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TRIESTE

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P. Budinich

1. INTRODUCTION

Infrared and ultraviolet divergences in relativistic quantum field theories are among the main difficulties encountered by theoretical physics of this century. The former appear when integrations are performed in flat Minkowski space-time $M = \mathbb{R}^{3,1}$. They could be naturally eliminated if, as often advocated [1], space-time M would be conformally compactified in M_c, which may be identified with the homogeneous space of the conformal group G:

$$M_c = \frac{G}{H_1} = \frac{S^3 \times S^1}{Z_2} \quad , \tag{1}$$

where the eleven parameter subgroup H_1 of G is given by: $H_1 = L \otimes D \times) K$, where L, D and K mean Lorentz, dilatations, special conformal transformations respectively. The Fourier dual to M_c, is an infinite lattice.

The ultraviolet divergences appear after integrations in flat momentum in space $P = \mathbb{R}^{3,1}$, the Fourier dual of Minkowski space-time M. They could also be naturally eliminated if P could be compactified. Now it happens that G has another eleven parameter subgroup $H_2 = L \otimes D \times) P_0$ obtained from H_1 by substituting K with Poincarè translation P_0 and then the homogenous space G/H_2 could be a natural candidate for P_c the conformal compactification of P:

$$P_c = \frac{G}{H_2} = \frac{S^3 \times S^1}{Z_2} \quad , \tag{2}$$

also because M_c and P_c are transformed in each other by the conformal inversion I. In such a compactified momentum space P_c any field theory would be naturally free from ultraviolet divergences. The corresponding Fourier dual (space-time) would be an infinite lattice.

In previous papers [2] the simultaneous compactification of M and P in M_c and P_c was conjectured. One should expect that in such a world any field theory would be naturally free from both infrared and ultraviolet divergences. Here we present some further arguments, deriving from spinor geometry and from phenomenological observations, on why M and P should be both conformally compactified. We further observe that, if both M_c and P_c are realized in nature, then to deal with field theories in such a world the concept of Fourier dual has to be redefined. We give a simple example (the space-time $M = R^{1,1}$) where M_c and P_c are represented by two <u>finite</u>, discrete, lattices, connected by a newly defined Fourier transform, and where divergences may not appear.

2. COMPACTIFIED MOMENTUM SPACE

2.1 Arguments from spinors geometry

The geometry of simple (or pure) spinors was discovered by E. Cartan, for our purpose, it may be summarized as follows [3]. Given a 2n-dimensional pseudoeuclidean space $W = \mathbb{R}^{n+1,n-1}$ and the corresponding Clifford algebra Cl(n+1, n-1) with generators γ_a, a = 1, 2,..., 2n, a spinor ϕ is a vector of the representation space S of Cl(n+1, n-1) = End S, defined by

$$p_a \gamma^a \phi = 0 \tag{3}$$

where p_a are the othonormal components of a vector $p \in W$. This vector is null since, for $\phi \neq 0$, eq. (3) implies $p_a p^a = 0$, defining the lightcone $Q \subset W$.

For fixed ϕ, all $p \in W$ satisfying (3) define a totally null plane $M(\phi) \subset W$, whose vectors are both null and mutually orthogonal. Let $\gamma_{n+1} = \gamma_1, \gamma_2, \ldots, \gamma_{2n}$ represent the volume element of Cl(n+1, n-1), then the spinors ϕ_+, ϕ_- obeying $\gamma_{2n+1}\, \phi_\pm = \pm \phi_\pm$ are named Weyl spinors and for them the defining equation (3) becomes

$$p_a\, \gamma^a (1 \pm \gamma_{2n+1})\, \phi_\pm = 0 \tag{4}$$

A Weyl spinor ϕ_+ or ϕ_- is named simple (or pure) if $M(\phi_\pm) \subset W$ is n-dimensional, that is, maximal. For $n \leq 3$ all Weyl spinors ϕ_+, ϕ_- are simple. For n = 3, the four component Weyl spinors $\phi_\pm$ were named twistors by Penrose.

The light cone $p_a p^a = 0$, considered as a projective quadric PQ, defines the 2n-2 compact space V_c:

$$PQ: \quad V_c = \frac{S^n \times S^{n-2}}{Z_2} \tag{5}$$

The pseudo - euclidean space $V = \mathbb{R}^{n,n-2}$ may be densely imbedded in V_c on which the conformal goup in V, represented linearly by 0(n+1, n-1) acting on W, acts globally and transitively. V_c may be considered as the conformal compactification of V.

For n = 2, $W = \mathbb{R}^{3,1}$ and eq. (4) becomes $p_\mu\, \gamma^\mu (1 \pm \gamma_5)\, \phi_\pm = 0$, $\mu = 1, 2, 3, 0$ which is the Weyl equation for massless neutrinos, in momentum space. Therefore $W = \mathbb{R}^{3,1}$ above may be conceived as the Fourier dual $P = \mathbb{R}^{3,1}$ to Minkowski space-time $M = \mathbb{R}^{3,1}$. Also Maxwell's equations in momentum space may be obtained from the Weyl equations (4) above [4].

The equations (4) for spinors associated with $\mathbb{R}^{4,2}$, or twistors, are: $p_a\, \gamma^a (1 \pm \Gamma_7)\, \phi_\pm = 0$. In this case the projective quadric $p_a p^a = 0$ defines the conformally compactified momentum space: $P_c = S^3 \times S^1/Z_2$ identical to eq. (2).

Another feature of spinor geometry, whose importance was underlined by Cartan, is that vectors and tensors of euclidean spaces may be obtained as bilinear polynomia of spinor components. In particular, he formulated the conjecture that, in view of the elegance of simple spinor geometry, it is on this geometry, or equivalently, on the geometry of totally null planes, that euclidean geometry is fundamentally based. In other words, simple spinor geometry is elementary while euclidean geometry derives from it despite the obvious fact that while the latter is accessible to our intuition, the former is not. We will adopt it and mention it as "Cartan's conjecture".

In our case we have that, for $W = \mathbb{R}^{n+1,n-1}$, its null vectors may be conceived, in the frame of Cartan's conjecture, as bilinear spinor polynomia. In fact, indicating by ψ an arbitrary spinor associated with W, the following proposition holds [5]:

Proposition 2.1: the vectors $p^\pm \in \mathbb{R}^{n+1,n-1}$ with components

$$p_\pm{}^a = \langle \psi^c B, \gamma^a \phi_\pm \rangle \qquad a = 1, 2, \ldots 2n, \tag{6}$$

are null if and only if the spinors $\phi_\pm = 1/2(1 \pm \gamma_{2n+1})\, \phi$ are simple. $p_\pm{}^a$ are real (or complex) if n is even (or odd). (We have indicated by ψ^c the charged conjugate of ψ defined by $\psi^c = C\bar{\psi}$, where

$\bar{\psi} \equiv$ complex conjugate of ψ and the matrices B and C are defined by $\gamma_a B = \gamma^t_a B$, $C\gamma_a = \bar{\gamma}_a C$, where $\gamma^t_a \equiv \gamma_a$ – transposed).

We have, therefore, that the null vectors p_a of pseudoeuclidean space $\mathbb{R}^{n+1,n-1}$, building up conformally compactified momentum spaces may be constructured, as bilinear spinor functions, from simple spinors, but then, because of the above proposition, the minimal simple spinors to build up real p_a constituting conformally compactified momentum space P_c would be the 8 component ones associated with $\mathbb{R}^{5,3}$, (which also obey triality) [5]. Therefore to arrive to P_c we should consider simple spinors associated with the imbeddings of the following pseudoeuclidean spaces: $\mathbb{R}^{3,1} \Rightarrow \mathbb{R}^{4,2} \Rightarrow \mathbb{R}^{5,3}$, and the conformally compactified P_c above should be conceived as imbedded in $V_c = S^4 \times S^2/Z_2$.

It may be shown [4] that real null vectors in the form (6) may be constructed in lorentzian pseudoeuclidean spaces $\mathbb{R}^{2n-1,1}$. In particular for $\mathbb{R}^{5,1}$ they give rise to the Dirac equation in the form: $(p^\mu \gamma_\mu \gamma_5 + m)\ \psi = 0$; and in $\mathbb{R}^{7,1}$ to the equation:

$$(p^\mu \gamma_\mu + \vec{\pi}.\vec{\sigma}\ \gamma_5 + iM)\ \Psi = 0 \qquad (7)$$

where $M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ and $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ with ψ_1, ψ_2 Dirac spinors and $\vec{\pi} = 1/8\ (B\ \psi_c \vec{\sigma} \times \gamma_5\ \psi)$, representing the equation for nuclons interacting with pseudoscalar pions in momentum space. Observe that the generators σ_1, σ_2, σ_3 of isospin internal symmetry here derive from generators of reflections with respect to the 5°, 6°, 7° axis [4].

The idea of considering the vectors of pseudo-Euclidean spaces, and in particular of Minkowski space-time, as derived (bilinearly) from underlying spinor geometry is an old dream which has interested several distinguished physicists (Heisenberg, Penrose). From the above considerations it would appear that simple spinors might indeed naturally build up compactified spaces, however for momentum space rather than for Minkowski space-time. Furthermore, it is remarkable that the null vectors of such momentum spaces, generated by simple spinors, naturally appear in most of the basic equations of relativistic quantum mechanics (Weyl, Maxwell, Dirac, etc.). Therefore, simple spinor geometry could be the underlying geometry not only for euclidean geometry (of momentum-space), as conjectured by Cartan, but also for quantum mechanics which, in turn, underlies classical mechanics, naturally described in configuration space. This genesis of quantum mechanics in conformally compactified momentum spaces could offer the advantage of eliminating at the origin the difficulties of ultraviolet divergences, which are the most severe ones of quantum field theory.

2.2 The hydrogen atom in momentum space

In 1935 Fock showed how the Schroedinger equation for the H atom in stationary states which, after a Fourier transformation, may be written as an integral equation in momentum space, may be formulated on S^3: one point compactification of $\mathbb{R}^3$-momentum space, after the inverse of a stereographic projection. On S^3 the Fock integral equation for the H-atom has elementary solutions of the form:

$$Y_{n\ell m}(p) = \frac{\lambda}{2\pi^2} \int_{S^3} \frac{Y_{n\ell m}(p')}{(p-p')^2} d^3 p' , \text{ with } \lambda = \frac{e^2}{\hbar c} \frac{mc}{p_0} = n+1 \qquad (9)$$

where p_0 is a unit of momentum. Assuming that $p_0 = (-2mE)^{1/2}$, Fock obtains for E the energy levels of the H atom. The corresponding wave functions $Y_{n\ell m}$ are the S^3-spherical harmonics.

If we now assume, following Cartan's conjecture proposed above, that the real components p_μ of the vectors of momentum space $P = \mathbb{R}^{3,1}$ densely contained in P_c have to be thought as spinor bilinears, then, as we have seen, the minimal spinors are those associated with $\mathbb{R}^{5,3}$ and that integral equations could be geometrically derived from the modality of the embeddings mentioned above. If furthermore one could derive from this modality also the value of λ ior $e^2/\hbar c$ in (9) then one could obtain a purely geometric genesis of quantization.

Eq. (9) represents the equations of motion of the H-atom in stationary states: a non relativistic quantum system. In the frame of our hypothesis of the conformal compactification of both space-time M and of momentum space P, there could correspond to it a system in ordinary space (compactified) presenting also an S0(4) symmetry; and it should be a classical system. Such a classical system might indeed exist in nature and it could be represented not only by planetary notions but also by the universe, as the recently observed large structures of galaxies seem to suggest.

3. CONFORMALLY COMPACTIFIED SPACE-TIME

Should space-time be represented by the homogeneous space M_c in eq.(1), then, as is often assumed, the space-time of cosmology could be: $M_{RW} = S^3 \times \mathbb{R}/Z_2$ where $\mathbb{R}$ represents the infinite covering of S^1 in eq.(1), which also represents Robertson-Walker space; the familiar space-time of several cosmological models (for a closed universe) whose metric may be unambiguously deduced from the universally accepted Cosmological Principle (stating the isotropy and uniformity of the universe). Since notoriously particular solutions of equations of motion may "spontaneously" violate the symmetry principles implied by the covariance group of those equations, if physical space-time results from the solution of field equations on M_{RW}, one could observe in the physical universe "spontaneous" violations of the Cosmological Principle, as seems indeed to be the case.

3.2 Possible spontaneous violations of the Cosmological Principle

Recent observations of galaxies distribution in the universe have revealed large-scale structures (of sizes larger than 50 Mpc/h where h = 0.5-1 stands for the Hubble's constant in units of 100 km s^{-1} Mpc^{-1}) which, as generally agreed, could not be due to gravity (at the origin of galaxies-clusters or-superclusters of the size up to 10-20 Mpc/h). They could then represent an unexpected violation of the uniformity and isotropy of the universe postulated by the Cosmological Principle. The most impressive are those discovered at large distances (up to 1000 Mpc/h) through "pencil beam surveys" which show striking regularity in galaxies distributions with a periodicity of 128 Mpc/h (for more than 11 peaks) [6]. It has been shown [7] that they may be explained through a scalar field representing eigenvibrations on S^3 represented by $Y_{n\ell m}$. We have taken the most symmetric form $Y_{n,0,0}$ (also because the model should be applicable to Cosmic Background Radiation (CBR) whose main characteristic is its high isotropy), having the simple form: $Y_{n,0,0} =$

$=k_n \sin (n+1)\, \rho/\sin \rho$, with $n = 46$ where k_n is a constant, ρ is the geodesic distance from the center O_c of the eigenmode and $n = 46$ was determined from the periodicity of 128 Mpc/h (in the hypothesis that $\rho_{max} = R\, \pi/2 = 3008$ Mpc/h). The observed structures of galaxies may be then well reproduced [7].

Should this model of the universe be confirmed by further observations, then we would have in nature two non relativistic systems, the Universe and the H-atom whose geometrical structure would be determined by the spherical harmonics $Y_{n\ell m}$ of S^3, however the first in ordinary space and the second in momentum space. This could suggest to consider those S^3 as the space and momentum sectors of the conformally compactified spaces M_c and P_c of eq. (1) and (2) and then to adopt the hypothesis discussed in the next section.

4. SIMULTANEOUS COMPACTIFICATION OF SPACE-TIME AND MOMENTUM SPACE

We know that the flat Minkowski space M and its Fourier dual P both isomorphic to $\mathbb{R}^{3,1}$ may be densely imbedded in M_c and P_c respectively. However, as such they are not flat but conformally-flat and in this way the standard formalism of Fourier transforms may not be adopted, and should be substituted with a new algorithm which is difficult to realize in general. It may be performed if $\mathbb{R}^3$-space is restricted to $\mathbb{R}^1$ and, correspondingly, if, in the compactified version, S^3 is restricted to S^1. In this case one may formulate an exactly soluble toy model from which it is possible to obtain interesting suggestions for the higher dimensional cases.

4.1 The toy model

Let us consider the space $W = \mathbb{R}^{2,1}$ and its projective light cone: PQ: $V_c = S^1/Z_2$ which is the one point conformal compactification of $V = \mathbb{R}^1$. The 3-parameter conformal group on V is: $G = D \times) T \times) K$ isomorphic to SO(2,1) in $W = \mathbb{R}^{2,1}$. V_c above may be identified with the homogeneous space

$$V_c = \frac{G}{H_1} = \frac{S^1}{Z_2} \qquad \text{where } H_1 = D \times) K . \tag{10}$$

Suppose S^1 has radius $\mathbb{R}$ and indicate with $x = R\,\phi$ the arc length on S^1, any single valued function f on S^1 may be expressed, as usual, through the infinite Fourier series:

$$f(x) = (2\pi)^{-\frac{1}{2}} \sum_{m=-\infty}^{+\infty} e^{i k_m x} F(k_m) \tag{11}$$

where $k_m = m/R$. Therefore conformal compactification V_c of $V = \mathbb{R}^1$ has restricted the dual momentum space $P = \mathbb{R}^1$ to the infinite lattice $k_m = m/R$; $-\infty < m < +\infty$, with spacing $\Delta k_m = 1/R$.

Let us now suppose that the dual momentum space $P = \mathbb{R}^1$ is compactified and represented by the homogeneous space: $P_c = G/H_2 = \tilde{S}^1/Z_2$ with $H_2 = D \times) P$. Repeating the previous procedure and supposing that M is the radius of $\tilde{S}^1$ we end up with

$$F(k) = (2\pi)^{-\frac{1}{2}} \sum_{m=-\infty}^{+\infty} e^{-i x_m k} f(x_m) \tag{11'}$$

with $x_m = m/M$ and the Fourier dual $V = \mathbb{R}^1$ of $P = \mathbb{R}^1$, has been restricted to an infinite lattice: $x_m = m/M$; $-\infty < m < +\infty$, with spacing: $\Delta x_m = 1/M$.

But now we may suppose the simultaneous compactifications of V and P in V_c and P_c respectively. We have only to impose that the number of points of the two dual lattices is equal to a number 2N of vertices of two regular poligons on S^1 and $\tilde{S}^1$: $2\pi M = 2N/R$ and $2\pi R = 2N/M$ which is satisfied for $N = \pi M R$ integer, (remember that if R is a length, M is the inverse of a length and $\pi M R$ is a pure number). Then we have that if $f(x_m)$ and $F(k_m)$ are the functions f (x) and F (k) restricted on the 2N point lattices on S^1 and $\tilde{S}^1$ the following equations hold:

$$f(x_n) = (2N)^{-\frac{1}{2}} \sum_{m=-N}^{N-1} e^{i k_m x_n} F(k_m)$$

$$F(k_m) = (2N)^{-\frac{1}{2}} \sum_{m=-N}^{N-1} e^{-i x_n k_m} f(x_n) \tag{12}$$

where $x_n k_m = nm/RM$. Eqs.(12) are true for any $N = \pi M R$ since

$$\frac{1}{2N} \sum_{n=-N}^{N-1} e^{in(m-m')\frac{\pi}{N}} = \delta_{mm'} \tag{13}$$

for any integer N, since this equation is equivalent to the theorem stating that the sum of the 2N roots of -1 equals zero.

We have then that the two 2N-point lattices L_M and M_P, obtained by restricting M_c and P_c to the vertices of regular poligons inscribed in them, may be defined, because of (12), as Fourier dual. It is easy to verify, in fact, that for $R \to \infty$ and $M \to \infty$ eq. (12) become the standard Fourier transforms on $\mathbb{R}^1$, which then justifies the name of "dual" assigned to the lattices L_M and L_P, and then to the interpretation of $P_c = G/H_2$ as compactified momentum-space.

The above means that the <u>finite</u> set of harmonics (periodic on the lattices):

$$\phi_m(x_n) = (2N)^{-\frac{1}{2}} e^{imn\frac{\pi}{N}} \qquad -N \le m, n \le N-1 \tag{14}$$

build up a complete orthonormal set for the representation of any single valued function $f(x_n)$ on the 2N-point lattice L_M in S^1. Consequently restricting S^1 and $\tilde{S}^1$ to the two 2N-point lattices L_M and M_P respectively is equivalent to the truncation to N of the usually infinite Fourier series for continuous single valued functions on S^1. This formalism may be easily extended to the two

dimensional Minkowski space $M = \mathbb{R}^{1,1}$. Here we have that while the homogeneous spaces determined by the postulated (conformal) symmetry on $M = \mathbb{R}^{1,1}$ are the smooth toruses M_c and P_c, the physical spaces are the 2N x 2N dual lattices L_M and L_P on them. Obviously in the physical spaces L_M, L_P any dynamical theory will be a priori free from divergences since the mere concepts of infinity and infinitesimal will be absent.

5. CONCLUSION

We have seen how, assuming the fundamental role of conformal symmetry and of simple (or pure)-spinor geometry several physical phenomena, ranging from quantum phenomena (spinor field equations, the H-atom) to classical ones (possibly large scale structures of galaxies) might be explained and correlated. In particular the appropriate space to describe the formers seems to be momentum space while configuration space remains the ideal one for the description of classical physical phenomena. And if, in particular, the mentioned Cartan's conjecture about the fundamental role of simple spinor-geometry is assumed, then those spaces should be assumed as conformally compactified. Not only but one could also extend Cartan's conjecture in assuming that null vectors of momentum space, bilinear in simple spinor, are not only at the basis of euclidean geometry of that space but also of quantum mechanics of fermions, which, in turn, is at the basis of classical mechanics well understood, and easily accessible to our intuition, in configuration space, where instead the formulation of quantum mechanics may present well-known difficulties to our every day intuition. In this frame the S0(4) symmetry of the Universe, of planetary motions of the H-atom would not be accidental, as sometimes assumed, but instead could be ascribed to the conformal compactification of both space-time and momentum space, and, as the toy model presented in section 4.1, for a two dimension space-time $\mathbb{R}^{1,1}$, seem to suggest, the conformal compactification of space-time and momentum space could naturally eliminate all kinds of divergences in such compactified spaces which would have the structure of finite, discrete lattices.

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