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LYAPUNOV EXPONENTS AND INSTABILITY OF N-BODY SYSTEMS

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Abstract.

The problem of numerical study of instability of N–body gravitating systems by means of Lyapunov characteristic exponents is considered. The discontinuity of Lyapunov exponents is shown for computer–created systems both with softened and, what is more interesting, unsoftened (i.e. pure Newtonian) potentials. Lyapunov technique thus cannot be considered as appropriate method of study of N–body systems, and physical and astrophysical interpretations of results of previous computer studies appear to be unfounded.

The numerical study of instability on N–body gravitating systems, since pioneering paper by [1] has become one of important areas of their computer simulations. It is partly determined by the relation of instability properties of that system with relaxation driving mechanisms in star clusters and galaxies. The latter factor became crucial after the proof of exponential instability of spherical N–body systems [2,3] and the evidence of essential consequences it can have for stellar dynamics.

Numerical studies performed to investigate this problem are based on the calculation of the growth of perturbations by time (see [4] summarizing previous studies and references therein), considered to be Lyapunov characteristic exponents of the system.

In the present paper, therefore, we investigate the problem of validity of Lyapunov exponent technique to the computer study of N–body problem. We show that the calculations of Lyapunov exponents for N-body systems with softened and even unsoftened (!) potential can have no any relation to properties of the corresponding real system.

We approach to this problem from the concept of theory of dynamical systems, enabling us to arrive general conclusions, valid for d–dimensional Hamiltonian systems with any potential.

Consider the space $\mathcal{D}$ of all dynamical systems $(M, \mathcal{B}(M), P, f)$ with topology (C^k, C^∞), where B(M) is the σ–field of d–dimensional manifold M, P is a complete measure, f^t is a group of diffeomorphisms on M with continuous $t \in R$ (or discrete $t \in Z$) time.

First let us argue the importance of a question: is Φ, defined in a following manner on $\mathcal{D}$:

$$\Phi : \mathcal{D} \to R,$$

a continuous function.

Note, that well known examples of such functions are Kolmogorov-Sinai (KS)–entropy and Lyapunov mean characteristic exponents:

$$\lambda_i = \int_M \lambda_i(x) P(dx), \quad i = 1, \ldots, q \ ,$$

where $\lambda_i(x)$ are Lyapunov characteristic exponents.

One can easily realize particular importance of continuity of the function Φ for computer studies.

Indeed, assume that one has to check the ergodicity of a function f_α when α is an irrational number, and non-ergodicity when α is rational.

Evidently one cannot solve this problem by computer methods since computer cannot deal with irrational numbers.

The reason of this fact is that at computer studies typically one is forced to consider not the transformation f^t but

$$f^t_\varepsilon = f^t + \varepsilon\psi(t),$$

because of the inevitable computing errors $\psi(t)$ arising at approximation of numbers.

As a result when Φ is not a continuous function all numerical calculations can completely lose their meaning, i.e. computer's $\Phi^c(f)$ does not coincide with real one $\Phi(f)$.

Such a situation can arise also as a result of deliberate change of the behavior of f to avoid "naughty" functions, as it happens at "softening" of Newtonian potential:

$$\frac{1}{r} \to \frac{1}{\sqrt{r^2 + \varepsilon^2}}.$$

Thus we arrive at the following two questions reflecting both fundamental properties of function Φ:

1. Is $\Phi(\varepsilon)$ close to its computer image $\Phi^c(\varepsilon)$? *(Stability)*;

2. Is $\Phi(0)$ the limit of $\Phi(\varepsilon)$ as ε (bifurcation parameter) tends to zero? *(Continuity).*

As a representation of particular interest of Φ we shall consider Lyapunov characteristic exponents.

First, recall the following quite remarkable results:

1a.*According to Mañé's theorem (see [5]) when M is a compact surface, C^1 area-preserving non-Anosov diffeomorphisms, all of whose Lyapunov exponents are equal to zero Lebesgue almost everywhere, are everywhere dense;*

1b. *In general, Lyapunov exponents are discontinuous functions of a bifurcation parameter* [6];

1c. *Topological entropy is proved to be discontinuous for dimM* ≥ 4 [7];

We see that Lyapunov exponents can be highly discontinuous.

The example considered above can be an illustration of this typical property of function Φ.

On the other hand one has the properties:

2a. *Topological entropy is continuous at dimM* $= 2$ [7];

2b. *For some dynamical systems it is proved that Lyapunov exponents are upper–continuous* [8];

2c. *Though typically Lyapunov exponents are highly discontinuous, there exists regular family of perturbations fulfilling conditions discussed in [5] making them stable.*

We see that while the properties 1(a–c) make doubtful the usefulness for computations of Lyapunov exponents the 2(a–c) ones leave some hope.

What is evident is the necessity of thorough consideration of this problem in any given particular case.

Turn now to the problem of stability of Lyapunov exponents in the case of N–body system.

First remind that the trajectories of Hamiltonian system

$$H(p,q) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + V(q),$$

in the region of configurational space $Q = \{q | V(q) < E\}$, can be represented as geodesics of Q with Riemannian metric

$$\mathbf{G} = [E - V(q)]\mathbf{g} \equiv W\mathbf{g},$$

and affine parameter $ds = \sqrt{2}W dt$.

It is also well known that the stability properties of trajectories can be determined by the behavior of Riemann ($Riem$), Ricci (Ric) and scalar (R) curvatures.

Indeed, from Jacobi equation

$$\frac{d^2\mathbf{n}}{ds^2} + Riem(\mathbf{n}, \mathbf{u})\mathbf{u} = 0,$$

after averaging by $\mathbf{u}$ and $\mathbf{n}$ one can arrive

$$\frac{d^2\mathbf{n}}{ds^2} + \frac{R}{d(d-1)}\mathbf{n} = 0,$$

since

$$Riem(\mathbf{n},\mathbf{u})\mathbf{u} = \frac{R}{d(d-1)}\left(n^a\|n\|^2 - u^a < u,n >\right) = \frac{R}{d(d-1)}\mathbf{n},$$

$$\|\mathbf{u}\|^2 = 1, < \mathbf{u},\mathbf{n} >= 0,$$

where vectors $\mathbf{u}$ and $\mathbf{n}$ denote the velocity on geodesics and their deviation.

One can see that the measure of average (in space and time) instability for the d-dimensional Hamiltonian system closely connected with the Lyapunov exponents is the following **instability mean index**

$$\lambda^2 \equiv -\frac{2RW^2}{d(d-1)} = \frac{2\Delta W}{d} + \left(\frac{1}{2} - \frac{3}{d}\right)\frac{\|dW\|^2}{W},$$

where time reparametrization is made and

$$\|dW\|^2 = g^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu},$$

$$\Delta W = g^{\mu\nu}\frac{\partial^2 W}{\partial q^\mu \partial q^\nu}.$$

Previously, in [9] we had introduced a measure of relative instability based on the value of Ricci curvature.

In the case of N-body gravitating system one has $d = 3N$ and

$$V(q) = -\sum_{a=1}^{N}\sum_{b=1}^{a-1} GM_aM_b\varphi(r_{ab})$$

$$r_{ab}^2 = (r_{ab}^1)^2 + (r_{ab}^2)^2 + (r_{ab}^3)^2$$

$$r_{ab}^i = r_a^i - r_b^i$$

where function φ is not specified yet, $a = 1,\ldots,N$ and $i = 1,\ldots,3$, $\mu = (a,i)$.

Calculating the instability mean index one has

$$\lambda^2 = \Lambda_1 + \Lambda_2,$$

where

$$\Lambda_1 = \frac{2}{3N}\sum_{a=1}^{N}\sum_{c=1,c\neq a}^{N} GM_c r_{ac}^{-2}(r_{ac}^2\varphi'(r_{ac}))',$$

$$\Lambda_2 = \left(\frac{1}{2} - \frac{1}{N}\right)\sum_{a=1}^{N}\frac{|F_a|^2}{M_a}\Big/\sum_{a=1}^{N}\frac{|P_a|^2}{M_a}$$

Here the following notations are used

$$|F_a|^2 = (F_a^1)^2 + (F_a^2)^2 + (F_a^3)^2,$$

$$F_a^i = \sum_{c=1,c\neq a}^{N} F_{ac}^i = \sum_{c=1,c\neq a}^{N} GM_aM_c\varphi'(r_{ac})\frac{r_{ac}^i}{r_{ac}}.$$

Now consider a class of potentials $\varphi(\varepsilon)$ containing the two main cases.

1.Newtonian potential ($\varepsilon = 0$):

$$\varphi(r) = \frac{1}{r};$$

2.Softened Newtonian potentials ($\varepsilon \neq 0$):

Case a.

$$\varphi_1(r,\varepsilon) = \frac{1}{\sqrt{r^2+\varepsilon^2}}.$$

Let us look for the behavior of λ when both r and ε are close to zero, i.e. for the continuity of mean index in physically most interesting case. For this purpose one has to obtain the following limits

$$\lim_{\varepsilon\to 0}\lim_{r\to 0}\lambda^2(r,\varepsilon) = \lim_{\varepsilon\to 0}\left\{-\frac{1}{N\varepsilon^3} + \left(\frac{1}{2}-\frac{1}{N}\right) * 0\right\} = -\infty,$$

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\lambda^2(r,\varepsilon) = \lim_{r\to 0}\left\{0 + \left(\frac{1}{2}-\frac{1}{N}\right)\frac{1}{r^3}\right\} = +\infty.$$

Case b.

$$\varphi_2(r,\varepsilon) = \frac{1}{r+\varepsilon}.$$

The corresponding limits yield

$$\lim_{\varepsilon\to 0}\lim_{r\to 0}\lambda^2(r,\varepsilon) = \lim_{\varepsilon\to 0}\left\{-\infty + \left(\frac{1}{2}-\frac{1}{N}\right)\frac{1}{\varepsilon^3}\right\} = -\infty,$$

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}\lambda^2(r,\varepsilon) = \lim_{r\to 0}\left\{0 + \left(\frac{1}{2}-\frac{1}{N}\right)\frac{1}{r^3}\right\} = +\infty.$$

When $\varepsilon = 0$, i.e. in the case of unsoftened potential the mean index is determined by Λ_2 and the system is exponentially instable, since

$$\lambda^2 \sim r^{-3} \quad \text{as} \quad r \to 0.$$

This limit corresponds to the close encounter of at least two particles.

The same limit when $\varepsilon \neq 0$, for both softened potentials reveals completely different behavior.Particularly in case **a.** the mean index is a complex number,

$$\lambda^2 \sim -\varepsilon^{-3} \quad \text{as} \quad \varepsilon \to 0,$$

since is determined by first member Λ_1; as a result the system is not instable any more. Similar behavior one has in case **b.**

We see that the mean instability index and hence Lyapunov exponents in the case of unsoftened potential are discontinuous

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon \neq \lambda_0.$$

Moreover the unsoftened system has quite different behavior, particularly in accord to point 2b of properties of Lyapunov exponents mentioned above, is more stable than the original Newtonian system.

Note that marked dependence of growth of initial errors on the parameter of softening has been noticed during computer simulations in [10].

Thus the calculations of Lyapunov exponents by means of computer methods for N–body systems cannot lead to any meaningful results.

Already this fact is enough to seriously influence the conclusions of numerous computer studies of instability of softened systems (see [4]). Other difficulties of those studies, particularly concerning the interpretation of relaxation–type effects, were outlined in [11].

However, the next conclusion of the present study is even more radical: the principal impossibility of investigation of instability of not only disturbed, but even $1/r$ potential N–body systems by computers.

These conclusions demonstrate the necessity of creation of new computer codes to describe the N–body system with phase trajectory close to the physical one for long enough time scales (in physical sense).

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