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SPECIAL LAGRANGIAN SUBMANIFOLDS OF MODULI SPACES OF VECTOR BUNDLES

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By means of a Fourier-Mukai transform we embed moduli spaces $\mathcal{M}_C$ of stable bundles on a curve C of genus 2 as Lagrangian subvarieties of moduli spaces $\mathcal{M}^{\mu}_{J(C)}$ of μ-stable bundles on the Jacobian variety $J(C)$. Since $\mathcal{M}^{\mu}_{J(C)}$ has a hyper-Kähler structure, after a suitable change of complex structure the image of $\mathcal{M}_C$ turns out to be a special Lagrangian submanifold of $\mathcal{M}^{\mu}_{J(C)}$. We briefly discuss the special Lagrangian deformations of $\mathcal{M}_C$.

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ABSTRACT. By means of a Fourier-Mukai transform we embed moduli spaces $\mathcal{M}_C$ of stable bundles on a curve C of genus 2 as Lagrangian subvarieties of moduli spaces $\mathcal{M}^{\mu}_{J(C)}$ of μ-stable bundles on the Jacobian variety $J(C)$. Since $\mathcal{M}^{\mu}_{J(C)}$ has a hyper-Kähler structure, after a suitable change of complex structure the image of $\mathcal{M}_C$ turns out to be a special Lagrangian submanifold of $\mathcal{M}^{\mu}_{J(C)}$. We briefly discuss the special Lagrangian deformations of $\mathcal{M}_C$.

1. INTRODUCTION

Special Lagragian submanifolds of a Calabi-Yau manifold provide an example of a *calibrated geometry* in the sense of Harvey and Lawson [4]. Special Lagrangian geometry has recently received quite a lot of attention in connection with mirror symmetry in string theory; according to a proposal of Strominger, Yau and Zaslow (SYZ)[19], the mirror partner to a Calabi-Yau manifold admitting a fibration in special Lagrangian tori should be identified with a (suitable compactification of a) moduli space parametrizing the fibres of the special Lagrangian fibration together with an additional datum (the specification of a flat $U(1)$ bundle on each fibre). A programme basically consisting in implementing the SYZ conjecture is being carried on by Gross [2, 3].

There are not many explicit examples of special Lagrangian submanifolds. One class of examples is provided by the fixed point sets of anti-holomorphic involutions of Calabi-Yau manifolds [1]. Recently Peng Lu has constructed special Lagrangian tori in the Borcea-Voisin threefolds [11]. Holomorphic Lagrangian

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submanifolds of a hyper-Kähler manifold, after a rotation of the complex structure inside the hyper-Kähler family, turn out to be special Lagrangian [6, 7]. In this paper we apply this "hyper-Kähler trick" to a situation arising in connection with moduli spaces of vector bundles. We consider a moduli space $\mathcal{M}_C(r,d)$ of stable vector bundles (of rank r and degree d) on a smooth algebraic curve C of genus 2, and show that via a Fourier-Mukai tranform $\mathcal{M}_C(r,d)$ can be embedded into a moduli space $\mathcal{M}^{\mu}_{J(C)}$ of μ-stable bundles on the Jacobian variety of C; moreover, the image of $\mathcal{M}_C(r,d)$ in $\mathcal{M}^{\mu}_{J(C)}$ turns out to be Lagrangian with respect to the Mukai holomorphic symplectic form on $\mathcal{M}^{\mu}_{J(C)}$. Thus in a suitable complex structure, $\mathcal{M}_C(r,d)$ is a special Lagrangian submanifold of $\mathcal{M}^{\mu}_{J(C)}$. By translating the image of C in $J(C)$ we obtain a family of deformations of $\mathcal{M}_C(r,d)$ as a special Lagrangian submanifold of $\mathcal{M}^{\mu}_{J(C)}$. When $r=1$, this describes $\mathcal{M}^{\mu}_{J(C)}$ as a fibration in special Lagrangian 4-tori.

In Section 2 we briefly review the basic definitions in special Lagrangian geometry. In 3 we describe the embedding $\mathcal{M}_C(r,d) \hookrightarrow \mathcal{M}^{\mu}_{J(C)}$, while in Section 4 we discuss the resulting special Lagrangian submanifolds.

2. Special Lagrangian submanifolds

Let X be a Calabi-Yau n-manifold with a Ricci-flat metric g and associated Kähler form ω. A compact real n-dimensional submanifold $j\colon Y \to X$ is said to be *special Lagrangian* if $j^*\omega = 0$, and a global nowhere vanishing holomorphic n-form Ω can be chosen on X so that $j^* \operatorname{Im}\Omega = 0$. Equivalently, up to changing Ω by a complex factor, one may require that $j^* \operatorname{Re}\Omega = \operatorname{vol}(Y)$, where $\operatorname{vol}(Y)$ is the volume form induced on Y by g: the special Lagrangian submanifolds are the submanifolds of X that are calibrated by the real part of Ω [4].

The deformations of Y through special Lagrangian submanifolds of X are unobstructed, and the tangent space at Y to the corresponding moduli space $\mathcal{M}$ can be identified with $H^1(Y,\mathbb{R})$. So the dimension of $\mathcal{M}$ is $b_1(Y)$ [12].

Consider a hyper-Kähler manifold X of complex dimension $2n$ with a given hyper-Kähler metric g. Let I, J, K be three complex structures compatible with g such that $IJ = K$, and denote by ω_I, ω_J and ω_K the three corresponding Kähler forms. If we fix the complex structure I, then the 2-form $\eta = \omega_J + i\omega_K$ is a holomorphic symplectic form, and $\Omega = \frac{1}{n!}\eta^n$ is a nowhere vanishing holomorphic $2n$-form.

Let us consider a holomorphic submanifold $j\colon M \to X$ which is Lagrangian with respect to η i.e., $j^*\eta = 0$. By Wirtinger's theorem, M is calibrated by the n-th power of the Kähler form, so that

$$\frac{1}{n!} j^*(\omega_I^n) = \mathrm{vol}(M)\,. \tag{1}$$

One easily checks that the submanifold M is special Lagrangian in the complex structure K. Just notice that $j^*\omega_K = 0$ since $j^*\eta = 0$ and that in the complex structure K we may take for global holomorphic $2n$-form the form

$$\Omega_K = \frac{1}{n!}(\omega_I + i\omega_J)^n$$

so that

$$j^* \operatorname{Re} \Omega_K = \mathrm{vol}(M)$$

by eq. (1).

3. A Lagrangian embedding of $\mathcal{M}_C(r,d)$

Let C be a smooth algebraic curve of genus 2, and denote by $J(C)$ its Jacobian variety and by $\Theta \in H^2(J(C),\mathbb{Z})$ the cohomology class of C embedded in $J(C)$ via the Abel-Jacobi map. Fix coprime positive integers r, d such that $d > 4r$, and let $\mathcal{M}_C(r,d)$ be the moduli space of stable vector bundles on C of Chern character (r,d). In this section we show how $\mathcal{M}_C(r,d)$ can be embedded as a Lagrangian holomorphic submanifold of $\mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$ — the moduli space of μ-stable vector bundles on $J(C)$ with Chern character $(d-r,-r\Theta,0)$.

We recall a few facts about the Fourier-Mukai transform in the context of Abelian varieties [13]. Let X be an Abelian variety (over $\mathbb{C}$) and $\widehat{X} = \mathrm{Pic}^0(X)$ its dual variety. Let $\mathcal{P}$ be the normalized Poincaré bundle on $X \times \widehat{X}$. The Mukai functor is defined as

$$\mathbf{R}\mathcal{F}\colon D(X) \to D(\widehat{X})$$
$$\mathbf{R}\mathcal{F}(-) = \mathbf{R}\pi_{\widehat{X},*}(\pi_X^*(-) \otimes \mathcal{P})$$

where $D(X)$ and $D(\widehat{X})$ are the bounded derived categories of coherent sheaves on X and $\widehat{X}$, respectively. Mukai has shown that the functor $\mathbf{R}\mathcal{F}$ is invertibile and preserves families of sheaves (cf. [13, 15]). If E is a WIT_i sheaf on X, that is, a sheaf whose transform is concentrated in degree i, then the functor $\mathbf{R}\mathcal{F}$ preserves the Ext groups:

$$\mathrm{Ext}^j_X(E,E) \cong \mathrm{Ext}^j_{\widehat{X}}(\hat{E},\hat{E}) \quad \text{for every } j,$$

where $\hat{E}$ indicates the transform of E.

Let C a smooth projective curve of genus $g > 1$ and $J(C)$ the Jacobian of C. If we fix a base point x_0 on C, and let $\alpha_{x_0}\colon C \to J(C)$ be the Abel-Jacobi embedding given by $\alpha_{x_0}(x) = \mathcal{O}_C(x - x_0)$, the normalized Poincaré bundle $\mathcal{P}_C$ on $C \times J(C)$ is the pullback of the Poincaré bundle on $J(C) \times J(C)$, where we identify $J(C)$ with $\widehat{J(C)}$ via the isomorphism $-\phi_\Theta\colon J(C) \to \widehat{J(C)}$. The Poincaré bundle on $C \times J(C)$ gives rise to a derived functor (which is not invertible):

$$\mathbf{R}\Phi_C\colon D(C) \to D(J(C))$$
$$\mathbf{R}\Phi_C(-) = \mathbf{R}\pi_{J(C),*}(\pi_C^*(-) \otimes \mathcal{P}_C)\,.$$

Since the morphism α_{x_0} is finite, we have the following relation between the functors $\mathbf{R}\Phi_C$ and $\mathbf{R}\mathcal{F}$:

$$\mathbf{R}\Phi_C \cong \mathbf{R}\mathcal{F} \circ \alpha_{x_0,*}. \tag{2}$$

Thus the study of the transfoms of bundles F on C with respect to $\mathbf{R}\Phi_C$ is equivalent to the study of transforms of sheaves of pure dimension 1 of the form $\alpha_{x_0,*}(F)$ with respect to $\mathbf{R}\mathcal{F}$. We recall the following fact which is proven in [10].

Proposition 3.1. *If E is a stable bundle on C of rank r and degree d such that $d > 2rg$, then*

1. *E is WIT_0, and*
2. *the transformed sheaf $\hat{E} = \mathbf{R}^0\Phi_C(E)$ is locally free and μ-stable with respect to the theta divisor on $J(C)$.*

Therefore, if we consider the moduli space $\mathcal{M}_C(r,d)$ of stable bundles of rank r and degree d on a projective smooth curve of genus 2 such that $d > 4r$ and r, d are coprime, the functor $\mathbf{R}\Phi_C$ gives rise to a morphism

$$\tilde{\jmath}\colon \mathcal{M}_C(r,d) \to \mathcal{M}^{\mu}_{J(C)}(d-r, -r\Theta, 0).$$

The moduli space $\mathcal{M}_C(r,d)$ is an irreducible projective variety of dimension r^2+1 and $\mathcal{M}^{\mu}_{J(C)}(d-r, -r\Theta, 0)$ is a (smooth) quasi-projective variety of dimension $2(r^2+1)$.

Before studying the morphism $\tilde{\jmath}$ we need to recall some elementary facts about the Yoneda product of Ext groups. Let $\mathcal{A}$ be an abelian category (with enough injectives). The elements of $\mathrm{Ext}^1_{\mathcal{A}}(E,E)$ are identified with equivalence classes of exact sequences $0 \to E \to F \to E \to 0$ with respect to the usual relation. This

can be generalized to the groups $\mathrm{Ext}^2_{\mathcal{A}}(E,E)$ as follows. We refer to [5] for proofs and details.

Consider the following diagram with exact rows:

$$\begin{array}{ccccccccccc} E: & 0 & \longrightarrow & B & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & A & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \mathrm{Id}_B} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}_A} & & \\ E': & 0 & \longrightarrow & B & \longrightarrow & G'_1 & \longrightarrow & G'_2 & \longrightarrow & A & \longrightarrow & 0. \end{array} \tag{3}$$

We write $E \twoheadrightarrow E'$ when such a diagram holds. The relation $\twoheadrightarrow$ is not symmetric, but it generates the following equivalence relation: $E \sim E'$ if and only if there exists a chain of sequences $E_0, E_1, \ldots, E_k$ such that

$$E = E_0 \twoheadrightarrow E_1 \twoheadleftarrow E_2 \twoheadrightarrow \ldots \twoheadleftarrow E_k = E'.$$

Let $\mathrm{Yext}^2_{\mathcal{A}}(-,-)$ the set of such equivalences. There is an isomorphism

$$\mathrm{Yext}^2_{\mathcal{A}}(-,-) \cong \mathrm{Ext}^2_{\mathcal{A}}(-,-).$$

Thus the sets $\mathrm{Yext}^2_{\mathcal{A}}(E,E')$ have a canonical group structure. From now on we shall identify the above groups. Observe that the identity of $\mathrm{Ext}^2_{\mathcal{A}}(A,B)$ is given by the class of the sequence

$$0 \longrightarrow B \xrightarrow{\mathrm{Id}_B} B \xrightarrow{0} A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0.$$

Moreover the Yoneda product

$$\mathrm{Ext}^1_{\mathcal{A}}(B,A) \times \mathrm{Ext}^1_{\mathcal{A}}(A,C) \to \mathrm{Ext}^2_{\mathcal{A}}(B,C)$$

is obtained in the following way: let E and E' be two elements of $\mathrm{Ext}^1_{\mathcal{A}}(B,A)$ and $\mathrm{Ext}^1_{\mathcal{A}}(A,C)$ represented respectively by the sequences

$$E: \quad 0 \longrightarrow A \xrightarrow{\nu} F \xrightarrow{p} B \longrightarrow 0$$

$$E': \quad 0 \longrightarrow C \xrightarrow{i} G \xrightarrow{\lambda} A \longrightarrow 0.$$

Then the class of the exact sequence

$$0 \longrightarrow C \xrightarrow{i} G \xrightarrow{\nu \circ \lambda} F \xrightarrow{p} B \longrightarrow 0$$

in $\mathrm{Ext}^2_{\mathcal{A}}(B,C)$ is the image of E, E' with respect to the Yoneda product.

We shall also need to introduce a moduli space of stable sheaves in Simpson's sense [18]. For simplicity we denote the Abel-Jacobi map as $j\colon C \to J(C)$. Observe that if E is a stable bundle on C then $j_*(E)$ is a stable sheaf of pure dimension 1 on $J(C)$ with respect to the polarization Θ. Let $\mathcal{M}_{J(C)}(0,\Theta,d-r)$ be the moduli space of all stable pure sheaves on $J(C)$ with Chern character

$(0, r\Theta, d-r)$. If $\mathcal{E}$ is a flat family of vector bundles on C parametrized by a Noetherian scheme S, then $j_{S,*}(\mathcal{E})$ is a flat family of sheaves on $J(C) \times S$ over S, where $j_S \colon C \times S \to J(C) \times S$ is the embedding $j \times \mathrm{Id}_S$. Therefore one has a morphism of moduli spaces

$$j_* : \mathcal{M}(r,d) \to \mathcal{M}_{J(C)}(0, r\Theta, d-r)\,. \tag{4}$$

Proposition 3.2. *The morphism $\tilde{j}\colon \mathcal{M}_C(r,d) \to M^{\mu}_{J(C)}(d-r, -r\Theta, 0)$ is an embedding.*

Proof. From the isomorphism given by Eq. (2) and recalling that the transform $\mathbf{R}\mathcal{F}$ preserves the Ext groups of WIT sheaves, it is enough to show that the same claim holds for the morphism (4). The injectivity of j_* is clear. By the very definition of the Kodaira-Spencer map, the tangent map to j_* may be identified with the map

$$\mathrm{Ext}^1_C(E,E) \overset{\phi}{\hookrightarrow} \mathrm{Ext}^1_{J(C)}(j_*(E), j_*(E))$$

obtained in the following way. Let

$$A: \quad 0 \longrightarrow E \longrightarrow F \longrightarrow E \longrightarrow 0 \tag{5}$$

be a sequence representing an element of $\mathrm{Ext}^1_C(E,E)$. If we apply the functor j_* to the above sequence we obtain the exact sequence

$$B: \quad 0 \longrightarrow j_*(E) \longrightarrow j_*(F) \longrightarrow j_*(E) \longrightarrow 0. \tag{6}$$

One checks immediately that the map $\phi([A]) = [B]$ is well defined. If $\phi([A]) = 0$ then $\phi([A])$ is represented by the extension

$$0 \longrightarrow j_*(E) \longrightarrow j_*(E) \oplus j_*(E) \longrightarrow j_*(E) \longrightarrow 0. \tag{7}$$

Now applying the functor j^* to the above sequence and recalling that $j^*(j_*(H)) \cong H$ for every vector bundle H on C we obtain the split exact sequence

$$0 \longrightarrow E \longrightarrow E \oplus E \longrightarrow E \longrightarrow 0. \tag{8}$$

Therefore $\phi([A]) = 0$ implies $[A] = 0$ and ϕ is injective. □

Mukai proved that the moduli space of simple sheaves on an abelian surface X is symplectic; more precisely, the Yoneda pairing

$$\mathrm{Ext}^1_X(E,E) \times \mathrm{Ext}^1_X(E,E) \to \mathrm{Ext}^2_X(E,E) \cong \mathbb{C}$$

defines a holomorphic, closed, nondegenerate 2-form on the moduli of simple sheaves on an abelian surface (cf. [14, 16]).

Proposition 3.3. *The subvarieties $\mathcal{M}_C(r,d)$ are Lagrangian with respect to the Mukai form of $\mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$.*

Proof. Let $E \in \mathcal{M}_C(r,d)$. It is enough to show that the Yoneda product

$$\mathrm{Ext}^1(j_*(E),j_*(E)) \times \mathrm{Ext}^1(j_*(E),j_*(E)) \longrightarrow \mathrm{Ext}^2(j_*(E),j_*(E))$$

vanishes when applied to pairs $([A],[B])$ of elements in $\mathrm{Ext}^1(j_*(E),j_*(E))$ where $[A]$ and $[B]$ are represented, respectively, by the sequences

$$A: \qquad 0 \longrightarrow j_*(E) \xrightarrow{\nu} j_*(F) \xrightarrow{p} j_*(E) \longrightarrow 0$$

with $F \in \mathcal{M}_C(r,d)$, and

$$B: \qquad 0 \longrightarrow j_*(E) \xrightarrow{i} j_*(G) \xrightarrow{\lambda} j_*(E) \longrightarrow 0$$

with $G \in \mathcal{M}_C(r,d)$. Equivalently one should show that the sequence

$$0 \longrightarrow j_*(E) \xrightarrow{i} j_*(G) \xrightarrow{\nu\circ\lambda} j_*(F) \xrightarrow{p} j_*(E) \longrightarrow 0 \tag{9}$$

represents the class of the identity. Observe that $j_*(G) \xrightarrow{\nu\circ\lambda} j_*(F)$ corresponds to a morphism $f \in \mathrm{Hom}_C(G,F)$. By hypothesis G and F are stable with the same Chern character, therefore there are two possibilities:

1) f is the zero morphism, and so $\nu \circ \lambda : j_*(G) \to j_*(F)$ is the zero morphism; therefore the sequence (9) is the identity in $\mathrm{Ext}^2_X(j_*(E),j_*(E))$.

2) f is an isomorphism, so that $\nu \circ \lambda$ is an isomorphism and one obtains the sequence

$$0 \longrightarrow j_*(E) \xrightarrow{0} j_*(G) \xrightarrow{\nu\circ\lambda} j_*(F) \xrightarrow{0} j_*(E) \longrightarrow 0$$

whence it follows that $j_*(E) = 0$ and therefore $E \cong 0$, which is impossible since $E \in \mathcal{M}_C(r,d)$. □

4. Special Lagrangian submanifolds of moduli spaces

In the last section we have seen how the Abel-Jacobi map and the Fourier-Mukai transform yield a holomorphic embedding

$$\tilde{\jmath}\colon \mathcal{M}_C(r,d) \to \mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$$

of the moduli space $\mathcal{M}_C(r,d)$ of μ-stable bundles of rank r and degree d on a genus 2 curve C into the moduli space $\mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$ of μ-stable bundles on the Jacobian of C. Furthermore, we proved that the image of $\tilde{\jmath}$ is a Lagrangian

submanifold of $\mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$ with respect to the Mukai holomorphic symplectic form on $\mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$.

In order to fit into the framework of Section 2 we notice that via the Hitchin-Kobayashi correspondence (which identifies μ-stable bundles on a Kähler manifold with irreducible Einstein-Hermite bundles, cf. [9]), the space $\mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$ acquires a hyper-Kähler structure, compatible with a Kähler form provided by the Weil-Petersson metric, and with a holomorphic symplectic form which may be identified with the Mukai form [8]. Therefore we obtain the following result.

Proposition 4.1. *The space $\mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$ has a complex structure such that $\tilde{\jmath}\colon \mathcal{M}_C(r,d) \to \mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$ is a special Lagrangian submanifold.*

The elements of the Jacobian variety $J(C)$ act on the embedding $j\colon C \to J(C)$ by translation, so that for every $x \in J(C)$ we have a special Lagrangian submanifold $\tilde{\jmath}_x\colon \mathcal{M}_C(r,d) \to \mathcal{M}^{\mu}_{J(C)}(d-r,-r\Theta,0)$. This provides a family of deformations of $\tilde{\jmath}(\mathcal{M}_C(r,d))$ through special Lagrangian submanifolds. One easily shows that this properly embeds $J(C)$ into the moduli space of special Lagrangian deformations of $\tilde{\jmath}(\mathcal{M}_C(r,d))$.

We can be more precise in the cases $r = 1, 2$.

For $r = 1$, we have $\mathcal{M}_C(1,d) \simeq J(C)$ and $\mathcal{M}^{\mu}_{J(C)}(d-1,-\Theta,0) \simeq J(C) \times J(C)$ by a result of Mukai [13]. In this case the map $\mathcal{M}_C(r,d) \times J(C) \to \mathcal{M}^{\mu}_{J(C)}(d-1,-\Theta,0)$ given by the action of $J(C)$ is an isomorphism, and the space $\mathcal{M}^{\mu}_{J(C)}(d-1,-\Theta,0)$ is thus fibred globally in special Lagrangian submanifolds.

In the case $r = 2$ and d odd (and thus $d > 8$) one has $b_1(\mathcal{M}_C(2,d)) = 4$ [17]. Hence the special Lagrangian deformations of $\tilde{\jmath}(\mathcal{M}_C(2,d))$ do not yield a fibration of $\mathcal{M}^{\mu}_{J(C)}(d-2,-2\Theta,0)$.

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