HILBERT SCHEMES OF POINTS ON SOME K3 SURFACES AND GIESEKER STABLE BUNDLES

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April 1995

By using a Fourier-Mukai transform for sheaves on K3 surfaces we show that for a wide class of K3 surfaces X the Hilbert schemes $\mathrm{Hilb}^n(X)$ can be identified for all $n \geq 1$ with moduli spaces of Gieseker stable vector bundles on X. We also introduce a new Fourier-Mukai type transform for such surfaces.

SISSA Ref. 34/95/GEO–FM

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ABSTRACT. By using a Fourier-Mukai transform for sheaves on K3 surfaces we show that for a wide class of K3 surfaces X the Hilbert schemes $\mathrm{Hilb}^n(X)$ can be identified for all $n \geqslant 1$ with moduli spaces of Gieseker stable vector bundles on X. We also introduce a new Fourier-Mukai type transform for such surfaces.

INTRODUCTION

Let (X, H) be a polarized K3 surface over $\mathbb{C}$ which also carries a divisor ℓ such that

$$H^2 = 2\,, \qquad \ell^2 = -12\,, \qquad H \cdot \ell = 0. \tag{0.1}$$

One must also include a technical condition which can be expressed in the form $H^0(\mathcal{O}(\ell + 2H)) = 0$. This will hold generically. There is an 18-dimensional family of such K3 surfaces which are called 'reflexive' in [3]; these include generic Kummer surfaces. For any sheaf $\mathcal{E}$ on X we denote its Mukai vector by $v(\mathcal{E})$. Recall that $v(\mathcal{E}) = (\mathrm{rk}\,\mathcal{E}, c_1(\mathcal{E}), s(\mathcal{E}))$, where

$$s(\mathcal{E}) = \mathrm{rk}\,\mathcal{E} + \mathrm{ch}_2(\mathcal{E})\,.$$

For such surfaces it can be shown that the moduli space $\widehat{X} = \mathcal{M}(2, \ell, -3)$ of Gieseker stable sheaves $\mathcal{E}$ on X with Mukai vector $v(\mathcal{E}) = (2, \ell, -3)$ is isomorphic to X, and is formed by locally-free μ-stable sheaves. The natural polarization on $\hat{X}$ is $\hat{H}$ which one can show is given by $2\ell + 5H$ if we identify $\hat{X}$ with X. There is a divisor $\hat{\ell}$ on

1991 *Mathematics Subject Classification.* 14D20, 14C05, 14J28, 14F05.

Key words and phrases. Moduli spaces, K3 surfaces, Hilbert schemes, Fourier-Mukai transforms.

Research partly supported by the National Group for Mathematical Physics of the Italian Research Council, by the Italian Ministry for University and Research through the research project 'Geometria delle varietà differenziabili,' the Engineering and Physical Sciences Research Council of United Kingdom and the Edinburgh Mathematical Society. The authors would like to thank the Vector Bundle's group of Europroj. The second author is a member of the VBAC group of Europroj.

$\hat{X}$ which plays the role of ℓ and one can show that it is given by $-5\ell - 12H$ if we identify $\hat{X}$ with X.

In this paper we show the following result.

Theorem 0.1. *For any $n \geqslant 1$, the Hilbert scheme* $\mathrm{Hilb}^n(X)$ *of length-n zero-dimensional subschemes of X is isomorphic, as an algebraic variety, to the moduli space $\mathcal{M}_n = \mathcal{M}(1+2n, -n\hat{\ell}, 1-3n)$ of Gieseker stable bundles on X with respect to $\hat{H}$.*

Notice that the theorem implies that all Gieseker stable sheaves are locally-free. It will also follow that they are never μ-stable with respect to $\hat{H}$. In particular, the underlying smooth bundles do not carry anti-self-dual connections with respect to the Kähler metric associated to $\hat{H}$.

The isomorphism $\mathrm{Hilb}^n(X) \simeq \mathcal{M}_n$ will be established by using a Fourier-Mukai (FM) transform of K3 surfaces (see [2]). This FM transform preserves the natural complex symplectic structures of the moduli spaces [5] and so $\mathrm{Hilb}^n(X)$ and $\mathcal{M}_n$ are isomorphic as complex hyperkähler manifolds as well.

Our results should be compared with those given in [9], where for a general polarized K3 surface (X, H) a birational map $\mathcal{M}(2, 0, -1-n^2H^2) \to \mathrm{Hilb}^{2n^2H^2+3}(X)$ is constructed. Further birational identifications can be found in [1]. A similar version of Theorem 0.1 can be found in [6] for the case of an abelian surface. Our theorem is consistent with the results of [4] which show that moduli spaces of semi-stable torsion-free sheaves have the same Hodge numbers as Hilbert schemes of points of the K3 surface.

The authors would like to thank Daniel Hernández Ruipérez and Claudio Bartocci for useful comments.

1. Fourier-Mukai Transforms for K3 Surfaces

Let us recall some definitions and properties related to the FM transform on X introduced in [8]. Let $\mathcal{Q}$ denote the universal sheaf on $X \times \widehat{X}$ normalized by $\mathbf{R}\hat{\pi}_*(\mathcal{Q}) \cong \mathcal{O}_{\widehat{X}}[-1]$. This gives rise to a Fourier-Mukai transform

$$\mathbf{R}\Phi(\mathcal{E}) = \mathbf{R}\hat{\pi}_*(\pi^*\mathcal{E} \otimes \mathcal{Q}).$$

In [2] this is shown to give an equivalence of derived categories between the derived category $D(X)$ of complexes of coherent sheaves on X and $D(\widehat{X})$. The inverse is given by

$$\mathbf{R}\hat{\Phi}(\mathcal{E}) = \mathbf{R}\pi_*(\hat{\pi}^*\mathcal{E} \otimes \mathcal{Q}^*)$$

up to a shift of complexes.

Definition 1.1. We say that a sheaf $\mathcal{E}$ on X is IT_k if $H^j(X, \mathcal{E} \otimes \mathcal{Q}_\xi) = 0$ for $j \neq k$, where $\mathcal{Q}_\xi = \mathcal{Q}|_{X \times \{\xi\}}$ for $\xi \in \widehat{X}$. We say that $\mathcal{E}$ is WIT_k if $R^j\hat{\pi}_*(\pi^*\mathcal{E} \otimes \mathcal{Q}) = 0$ for $j \neq k$, where π and $\hat{\pi}$ are the projections of $X \times \widehat{X}$ onto the two factors. Obviously, any IT_k sheaf is WIT_k.

For any WIT_k sheaf $\mathcal{E}$ on X we denote its FM transform as the sheaf on $\widehat{X}$

$$\widehat{\mathcal{E}} = R^k\Phi(\mathcal{E}).$$

Note that $\mathcal{E}$ is IT_k if and only if it is WIT_k and its FM transform is locally free.

The main properties of this FM transform are summarized as follows (compare [2]).

Proposition 1.2. *Let $\mathcal{E}$ be a sheaf on X. Then the Chern character of the transform of $\mathcal{E}$ is given by*

$$\begin{aligned}
\mathrm{ch}_0 &= -\,\mathrm{ch}_0(\mathcal{E}) + c_1(\mathcal{E}) \cdot \ell + 2\,\mathrm{ch}_2(\mathcal{E}) \\
\mathrm{ch}_1 &= -c_1(\mathcal{E}) + (c_1(\mathcal{E}) \cdot H - \mathrm{ch}_2(\mathcal{E}))\hat{\ell} + c_1(\mathcal{E}) \cdot (\ell + 2H)\hat{H} \\
\mathrm{ch}_2 &= -5\,\mathrm{ch}_2(\mathcal{E}) - 2c_1 \cdot \ell.
\end{aligned}$$

From this it follows that $\deg \mathbf{R}\Phi\mathcal{E} = -\deg \mathcal{E}$ *and* $\chi(\mathbf{R}\Phi\mathcal{E}) = -\chi(\mathcal{F})$.

Proposition 1.3. *(Invertibility) Let $\mathcal{E}$ be a WIT_k sheaf on X. Then its FM transform $\widehat{\mathcal{E}} = R^k\hat{\pi}_*(\pi^*\mathcal{E} \otimes \mathcal{Q})$ is a WIT_{2-k} sheaf on $\widehat{X}$, whose inverse FM transform $R^{2-k}\pi_*(\hat{\pi}^*\widehat{\mathcal{E}} \otimes \mathcal{Q}^*)$ is isomorphic to $\mathcal{F}$.*

2. The isomorphism between the Hilbert scheme and the moduli space

We give a definition that extends to K3 surfaces the notion of homogeneous bundle on an abelian variety (cf. [7] [3]).

Definition 2.1. A coherent sheaf $\mathcal{F}$ on $\widehat{X}$ is *quasi-homogeneous* if it has a filtration by sheaves of the type $\widehat{\mathcal{O}_Y}$, where the Y's are zero-dimensional subschemes of X, so that the associated grading is of the form $\oplus_k \mathcal{Q}_{p_k}$, with $p_k \in X$.

Let W be a zero-dimensional subscheme of X; the structure sheaf $\mathcal{O}_W$ is IT_0, and its FM transform $\widehat{\mathcal{O}}_W$ is a quasi-homogeneous locally free sheaf. $\widehat{\mathcal{O}}_W$ is μ-semistable, due to the following result.

Proposition 2.2. *Let $p \in X$. Any nontrivial extension*

$$0 \longrightarrow \mathcal{Q}_p \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q}_p \longrightarrow 0 \tag{2.1}$$

is μ-semistable. Any destabilizing μ-semistable subsheaf of $\mathcal{F}$ is isomorphic to $\mathcal{Q}_p$.

Proof. The first part is standard. For the second part one observes that any torsion-free destabilising sheaf $\mathcal{Q}'$ of $\mathcal{F}$ which is μ-semistable must have Chern character $(2, \ell, -5)$ because both $\mathcal{Q}'$ and $\mathcal{F}/\mathcal{Q}'$ must satisfy the Bogomolov inequality. Then it must also be locally-free by the Bogomolov inequality applied to $\mathcal{Q}'^{**}$. □

Lemma 2.3. *The ideal sheaf $\mathcal{I}_W$ is IT_1.*

Proof. Let $\xi \in \widehat{X}$. Then $H^0(X, \mathcal{I}_W \otimes \mathcal{Q}_\xi) \hookrightarrow H^0(X, \mathcal{Q}_\xi) = 0$ because $\mathcal{Q}_\xi$ is μ-stable, and

$$H^2(X, \mathcal{I}_W \otimes \mathcal{Q}_\xi)^* \simeq \operatorname{Ext}^0(\mathcal{I}_W \otimes \mathcal{Q}_\xi, \mathcal{O}_X) \simeq \operatorname{Hom}(\mathcal{I}_W, \mathcal{Q}_\xi^*) = 0$$

since $\mathcal{Q}_\xi$ is locally free. □

So by applying the FM transform to the sequence

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_W \longrightarrow 0$$

we get

$$0 \longrightarrow \widehat{\mathcal{O}}_W \longrightarrow \widehat{\mathcal{I}}_W \longrightarrow \mathcal{O}_{\widehat{X}} \longrightarrow 0. \tag{2.2}$$

Proposition 2.4. *The FM transform $\widehat{\mathcal{I}}_W$ is Gieseker stable.*

Proof. Since $\operatorname{ch}(\widehat{\mathcal{I}}_W) = (1, 0, -n)$, by the formulas in Proposition 1.2 we obtain

$$\operatorname{rk} \widehat{\mathcal{I}}_W = 1 + 2n, \quad \operatorname{ch}_2 \widehat{\mathcal{I}}_W = -5n, \quad \tilde{p}(\widehat{\mathcal{I}}_W) = \frac{\chi(\widehat{\mathcal{I}}_W)}{\operatorname{rk} \widehat{\mathcal{I}}_W} = \frac{2-n}{1+2n} > -\frac{1}{2}.$$

Let $\mathcal{A}$ be a destabilizing subsheaf of $\widehat{\mathcal{I}}_W$, that we may assume to be Gieseker stable with a torsion-free quotient. Then we have $\tilde{p}(\mathcal{A}) \geqslant \tilde{p}(\widehat{\mathcal{I}}_W) > -\frac{1}{2}$. Let f denote the composite $\mathcal{A} \to \widehat{\mathcal{I}}_W \to \mathcal{O}_{\widehat{X}}$.

There are two cases:

Case (i) $f = 0$. Then there is a map $\mathcal{A} \to \widehat{\mathcal{O}}_W$. Let $g_k \colon \mathcal{A} \to \mathcal{Q}_{p_k}$ be the composition of this map with the canonical projection onto $\mathcal{Q}_{p_k}$. Since $\tilde{p}(\mathcal{A}) > -\frac{1}{2} = \tilde{p}(\mathcal{Q}_{p_k})$ and both sheaves are Gieseker stable, we obtain $g_k = 0$ for all k, which is absurd.

Case (ii) $f \neq 0$. We divide this into two further cases: $\operatorname{rk} \mathcal{A} = 1$ and $\operatorname{rk} \mathcal{A} > 1$.

If $\operatorname{rk} \mathcal{A} = 1$ we have $\mathcal{A}^* \simeq \mathcal{O}_{\widehat{X}}$; hence the sequence (2.2) splits, which contradicts the inversion theorem $\widehat{\widehat{\mathcal{I}}_W} \simeq \mathcal{I}_W$.

If $\operatorname{rk} \mathcal{A} > 1$ we consider the exact sequences

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{A} \xrightarrow{h} \mathcal{B} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{O}_{\widehat{X}} \longrightarrow \mathcal{K}_2 \longrightarrow 0,$$

where $\operatorname{rk}\mathcal{K}_2 = 0, 1$. If $\operatorname{rk}\mathcal{K}_2 = 1$ then $\mathcal{B} = 0$, i.e. $f = 0$ which is absurd, so that $\operatorname{rk}\mathcal{K}_2 = 0$, and $\mathcal{B}$ has rank one. We have an exact commuting diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathcal{K}_3 \arrow[r] \arrow[u] & \mathcal{K}_4 \arrow[r] \arrow[u] & \mathcal{K}_2 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \widehat{\mathcal{O}}_W \arrow[r] \arrow[u] & \widehat{\mathcal{I}}_W \arrow[r] \arrow[u] & \mathcal{O}_{\widehat{X}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{K}_1 \arrow[r] \arrow[u, "g"] & \mathcal{A} \arrow[r, "h"] \arrow[u] & \mathcal{B} \arrow[r] \arrow[u, "h'"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{2.3}$$

with $\mu(\mathcal{K}_1) = 0$, $0 < \operatorname{rk}\mathcal{K}_1 < 2n$ and $f = h' \circ h$.

If $n = 1$ then $\widehat{\mathcal{O}}_W$ is μ-stable, but this is a contradiction. For $n > 1$, we may assume that $\mathcal{K}_1$ is μ-semistable so that it is a direct summand of $\widehat{\mathcal{O}}_W$. Then $\mathcal{K}_3$ is locally free and $\operatorname{rk}\mathcal{K}_1 \geqslant 2$. Moreover, $\mu(\mathcal{B}) \leqslant 0$ because $\mathcal{B}$ injects into $\mathcal{O}_{\widehat{X}}$, and $\mu(\mathcal{B}) \geqslant 0$ because $\mu(\mathcal{K}_1) \leqslant 0$. Then $\mu(\mathcal{B}) = \mu(\mathcal{K}_1) = 0$. Since $\mathcal{K}_3$ is locally free the support of $\mathcal{K}_2$ is not zero-dimensional. So $\mu(\mathcal{B}) = 0$ implies $\mathcal{K}_2 = 0$ and $\mathcal{K}_3 \simeq \mathcal{K}_4$.

Finally, we consider the middle column in (2.3). The sheaf $\mathcal{A}$ has rank greater than 2, and is Gieseker stable, so that it is IT_1. But $\widehat{\mathcal{I}}_W$ is WIT_1 while $\mathcal{K}_4$ is WIT_2. Then $\mathcal{A} \simeq \widehat{\mathcal{I}}_W$, but this is a contradiction. $\square$

Note that $\widehat{\mathcal{I}}_W$ is never μ-stable because (2.2) destabilizes it.

Let $\mathcal{M}_n$ be the moduli space $\mathcal{M}(1+2n, -n\hat{\ell}, 1-3n)$ of Gieseker stable sheaves on $\widehat{X}$. The previous construction yields a map $\operatorname{Hilb}^n(X) \to \mathcal{M}_n$. This map is algebraic because the Fourier-Mukai transform is functorial and so preserves the Zariski tangent spaces (see [5]). Another way to see this is to observe that the Fourier-Mukai transforms give a natural isomorphism of moduli functors and so give rise to an isomorphism of (coarse or fine) moduli schemes.[1] We shall now show that the Fourier-Mukai transform is a surjection up to isomorphism.

Lemma 2.5. *Any element $\mathcal{F} \in \mathcal{M}_n$ is WIT_1.*

[1] We would like to thank Daniel Hernández Ruipérez for this observation.

Proof. Since $\tilde{p}(\mathcal{F}) > -\frac{1}{2}$ and $\tilde{p}(\mathcal{Q}_p) = -\frac{1}{2}$ there is no map $\mathcal{F} \to \mathcal{Q}_p$. This means that $H^2(\widehat{X}, \mathcal{F} \otimes \mathcal{Q}_p^*) = 0$.

We consider now nonzero morphisms $\mathcal{Q}_p \to \mathcal{F}$. Any such map is injective; otherwise it would factorize through a rank-one torsion-free sheaf $\mathcal{B}$ with $\mu(\mathcal{B}) > 0$ (because $\mathcal{Q}_p$ is μ-stable) and $\mu(\mathcal{B}) \leqslant 0$ (because $\mathcal{F}$ is μ-semistable), which is impossible. Then $\mathcal{Q}_p$ is a locally free element of a Jordan-Holder filtration of $\mathcal{F}$. Since any such filtration has only a finite number of terms, and the associated grading $\operatorname{gr}(\mathcal{F})^{**}$ is unique, there is only a finite number of p's giving rise to nontrivial morphisms, i.e. $\operatorname{Hom}(\mathcal{Q}_p, \mathcal{F}) \simeq H^0(X, \mathcal{F} \otimes \mathcal{Q}_p^*)$ does not vanish only for a finite set of points p. This suffices to prove that $\mathcal{F}$ is WIT_1 due to Proposition 2.26 of [8]. □

Proposition 2.6. *The FM transform $\hat{\mathcal{F}}$ of $\mathcal{F}$ is torsion-free.*

Proof. Let $\mathcal{T}$ be the torsion subsheaf of $\hat{\mathcal{F}}$, so that one has an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \hat{\mathcal{F}} \longrightarrow \mathcal{G} \longrightarrow 0. \tag{2.4}$$

Since $\mathcal{T}$ is supported at most by a divisor, and $\hat{\mathcal{F}}$ is WIT_1, the sheaf $\mathcal{T}$ is WIT_1 as well. Moreover $\deg(\mathcal{T}) \geqslant 0$. If $\deg \mathcal{T} = 0$ then $\mathcal{T}$ is IT_0, i.e. $\mathcal{T} = 0$.

Hence, we assume $\deg(\mathcal{T}) > 0$. The rank-one sheaf $\mathcal{G}$ is torsion-free and, by imbedding it into its double dual, we see that it is IT_1. Then, applying $\mathbf{R}\hat{\mathbf{\Phi}}$ to (2.4), we get

$$0 \longrightarrow \hat{\mathcal{T}} \longrightarrow \mathcal{F} \longrightarrow \hat{\mathcal{G}} \longrightarrow 0.$$

Since $\mathcal{F}$ is μ-semistable we see that

$$\deg \mathcal{T} = \deg \hat{\mathcal{T}} \leqslant 0,$$

which is a contradiction. □

Now the Chern character of $\hat{\mathcal{F}}$ is $(1, 0, -n)$, so that it is the ideal sheaf of a zero-dimensional subscheme of X of length n. We have therefore shown that the Fourier-Mukai transform surjects as a map $\operatorname{Hilb}^n X \to \mathcal{M}_n$. The inversion theorem for $\mathrm{R}\Phi$ therefore implies that the transform gives an isomorphism of smooth varieties. This establishes Theorem 0.1.

3. Another Fourier-Mukai Transform

We shall now show that $\mathcal{M}_1$ gives rise to another FM transform.

Lemma 3.1. $\dim H^1(\mathcal{Q}) = 1$.

Proof. This follows immediately from the degeneration of the Leray spectral sequence applied to $\hat{\pi}$ and the fact that $\mathbf{R}\hat{\pi}_*(\mathcal{Q}) = \mathcal{O}_{\widehat{X}}[-1]$. $\square$

This lemma shows that there is a unique extension

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{X\times\widehat{X}} \longrightarrow 0. \tag{3.1}$$

Both of the restrictions of $\mathcal{E}$ to the factors of $X \times \widehat{X}$ give families of Gieseker stable bundles. Then the general theory of FM transforms (see [6]) implies that $\mathcal{E}$ gives rise to an FM transform which we denote by $\mathbf{R}\Psi$.

Proposition 3.2. *The FM transform* $\mathbf{R}\Psi$ *and its inverse* $\mathbf{R}\hat{\Psi}$ *satisfy the following:*

(1) $\mathbf{R}\Psi\mathcal{O}_X \cong \mathcal{O}_{\widehat{X}}[-2]$,
(2) $\mathrm{ch}(\mathbf{R}\Psi\mathcal{F}) = \chi(\mathcal{F})\,\mathrm{ch}(\mathcal{O}_{\widehat{X}}) + \mathrm{ch}(\mathbf{R}\Phi\mathcal{F})$, *where* $\mathcal{F}$ *is any sheaf on* X *and*
(3) $\mathbf{R}\hat{\Psi}\mathcal{I}_p \cong \mathcal{Q}_p^*[-1]$, *for* $p \in \widehat{X}$.

Proof. Apply $\mathbf{R}\hat{\pi}_*$ to the short exact sequence 3.1 to obtain a long exact sequence. Note that $\mathbf{R}\hat{\pi}_*\mathcal{O}_{X\times\widehat{X}}$ is concentrated in the 0th and 2nd positions and $\mathbf{R}\Phi\mathcal{Q} = \mathcal{O}_{\widehat{X}}[-1]$. Then part (1) follows immediately.

For the second part just twist 3.1 by $\pi^*\mathcal{F}$ and apply $\mathbf{R}\hat{\pi}_*$. Then the formula follows from the facts that the Chern character is additive with respect to triangles in $D(\widehat{X})$ and

$$\mathbf{R}\hat{\pi}_*(\pi^*\mathcal{E}) = \mathbf{R}\Gamma(X,\mathcal{E}) \otimes \mathcal{O}_{\widehat{X}}$$

by the projection formula, where Γ denotes the sections functor.

By (1) we have $\mathbf{R}\hat{\Psi}\mathcal{O}_{\widehat{X}} = \mathcal{O}_X$. We also have $\mathbf{R}\hat{\Psi}\mathcal{O}_p \cong \mathcal{E}_p^*$ from the definition of $\mathbf{R}\hat{\Psi}$. Then when we apply $\mathbf{R}\hat{\Psi}$ to the structure sequence of p we obtain the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}_p^* \longrightarrow R^1\hat{\Psi}\mathcal{I}_p \longrightarrow 0.$$

This is just the dual of (2.2). This completes the proof. $\square$

Note that $\mathcal{I}_{\Pi^\cdot}$ does not satisfy WIT with respect to $\mathbf{R}\Psi$ but $\mathcal{I}_{\widehat{\Pi}^\cdot}$ does satisfy WIT_1 with respect to $\mathbf{R}\hat{\Psi}$ and the transform is just the dual of the corresponding quasi-homogeneous bundle.

4. Concluding remarks

Since both the moduli spaces and the punctual Hilbert schemes have complex symplectic structures which are given by the cup product on $\mathrm{Ext}^1(E,E)$ the FM transforms will preserve the symplectic structures and so are complex symplectic isomorphisms. It follows immediately that the spaces are hyperkähler isometric as well.

Theorem 0.1 has several immediate consequences which we can state in the following theorem.

Theorem 4.1. *Let X be a reflexive K3 surface.*

(1) *The moduli space $\mathcal{M}_n$ of Gieseker stable sheaves on X is connected and projective. All points of $\mathcal{M}_n$ are locally-free.*
(2) *$\mathcal{M}_n$ contains no μ-stable sheaves and so the moduli space of irreducible $U(2n+1)$-instantons, with fixed determinant $\mathcal{O}(\hat{\ell})^{-n}$ and second Chern character $-5n$, is empty.*
(3) *The moduli space of all instantons with this type is isomorphic to the n^{th} symmetric product S^nX.*

For the last part one uses the fact that any μ-semistable sheaf of the given Chern character admits a surjection to $\mathcal{O}_X$ and so fits into a sequence of the form (2.2).

References

1. Ballico E., Chiantini L., On some moduli space of rank 2 bundles over K3 surfaces, *Boll. U.M.I.*, **7-A** (1993) 279–287.
2. Bartocci C. Bruzzo U. Hernández Ruipérez D., *A Fourier-Mukai transform for stable bundles on K3 surfaces*, Preprint DIMA 261/1994; Alg-Geom Preprint # 9405006.
3. ———, Existence of μ-stables bundles on K3 surfaces and the Fourier-Mukai transform, to appear
4. Göttsche L. & Huybrechts D., *Hodge numbers of moduli spaces of stable bundles on K3 surfaces*, Preprint, M.P.I. (1994).
5. Maciocia A., The determinant line bundle over moduli spaces of instantons on abelian surfaces, *Math. Z.*, **217** (1994) 317–333. MAZEA
6. ———, *Generalised Fourier-Mukai Transforms*, Preprint, Edinburgh University (1994).
7. Mukai S., Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.*, **81** (1981) 153–175. NGMJA
8. ———, On the moduli space of bundles on a K3 surface I, in *Vector bundles on algebraic varieties*, ed. M.F.Atiyah et. al., Tata Institute of Fundamental Research, Oxford University Press, Bombay and London , (1987)
9. Zuo K., The moduli spaces of some rank-2 stable vector bundles over algebraic K3-surfaces, *Duke Math. J.*, **64** (1991) 403–408. DUMJA

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