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Bifurcation diagram of a one-parameter family of one-dimensional nonlinear dispersive waves

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Tamara Grava

SISSA, Via Beirut 2/4, I-34014 Trieste, Italy
E–mail: grava@sissa.it

Abstract

The KdV equation with small dispersion is a model for the formation and propagation of dispersive shock waves. Dispersive shock waves are characterized by the appearance of modulated oscillations nearby the breaking point. The modulation in time and space of the amplitude, the frequencies and the wave-numbers of these oscillations is described by the g-phase Whitham equations. We study the initial value problem of the g-phase Whitham equations for a one-parameter family of monotone decreasing initial data. We use a variational principle for the g-phase Whitham equations recently proposed by Dubrovin: the minimizer of a functional on a certain infinite-dimensional space formally solves the initial value problem for each point of the (x, t) plane. For each value of the parameter of the initial data, we study the number of phases involved in the solution of the initial value problem and we classify the topological type of bifurcation diagram of the genus $g(x, t)$.

1 Introduction

The Korteweg de Vries (KdV) equation

$$\begin{cases} u_t + 6uu_x + \epsilon^2 u_{xxx} = 0 \\ \\ u(x, t = 0, \epsilon) = u_0(x) \end{cases} \tag{1}$$

with a small parameter ϵ is a model for the formation and propagation of dispersive shock waves in one dimension.

As $\epsilon \to 0$ equation (1) becomes the Burgers equation

$$\begin{cases} u_t + 6uu_x = 0 \\ \\ u(x, 0) = u_0(x) \end{cases} \tag{2}$$

whose solution $u(x, t)$ is given in implicit form by the method of characteristics

$$\begin{cases} u(x, t) = u_0(\xi) \\ x = 6t\, u_0(\xi) + \xi \end{cases} \tag{3}$$

(here ξ is a parameter). Taking the x-derivative of both equations (3) we obtain

$$u_x = \frac{1}{6t + (u_0'(\xi))^{-1}}\,. \tag{4}$$

It can be seen that, if the initial data is decreasing somewhere, there always exists a time $t_0 > 0$,

$$t_0 = \frac{1}{6} \min_{\xi \in \mathbb{R}} \left[-\left(u_0'(\xi)\right)^{-1} \mid u_0'(\xi) < 0 \right]$$

for which $u_x \to \infty$. The solution $u(x, t)$ of equation (2) is globally well defined up to the point of gradient catastrophe (x_0, t_0), where an infinite derivative develops.

The solution $u(x, t, \epsilon)$ of equation (1) in the limit $\epsilon \to 0$ has first been investigated by Lax and Levermore [LL] and Venakides [V]. Lax and Levermore used inverse scattering method to determine the weak limit $\tilde{u}(x, t)$ of the solution $u(x, t, \epsilon)$ of the Cauchy

problem (1) as $\epsilon \to 0$. They considered only initial data with one positive hump which tends to zero sufficiently fast as $\mid x \mid \to \infty$. Venakides later considered more general types of boundary conditions at $\mid x \mid \to \infty$. The weak limit is characterized by a variational problem with constrains and they showed that the solution to this variational problem is unique. The solution of the variational problem can be reduced to the solution of a Riemann-Hilbert problem. Function theoretic methods are then used to solve the Riemann-Hilbert problem. They showed that the limit of the solution $u(x,t,\epsilon)$ of (1) as $\epsilon \to 0$ exists strongly and satisfies the Burgers equation (zero-phase averaged equation) until the breaking time t_0 for the Burgers equation.

At later times, only a weak limit of the solution $u(x,t,\epsilon)$ of (1) exists as $\epsilon \to 0$. After the time of gradient catastrophe, the solution can be approximately described by an oscillatory wave whose wavenumbers, frequencies and amplitude are slowly varying function of time and space. (The idea and first example of such an approximate description were proposed by physicists Gurevich and Pitaevski [GP].)

The slow modulation of wavenumbers, frequencies and amplitude parameters can be described by Whitham equations as follows. For almost all (x,t) there exists an integer g such that the weak limit is determined by a $(2g+1)$-dimensional real valued function $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$ with the ordering condition

$$u_1 > u_2 > \ldots > u_{2g+1}$$

and satisfying the first order quasi-linear PDEs of the form:

$$\frac{\partial u_i}{\partial t} + \lambda_i(\vec{u})\frac{\partial u_i}{\partial x} = 0\,, \quad i = 1, ..., 2g+1\,, \quad g \geq 0\,, \tag{5}$$

where the λ_i's given in the next section, are expressed via certain complete hyperelliptic integrals on the Riemann surface of genus g

$$\Gamma_g := \left\{\mu^2 = (r-u_1)(r-u_2)\ldots(r-u_{2g+1})\right\}\,.$$

The weak limit $\tilde{u}(x,t)$ is given in term of the vector $\vec{u}$ by the formula

$$\tilde{u}(x,t) = \sum_{i=1}^{2g+1} u_i + 2\alpha_1(\vec{u})$$

where the form of the function $\alpha_1(\vec{u})$ will also be given in the next section.

The solution $u(x,t,\epsilon)$ can be approximately described by the formula

$$u(x,t,\epsilon) = U\left(\frac{S_1(x,t)}{\epsilon}, \ldots, \frac{S_g(x,t)}{\epsilon}; u_1(x,t), \ldots, u_{2g+1}(x,t)\right),$$

where the functions $S_j(x,t)$ satisfy the equations

$$\frac{\partial S_j}{\partial x} = k_j(\vec{u}(x,t)), \quad \frac{\partial S_j}{\partial t} = \omega_j(\vec{u}(x,t)), \quad j = 1, \ldots, g,$$

and the formula

$$u(x,t) = U\left(k_1 x + \omega_1 t + \phi_1, \ldots, k_g x + \omega_g t + \phi_g; u_1, \ldots, u_{2g+1}\right)$$

for constant values of the parameters $u_1, \ldots, u_{2g+1}$, $k_j = k_j(\vec{u})$ and $\omega_j = \omega_j(\vec{u})$ and for arbitrary ϕ_j, $j = 1, \ldots, g$, gives the family of the so called g-gap exact solutions of KdV.

We recall that, in this formula, the wavenumbers $k_j = k_j(\vec{u})$ and the frequencies $\omega_j = \omega_j(\vec{u})$ are certain hyperelliptic integrals of genus g; the 2π-periodic w.r.t $\phi_1, \ldots, \phi_g$ function $U(\phi_1, \ldots, \phi_g; u_1, \ldots, u_{2g+1})$ can be expressed via theta-functions (see, e.g., [D3]).

For $g = 0$ equation (5) turns out to be the Burgers equation. When $g = 1$, equations (5) are identical to the Whitham's modulation equations for periodic cnoidal waves [W], hence (5) are also referred to as Whitham equations. The algebraic geometric description of these equations for $g > 1$, was first derived by Flaschka Forest and McLaughlin [FFM] applying the Whitham averaging procedure to the family of g-gap quasi-periodic solutions of KdV. Hence these equations are also referred to as g-phase averaged equations. Observe that for $g = 0$ the initial data of equation (5) is the same

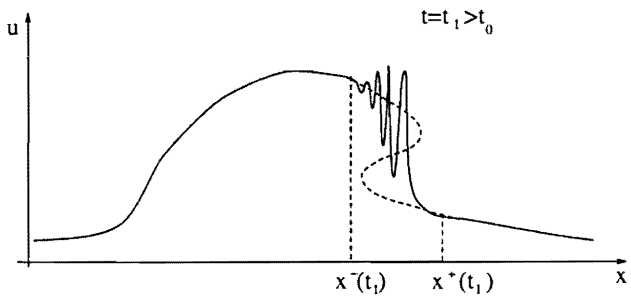


Figure 1: The dashed line represents the formal solution of the Burger equation after the time of gradient catastrophe $t = t_0$. The oscillations on the picture are close to a modulated periodic wave.

as the KdV initial data, for $g > 0$ instead, one has a free boundary problem (see below). The global picture of the solution is as follows. For $t \leq t_0$ where t_0 is the time of gradient catastrophe for the Burgers equation one finds $g = 0$. The solution $u(x, t, \epsilon)$ converges strongly to $u(x, t)$ which is the solution of the Burgers equation (2). After the time of gradient catastrophe the formal solution of the Burgers equation becomes multivalued. Oscillations appear near the point of gradient catastrophe as shown in Figure 1. In this oscillation zone the solution of the initial value problem is described

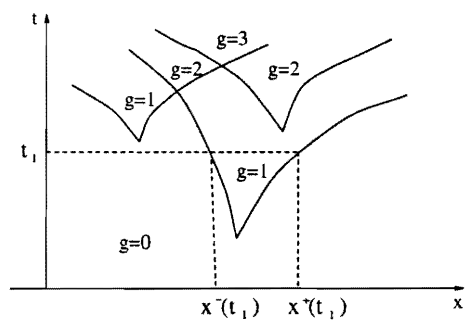


Figure 2: Bifurcation diagram in the (x, t) plane

by the $g = 1$ Whitham equations. The graph of the solution $(u_1(x, t), u_2(x, t), u_3(x, t))$ for any t can be plotted on the same (x, u) plane like branches of a multivalued function

with the different levels of the fold given by $u_1 > u_2 > u_3$. This curve can evolve further developing more folds that are governed in a similar way by the Whitham equations for higher g. The genus $g(x,t)$ is a piecewise constant (see Figure 2) that gives the number of folds over the (x,t) plane, each new fold adding two new levels. The main problem is to glue together the solutions $u_i(x,t)$ of the Whitham equations for different g in order to produce a C^1-smooth curve evolving smoothly with t (see Figure 3). This problem is referred to as initial value problem of the Whitham averaged equations.

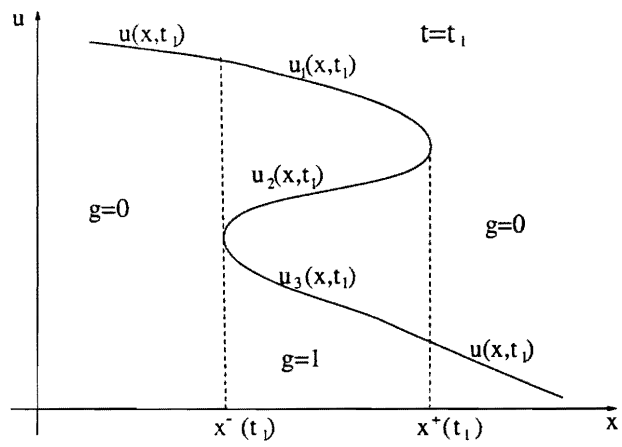


Figure 3: On the picture, $u(x,t)$ is the solution of (2), $(u_1(x,t)\,,\, u_2(x,t), u_3(x,t))$ is the solution of the Whitham equations (5) for $g = 1$. This solution and the position of the boundaries $x^-(t)$, $x^+(t)$ of the $g = 1$ oscillation zone are to be determined from the conditions $u(x^-(t),t) = u_1(x^-(t),t)$, $\quad u_x(x^-(t),t) = u_{1x}(x^-(t),t)$, $\quad u(x^+(t),t) = u_3(x^+(t),t)$, $\quad u_x(x^+(t),t) = u_{3x}(x^+(t),t)$.

The investigation of the problem of dispersive shock waves using the Whitham averaged system began with Gurevich and Pitaevskii [GP]. They studied concrete problems of the physics of weakly dispersive media and in particular they studied the dispersive analogue of a shock wave of ordinary hydrodynamics. In ordinary hydrodynamics, there arises after the breaking of the front wave a second-order discontinuity, or the so called shock-wave. In equation (1) dispersion becomes important in the vicinity of the

breaking point and after the breaking there is formed an expanding region filled with oscillations. GP described these oscillations as a modulated periodic wave. The modulation in time and space of wave number, frequency and amplitude of these oscillations is given by the one-phase Whitham equations.

Using invariance of KdV equation under the transformations

$$\begin{cases} t \to t \\ x \to x + 6\,C\,t \quad C \neq 0 \\ u \to u + C \end{cases} \tag{6}$$

and

$$\begin{cases} t \to t + C_1 \\ x \to x + C_2 \end{cases} \tag{7}$$

where C, C_1 and C_2 are constants, GP showed that it is always possible to choose the breaking point (x_0, t_0, u_0) of the Burgers equation such that $x_0 = t_0 = u_0 = 0$. Expanding the initial data with a cubic inflection point near this point one obtains in a first approximation $x = -\rho\, u^3$. It is always possible to choose $\rho = 1$ using the invariance of KdV under the transformation

$$\begin{cases} x \to C^{\frac{1}{3}}\, x \quad C \neq 0 \\ t \to C\, t \\ u \to C^{-\frac{2}{3}}\, u\,. \end{cases} \tag{8}$$

For the initial data $x = -u^3$, GP showed numerically that the (x, t) plane has just a zero phase domain and a one phase domain. Whitham equations with cubic initial data are invariant under the similarity transformation

$$z = \frac{x}{t^{\frac{3}{2}}}\,, \quad u(x,t) = t^{\frac{1}{2}}\Theta(z)\,.$$

Using the above transformation GP showed that the one phase domain grows proportional to $t^{\frac{3}{2}}$. Later Avilov and Novikov [AN] showed numerically that the one phase solution is stable.

Numerical investigations on the development of quasi-periodic modulated oscillations nearby the point of gradient catastrophe have not yet been done. This problem would require at least the study of the 2-phase Whitham equations whose solution is non trivial.

The algebraic and geometric structure of equations (5) was elucidated only in the last decade. Dubrovin and Novikov [DN] developed a geometric-Hamiltonian theory for the Whitham equations. Based on this theory, Tsarev [T] was able to prove that, for each g, equations (5) can be solved by a hodograph method. This method was put into an algebro-geometric setting by Krichever [K]. Potemin [P] gave an analytical solution of the initial value problem of the Whitham equations for cubic initial data which confirmed the results obtained much earlier by Gurevich and Pitaevskii.

In order to study the initial value problem for generic monotone decreasing initial data $x = f(u)$ in the vicinity of a cubic inflection point ξ where $f'''(\xi) \neq 0$, let us consider the Taylor series

$$x = -\sum_{j=0}^{5} c_j (u-\xi)^j + O((u-\xi)^6)\,, \quad c_3 \neq 0\,. \tag{9}$$

Since the KdV equations are invariant under the groups of transformations (6) and (7) we can reduce the generic initial data (9) to the form

$$x = -c_3 u^3 - c_4 u^4 - c_5 u^5 + O(u^6)\,. \tag{10}$$

For this initial data the point of gradient catastrophe for the Burgers equation is ($x_0 = 0$, $t_0 = 0$, $u_0 = 0$). Gurevich and Pitaevskii approximate the initial data (10) in the vicinity of the breaking point retaining only the cubic term. The parameter c_3 can be reduced to $c_3 = 1$ using invariance of KdV under the rescaling (8).

In this paper we study the behaviour of a generic one-parameter deformation of the GP solution. At this end we approximate the initial data $x = f(u)$ by the quintic Taylor polynomial assuming that $f^V \neq 0$ in the point of gradient catastrophe ξ:

$$x = -(c_3\, u^3 + c_4\, u^4 + c_5\, u^5)\,, \quad c_5 \neq 0\,. \tag{11}$$

The parameter space (c_3, c_4, c_5) can be reduced exploiting the invariance of KdV equation under the following groups of transformations

$$\begin{cases} x \to k^3 x\,, & k \neq 0 \\ t \to k^2 t \\ u \to k u\,, \end{cases}$$

and

$$\begin{cases} x \to \alpha x\,, & \alpha \neq 0 \\ t \to \alpha t \end{cases}$$

(these transformations change, however, the value of the small parameter ϵ). Taking $k = \sqrt{c_3/c_5}$ and $\alpha = c_3$, the initial data (11) can be reduced to the form

$$x = -u^3 - c\,u^4 - u^5\,, \tag{12}$$

where the dimensionless parameter c is chosen in the form

$$c = \frac{c_4}{\sqrt{c_3\,c_5}} = \frac{\sqrt{5}}{2}\frac{f^{IV}(\xi)}{\sqrt{f'''(\xi)f^{V}(\xi)}}\,, \qquad c^2 \leq \frac{15}{4}\,.$$

We study the initial value problem of the Whitham equations for the one parameter family of initial data (12). Monotonicity condition on (12) requires $c^2 \leq \frac{15}{4}$.

Using Krichever algebro-geometric scheme, we prove that Whitham equations develop at most a 2-phase solutions for the initial data (12) (in fact, we derived an upper estimate for the number g of the phases for arbitrary polynomial initial data).

In this case the (x, t) plane has zero phase domain, a one phase domain and at most a 2-phase domain. For the values of the parameter $c^2 \leq \frac{15}{4}$ we classify the various topological types of bifurcation diagrams in the (x, t) plane.

From the above analysis it is natural to conjecture that solutions of Whitham equations with initial data $f(u)$ that can be well approximated by a quintic polynomial nearby the point of gradient catastrophe and that have the same ratio $\dfrac{f^{IV}(\xi)}{\sqrt{f'''(\xi)f^{V}(\xi)}}$,

$f'''(\xi) \neq 0$, $f^{V}(\xi) \neq 0$ exhibit, till a certain time, the same topological type of bifurcation diagram in the (x,t) plane. In a subsequent publication we are going to further analyze this conjecture and the relationship between the bifurcation diagram of solutions of Whitham equations and the development of quasi-periodic modulated oscillations near a point of gradient catastrophe of a generic one-parameter family of initial value problems for KdV.

The study of the initial value problem (12) has been investigated through a variational principle for the Whitham equations recently proposed by Dubrovin. In [D1] the minimizer of a functional on a certain infinite-dimensional space formally solves the initial value problem for each point of the (x,t) plane.

This paper is organized as follows:

In Sect.2 we described the initial value problem for the Whitham-type equations (5) and the Tsarev hodograph method.

in Sect.3 we describe the Krichever's algebro-geometric scheme;

in Sect. 4 we describe Dubrovin's variational principle;

in Sect. 5 we study the initial value problem of Whitham averaged equations for the one parameter family of initial data (12);

in Sect. 6 we draw the various topological type of bifurcation diagrams in the (x,t) plane obtained from the study of above initial value problem.

2 The initial value problem

In this section we describe the initial value problem of the Whitham averaged system (5) for monotone decreasing initial data $u_0(x)$. The solution of the Burgers equation (2) satisfies the characteristic equation:

$$x = 6tu + f(u) \tag{13}$$

where $f(u)$ is the inverse function of the decreasing initial data $u_0(x)$. The curve $u = u(x,t)$ implicitly defined by (13) is single-valued until a certain time. For example, for $f(u) = -u^3$, $u(x,t)$ for $t > 0$ is uniquely defined only for $\mid x \mid > 4\sqrt{4}\, t^{\frac{3}{2}}$. At later times, the evolving curve can formally be given by a multi-valued function with an odd number of branches: $u_k = u_k(x,t)$, $k = 1, \ldots, 2g+1$ where the u_k's satisfy equations (5). Solutions for different g's are matched on the phase transition boundaries in order to produce a C^1-smooth curve evolving smoothly with time.

The g-phase Whitham equations (5) are built in the following way. On the Riemann surface

$$\Gamma_g := \left\{\mu^2 = (r-u_1)(r-u_2)\ldots(r-u_{2g+1})\right\} \ , \quad u_1 > u_2 > \ldots > u_{2g+1} \tag{14}$$

define the abelian differentials of the second kind

$$dp = \frac{P_g(r,\vec{u})}{2\mu} dr \ , \quad dq = \frac{Q_g(r,\vec{u})}{2\mu} dr \tag{15}$$

where

$$P_g(r,\vec{u}) = r^g + \alpha_1 r^{g-1} + \ldots + \alpha_g \, , \tag{16}$$

$$Q_g(r,\vec{u}) = 12r^{g+1} - 6\left(\sum_{i=1}^{2g+1} u_k\right) r^g + \beta_1 r^{g-1} + \ldots + \beta_g \, . \tag{17}$$

The coefficients $\alpha_i = \alpha_i(\vec{u})$, and $\beta_i = \beta_i(\vec{u})$, $i = 1, 2, \ldots, g$ are uniquely determined by the normalization conditions:

$$\int_{u_{2k+1}}^{u_{2k}} dp = 0 \ , \qquad \int_{u_{2k+1}}^{u_{2k}} dq = 0 \ , \quad k = 1, 2, \ldots, g \, . \tag{18}$$

In literature the differential dp is called quasi-momentum and the differential dq quasi-energy.

The speeds λ_i of the g-phase Whitham equations (5) are given by the ratio:

$$\lambda_i(\vec{u}) = \frac{Q_g(u_i,\vec{u})}{P_g(u_i,\vec{u})} \ , \quad i = 1, 2 \ldots 2g+1 \ , \tag{19}$$

In the case $g = 0$, (16) and (17) become $P_0(r) = 1$ and $Q_0(r) = 12r - 6u$ respectively, so that the zero-phase Whitham equation (5) turns out to be the Burgers equation (2).

The striking fact about the Whitham equations (5) is that they can be integrated using a hodograph transform which is the generalization of the characteristic method (13). More precisely we have the following result due to Tsarev [T].

Theorem 1 *If $w_i(\vec{u})$ solves the linear over-determined system*

$$\frac{\partial w_i}{\partial u_j} = \frac{1}{\lambda_i - \lambda_j}\frac{\partial \lambda_i}{\partial u_j}[w_i - w_j] \ , \quad i, j = 1, 2 \ldots 2g+1 \ , i \neq j \ , \tag{20}$$

then the solution $u_1(x,t), u_2(x,t), \ldots, u_{2g+1}(x,t)$ of the hodograph equations

$$x = \lambda_i(\vec{u})t + w_i(\vec{u}) \ , i = 1, 2, \ldots, 2g+1 \tag{21}$$

satisfies the Whitham equations (5). Conversely, any solution $(u_1, u_2, \ldots, u_{2g+1})$ of the Whitham equations (5) can be obtained in this way in the neighborhood of (x_0, t_0) at which the u_{ix}'s are not vanishing.

The linear over-determined system (20) can be shown to satisfy compatibility conditions and therefore it has local solutions. Since equation (20) is singular on the phase transition boundaries, the solutions cannot be extended globally.

3 Krichever's algebro-geometric scheme

In this section we analyze Krichever's scheme (see [DN][K][FFM]) of finding solutions to the Whitham equations.

Let dp and dq be the abelian differentials defined by equation (15) on the Riemann surface Γ_g. Then the g-phase Whitham equations (5) are equivalent to the equation (see [FFM]):

$$(dp)_t + (dq)_x = 0 \, . \tag{22}$$

Now let us consider the abelian differential ds_n $(n \geq g)$ on the Riemann surface Γ_g:

$$ds_n = \frac{S_n(r, \vec{u})}{2\mu} dr \ , \quad S_n(r, \vec{u}) = r^n + \gamma_1 r^{n-1} + \ldots + \gamma_n \ . \tag{23}$$

The constants $\gamma_i = \gamma_i(\vec{u})$, $i = 1, 2, \ldots, n$ are uniquely determined by the normalization conditions:

$$\int_{u_{2k+1}}^{u_{2k}} ds_n = 0, \quad k = 1, 2, \ldots, g \tag{24}$$

and by imposing the asymptotic behaviour:

$$ds_n = \left[\frac{r^{n-g-\frac{1}{2}}}{2} + O(r^{-\frac{3}{2}}) \right] dr \ , \quad \text{for large } \mid r \mid \ . \tag{25}$$

We have the following theorem (see [FFM] and [DN]).

Theorem 2 *The equation*

$$(dp)_y + (ds_n)_x = 0 \tag{26}$$

holds if and only if

$$u_{iy} + W_i(\vec{u}) u_{ix} = 0 \ , \quad i = 1, 2, \ldots, 2g+1 \tag{27}$$

where the characteristic velocities $W_i(\vec{u})$ are given by

$$W_i(\vec{u}) = \frac{ds_n}{dp} \mid_{r=u_i} \ , i = 1, 2, \ldots, 2g+1 \ . \tag{28}$$

The characteristic velocities $W_i(\vec{u})$, $i = 1, \ldots, 2g+1$, satisfy equations (20).

Finally we combine Theorem 1 and Theorem 2, to construct the transform

$$x \ = \ \lambda_i(\vec{u})\, t + W_i(\vec{u}) = \left(\frac{dq}{dp} t + \frac{ds_n}{dp} \right) \bigg|_{u=u_i} \ , \quad i = 1, 2, \ldots, 2g+1 \ . \tag{29}$$

Now suppose that the initial data of the Whitham equations has the form

$$f(u) = -(c_0 + c_1 u + \ldots + c_{2N} u^{2N} + c_{2N+1} u^{2N+1}) \ , \tag{30}$$

and construct on the Riemann surface Γ_g the differential

$$ds = \sum_{k=0}^{k=2N+1} \frac{2^k k!}{(2k-1)!!} c_k ds_{k+g} \ . \tag{31}$$

Then, if it exists, the solution of the g-phase equations with initial data (30) is given by the equations [FRT2]:

$$(xdp - tdq + ds) \mid_{r=u_i} = 0 \ , \quad i = 1, 2 \ldots, 2g+1 \ . \tag{32}$$

The following Lemma will provide an upper estimate to the number g of phases of the Whitham equations for arbitrary polynomial initial data.

Lemma 1 *Let be*

$$f(u) = -(c_0 + c_1 u + \ldots + c_{2N} u^{2N} + c_{2N+1} u^{2N+1}) \ , \quad N \geq 1$$

the initial data of the Whitham equations (5). Then equations (5) admit at most a N-phase solution.

Proof: For $i = 1, 2, \ldots, 2g+1$ we write the g-phase equations (32) in the form

$$\left(x \frac{P_g(r, \vec{u})}{2\mu} - t \frac{Q_g(r, \vec{u})}{2\mu} + \sum_{k=0}^{2N+1} \frac{2^k k!}{(2k-1)!!} c_k \frac{S_{k+g}(r, \vec{u})}{2\mu} \right) \Bigg|_{r=u_i} = 0 \ . \tag{33}$$

We call $Z(r, \vec{u})$ the numerator of the above differential

$$Z(r, \vec{u}) = x\, P_g(r, \vec{u}) - t\, Q_g(r, \vec{u}) + \sum_{k=0}^{2N+1} \frac{2^k k!}{(2k-1)!!} c_k\, S_{k+g}(r, \vec{u}) \ . \tag{34}$$

$Z(r, \vec{u})$ is a polynomial of degree $2N+1+g$ and because of (33) it has $2g+1$ zeros at $u_1, u_2, \ldots, u_{2g+1}$. By the normalization conditions (18) and (24), $Z(r, \vec{u})$ must have at least one zero in each of the intervals (u_{2k+1}, u_{2k}), $k = 1, 2, \ldots, g$. So the total number of zeros of the polynomial (34) must be at least $3g+1$. This implies the inequality $g \leq N$. □

4 Variational principle for Whitham equations

In the following we write Whitham equations as the minimizer of a functional defined on a certain infinite-dimensional space. Let us first consider the zero-phase equation. The characteristic equation $x = 6tu + f(u)$, where $f(u)$ is given by (30), can be consider as the minimum of the function

$$G_{[x,t]}(u) = xu - 3tu^2 - F(u) \tag{35}$$

for $F'(u) = f(u)$. The minimization problem is well defined up to the point of gradient catastrophe (x_0, t_0) where the function (35) fails to have a unique minimum. We want to extend the function of type (35) onto a certain infinite dimensional space in such a way that it has a unique C^1-smooth minimum in this space. This minimum will give the solution of the initial value problem of the Whitham equations (5).

First we define this functional on a Riemann surface of genus g in such a way that its minimizer solves the g-phase Whitham equations. Then we extend it to a smooth functional on the moduli space of the hyperelliptic curves and their degeneration [D1].

Let us consider the asymptotic expansion of the quasi-momentum dp defined on the Riemann surface Γ_g:

$$dp = \left[\frac{1}{2\sqrt{r}} - \frac{1}{2\sqrt{r}} \sum_{k=0}^{\infty} \frac{(2k+1)I_k}{2^{2k+1} r^{k+1}} \right] dr \tag{36}$$

where the coefficient $I_k = I_k(u_1, u_2, \ldots, u_{2g+1})$ are the so called KdV integrals and are smooth functions on the Riemann surface Γ_g.

Theorem 3 *Let us consider the functional*

$$G^g_{[x,t,c_0,\ldots,c_{2N+1}]}(u_1, u_2, \ldots, u_{2g+1}) = -xI_0 + 3tI_1 - \sum_{k=0}^{2N+1} \frac{k!}{2^k (2k-1)!!} c_k I_k \,, \tag{37}$$

where $I_k = I_k(u_1, u_2, \ldots, u_{2g+1})$.

Then the equations

$$\frac{\partial}{\partial u_i} G^g_{[x,t,c_0,\ldots,c_{2N+1}]}(u_1, u_2, \ldots, u_{2g+1}) = 0 \,, \quad i = 1, 2 \ldots, 2g+1 \,, \tag{38}$$

are equivalent to the equations

$$(x\, dp - t\, dq + ds)|_{r=u_i} = 0 \,, \quad i = 1, 2 \ldots, 2g+1 \,, \tag{39}$$

where dp, dq and ds have been defined by equations (15) and (31) respectively.

The proof is based on the following lemma [D2].

Lemma 2 *Let ds^1 and ds^2 be normalized abelian differentials of the second kind on a Riemann surface Γ_g with pole at infinity and with asymptotic behaviour*

$$ds^1 = \left[\frac{r^{N_1 - g - \frac{1}{2}}}{2} + O(r^{-\frac{3}{2}}) \right] dr \,, \quad N_1 \geq g \,,$$

$$ds^2 = \left[\frac{r^{N_2 - g - \frac{1}{2}}}{2} + O(r^{-\frac{3}{2}}) \right] dr \,, \quad N_2 \geq g \,.$$

Let consider their asymptotic expansion

$$ds^1 = \frac{1}{2\sqrt{r}} \sum_k \frac{a^1_k}{r^{k+1}} \,, \quad \textit{for large } \mid r \mid \,, \tag{40}$$

$$ds^2 = \frac{1}{2\sqrt{r}} \sum_k \frac{a^2_k}{r^{k+1}} \,, \quad \textit{for large } \mid r \mid \,. \tag{41}$$

Then

$$Res_{u_j} \frac{ds^1 ds^2}{dr} = \partial_{u_j} V_{dp^1 dp^2} \,, \quad j = 1, 2, \ldots, 2g+1 \,, \tag{42}$$

where

$$V_{dp^1 dp^2} = \frac{1}{2} \sum_{k \geq 0} \frac{a^1_{-k-1} a^2_k}{2k+1} \,. \tag{43}$$

Proof of Theorem 3.

From Lemma 2 it can be easily checked that for $j = 1, 2, \ldots, 2g+1$:

$$\frac{\partial I_0}{\partial u_j} = -4\, Res_{u_j} \frac{dp\, dp}{dr}$$

$$\frac{\partial I_1}{\partial u_j} = -\frac{4}{3}\, Res_{u_j} \frac{dp\, dq}{dr} \tag{44}$$

$$\frac{\partial I_n}{\partial u_j} = -2^{(2n+2)}\, Res_{u_j} \frac{dp\, ds_n}{dr}\,, \quad n > 1\,,$$

and Theorem3 follows. □

To extend the functional (37) defined on the hyperelliptic surfaces of genus g to the infinite dimensional space M of all hyperelliptic Riemann surfaces Γ_g $g \geq 0$ with real branch points $u_1, u_2, \ldots, u_{2g+1}$ and their degeneration, we refer to [D1].

Construct the space M inductively starting from

$$M_0 = \mathbb{R}.$$

We denote u the coordinate in M_0.

Define now

$$M_g = M_g^0 \cup_{j=1}^g M_{g-1}^1(j) \cup_{j=1}^g M_{g-1}^2(j)$$

where

$$M_g^0 = \left\{(u_1, u_2, \ldots, u_{2g+1}) \in \mathbb{R}^{2g+1} |\; u_1 > u_2 > \ldots > u_{2g+1}\right\}$$

and any of the spaces $M_{g-1}^{1,2}(j)$ is isomorphic to M_{g-1} assumed to be already constructed. The space $M_{g-1}^1(j)$ is attached to the component of the boundary of M_g^0 where

$$u_{2j} - u_{2j+1} \to 0\,, \quad j = 1, 2 \ldots, g\,;$$

the space M_{g-1}^2 is attached to the component of the boundary of M_g^0 where

$$u_{2j-1} - u_{2j} \to 0\,, \quad j = 1, 2, \ldots, g\,.$$

Definition 1 *A C^k-smooth functional f on M is a sequence $f_g(u_1, \ldots, u_{2g+1})$ of functions defined on every M_g satisfying the following properties.*

1) On the open part M_g^0 this is a C^k-smooth function of the variables $u_1 > u_2 > \ldots > u_{2g+1}$.

2) Near an inner point of $M_{g-1}^1(j)$ this function can be represented as

$$f_g(u_1, \ldots, u_{2g+1}) = f_{g-1}(u_1, \ldots, \hat{u}_{2j}, \hat{u}_{2j+}, \ldots, u_{2g+1}) \\ +\epsilon f_{g,j}^1(u_1, \ldots, \hat{u}_{2j}, \hat{u}_{2j+1}, \ldots, u_{2g+1}; v, \epsilon)$$

for a C^k-smooth function $f_{g,j}^1$ of $u_1 > \ldots > u_{2j-1} > v > u_{2j+2} > \ldots > u_{2g+1}$ and $\epsilon > 0$. Here

$$v = \frac{u_{2j} + u_{2j+1}}{2}, \quad \epsilon = \frac{(u_{2j} - u_{2j+1})^2}{4}. \tag{45}$$

Here and below the hat above a letter means that the correspondent coordinate is omitted.

3) Near an inner point of $M_{g-1}^2(j)$ the function can be represented as

$$f_g(u_1, \ldots, u_{2g+1}) = f_{g-1}(u_1, \ldots, \hat{u}_{2j-1}, \hat{u}_{2j}, \ldots, u_{2g+1}) \\ +\delta f_{g,j}^2(u_1, \ldots, \hat{u}_{2j-1}, \hat{u}_{2j}, \ldots, u_{2g+1}; v, \delta)$$

for a C^k-smooth function $f_{g,j}^2$ of $u_1 > \ldots > u_{2j-2} > v > u_{2j+1} > \ldots > u_{2g+1}$ and $\delta > 0$. Here

$$v = \frac{u_{2j-1} + u_{2j}}{2}, \quad \delta = \left[\log \frac{4}{(u_{2j-1} - u_{2j})^2}\right]^{-1}. \tag{46}$$

Remark The inner part of M_g parameterizes isospectral classes of g-gap potential $u(x)$ of the Sturm-Liouville operator

$$L = \frac{\partial^2}{\partial x^2} + u(x). \tag{47}$$

Any such potential is a certain quasi-periodic analytic function of x. Generically it has g independent periods. For a g-gap potential, the spectrum of the operator L consists

of the segments

$$\text{spectrum} = (\infty, u_{2g+1}] \cup [u_{2g}, u_{2g-1}] \cup \ldots \cup [u_2, u_1]$$

which are called bands of the spectrum. The segments $[u_{2g+1}, u_{2g}] \cup \ldots \cup [u_3, u_2]$ are called gaps of the spectrum.

In the inner part of M_g the quasi momentum $dp = \dfrac{P_g(r, \vec{u})}{2\mu} dr$ is well defined. To extend it on the whole space M we have the following proposition:

Proposition 1 *For any fixed non-real r the function*

$$\frac{P_g(r, \vec{u})}{2\mu} \tag{48}$$

can be extended from M_g^0 to a C^∞-smooth functional on M.

The proof is based on a result of section III of [F].

Corollary 1 *The coefficients $I = I_k(u_1, u_2, \ldots, u_{2g+1})$ of the expansion*

$$dp = \left[\frac{1}{2\sqrt{r}} - \frac{1}{2\sqrt{r}} \sum_{k=0}^{\infty} \frac{(2k+1) I_k}{2^{2k+1} r^{k+1}} \right] dr \,,$$

are C^∞-smooth functionals on M.

From Proposition 1 and Corollary 1 we have

Theorem 4 *The functional*

$$G_{[x,t,c_0,\ldots,c_{2N+1}]} = -xI_0 + 3tI_1 - \sum_{k=0}^{2N+1} \frac{k!}{2^k (2k-1)!!} c_k I_k \tag{49}$$

is C^∞ smooth functional on M. Its minimizer is a C^1-smooth multi-valued function of x depending C^1-smoothly on the parameters $t, c_1, \ldots, c_{2N+1}$.

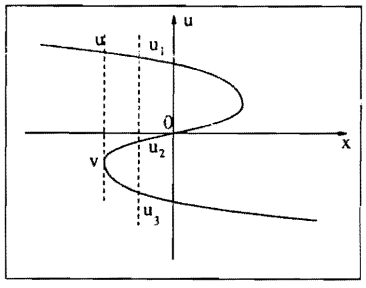


Figure 4: Trailing edge

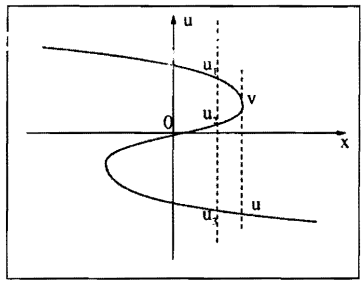


Figure 5: Leading edge

Example. We study the quasi-momentum dp on the boundaries of the space M_1. The space M_1 has two boundary components M_0^1 and M_0^2. We call trailing edge the boundary component M_0^1 (see Figure 4) that corresponds to the opening of a gap in the spectrum of the Sturm-Liouville operator (47) and leading edge the boundary component M_0^2 (see Figure 5) that corresponds to the opening of a band in the spectrum.

Near M_0^1 one has

$$dp = \frac{dr}{2\sqrt{r-u}} + \epsilon \frac{dr}{8(v-u)} \frac{r+v-2u}{(r-v)^2\sqrt{r-u}} dr + O(\epsilon^2) \,, \tag{50}$$

where

$$u_1 = u \,, \quad v = \frac{u_2+u_3}{2} \,, \quad \epsilon = \frac{(u_2-u_3)^2}{4} \,. \tag{51}$$

Near M_0^2

$$dp = \frac{dr}{2\sqrt{r-u}} + 2\delta \frac{(v-u)dr}{(r-v)\sqrt{r-u}} dr + O(\delta^2) \,, \tag{52}$$

where

$$v = \frac{u_1+u_2}{2}, \quad \delta = \left[\log \frac{4}{(u_1-u_2)^2}\right]^{-1} \,, \quad u_3 = u \,. \tag{53}$$

5 Example

We are going to study the initial value problem of the Whitham equations for the monotone decreasing initial data

$$f(u) = -(u^3 + c\,u^4 + u^5)\,, \quad c^2 \leq \frac{15}{4}\,. \tag{54}$$

The characteristic equation

$$x = 6ut - u^3 - cu^4 - u^5\,, \tag{55}$$

solves the Burgers equation until the time $t = 0$ of gradient catastrophe.

In principle the Burgers equation can have another point of gradient catastrophe at $t \geq 0$, since the curve (55) can have another inflection point different from $(x_0 = 0, t_0 = 0, u_0 = 0)$. For $c = \pm\frac{1}{2}\sqrt{15}$ there is another point of gradient catastrophe of the Burgers at the time $t = 0$ in $(x_1 = \pm\frac{93}{50}\sqrt{\frac{3}{5}}, u_1 = \mp\sqrt{\frac{3}{5}})$. After the breakdown, an oscillation zone develops around both points (x_0, u_0) and (x_1, u_1). We describe qualitatively the evolving behaviour of both zones. In the reference frame $(x + 6\,t\,u_1, u + u_1, t)$ the oscillation zone correspondent to the breakpoint (x_1, u_1), in first approximation, behaves locally like the oscillation zone developing around the point (x_0, u_0). This means that the leading edges and the trailing edges of both zones expand locally with almost the same velocity. For $c = -\frac{\sqrt{15}}{2}$, (see Figure 6) the reference frame $(x + 6\,t\,u_1, u + u_1, t)$ moves with positive velocity with respect to the point $(x = 0, u = 0)$. This means that the leading edge of the oscillation zone expanding around the point (x_1, u_1) has a higher absolute velocity than the leading edge of the zone expanding around the point (x_0, u_0). Instead, the trailing edge of the oscillation zone expanding around the point (x_1, u_1) has a smaller absolute velocity than the the trailing edge expanding around the point (x_0, u_0).

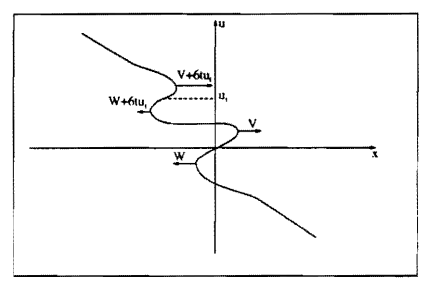


Figure 6: $c = -15/2^{1/2}$

There will be a time $t > 0$ in which the trailing edge of one zone will meet the leading edge of the other. From the above considerations we can conjecture that there will be a time in which the two trailing edges will have the same x-coordinate (double trailing edge) and there will be also a time in which the two leading edges will have the same x-coordinate (double leading edge). For $c = \frac{\sqrt{15}}{2}$ we can make analogous considerations taking care that in this case the reference frame $(x + 6\,t\,u_1, u + u_1, t)$ moves with negative velocity. In the case $c^2 < \frac{15}{4}$ there is a range of values of the parameter c for which there exists a second breakpoint (x_1, u_1) of the Burgers equation at the time $t_1 > 0$. The qualitative behaviour of the two expanding oscillation zones is the same as the one described in the cases $c = \pm\frac{\sqrt{15}}{2}$ (see Figure 7, 8, 9). If there is a

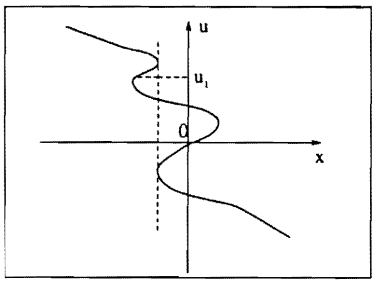


Figure 7: Leading trailing edge, $c < 0$, $t > 0$

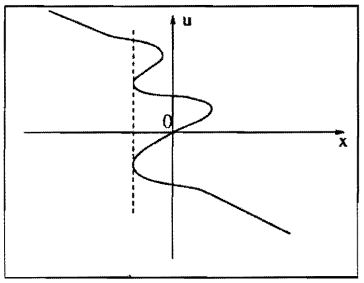


Figure 8: Double trailing edge

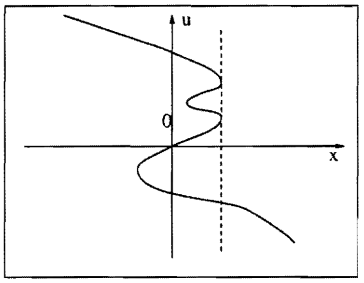


Figure 9: Double leading edge

point of gradient catastrophe inside the first oscillation zone, apriori we can say very little about the qualitative evolution of the oscillations. In the further analysis we are going to study also this case.

We start the analysis of the initial value problem (58) first giving an analytical expression for the second breakpoint of the Burgers equations, then we are going to study the conditions of existence of a double leading edge, a double trailing edge and a leading-trailing edge. Finally we write the equations which determine the point of gradient catastrophe of the one-phase Whitham equations and we study their solutions.

We tackle the analysis considering the functional

$$G_{[x,t,c]} = -xI_0 + 3tI_1 - \frac{1}{20}I_3 - \frac{1}{70}\,cI_4 - \frac{1}{252}I_5\,. \tag{56}$$

The restriction of the functional $G_{[x,t,c]}$ on M_0 has the form

$$G^0_{[x,t,c]}(u) = xu - 3tu^2 + \frac{u^4}{4} + c\frac{u^5}{5} + \frac{u^6}{6}\,, \tag{57}$$

thus the minimizer given by

$$x = 6ut - u^3 - cu^4 - u^5 \tag{58}$$

solves the Burgers equation until the breaking time $t = 0$.

At later times the minimizer of $G_{[x,t,c]}$ may belong to M_0, M_1 or M_2.

5.1 Second breakpoint for the Burgers equation

If a second breakpoint of the Burgers equation exists, it belongs to the component $M_0^1 \cap M_0^2$ of the boundary of the space M_1. In this case a band and a gap open in the spectrum at the same time. We put

$$u_1 = u + \sqrt{\rho}\,, \quad u_2 = u\,, \quad u_3 = u - \sqrt{\epsilon}\,, \quad \epsilon, \rho > 0\,. \tag{59}$$

The expansion for small ϵ and small ρ of the quasi-momentum dp, defined on the elliptic curve $\Gamma_1 = \{\mu^2 = (r-u_1)(r-u_2)(r-u_3)\}$, reads

$$dp = \frac{dr}{2\sqrt{r-u}} + \frac{a_1\,dr}{2(r-u)\sqrt{r-u}} + \frac{a_2\,dr}{2(r-u)^2\sqrt{r-u}} + \sum_{l=0}^{3} O(\epsilon^{\frac{l}{2}}\rho^{\frac{3-l}{2}})\,, \tag{60}$$

where

$$a_1 = (\sqrt{\epsilon} + \sqrt{\rho})\left(\frac{E(k)}{K(k)} - \frac{1}{2}\right)\,, \tag{61}$$

$$a_2 = \frac{3}{8}\epsilon - \frac{1}{8}\rho + \frac{1}{4}\sqrt{\epsilon\rho} - \frac{1}{2}(\epsilon - \rho)\frac{E(k)}{K(k)} \tag{62}$$

and $k = \frac{\sqrt{\epsilon}}{\sqrt{\epsilon}+\sqrt{\rho}}$, $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind respectively. Observe that for $\epsilon \simeq \rho$, $\epsilon \ll \rho$ and $\rho \ll \epsilon$ the quantity a_1 and a_2 remain finite.

From formula (60) one obtains the following expression of the functional $G_{[x,t,c]}$ near $M_0^1 \cap M_0^2$:

$$\begin{aligned} G_{[x,t,c]}(u) &= xu - 3tu^2 - F(u) + a_1\,[x - 6tu - f(u)] \\ &\quad + a_2\,[-6t - f'(u)] + \textstyle\sum_{l=0}^{3} O(\epsilon^{\frac{l}{2}}\rho^{\frac{3-l}{2}})\,, \end{aligned} \tag{63}$$

where $f(u)$ is the initial data (54) and $F'(u) = f(u)$.

Near the minimizer (58) expression (63) becomes

$$G_{[x,t,c]}(u) = xu - 3tu^2 - F(u) + \left(a_2 - a_1^2\right)[-6t - f'(u)] + \sum_{l=0}^{3} O(\epsilon^{\frac{l}{2}}\rho^{\frac{3-l}{2}})\,, \tag{64}$$

where $a_2 - a_1^2 > 0 \;\; \forall\, \epsilon, \rho > 0$.

For given (x,t) a point u is on the boundary $M_0^1 \cap M_0^2$, if (x,t,u) minimize the functional (56) and if t and u are zero and a local minima of the term $(-6t - f'(u))$ in expression (64), that is they satisfy the equations:

$$\begin{cases} -6t + 3u^2 + 4cu^3 + 5u^4 = 0 \\ -f''(u) = 6u + 12cu^2 + 20u^3 = 0 \\ -f'''(u) > 0 \end{cases} \tag{65}$$

whose solutions are obviously $u_0 = 0,\; t_0 = 0$ and

$$\begin{cases} u_1 = -\frac{3}{10}(c + \sqrt{c^2 - \frac{10}{3}}) \text{ for } c > 0\,, \\ u_1 = -\frac{3}{10}(c - \sqrt{c^2 - \frac{10}{3}}) \text{ for } c < 0\,, \\ t_1 = \frac{1}{6}(3u_1^2 + 4cu_1^3 + 5u_1^4) > 0\,. \end{cases} \tag{66}$$

The space coordinate x_1 relative to (t_1, u_1) is recovered from equation (58).

Observation The second breakpoint is real if $c^2 \geq \frac{10}{3}$. In the previous analysis we have not checked if the breakpoint (x_1, t_1, u_1) minimizes the functional $G_{[x,t,c]}$. It could happen that for such (x_1, t_1, u_1) the minimizer of $G_{[x,t,c]}$ belongs to the inner part of M_1.

5.2 Trailing edges

The trailing edge of the 1-phase oscillatory zone belongs to the the component M_0^1 of the boundary of the space M_1. To study the functional $G_{[x,t,c]}$ near M_0^1 we use the expression of the quasi momentum (50), when small gap of width $2\sqrt{\epsilon}$ opens in the spectrum near the point v for some $v < u$.

We obtain the following expression:

$$G_{[x,t,c]}(u,v) = G^0_{[x,t,c]}(u) + \epsilon \frac{-6t(2v-u) + x + B_c(u,v)}{2(v-u)} + O(\epsilon^2)\,, \tag{67}$$

where $G^0_{[x,t,c]}$ has been defined in (57), and $B_c(u,v)$ is a fifth degree polynomial in u and v. Near the minimizer (58) the ϵ correction of (67) can be written in the form

$$(-6t + P_c(u,v)) + O(\epsilon)\,, \tag{68}$$

where

$$P_c(u,v) = (189\,u^2 + 180\,c\,u^3 + 175\,u^4 + 252\,u\,v + 216\,c\,u^2\,v + 200\,u^3\,v$$

$$+504\,v^2 + 288\,c\,u\,v^2 + 240\,u^2\,v^2 + 576\,c\,v^3 + 320\,u\,v^3 + 640\,v^4)/315\,.$$

Observe that the expression (68) in the limit $v \to u$ becomes $(-6t - f'(u)) + O(\epsilon)$ and this shows that the functional (67) is continuous on the intersection $M_0^1 \cap M_0^2$.

For (u,t) in the set

$$S_g = \{(u,t) \in \mathbb{R} \times \mathbb{R} \mid (-6t + P_c(u,v)) > 0 \;\forall\, v \in \mathbb{R}\}\,, \tag{69}$$

the minimizer belongs to M_0.

For (u,v,t) such that $(-6t + P_c(u,v)) < 0$ the minimizer moves into the inner part of M_1. The points $v(t,c) < u(t,c)$ on the boundary M_0^1 are recovered from the equations:

$$\begin{cases} -6t + P_c(u,v) = 0 \\ \\ \partial_v P_c(u,v) = 0\,. \end{cases} \tag{70}$$

where $\partial_v = \dfrac{\partial}{\partial v}$.

Since $P_c(u,v)$ is a fourth degree polynomial, the real solutions $v(u,t,c)$ of system (70) for fixed t and fixed u can not be more than two. To have two such solutions $v(u,t,c)$ and $w(u,t,c)$, $v(u,t,c) \neq w(u,t,c)$, we should require the existence of real solutions of the system:

$$\begin{cases} -6t + P_c(u,v) = 0 \\ \\ \partial_v P_c(u,v) = 0 \\ \\ -6t + P_c(u,w) = 0 \\ \\ \partial_w P_c(u,w) = 0 \ , \end{cases} \tag{71}$$

where the unknowns are t, u, v, w and c is a parameter. The solutions $u(c), v(c)$ and $w(c)$ of (71) belong to the component $M_1^1(1) \cup M_1^1(2)$ of the boundary of the space M_2, (see Figure 10). In this case we have a double trailing edge. System (71) is equivalent to the system

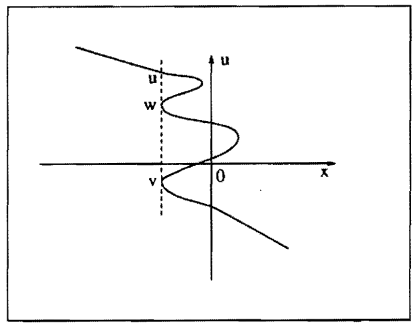


Figure 10: $M_1^1(1) \cup M_1^1(2)$

$$\begin{cases} -6t + P_c(u,v) = 0 \\ \\ \partial_v P_c(u,v) + \partial_w P_c(u,w) = 0 \\ \\ \dfrac{\partial_v P_c(u,v) - \partial_w P_c(u,w)}{w-v} = 0 \\ \\ \dfrac{\dfrac{\partial_v P_c(u,v) - \dfrac{P_c(u,v) - P_c(u,w)}{v-w}}{w-v} + \dfrac{\partial_v P_c(u,v) - \partial_w P_c(u,w)}{2(w-v)}}{w-v} = 0 \ . \end{cases} \tag{72}$$

where the last two equations reads

$$\begin{cases} -6t + P_c(u,v) = 0 \\ \\ \partial_v P_c(u,v) + \partial_w P_c(u,w) = 0 \\ \\ \dfrac{63}{10} + \dfrac{18}{5}cu + 3u^2 + (6u + \dfrac{54}{5}c)(w+v) + 16(w^2+v^2) + 16wv = 0 \\ \\ \dfrac{9}{10}c + \dfrac{u}{2} + 2w + 2v = 0\,. \end{cases} \tag{73}$$

It is clear that the degree of the problem has been reduced. We will now study the range of the parameter c guaranteeing reality of the solutions $v < w < u$ of the above system for $t > 0$.

Observation: The system (72) is well defined in the limits $w \to v$ and $w \to u$, so we first study system (73) in these limiting cases. In the limiting case $w \to v$, (see Figure 11) the two trailing edges coincide. In this case we are considering the embedding of M_0 as the component $M_1^1(1) \cap M_1^1(2) \cap M_1^2(2)$ of the boundary of M_2. System (73) becomes a system of four equations in the four unknowns u, v, t, c and reads:

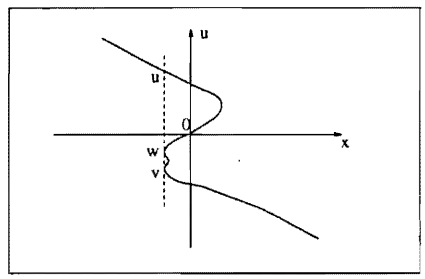


Figure 11: $c = \nu_3 + \Delta\,,\ \Delta > 0$

$$\begin{cases} -6t + P_c(u,v) = 0 \\ \\ 2\partial_v P_c(u,v) = 0 \\ \\ \dfrac{63}{10} + \dfrac{18}{5}cu + 3u^2 + \dfrac{108}{5}cv + 12uv + 48v^2 = 0 \\ \\ \dfrac{9}{10}c + \dfrac{u}{2} + 4v = 0\,, \end{cases} \tag{74}$$

Eliminating u and v from the last three equations of (74) we obtain

$$-42875 + 40200\,c^2 - 14580\,c^4 + 1944\,c^6 = 0\,.$$

The only real solution of the above equation compatible with the conditions $t > 0$, $v < u$, $u > 0$ and $c^2 < \frac{15}{4}$ is obtained for $c = \nu_3$ where

$$\nu_3 = +\sqrt{\frac{5}{2} + \frac{5}{18}\left(\frac{25}{18}\right)^{\frac{1}{3}}\left((27 - 7\sqrt{21})^{\frac{1}{3}} + (27 + 7\sqrt{21})^{\frac{1}{3}}\right)} \simeq 1.78167\,. \tag{75}$$

In the limiting case $w \to u$ (see Figure 12) the Burgers equation has a point of gradient catastrophe in correspondence of the trailing edge. Technically speaking we are just considering the embedding of M_0 as the component $M_1^1(2) \cup (M_1^1(1) \cap M_1^2(1))$ of the boundary of M_2. System (73) becomes

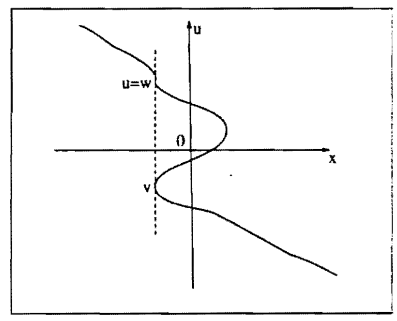


Figure 12: $c = \nu_1$

$$\begin{cases} -6t + P_c(u,v) = 0 \\ \partial_v P_c(u,v) + \dfrac{\partial}{\partial w} P_c(u,w)\,|_{w=u} = 0 \\ \dfrac{63}{10} + \dfrac{72}{5}cu + 25u^2 + \dfrac{54}{5}cv + 22uv + 16v^2 = 0 \\ \dfrac{9}{10}c + \dfrac{5u}{2} + 2v = 0\,, \end{cases}$$

Eliminating u and v from the above system we obtain the equation

$$875 + 27075\,c^2 - 14580\,c^4 + 1944\,c^6 = 0\,.$$

The only real solution of the above equation compatible with the conditions $t > 0$, $v < u$, $u > 0$ and $c^2 < \frac{15}{4}$ is given by $c = \nu_1$ where

$$\nu_1 = -\sqrt{\frac{5}{36}(13 + 5\sqrt{7})} \simeq -1.90863\,. \tag{76}$$

We infer that for $-\frac{\sqrt{15}}{2} \leq c < \nu_1$ the second breakpoint of the Burgers equation given by (66) lies in M_0, for $c > \nu_1$ we should make an analysis of the conditions of breakdown in the inner part of M_1.

In the general case, system (73) admits real solutions $t > 0$, $v < w < u$, and $u > 0$ for c in the intervals

$$-\sqrt{\frac{15}{4}} \leq c < \nu_1 \,, \quad \nu_3 < c \leq \sqrt{\frac{15}{4}} \,,$$

where $c^2 \leq \dfrac{15}{4}$ defines the region of monotonicity of the initial data.

For c in the interval $\nu_1 < c < \nu_3$ there exists just one real solution $v(t,c) < u(t,c)$ of system (70) for all $t > 0$. This corresponds to the existence of just a single trailing edge.

5.3 Leading edges

The leading edge of the 1-phase oscillatory zone corresponds to the embedding of M_0 as the component M_0^2 of the boundary of the space M_1. From the expression of the quasi-momentum dp given in (52) when a small band of width $2e^{-\frac{1}{2\delta}}$ opens in the spectrum near the point v for some $v > u$, one obtains the expression of the functional $G_{[x,t,c]}$ near M_0^2;

$$G_{[x,t,c]} = G^0_{[x,t,c]}(u) + 4\delta(v-u)[-2t(2v+u) + x + D_c(u,v)] + O(\delta^2), \tag{77}$$

where $D_c(u,v)$ is a fifth degree polynomial in u and v. Near the minimizer (58) the coefficient in front of δ can be written in the form

$$8(v-u)^2(-2t + Q_c(u,v)) + O(\delta)\,, \tag{78}$$

where

$$Q_c(u,v) = \frac{3}{7}u^2 + \frac{4}{9}c\,u^3 + \frac{5}{11}u^4 + \frac{12}{35}u\,v + \frac{8}{21}c\,u^2\,v + \frac{40}{99}u^3\,vx$$

$$+\frac{8}{35}v^2 + \frac{32}{105}c\,u\,v^2 + \frac{80}{231}u^2\,v^2 + \frac{64}{315}c\,v^3 + \frac{64}{231}u\,v^3 + \frac{128}{693}v^4\,.$$

For (u,t) in the set

$$S_b = \{(u,t) \mid (-2t + Q_c(u,v)) > 0 \; \forall \, v \in \mathbb{R}\} \; , \tag{79}$$

the minimizer belongs to curves of genus zero.

For (u,v,t) such that $(-2t + Q_c(u,v)) < 0$ the minimizer moves into the inner part of M_1. The points $v(t,c) > u(t,c)$ on the boundary M_0^2 are recovered from the equations:

$$\begin{cases} -2t + Q_c(u,v) = 0 \\ \\ \partial_v Q_c(u,v) = 0 \; . \end{cases} \tag{80}$$

First we consider the situation of a double leading edge, that is we study multiple solutions of the system (80). These solutions belong to the boundary component $M_1^2(1) \cup M_1^2(2)$ of the space M_2 (see Figure 13). The analogous of system (72) turns out to be

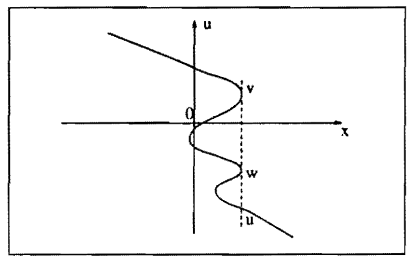


Figure 13: Double leading edge

$$\begin{cases} -2t + Q_c(u,v) = 0 \\ \\ \partial_v Q_c(u,v) + \partial_w Q_c(u,w) = 0 \\ \\ 11c + 15u + 20(v+w) = 0 \\ \\ \dfrac{99}{10} + \dfrac{66}{5}cu + 15u^2 + (18u + \dfrac{66}{5}c)(v+w) + 16(v^2+w^2) + 16vw = 0. \end{cases} \tag{81}$$

In the limiting case $w \to v$ the two leading edges coincide (see Figure 14).

In other words we are considering the embedding of M_0 as the component $M_1^2(1) \cap M_1^2(2) \cap M_1^1(1)$ of the boundary of M_2. System (81) turns out to be a system of four

equations in the four unknowns u, v, t, c and the only real solution compatible with the conditions $t > 0$, $v > u$, $u < 0$ and $c^2 < \dfrac{15}{4}$ is obtained for $c = \nu_2$,

$$\nu_2 = -\sqrt{\frac{5}{2} - \frac{5}{22}\left(\frac{49}{2}\right)^{\frac{1}{3}}\left(e^{\frac{i\pi}{3}}\left(-11+5\,i\sqrt{3}\right)^{\frac{1}{3}} + e^{-\frac{i\pi}{3}}(-11-5\,i\sqrt{3})^{\frac{1}{3}}\right)}\,, \tag{82}$$

$\nu_2 \simeq -1.85585$.

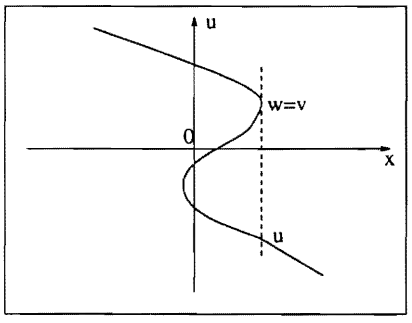


Figure 14: $M_1^2(1) \cap M_1^2(2) \cap M_1^1(1)$

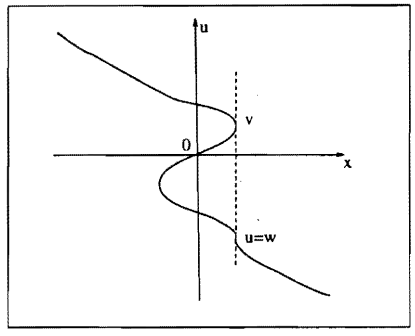


Figure 15: $M_1^2(1) \cup (M_1^2(2) \cap M_1^1(2))$

In the limiting case $w \to u$, there exists a point of gradient catastrophe of the Burgers equation in correspondence (see Figure 15) of the leading edge. System (81) turns out to be a system of four equations in the four unknowns u, v, t, c and the only real solution compatible with the conditions $t > 0$, $v > u$ and $u < 0$ is obtained for $c = \nu_4$,

$$\nu_4 = \sqrt{\frac{1}{44}(75 + 21\sqrt{15})} \simeq 1.88494\,. \tag{83}$$

For $\nu_4 < c \leq \frac{\sqrt{15}}{2}$, the second breakpoint of the Burgers equation lies in M_0. For $c < \nu_4$ we should make an analysis of the conditions of breakdown in the inner part of M_1.

In the general case, system (73) admits real solutions $t > 0$, and $u < w < v$ for the following values of the parameter c:

$$\nu_4 < c \leq \sqrt{\frac{15}{4}}\,, \quad -\sqrt{\frac{15}{4}} \leq c < \nu_2\,.$$

For the values $\nu_2 < c < \nu_4$ there exists just one real solution $v(t,c) > u(t,c)$ of system (70) for all $t > 0$. In this case there exists just a single leading edge for all $t > 0$.

5.4 Leading-trailing edge

We call leading-trailing edge the situation in which a leading edge and a trailing edge have the same (x,t) coordinates.

A leading-trailing edge corresponds to the embedding of M_0 as the component $M_1^2(1) \cup M_1^1(2)$ of M_2. In this case a band and a gap open in the spectrum at the same time and in correspondence of the same x coordinate. If the gap opens near the point $v < u$ and the band opens near the point $w > u$, the point u, v, w on the boundary $M_1^2(1) \cup M_1^1(2)$ are recovered from the equations:

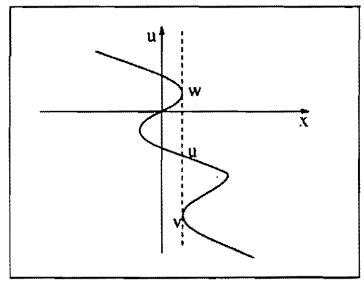


Figure 16: $M_1^2(1) \cup M_1^1(2)$

$$\begin{cases} -6t + P_c(u,v) = 0 \\ \\ \partial_v P_c(u,v) = 0 \\ \\ -2t + Q_c(u,w) = 0 \\ \\ \partial_w Q_c(u,w) = 0\,. \end{cases} \tag{84}$$

The degenerate cases are $w \to u$ and $v \to u$.

The first case, which corresponds to the boundary $M_1^1(2) \cup (M_1^1(1) \cap M_1^2(1))$, has already been considered, and a solution of system (84) exists just for the value $c = \nu_4$ defined in (83).

The second case has also been already considered and corresponds to the boundary $M_1^2(1) \cup (M_1^1(2) \cap M_1^2(2))$. System (84) admits solution for the value of the parameter $c = \nu_1$ defined in (76). In the general case $v < u < w$ system (84) is not reducible to a

simpler one in order to get analytical solutions. We have checked numerically that for c in the intervals

$$-\sqrt{\frac{15}{4}} \leq c < \nu_1 \,, \quad \nu_4 < c \leq \sqrt{\frac{15}{4}} \,,$$

compatible real solutions exist.

Remark. From the above analysis it is clear that, for the values of the parameter c for which there exists an embedding of M_0 as some component of M_2, there should also exist a solution of two 2-phase Whitham equations. The (x,t) plane should have a zero-phase domain, a 1-phase domain, and a 2-phase domain.

It is not clear if, for the values of the parameter c for which there is not an embedding of M_0 as some component of the boundary of M_2, there exists a solution of the 2-phase Whitham equations.

5.5 Point of gradient catastrophe for the one-phase equations

In the following, we analyze the conditions for the existence of a point of gradient catastrophe for the 1-phase Whitham equations. We are going to consider monotone decreasing initial data

$$f(u) = -(u^3 + \ldots + c_{2N}u^{2N} + c_{2N+1}u^{2N+1}) \,. \tag{85}$$

We call dp_1 the quasi-momentum dp restricted to M_1,

$$dp_1 = \frac{r + \alpha_0}{2\sqrt{(r-u_1)(r-u_2)(r-u_3)}} dr \,, \tag{86}$$

where α_0 is determined by the normalization condition

$$\int_{u_3}^{u_2} \frac{r + \alpha_0}{2\sqrt{(r-u_1)(r-u_2)(r-u_3)}} dr = 0 \,. \tag{87}$$

The restriction on M_1 of the functional $G_{[x,t,c_0,\ldots,c_{2N+1}]}$ defined by (49) has the form

$$\begin{aligned} G^1_{[x,t,c_0,\ldots,c_{2N+1}]}(u_1,u_2,u_3) &= 2x(\alpha_0\Gamma_0 + \Gamma_1) - 8t(\alpha_0\Gamma_1 + \Gamma_2) \\ &\quad + \textstyle\sum_{k=3}^{2N+1} \dfrac{2^{k+1}k!}{(2k+1)!!} c_k(\alpha_0\Gamma_k + \Gamma_{k+1}) \,, \end{aligned} \tag{88}$$

where the Γ_k's are the coefficients of the expansion for $r \to \infty$ of

$$\frac{1}{\sqrt{(r-u_1)(r-u_2)(r-u_3)}} = \frac{1}{r^{\frac{3}{2}}}(\Gamma_0 + \frac{\Gamma_1}{r} + \frac{\Gamma_2}{r^2} + \frac{\Gamma_3}{r^3} + \ldots + \frac{\Gamma_k}{r^k} + \ldots). \tag{89}$$

Proposition 2 *The critical points of (88) on the space of elliptic curves are given by the equations for $u_1 > u_2 > u_3$*

$$x = \lambda_i t + w_i\,, \quad i = 1,2,3 \tag{90}$$

where

$$\lambda_i = 2(u_1+u_2+u_3) + 4\frac{\prod_{j\neq i}(u_i-u_j)}{\alpha_0+u_i}\,, \quad w_i = 2\frac{\prod_{j\neq i}(u_i-u_j)}{\alpha_0+u_i}\partial_{u_i}\sigma + \sigma \tag{91}$$

and

$$\sigma = -\sum_{k=3}^{2N+1} \frac{2^k k!}{(2k+1)!!} c_k \Gamma_k. \tag{92}$$

Proof: Using the following formula obtained from Lemma 2

$$\frac{\partial}{\partial u_i}\alpha_0 = -\frac{1}{2} + \frac{1}{2}\frac{(\alpha_0+u_i)^2}{\prod_{j\neq i}(u_i-u_j)} \quad i = 1,2,3 \tag{93}$$

and the identity obtained from expression (89)

$$\frac{\partial}{\partial u_i}\Gamma_{k+1} = \frac{\Gamma_k}{2} + u_i\frac{\partial}{\partial u_i}\Gamma_k \quad i = 1,2,3 \tag{94}$$

equations (90) are recovered straightforward. □

Equations (90) solve the 1-phase Whitham equations with initial data (85) (see [KS],[FRT1]) and are well defined for $t > 0$ and $u_1 > u_2 > u_3$ up to the time of gradient catastrophe when one of the $\partial_x u_i$, $1 \leq i \leq 3$ becomes infinite.

We want to study the functional (49) near this point of gradient catastrophe. For the purpose we study the functional (49) when a small gap opens in the spectrum in the 1-phase zone, technically this corresponds to study the functional (49) near the embedding of M_1 as the component $M_1^1(1)$ or $M_1^1(2)$ of M_2, (see Figure 17 and

Figure 18). First we let a gap open in the spectrum near a point v and we study the functional (49) in this case, and then for a chosen i, $1 \leq i \leq 3$, we let $v \to u_i$. Thus we get the behaviour of the functional (49) near one of the boundaries $M_1^1(1) \cap M_1^2(1)$, $M_1^1(1) \cap M_1^2(2)$ or $M_1^1(2) \cap M_1^2(2)$. If the functional (49) has a minimum on one of these boundaries, then the one-phase Whitham equations have a point of gradient catastrophe.

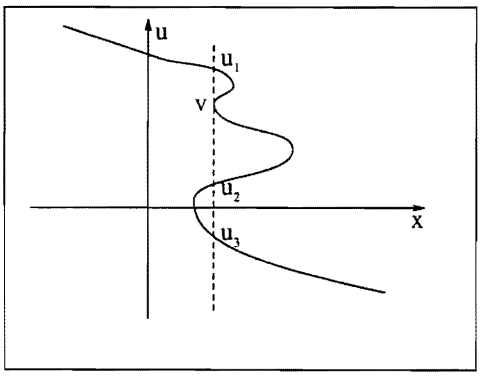


Figure 17: $M_1^1(1)$

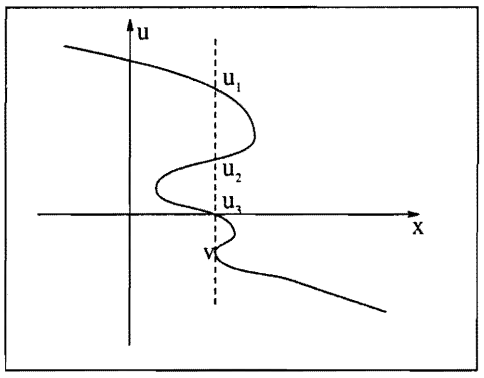


Figure 18: $M_1^1(2)$

If a gap of width $2\sqrt{\epsilon}$ opens near a point v the quasi-momentum dp becomes

$$dp = dp_1 + \frac{\epsilon}{4} \frac{(\alpha r + \beta)(r - v) + r + \alpha_0}{(r - v)^2 \sqrt{(r - u_1)(r - u_2)(r - u_3)}} + O(\epsilon^2)\,, \tag{95}$$

where dp_1 has been defined in (86) and

$$\alpha = \frac{(\alpha_0 + v)^2}{2\prod_{i=1}^3 (v - u_i)}\,, \quad \beta = -2\alpha\, v - 1 + \frac{\alpha_0 + v}{2} \sum_{i=1}^{3} \frac{1}{v - u_i}\,.$$

Near one of the boundaries $M_1^1(1)$ or $M_1^2(1)$ the functional (49) reads:

$$\begin{aligned} G_{[x,t,c_3,\ldots,c_{2N+1}]}(u_1, u_2, u_3, v) &= G^1_{[x,t,c_3,\ldots,c_{2N+1}]} + \frac{\epsilon}{2}\,[\alpha(x - 4\,t\,a - 2\sigma) \\ &+ (\alpha\, v + \beta + \frac{1}{2})(-8t - 2\sigma^v) - 2(\alpha_0 + v)\partial_v \sigma^v] + O(\epsilon^2)\,, \end{aligned} \tag{96}$$

where

$$\sigma^v = -\sum_{k=3}^{2N+1} \frac{2^k k!}{(2k+1)!!} c_k \Gamma^v_{k-1} \tag{97}$$

and the Γ^v_k's are the coefficients of the expansion

$$\frac{1}{(r-v)\sqrt{(r-u_1)(r-u_2)(r-u_3)}} = \frac{1}{r^{\frac{5}{2}}}(\Gamma^v_0 + \frac{\Gamma^v_1}{r} + \frac{\Gamma^v_2}{r^2} + \ldots + \frac{\Gamma^v_k}{r^k} + \ldots) . \tag{98}$$

For fixed i we can simplify the ϵ correction of (96) near the minimizer $x = \lambda_i t + w_i$ obtaining:

$$G_{[x,t,c_3,\ldots,c_{2N+1}]}(u_1, u_2, u_3, v) = G^1_{[x,t,c_0,\ldots,c_{2N+1}]} - \frac{\epsilon}{2}\left[\sum_{j=1}^{3} \frac{v-u_i}{v-u_j}\partial_{u_i}\sigma^v + \right.$$

$$\partial_v\sigma^v + 2\left(t + \tfrac{1}{2}\partial_{u_i}\sigma\right)\left(\sum_{j=1}^{3}\frac{1}{v-u_j} - \frac{1}{\alpha_0+u_i}\frac{\prod_{j\neq i}(u_i-u_j)}{\prod_{j\neq i}(v-u_j)} - \right. \tag{99}$$

$$\left.\left.\frac{\prod_{j\neq i}(u_i-u_j)}{\prod_{j=1}^{3}(v-u_j)}\right)\right] + O(\epsilon^2) .$$

To get the behaviour of the functional (49) near one of the boundaries $M^1_1(1) \cap M^2_1(1)$, $M^1_1(1) \cap M^2_1(2)$ or $M^1_1(2) \cap M^2_1(2)$, we should let $v \to u_i$, $1 \leq i \leq 3$ in the expression (99). We have the following result.

Lemma 3 *In the limit $v \to u_i$, $1 \leq i \leq 3$, the functional (99) becomes*

$$G_{[x,t,c_3,\ldots,c_{2N+1}]}(u_1, u_2, u_3) = G^1_{[x,t,c_0,\ldots,c_{2N+1}]} -$$

$$\frac{\epsilon}{2}\frac{(\alpha_0+u_i)^2}{\prod_{j\neq i}(u_i-u_j)}\frac{\partial}{\partial u_i}(\lambda_i t + w_i) + O(\epsilon^2) . \tag{100}$$

From the ϵ-correction of the functional (100) it holds:

Lemma 4 *If the equations $x = \lambda_i t + w_i$, $i = 1,2,3$, given in (90) minimize the functional*

$G_{[x,t,c_3,\ldots,c_{2N+1}]}$, *then on the minimizer the following inequalities holds:*

$$\frac{\partial}{\partial u_1}(\lambda_1 t + w_1) < 0 \,, \tag{101}$$

$$\frac{\partial}{\partial u_2}(\lambda_2 t + w_2) > 0 \,, \tag{102}$$

$$\frac{\partial}{\partial u_3}(\lambda_3 t + w_3) < 0 \,. \tag{103}$$

From the above lemma it follows that on the solution of the one-phase equations it holds $\partial_x u_i < 0$ for $i = 1,3$ and $\partial_x u_2 > 0$.

The point of gradient catastrophe of the one-phase equations should be a zero and a local minima of the ϵ correction of (100). If the point of gradient catastrophe in on the branch u_i, $1 \leq i \leq 3$, it is determined by the system of equations:

$$\begin{cases} \dfrac{\partial}{\partial u_i}(\lambda_i t + w_i) = 0 \,, \\[2ex] \dfrac{\partial^2}{\partial u_j u_i}(\lambda_i t + w_i) = 0 \,, \quad j = 1,2,3 \,, \end{cases} \tag{104}$$

with the constrains $\dfrac{\partial^3}{\partial u_k u_j u_i}(\lambda_i t + w_i) < 0$, $k,j = 1,2,3$, $i = 1,3$ and $\dfrac{\partial^3}{\partial u_k u_j u_2}(\lambda_2 t + w_2) > 0$, $k,j = 1,2,3$.

All the above derivatives are identically zero for $j \neq i$, $k \neq j \neq i$ and $k \neq j \neq 2$, (for a proof see [FRT1]) and system (104) becomes:

$$\begin{cases} \dfrac{\partial}{\partial u_i}(\lambda_i t + w_i) = 0 \,, \quad 1 < i < 3 \,, \\[2ex] \dfrac{\partial^2}{\partial u_i^2}(\lambda_i t + w_i) = 0 \,, \quad 1 < i < 3 \,, \end{cases} \tag{105}$$

with the constrain $\dfrac{\partial^3}{\partial u_i^3}(\lambda_i t + w_i) < 0$ for $i = 1,3$ and $\dfrac{\partial^3}{\partial u_2^3}(\lambda_2 t + w_2) > 0$.

System (105) together with the one phase equations (90), gives a system of five equations in the five unknowns $x, t, u_1 > u_2 > u_3$:

$$\begin{cases} -x + \lambda_1 t + w_1 = 0 \\ -x + \lambda_2 t + w_2 = 0 \\ -x + \lambda_3 t + w_3 = 0 \\ \dfrac{\partial}{\partial u_i}(\lambda_i\, t + w_i) = 0 \\ \\ \dfrac{\partial^2}{\partial u_i^2}(\lambda_i\, t + w_i) = 0\,, \quad 1 \le i \le 3\,. \end{cases} \tag{106}$$

with the constrain $\dfrac{\partial^3}{\partial u_i^3}(\lambda_i\, t + w_i) < 0$ for $i = 1, 3$ and $\dfrac{\partial^3}{\partial u_i^3}(\lambda_i\, t + w_i) > 0$ for $i = 2$.

Lemma 5 *System (106) is equivalent to the following system*

$$\begin{cases} -x + \lambda_i t + w_i = 0 \\ \\ \prod_{j\neq i}(u_i - u_j)\left[\left(2\sum_{j\neq i}\dfrac{1}{u_i - u_j} - \dfrac{1}{\alpha_0 + u_i}\right)(t + \dfrac{1}{2}\partial_{u_i}\sigma) + \partial^2_{u_i}\sigma\right] = 0 \\ \\ 3(t + \dfrac{1}{2}\partial_{u_i}\sigma) - \displaystyle\sum_{j\neq i}(u_i - u_j)\partial^2_{u_i u_j}\sigma = 0 \\ \\ \partial_{u_i}(\partial_{u_1}\sigma + \partial_{u_2}\sigma + \partial_{u_3}\sigma) = 0 \\ \\ \partial^2_{u_i}(\partial_{u_1}\sigma + \partial_{u_2}\sigma + \partial_{u_3}\sigma) = 0\,, \end{cases} \tag{107}$$

with the constrain $\partial^3_{u_i}(\partial_{u_1}\sigma + \partial_{u_2}\sigma + \partial_{u_3}\sigma) < 0$ *for* $i = 1, 2$ *and* $\partial^3_{u_i}(\partial_{u_1}\sigma + \partial_{u_2}\sigma + \partial_{u_3}\sigma) > 0$ *for* $i = 3$.

For a proof see appendix A.

Observe that the last two equations of system (107) are algebraic in u_1, u_2, u_3 thus we can give some algebraic constrain to the non existence of solutions of (107).

Theorem 5 *If the initial data (85) satisfies $f'''(u) < 0$, then the one phase equations do not have any point of gradient catastrophe.*

Proof: It is sufficient to prove that the equations

$$\partial^2_{u_i}(\partial_{u_1}\sigma + \partial_{u_2}\sigma + \partial_{u_3}\sigma) = 0\,, \quad i = 1, 2, 3\,, \tag{108}$$

have no real solutions. For the purpose we write an expression [FRT1] of σ which relates it to the initial data $f(u)$:

$$\sigma = \frac{1}{2\sqrt{2}\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{f\left(\frac{1+s}{2}\frac{1+t}{2}u_1 + \frac{1-s}{2}\frac{1+t}{2}u_2 + \frac{1-t}{2}u_3\right)}{\sqrt{(1-t)(1-s^2)}}dt\,ds\,. \tag{109}$$

Taking triple derivative with respect to u_1, u_2, u_3 we obtain

$$\frac{\partial^3\sigma}{\partial u_1^i \partial u_2^j \partial u_3^k} = C\int_{u_2}^{u_1}\frac{\int_{u_3}^{y} f'''(z)(z-u_3)^{i+j}(y-z)^{k-\frac{1}{2}}dz}{(y-u_3)^{\frac{7}{2}}(u_1-y)^{\frac{1}{2}-j}(z-u_2)^{\frac{1}{2}-i}}dy\,,$$

where $i + j + k = 3$ and $C = \dfrac{1}{2\pi(u_1-u_2)^{i+j}}$.

For $f'''(u) < 0$ it holds $\dfrac{\partial^3\sigma}{\partial u_1^i \partial u_2^j \partial u_3^k} < 0$, hence equations (108) have no real solutions.

□

Remark: In [FRT1] Tian has proved that for smooth monotone decreasing initial data satisfying the condition $f'''(u) < 0$, the solution of the one phase Whitham equations exists for all $t > 0$. Here we have proved that the one phase Whitham equations have no points of gradient catastrophe. From the results of Sec. 5.1, also the Burgers equation has no points of gradient catastrophe but $(x = 0, t = 0, u = 0)$ for $f'''(u) < 0$. Hence the solution of the one-phase equations exists for all $t > 0$.

5.5.1 Initial data $f(u) = -u^3 - cu^4 - u^5$

In this section we are going to study the solutions of system (107) for the one parameter family of initial data (54). From Theorem 5 it follows that for $c^2 < \frac{5}{2}$, there is no point

of gradient catastrophe for the one-phase Whitham equations. We have already seen that for c in the intervals

$$-\frac{\sqrt{15}}{2} \leq c \leq \nu_1 \,, \quad \nu_4 \leq c \leq \frac{\sqrt{15}}{2} \,,$$

where ν_1 and ν_4 have been defined in (76) and (83) respectively, the Burgers equation admits two points of gradient catastrophe and there also exists an embedding of M_0 as some component of M_2. From Lemma 1 it follows that the one-phase Whitham equations cannot have a point of gradient catastrophe. Hence a point of gradient catastrophe of the one phase Whitham equations could exist only for c in the intervals:

$$\nu_1 < c \leq -\sqrt{\frac{5}{2}} \,, \quad \sqrt{\frac{5}{2}} \leq c < \nu_4 \,.$$

In order to study the solutions of the system (107) for $f(u) = -u^3 - cu^4 - u^5$, we first consider its last two equations:

$$\begin{cases} 5\sigma_0^3 - 12\sigma_0\sigma_1 + \dfrac{63}{10}\sigma_0 + \dfrac{27}{5}c\,\sigma_0^2 - \dfrac{36}{5}c\,\sigma_1 + 9\sigma_0^2 u_i - 12\sigma_1 u_i + \\ \dfrac{189}{10}u_i + \dfrac{54}{5}c\,\sigma_0\,u_i + 15\sigma_0 u_i^2 + 27c\,u_i^2 + 35u_i^3 = 0 \\ 3\sigma_0^2 - 4\sigma_1 + \dfrac{63}{10} + \dfrac{18}{5}c\,\sigma_0 + 10\sigma_0 u_i + 18c\,u_i + 35u_i^2 = 0 \,, \end{cases} \tag{110}$$

with the constrain $-\sigma_0 - 7\,u_i - \frac{9}{5}\,c < 0$ for $i = 1, 2$ and $-\sigma_0 - 7\,u_i - \frac{9}{5}\,c > 0$ for $i = 3$, where $\sigma_0 = \sum_{j \neq i} u_j$ and $\sigma_1 = \prod_{j \neq i} u_j$, $\sigma_0^2 - 4\sigma_1 > 0$. Eliminating σ_1 in (110) and in $\sigma_0^2 - 4\sigma_1 \geq 0$ we obtain

$$\begin{cases} 16\sigma_0^3 + \dfrac{252}{5}\sigma_0 + \dfrac{216}{5}c\,\sigma_0^2 + \dfrac{1134}{25}c + \dfrac{648}{25}c^2\,\sigma_0 + \\ 120\sigma_0^2 u_i + 288c\,\sigma_0 u_i + \dfrac{648}{5}c^2 u_i + 480\sigma_0 u_i^2 + 360c\,u_i^2 + 280u_i^3 = 0 \\ -2\sigma_0^2 - \dfrac{63}{10} - \dfrac{18}{5}c\,\sigma_0 - 10\sigma_0 u_i - 18c\,u_i - 35u_i^2 \geq 0 \,, \end{cases} \tag{111}$$

with the constrain $-\sigma_0 - 7\,u_i - \frac{9}{5}\,c < 0$ for $i = 1, 2$ and $-\sigma_0 - 7\,u_i - \frac{9}{5}\,c > 0$ for $i = 3$.

The plot of the equations (111) (a cubic and an ellipse respectively) and of the line $-\sigma_0 - 7\,u_i - \frac{9}{5}\,c = 0$ in the (u_i, σ_0) plane for c in the range $\nu_1 < c < -\sqrt{5}/2$ and $\sqrt{5}/2 < c < \nu_4$ looks like Figure 19. Real solutions (u_1, u_2, u_3) of the system (110) exist only when the cubic lies inside the ellipse. From the picture it can be seen that only the compact component of the cubic lies inside the ellipse and it becomes a single point for $c = \pm\nu_3$, where ν_3 has been defined in (75). System (110) has no real solutions for $-\nu_3 < c < \nu_3$. Hence we can restrict ourself to look for solutions of system (107) for c in the intervals $\nu_3 \leq c < \nu_4$ and $\nu_1 < c \leq -\nu_3$.

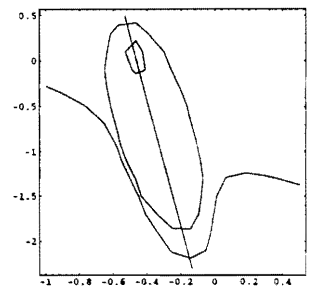


Figure 19: $c = 1.79$

From numerical analysis it turns out that there exists a point of gradient catastrophe on the u_1-branch for c in the interval

$$\nu_1 < c < \nu_2\,.$$

Even though for $c = \nu_2$ system (107) admits a solution, there does not exist a point of gradient catastrophe on u_1 because it also holds $\partial^3_{u_1^3}(\lambda_1\,t + w_1) = 0$.

There exists a point of gradient catastrophe on the u_3-branch for c in the interval

$$\nu_3 < c < \nu_4\,.$$

Even though for $c = \nu_3$ system (107) admits a solution, there does not exist a point of gradient catastrophe on u_3 because it also holds $\partial^3_{u_3^3}(\lambda_3\,t + w_3) = 0$.

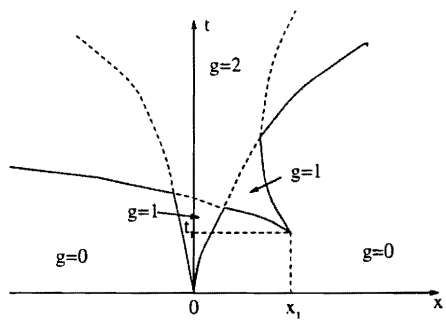
From numerical analysis it turns out that there exists no point of gradient catastrophe on the u_2-branch for the 1-phase Whitham equations.

Observe that from numerical analysis, the interval in which the one-phase equations do not have a point of gradient catastrophe is wider than the interval obtained applying Theorem 4.

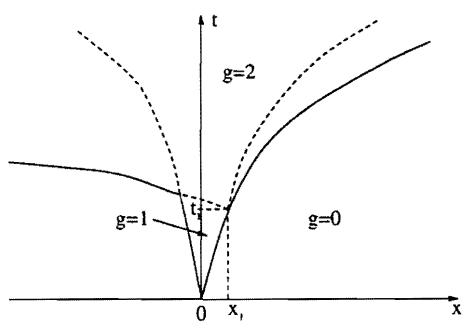
6 Bifurcation diagrams in the $x - t$ plane

From the above analysis we can draw the various topological types of boundaries M_0^1, M_0^2 in the (x, t) plane. We draw with a dashed line the boundaries of M_2.

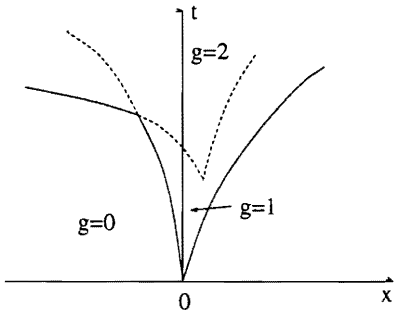
1) $\nu_4 < c \leq \sqrt{\frac{15}{4}}$; there exists a second breakpoint for the Burgers equation in $x_1 > 0$, $t_1 > 0$ there exist a double trailing edge, a leading-trailing edge and a double leading edge hence the bifurcation diagram of the genus $g(x, t)$ is



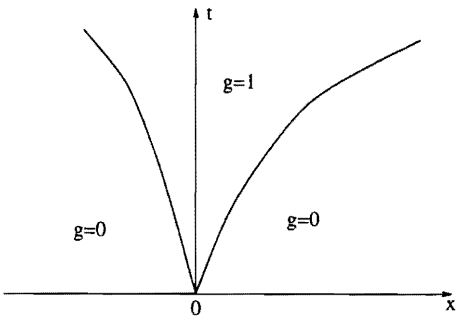
2) $c = \nu_4$; the Burgers equation has a second point of gradient catastrophe in correspondence of the leading edge and there exists a double trailing edge.



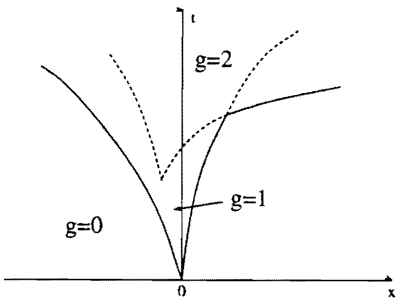
3) $\nu_3 < c < \nu_4$; there exist a point of gradient catastrophe for the one-phase Whitham equations for $t > 0$ and there exist a double trailing edge.



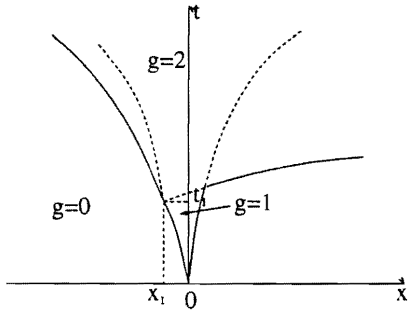
5) $\nu_2 \leq c \leq \nu_3$; there exists just the point $x = t = u = 0$ of gradient catastrophe for the Burgers equation and the one-phase Whitham equations are stable.



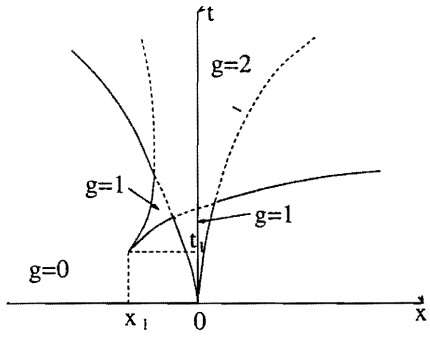
6) $\nu_1 < c < \nu_2$; there exist a point of gradient catastrophe for the one-phase Whitham equations for $t > 0$ and there exist a double leading edge.



7) $c = \nu_1$; the Burgers equation has a second point of gradient catastrophe in correspondence of the trailing edge, there exists a double leading edge.



8) $-\sqrt{\frac{15}{4}} \leq c < \nu_1$; there exists a second breakpoint of the Burgers equation for $x_1 < 0$ and $t_1 > 0$. There exist a double leading edge, a leading-trailing edge and a double trailing edge.



A Proof of Lemma 5

For simplicity let us choose $i = 3$ in system (106). System (106) is equivalent to the system

$$\begin{cases} -x + \lambda_3 t + w_3 = 0 \\ (\lambda_1 - \lambda_3)t + w_1 - w_3 = 0 \\ (\lambda_2 - \lambda_3)t + w_2 - w_3 = 0 \\ \partial_{u_3}(\lambda_3\, t + w_3) = 0 \\ \partial^2_{u_3^2}(\lambda_3\, t + w_3) = 0 \\ \partial^3_{u_3^3}(\lambda_3\, t + w_3) < 0\,. \end{cases} \tag{112}$$

Using (91), (92) we have the following identities:

$$\begin{aligned} F_j &= \frac{\alpha_0 + u_j}{4(u_j - u_3)}[(\lambda_j - \lambda_3)t + w_j - w_3] = 0 \quad j = 1, 2 \\ &= \left(\frac{\Pi_j}{u_j - u_3} - \frac{\Pi_3}{u_j - u_3} - \frac{\Pi_3}{\alpha_0 + u_3}\right)(t + \frac{1}{2}\partial_{u_3}\sigma) + \Pi_j\, \partial^2_{u_j u_3}\sigma\,, \end{aligned} \tag{113}$$

$$\begin{aligned} F_3 &= \tfrac{1}{2}\partial_{u_3}(\lambda_3\, t + w_3)(\alpha_0 + u_3) = 0 \\ &= \left(2\,\Sigma_3 - \frac{\Pi_3}{\alpha_0 + u_3}\right)(t + \frac{1}{2}\partial_{u_3}\sigma) + \Pi_3\partial^2_{u_3^2}\sigma\,, \end{aligned} \tag{114}$$

$$\begin{aligned} F_4 &= \tfrac{1}{2}\partial^2_{u_3^2}(\lambda_3\, t + w_3)(\alpha_0 + u_3) = 0 \\ &= \left(\frac{9}{2} - 4\frac{\Sigma_3^2}{\Pi_3} + \frac{2\,\Sigma_3}{\alpha_0 + u_3}\right)(t + \frac{1}{2}\partial_{u_3}\sigma) - \Pi_3\partial^3_{u_3^3}\sigma\,, \end{aligned} \tag{115}$$

$$\begin{aligned} F_5 &= \tfrac{1}{2}\partial^3_{u_3^3}(\lambda_3\, t + w_3)(\alpha_0 + u_3) > 0 \\ &= \left(\frac{12\Sigma_3^3}{\Pi_3^2} - \frac{26\Sigma_3}{\Pi_3} + \frac{1}{\alpha_0 + u_3}\left(\frac{25}{4} - 6\frac{\sigma_3^2}{\Pi_3}\right)\right)(t + \frac{1}{2}\partial_{u_3}\sigma) - \Pi_3\partial^4_{u_3^4}\sigma\,, \end{aligned} \tag{116}$$

where $\Pi_i = \prod_{j\neq i}(u_i - u_j)$ and $\Sigma_i = \sum_{j\neq i}(u_i - u_j)$ for $i, j = 1, 2, 3$.

From the equalities (113), (114), (115) and (116), system (112) is equivalent to the system

$$\begin{cases} -x + \lambda_3 t + w_3 = 0 \\ F_3 = 0 \\ F_2 - F_1 = 0 \\ \dfrac{F_2 - F_1}{u_2 - u_1} - \dfrac{F_2 - F_3}{u_2 - u_3} = 0 \\ -3\dfrac{\dfrac{F_2 - F_1}{u_2 - u_1} - \dfrac{F_2 - F_3}{u_2 - u_3}}{u_1 - u_3} + \dfrac{3\dfrac{F_2 - F_3}{u_2 - u_3} - 2\dfrac{(F_4\,\Pi_3 + 2F_3\,\Sigma_3)}{\Pi_3}}{u_2 - u_3} = 0 \\ \dfrac{F_5 + 3\dfrac{\Sigma_3}{\Pi_3}F_4 + \dfrac{25}{4\,\Pi_3}F_3}{\Pi_3} > 0\,. \end{cases} \tag{117}$$

Substituting the explicit formula of F_1, F_2, F_3, F_4 and F_5 in the above system one obtains system (107) for $i = 3$. For $i = 1, 2$ analogous computations have to be done. □

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