m-2$ curve Y, which is a multiplicity two structure on a degree $m-1$ plane curve;

2) $h^0(E(m)) \geq m(m+1)2-1$ and if $s \in H^0(E(m))$ is general, then it vanishes in codimension 2 and its zero locus is a curve C which is the disjoint union of a degree $2m-2$ plane curve with a complete intersection (m,m-1). $\diamondsuit$

We begin proving the following result:

Proposition 2.2: For any $m \geq 2$, the moduli space of rank 2 semistable reflexive sheaves with Chern classes $(0, r, r^2+r)$ (c_3 is maximal for semistable sheaves [H1;8.2]) is irreducible and non singular of dimension r^2+4r+2.

The proof of proposition 2.2 will follow after several lemmas.

Lemma 2.3: Let F be any sheaf as in 2.2, then F has an instable plane H of order r. The reduction step with respect to H gives us F as an extension

$$0 \longrightarrow O \oplus O(-1) \longrightarrow F \longrightarrow I_{Y,H} \longrightarrow 0 \qquad (2.1)$$

where Y is a degree r plane curve.

Proof. The proof of the existence of H goes along the same lines of lemma 1.2. Reduction step gives

$$0 \longrightarrow F' \longrightarrow F \longrightarrow I_{W,H}(-r) \longrightarrow 0 \qquad (\hat{o})$$

The Chern classes of F' are $(-1, 0, c_3)$. From $(\hat{o})$ and the semistability of F, we see that F' has order of instability -1 (see[S]). Proposition 3.8 of the paper [S] says $c_3 \leq c_2(c_2+1) = 0$, thus F' is a vector bundle. Moreover as $h^0(F') \neq 0, h^0(F'(-1)) = 0$ we see that F' is given by the extension

$$0 \longrightarrow O \longrightarrow F' \longrightarrow I_X(-1) \longrightarrow 0 \qquad (\tilde{o})$$

with X a curve with $degX = c_2' = 0$. But then $X = \Phi$, $I_X(-1) = O(-1)$ and $(\tilde{o})$ splits. Finally putting $s := lengthO_W$, we have ([H2;9.1]) $2s = c_3' = 0$, $s = 0$, $W = \Phi$ and $(\hat{o})$ becomes the extension of the statement. $\diamondsuit$

Lemma 2.4: Every semistable rank 2 reflexive sheaf F with Chern classes $(0, r, r^2+r)$ is given by an extension $0 \to O \to F \to I_Y \to 0$ where Y is a degree r plane curve. Conversely every such an extension gives a reflexive sheaf with the right Chern classes. This construction gives us a family which is irreducible of dimension $r^2 + 4r + 2$.

Proof. [H1;4.1] implies that the extension of the statement gives a semistable reflexive sheaf with Chern classes $(0, r, r^2+r)$. Conversely lemma 2.3 implies F to be given by (2.1) where Y was a degree r plane curve and so F has a section whose zero locus is Y. Dualizing the reduction step sequence (2.1) we get $0 \to F \to O \oplus O(1) \to I_{Z,H}(r+1) \to 0$, where Z is a zero dimensional scheme s.t. $lengthZ = r^2 + r$ ([H2;9.1]). To prove the statement concerning the family it is sufficient to note that giving F is equivalent to giving a plane H and two plane curves without common components with degrees r and $r+1$. $\diamond$

Proof of the proposition 2.2. To complete the proof we are only concerned with the smoothness. What we have to check is the equality between the dimension of the family above and the dimension of the Zariski tangent space to the moduli space at a sheaf F. The Zariski tangent space at F is $Ext^1(F,F)$ and we have to show $dimExt^1(F,F) = r^2+4r+2$ if F is given by (2.1). To calculate $dimExt^1(F,F)$ one can apply the functor $Hom(\cdot, F)$ to (2.1)

$$0 \to Hom(F,F) \to Hom(O(-1),F) \to Ext^1(O_H(-r),F) \to Ext^1(F,F) \to 0$$

$Ext^1(O_H(-r),F)$ can be found applying the functor $Hom(\cdot, F)$ to the resolution sequence of $O_H(r)$, $Ext^1(O(-i),F) = H^1(F(i))$ and $H^0(F(i))$ are easily calculated from (2.1). To find $dimHom(F,F)$ we can proceed as follows. Firstly we have $dimHom(F,F) \geq 2$ because F is not stable. Then applying $Hom(F,\cdot)$ to (2.1) we find $dimHom(F,F) \leq 1 + dimHom(F, O_H(-r))$ and applying $Hom(\cdot, O_H(-r))$ to (2.1) we get

$dimHom(F, O_H(-r)) = 1$ so $dimHom(F,F) = 2$ and the proposition follows. $\diamond$

Taking into account 2.3, 2.4, the proof of the following lemmas goes exactly as the

one of lemmas 9.11-12 of [H2]:

Lemma 2.5: Let E be as the statement of the theorem. Then E has an unique unstable plane H of order $m-1$. Performing the reduction step with respect to H gives

$$0 \longrightarrow E'(-1) \longrightarrow E \longrightarrow I_{Z,H}(1-m) \longrightarrow 0 \qquad (2.2)$$

where E' is a stable reflexive sheaf with Chern classes $(0, m-1, m^2-m)$; H is also the unique instable plane of order $m-1$ of E'. $\diamondsuit$

Lemma 2.6: The restriction of E and E' to H is described by

$$0 \longrightarrow O_H(m-2) \longrightarrow E_H \longrightarrow I_{Z,H}(1-m) \longrightarrow 0$$

and $E'_H \simeq I_{Z,H}(m-1) \oplus O_H(1-m)$. $\diamondsuit$

Proof of the theorem 2.1. Dualizing the sequence (2.2) we get a surjection v :E'$_H \simeq I_{Z,H}(m-1) \oplus O_H(1-m) \to O_H(m-1) \to 0$ where v is given by a form f of degree $2m-2$ and not vanishing on Z on the second factor and a scalar on the first. With a simple counting the parameters we conclude the statement concerning the dimension of the family (see [H2;9.13-14]).

Proof of 1). We have $h^0(E(1)) = h^0(E') = h^0(O(-1) \oplus O) = 1$ as follows from the sequences of lemmas 2.3 and 2.5. Giving $s \in H^0(E(1))$ is equivalent to giving $E(1)$ as an extension

$$0 \longrightarrow O \longrightarrow E(1) \longrightarrow I_Y(1) \longrightarrow 0$$

where Y is a degree $2m-2$ scheme of codimension 2 s.t. $\omega_Y = O_Y(-3)$. As the sheaves $E(1)$, E' and $O \oplus O(-1)$ are isomorphic outside the unstable plane H, it is clear that the support of Y is in H. From the sequence of lemma 2.6 we have $H^0(E_H(1)) = H^0(O_H(m-1))$, so the restriction of Y to H is a plane curve Y_0 of degree $m-1$. Because of $\omega_Y = O_Y(-3)$, no component of Y can be contained in H and so the conclusion comes from the degrees.

Proof of 2). From the sequences of Lemmas 2.1 and 2.5 we get $H^0(E(l)) \simeq H^0(E'(l-1)) \simeq H^0(O(l-2) \oplus O(l-1))$ if $l < m$ and $h^0(E(m)) > h^0(O(m-2) \oplus O(m-1))$. Therefore $E(m)$ has sections vanishing in codimension 2 . Let s be such a section. Its zero locus will be a degree $m^2 + m - 2$ curve C s.t. $\omega_C = O_C(2m-5)$. From the sequence of lemma 2.6 we see that the restriction s_H of s to H is the product of the section of $E_H(-m+2)$ with a form of degree $m-2$ so its zero locus is the union of a degree $2m-2$ plane curve P with Z. Arguing as in the proof of lemma 9.16 of the paper [H1], it can be easily seen that $Z \cap P = \Phi$. On the other hand the sequence

$$0 \longrightarrow E' \longrightarrow O \oplus O(-1) \longrightarrow I_{Z,H} \longrightarrow 0$$

obtained dualizing the reduction step (2.1) (see [H1;9.1]) shows that, outside H, s is a complete intersection $(m, m-1)$. $\diamond$

Corollary 2.7: Let E be as in theorem 2.1. Then $h^1(E(p)) = 0$ if $p \geq \frac{3c_2}{2} - 1$

Proof. As the the proof is very similar to that of corollary 1.4 we are going to be very sketchy.

Let E' be the sheaf arising from the sequence (2.2). Dualizing the defining sequence of E' (see (2.1))

$$0 \longrightarrow O \oplus O(-1) \longrightarrow E' \longrightarrow O_H(-m+1) \longrightarrow 0 \qquad (\hat{\cdot})$$

we find ([H2;9.1])

$$0 \longrightarrow E' \longrightarrow O \oplus O(1) \longrightarrow I_{Z,H}(m) \longrightarrow 0 \qquad (\circ)$$

where Z is the same scheme of (2.2). From $(\hat{\cdot})$ we see $h^1(E'(l)) = 0$ if $l \in \mathbf{Z}$. From the properties of the spectrum we find $h^2(E'(l)) = h^1(\oplus O(k_i + l + 1)) = 0$ if $l \geq c_2(E') - 2 = m-3$ ($sp(E') := \{k_i\}$). Finally from $(\circ)$ and (2.2) we conclude $h^1(I_{Z,H}(r)) = 0$ if $r \geq 2m-3$ and $h^1(E(p)) = 0$ if $p \geq 3m - 4 = \frac{3c_2}{2} - 1$. $\diamond$

III) A vanishing theorem

We start recalling the following facts (see [E],[HR] and ref. therein).

Proposition 3.1: Let E a rank 2 vector bundle on $\mathbf{P}^3$, with Chern classes c_1, c_2, $-1 \leq c_1 \leq 0$. Let M denote the graded module $H^1(E(*))$. If

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \xrightarrow{\xi} L_0 \to M \to 0$$

is a minimal resolution of M, then $rk(L_1) = 2rk(L_0) + 2$, and there exists an isomorphism $\phi : L_1^*(c_1) \to L_1$ s.t. $\xi\phi\xi^*(c_1) = 0$ and which induces a minimal monad for E:

$$0 \to \tilde{L}_0^*(c_1) \xrightarrow{\phi\xi^*(c_1)} \tilde{L}_1 \xrightarrow{\xi} \tilde{L}_0 \to 0.$$

Moreover $\tilde{L}_0^*(c_1)$ is direct summand in L_2 ([HR;3.2]). $\diamond$

In the sequel we will put: $\tilde{L}_0 = \oplus_{1\leq i\leq r} O(-\alpha_i)$, $\tilde{L}_1 = \oplus_{1\leq j\leq 2r+2} O(-\beta_j)$, $\alpha_1 \leq ... \leq \alpha_r$; $\beta_1 \leq ... \leq \beta_{2r+2}$.

If moreover E is such that $h^0(E(-1)) = 0$, we denote by $sp(E) = \{k_i\}_{1\leq i\leq c_2}$, the spectrum of E. Furthermore we set $k^+ := max\{k_i\}$, $s(n) := \#\{k_i | k_i = n\}$. Finally we denote by $r(n)$ the number of minimal generators of degree n of the module M. Then the following properties hold ($m_i := dim M_i$):

1) $r(m) = 0$, $m \geq k^+ - c_1$, $\alpha_r \leq k^+ - 1 - c_1$;

2) $r(m) = 0$, $m \leq -2 - k^+$, $\alpha_1 = -1 - k^+$, $s(k^+) = \#\{i | \alpha_i = -1 - k^+\}$;

3) $\beta_{2r+2} \geq \alpha_r + 1$, $\beta_1 > \alpha_1$;

4)$\{-\beta_j\} = \{\beta_j + c_1\}$;

5) $m_{-1-i} = \sum_{j\geq i} s(j)(j - i + 1)$, $i \geq 0$;

6) $s(i) - 2\sum_{j\geq i+1} s(j) \leq r(-i-1) \leq s(i) - 1$ for $0 \leq i < k^+$, $r(-1-k^+) = s(k)$.

Lemma 3.2: Using the above notations, suppose L_0 has r summands with degrees $\leq l$. Then L_1 must contain at least $r + 3$ summands with degrees $\geq 1 - l$ if $c_1 = 0$, $\geq -l$ if $c_1 = -1$ (see [HR] prop 3.3). $\diamond$

Lemma 3.3: Using the above notations, $h^1(E(m)) = 0$, if $m \geq (\beta_r - \alpha_1) + (\beta_{r+1} - \alpha_2) + ... + (\beta_{2r-1} - \alpha_r) + (\beta_{2r} - \alpha_1) + (\beta_{2r+1} - \alpha_1) + (\beta_{2r+2} - 3)$ (see [E] III.5). $\diamondsuit$

The aim of this section is to prove the following improvement of Corollary IV.6 of [E]:

Proposition 3.4: Let E be a stable vector bundle with $c_2 \leq 6$. Then $h^1(E(m)) = 0$ if $m \geq 2c_2 + 2 - (4c_2 + 5)^{1/2}$ and $c_1 = 0$ or $m \geq 2c_2 + 1/2 - (c_2 + 1/4)^{1/2}$ and $c_1 = -1$.

Let us start by proving three lemmas

Lemma 3.5: Let E be a stable vector bundle with Chern classes (-1,6) and spectrum $(-2, -1^2, 0^2, 1)$. Then E is the cohomology of a monad of one of the following types:

$$0 \longrightarrow O(-3) \longrightarrow 2O(-1) \oplus 2O \longrightarrow O(2) \longrightarrow 0$$

$$0 \longrightarrow O(-3) \oplus O(-2) \longrightarrow O(-2) \oplus 2O(-1) \oplus 2O \oplus O(1) \longrightarrow O(1) \oplus O(2) \longrightarrow 0$$

In particular, applying lemma 3.3, we get $h^1(E(m)) = 0$ if $m \geq 8$.

Proof. From the properties of the spectrum we find $m_{-2} = 1$, $m_{-1} = 4$, $r(-2) = 1$, $0 \leq r(-1) \leq 1$.

a) $r(-1) = 0$: none of the α's is equal to -1, moreover the generator in M_{-2} has no relations in degree -1 so none of the β's is equal to -1. By the symmetry property 4) we have $\{\beta_i\} = (0^{a+2}, 1^{a+2})$. From the constraint on the ranks and the property 3) above we have $\{\alpha_i\} = (-2, 0^a)$. Finally lemma 3.2 with $l = 0$ implies $a = 0$;

b) $r(-1) = 1$: now one of the α's is equal to -1 and the generator of M_{-2} has one relation in degree -1 thus property 4) implies $\{\beta_i\} = (-1, 0^c, 1^c, 2)$. By property 3) we have $\{\alpha_i\} = (-2, -1, 0^a, 1^b)$ and from the constraint on the ranks $c = a + b + 2$. Again lemma 3.2 with $l = -1$ gives $b = 0$ and with $l = 0$ gives $a = 0$. $\diamondsuit$

Lemma 3.6: Let E be a stable vector bundle with Chern classes $(-1, c_2)$ and spec-

trum $(-1^{c_2/2}, 0^{c_2/2})$. Then E is the cohomology of a monad of the following type:

$$0 \to (c_2/2)O(-2) \oplus uO(-1) \to (c_2/2+1+u)O(-1) \oplus (c_2/2+1+u)O \to uO \oplus (c_2/2)O(1) \to 0$$

where $u \leq c_2 - 5$. In particular, applying lemma 3.3, we get $h^1(E(m)) = 0$ if $m \geq max(c_2 - 5, c_2/2)$.

Proof. We have $m_{-1} = c_2/2$, $r(-1) = c_2/2$. From the properties 3), 4) we find $\{\beta_i\} = (0^{c_2/2+1+u}, 1^{c_2/2+1+u})$, $\{\alpha_i\} = (-1^{c_2/2}, 0^u)$. Let us consider the following sequence

$$0 \longrightarrow K \longrightarrow (c_2/2+1+u)O \xrightarrow{j} (c_2/2)O(1) \qquad (3.1)$$

Suppose now the map j to be generically surjective and define $G := coker(uO(-1) \to (c_2/2+1+u)O)$. From the Eagon-Northcott complex associated to $\bar{j} : G \to (c_2/2)O(1)$ ($\bar{j}$ is the map induced by j)

$$0 \longrightarrow O(u) \longrightarrow G \otimes O(\frac{c_2}{2}) \xrightarrow{\bar{j} \otimes det\bar{j}} (c_2/2)O(1) \otimes O(\frac{c_2}{2}) \longrightarrow 0$$

we infer $u \leq c_2/2$ otherwise G would have a section contained in the kernel of $\bar{j}$ and this cannot happen as follows from the sequence $0 \to (c_2/2)O(-2) \to Ker\bar{j} \oplus \mathcal{F} \to E \to 0$ ($\mathcal{F} := Ker((\frac{c_2}{2}+1+u)O(-1) \to \tilde{L}_0)$), recalling that $h^0(E) = 0$ by stability. Suppose now the map j of (3.1) to be not generically surjective. Then lemma 3.7 of [HR] implies $h^0(K(1)) \geq 4(c_2/2+1+u) - 10(c_2/2-1)$ and $h^0(E(1)) \geq 4(c_2/2+1+u) - 10(c_2/2-1) - u$. But $h^0(E(1)) \leq 1$ otherwise the zero locus of a section of $E(1)$ would be a plane curve Y with $\omega_Y = O_Y(-3)$ which is impossible. This gives $u \leq c_2 - 5$. $\diamond$

Lemma 3.7: Let E be a stable vector bundle with Chern classes $(-1, 6)$ and spectrum $(-2^2, -1, 0, 1^2)$. Then E is the cohomology of a monad of the following type:

$$0 \longrightarrow 2O(-3) \longrightarrow 3O(-2) \oplus 3O(1) \longrightarrow 2O(2) \longrightarrow 0$$

In particular, applying lemma 3.3, we get $h^1(E(m)) = 0$ if $m \geq 4$.

Proof. We have $m_{-2} = 2$, $m_{-1} = 5$, $r(-1) = 0$. Arguing as above we find $\{\beta_i\} = (-1^3, 0^{a+b}, 1^{a+b}, 2^3)$, $\{\alpha_i\} = (-2^2, 0^a, 1^b)$. From lemma 3.2 with $l = -1$ we get $b = 0$. Let us consider the following sequence

$$0 \longrightarrow K \longrightarrow aO \oplus 3O(1) \xrightarrow{k} 2O(2) \qquad (3.2)$$

The map k cannot have rank < 2 otherwise lemma 3.7 of [HR] would imply $h^0(E) \geq h^0(K) \geq a + 12 - 10 = a + 2$. Setting $G := coker(aO(-1) \to aO \oplus 3O(1))$ we have from the Eagon-Northcott complex associated to $\overline{k} : G \to 2O(2)$ ($\overline{k}$ is the map induced by k)

$$0 \longrightarrow O(a-1) \longrightarrow G \longrightarrow 2O(2) \longrightarrow 0$$

Again, we must have $a - 1 < 0$ otherwise E would have a section. $\diamondsuit$

Proof of proposition 3.4. a) $c_1 = 0$: all the allowed monads are listed in tab. 5.3 of the paper [HR]. For all but the 6(4) it is sufficient to apply lemma 3.3 to the monad. The case 6(4) is treated in corollary 1.4.

b) $c_1 = -1$: all the possibilities are covered by lemmas 3.5-7 and corollary 2.7. $\diamondsuit$

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