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CONNECTIONS ON BIMODULES IN NONCOMMUTATIVE GEOMETRY

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SISSA-179-96-FM

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Abstract

Properties of pairs consisting of left and right connections are studied on the bimodules of differential 1-forms. An example is given in terms of an $SL_q(2,\mathbb{C})$-covariant calculus of the quantum plane at a generic q and at the cubic root of unity. A compatibility condition between a left and right connection is discussed. Consequences entailed by reducing to the centre of a bimodule the domain of those conditions are investigated. Alternative ways of relating left and right connections are considered.

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1 Introduction

When Noncommutative Geometry[1] meets General Relativity a natural question arises about a satisfactory generalization of the notion of a *linear* connection.
A possible approach is the one that focuses its attention on the fact that the spaces of one-forms, both in commutative and in non commutative settings, have a natural bimodule structure over the algebra defining the (eventually noncommutative) space-time. We recall that, from an algebraic point of view, a connection is described in terms of a covariant derivative, that is a linear map defined on a (left or right) projective module E over some algebra A, having values in the tensor product over A of the module with a first order differential calculus over the algebra, satisfying a (left or right) Liebnitz rule. The archetype of this construction is given by the commutative case, where A is the commutative algebra of functions on a manifold. E is the space of the sections of some vector bundle, and the first order differential calculus is given by the ordinary one-forms. All this structure, however, remains perfectly meaningful even after dropping the hypothesis of commutativity.
In the framework of G.R., one must consider *linear* connections, i.e. connections on vector bundles associated with the principal bundle of linear frames, e.g. tangent bundle, cotangent bundle and so on. In Noncommutative geometry, it is more natural to deal with forms, rather than with vector fields (derivations of the algebra), so let's assume that E itself is a first order differential calculus over A. In this case E has a natural bimodule (left and right module) structure over A, so that, in the hypothesis of left and right projectivity, we can independently consider left and right connections. The idea, now, is to define a *linear* connection on a bimodule as a couple consisting of a left and a right connection satisfying some sort of compatibility requirement.
In the following, guided by the commutative situation, we will give more precise definitions and will see how this idea has been developed in[3, 2]. At the end, we will exhibit a particular example based on Manin's quantum plane.

2 General Definitions

Let A be a unital associative algebra over a field k, let (A^1, d) be a first order differential calculus on A and E be a left and right projective A-bimodule (which in principle may coincide with A^1, but that we keep distinct in order to avoid confusion). We give the following

Definition 1 *A left connection ∇^L is a linear map from E to $A^1 \otimes_A E$ satisfying the left Leibniz rule: $\nabla^L(a\zeta) = da \otimes_A \zeta + a\nabla^L\zeta$, $\forall\, a \in A$, $\zeta \in E$.*
A right connection ∇^R is a linear map from E to $E \otimes_A A^1$ fulfilling the right Leibniz rule: $\nabla^R(\zeta a) = (\nabla^R\zeta)a + \zeta \otimes_A da$, $\forall\, a \in A$, $\zeta \in E$.

If we consider the commutative case, i.e. $A = \mathcal{F}(M)$, $A^1 = E = \Lambda^1(M)$, with M some given manifold, we can see that due to the commutativity of the algebra and to the fact that left and right actions of $\mathcal{F}(M)$ on the ordinary de Rham one-forms $\Lambda^1(M)$ coincide, these two definitions are totally equivalent in the sense that given any right connection ∇^R we can build up a left connection ∇^L defined as

$$\nabla^L \doteq \sigma \circ \nabla^R \tag{1}$$

where $\sigma : E \otimes_A A^1 \to A^1 \otimes_A E$ is the usual tensor product flip.
This can be stated equivalently by saying that, in the commutative case, any left connection ∇^L also satisfies an additional σ- right Liebnitz rule:

$$\nabla^L(\zeta a) = (\nabla^L \zeta)a + \sigma(\zeta \otimes_A da)\,,\ \forall\, a \in A,\ \zeta \in E, \tag{2}$$

In the classical case, then, the spaces of left and right connections are isomorphic, and choosing one or the other is simply a matter of convention.
In the noncommutative setup the formula in Eq. 1 is no longer automatically true: firstly, as shown in Ref. [5], the usual tensor product flip is now not well defined on $E \otimes_A A^1$ and must be substituted with some other braiding operator; secondly, one cannnot arbitrarily shift elements of the algebra through elements of E or A^1.
Nevertheless, in analogy with the commutative case, one can select in the spaces of left and right connections the couples that fulfill something similar to Eq. 1. More precisely, we give the following [3, 6]

Definition 2 *Let $\sigma : E \otimes_A A^1 \to A^1 \otimes_A E$ be a bimodule isomorphism. A couple (∇^L, ∇^R) consisting of a left and a right connection is called σ-compatible iff*
$\nabla^L \zeta = (\sigma \circ \nabla^R)\zeta$, $\forall \zeta \in E$

We remind, again, that this is equivalent to select all the left connections ∇^L satisfying the extra rule of Eq. 2. The bimodule isomorphism σ plays the role of a generalized permutation, and must be given by the particular cases under consideration.
Apart from the analogy with the commutative case, this definition has another deep motivation: in fact, it is shown [7] that, denoted with A^1_u the first order universal calculus on A, for any left connection ∇^L on a bimodule E, there is a unique bimodule homomorphism $\sigma_u(\nabla^L)$ of $E \otimes_A A^1_u$ into $A^1 \otimes_A E$, such that

$$\nabla^L(\zeta a) = (\nabla^L \zeta)a + \sigma_u(\nabla^L)(\zeta \otimes_A d_u a)\,,\ \forall\, a \in A,\ \zeta \in E\ .$$

So we have two facts: first, any left connection having values in $A^1_u \otimes_A E$ verifies Eq. 2 for a particular $\sigma_u(\nabla^L)$; furtherly, for a connection having values in an arbitrary first order calculus A^1, Eq. 2 is equivalent to the requirement that $\sigma_u(\nabla^L)$ factorizes through a bimodule homomorphism σ of $E \otimes_A A^1$ into $A^1 \otimes_A E$.
This definition, although being deeply theoretically motivated, has the drawback of being very restrictive [2]: connections satisfying Eq. 2 are very *commutative*, i.e. they typically have Christoffel symbols belonging to the center of the algebra or parametrized by it. This is why the following *softer* definition appeared [2]:

Definition 3 *Let $\sigma : E \otimes_A A^1 \to A^1 \otimes_A E$ be a bimodule isomorphism. A couple (∇^L, ∇^R) is called (softly) σ-compatible iff $\nabla^L \zeta = (\sigma \circ \nabla^R)\zeta$, $\forall \zeta \in Z(E)$. where $Z(E) \doteq \{\zeta \in E \mid a\zeta = \zeta a, \forall a \in A\}$ is the centre of E.*

In order to clarify the differences of these two definitions, let us consider the following example.

3 An Example: Quantum Plane at root of unity

The example that we study regards a bimodule of differential 1-forms ($E = A^1$) on the quantum plane. We choose as our space of differential 1-forms [3] the grade one part of the differential algebra $\Omega(A) = A \oplus A^1 \oplus A^2$ that is given by the generators $1, x, y, \xi, \eta$, where $\xi = \theta^1 = dx$, $\eta = \theta^2 = dy$, and relations

$$\begin{aligned} & xy = qyx \ , \\ & x\xi = q^2\xi x \ , \quad x\eta = q\eta x + (q^2-1)\xi y \ , \quad y\xi = q\xi y \ , \quad y\eta = q^2\eta y \ . \\ & \eta\xi + q\xi\eta = 0 \ , \quad \xi^2 = 0 \ , \quad \eta^2 = 0 \ . \end{aligned} \tag{3}$$

For our bimodule automorphism σ it is natural (see the paragraph between (2.11) and (2.12) in Ref. [3]) to take the map defined by

$$\begin{aligned} \sigma(\xi \otimes_A \xi) &= q^{-2}\xi \otimes_A \xi \ , & \sigma(\xi \otimes_A \eta) &= q^{-1}\eta \otimes_A \xi \ . \\ \sigma(\eta \otimes_A \xi) &= q^{-1}\xi \otimes_A \eta - (1-q^{-2})\eta \otimes_A \xi \ , & \sigma(\eta \otimes_A \eta) &= q^{-2}\eta \otimes_A \eta \ . \end{aligned} \tag{4}$$

First we consider the case of a generic q. Then the centre of A is $\mathbb{C}$ and, as can be checked by a direct computation, the centre of A^1 is zero. It is shown [2, 3] that for the σ-compatibility condition considered over the whole A^1 there exists only a one-parameter family of solutions, parametrized by a complex number, whereas for the softer definition we have that solutions are trivially given by any pair (∇^L, ∇^R) of independent and unrestricted left and right connections.
More interesting things happen when q is a root of unity (for simplicity we choose $q^3 = 1$). The following statements are then true:

$$Z(A) = \{a_{ij}x^{3i}y^{3j} \mid a_{ij} \in \mathbb{C}\}, \tag{5}$$

$$\forall a \in Z(A), \alpha \in A^1 : \ a\alpha = \alpha a, \tag{6}$$

$$Z(A^1) = \{c_i\theta^i \in A^1 \mid c_1 = axy - bxy^3, \ c_2 = bx^2y^2, \ a, b \in Z(A)\}. \tag{7}$$

Defining left and right Christoffel symbols as: $\nabla^L\theta^i = \theta^j \otimes_A \theta^k \Gamma^i_{jk}$, $\nabla^R\theta^i = \theta^j \otimes_A \theta^k \widetilde{\Gamma}^i_{jk}$, it turns out [2, 3] that requiring the condition of Eq. 2 implies that the Γ^i_{jk} and $\widetilde{\Gamma}^i_{jk}$ must have particular expressions parametrized by $Z(A)$, whereas [2] imposing the condition in Eq. 3 implies, more like in the classical case, that the $\widetilde{\Gamma}^i_{jk}$ can be expressed in terms of the Γ^i_{jk}, if the following requirements are fulfilled:

$$\begin{aligned} & \Gamma^1_{11} \in A \ , \qquad \Gamma^1_{12} \in xA \ , \qquad \Gamma^1_{21} \in xA \ , \qquad \Gamma^1_{22} \in x^2A \ , \\ & \Gamma^2_{11} \in A \ , \qquad (q-1)\Gamma^2_{12} + (1-q^2)\Gamma^2_{21} + 3q^2yx^{-1}\Gamma^2_{22} \in xA \ , \qquad \Gamma^2_{22} \in xA \ . \end{aligned} \tag{8}$$

4 Final Remarks

Although not going deep in computations, for which we refer to the Ref. [2, 3], we have shown that there are two possible noncommutative extensions of the notion of a linear connection on a bimodule, both based on a generalization of the usual tensor product permutation. It is shown[2] that both such definitions fail to be gauge invariants under arbitrary changes of coordinates (left or right module maps), being invariants only under bimodule maps, which in general are very few (e.g. for the quantum plane, the space of bimodule automorphisms $Aut_A^A(A^1)$ is simply C^*).
Let us recall that exist in literature different approaches to connections on bimodules not making use of generalized permutations. For example, in Section 8 of Ref. [4] a connection on a bimodule is also defined as a pair consisting of a left and right connection. There, however, instead of a σ-compatibility condition, the condition of ∇^L being a right A-homomorphism and ∇^R being a left A-homomorphism is imposed. The latter condition, albeit it permits for an interesting algebraic theory, does not collapse to the right notion in the commutative case.
Another possible approach is via *-structures (We owe this point to Andrzej Sitarz, as reported by Piotr Hajac): If A is a *-algebra and A^1 is a *-differential first order calculus[5] and if there exist a *-structure on the bimodule E[8], one can pass without restrictions from a left connection ∇^L to a right one ∇^R by defining $\nabla^R = \tilde{*} \circ \nabla^L \circ *$, where $(\omega \otimes_A \zeta)^{\tilde{*}} = \zeta^* \otimes_A \omega^*$. In this approach, which is also closer to Connes' set-up, problems are shifted to the existence of such *-structures.

Acknowledgements

This communication is based on the work[2], in collaboration with Ludwik Dabrowski, Piotr Hajac and Gianni Landi.

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