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DYNAMICS OF SILENT UNIVERSES WITH AND WITHOUT Λ* lambda

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Abstract

We present results on the non–linear dynamics of a wide class of inhomogeneous cosmological models with irrotational dust and vanishing magnetic part of the Weyl tensor, with and without a cosmological constant Λ. For $\Lambda \neq 0$, the study of these models is relevant for the early universe: we find that expanding regions always inflate and finally isotropize, i.e. de Sitter is the unique attractor for expansion with $\Lambda \neq 0$. There are however regions that can recollapse, even after an inflationary phase. In this case Λ becomes negligible, and the attractor is the same as for the $\Lambda = 0$ case: we find that collapsing regions always fall in filamentary spindle singularities. For $\Lambda = 0$, we also find that expanding regions tend to locally spherical voids. We end with a brief discussion of how these results with $\Lambda = 0$ could be relevant for structure formation in the present matter dominated era.

Ref. SISSA (171/94/A)
094/09/26.5

*Talk given by M. Bruni at the XI Italian Relativity Meeting, Trieste, September 1994.

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Two major problems in contemporary cosmology arise from the observational evidence that the observable universe passed through a long period of almost perfect homogeneity and isotropy, at least between nucleosynthesis and decoupling, when the cosmic microwave background radiation was released. Then the first problem arises in the early universe, as one would like *to explain* - rather than postulate *ab initio* - this isotropy, and the second one regards post–decoupling times, as we have to explain the large scale structure observed in the distribution of matter.

In this contribution we want to present some results obtained in tackling these problems with the study of the non–linear dynamics of a wide class of exact solutions of Einstein equations, with cosmological constant Λ taken to be non–vanishing in the early universe[1], and vanishing otherwise[2]. It may perhaps appears odd to study the large scale structure formation with relativistic exact solutions, but ultimately gravity is described by general relativity - and only approximately by Newtonian theory - so that the reason given below justifies our approach. On the other hand an inflationary era is called to explain the isotropy of the universe, and in this scenario it is standard to take Λ to represent the local value of the vacuum energy density in the early universe, in order to study the possible isotropization from inhomogeneous initial conditions (cosmic no–hair theorem, see [3, 4, 5] and references therein).

These cosmological models are characterized by three basic assumptions: *i)* the matter content is given by collisionless cold matter (dust, $p = 0$); *ii)* the fluid flow is irrotational,[6] $\omega_{ab} = 0$; *iii)* the magnetic part of the Weyl tensor vanishes, $H_{ab} = 0$. Assumption *i)* is well justified in the period relevant for structure formation, when the universe appears matter dominated, while in the early universe - as it will be clear later - the vacuum energy density is the most important component, with latter playing a secondary role, and only for a transient period of time. Assumption *ii)* is well justified if we are not interested in the smaller scales, where vorticity could be important, plus the fact that it is well known that an initially irrotational fluid stays irrotational. Finally, assumption *iii)* can be justified on the grounds that H_{ab} is negligible in linear scalar Robertson–Walker perturbations[7] (those relevant for structure formation), and although is non–zero to second order[8], it becomes negligible on scales larger than the Hubble radius H^{-1}. Thus these cosmological models seem at least adequate to provide a coarse grained view of the universe on this sort of scales, while they could be relevant also on smaller scales, possibly giving a fair enough picture of the late stages of the gravitational collapse of irrotational dust. The assumption $H_{ab} = 0$ is further justified if one considers its physical significance: basically, to take $H_{ab} = 0$ means to neglect

gravitational waves[9] and gravitational induction[10], and if we take H^{-1} to be the relevant time scale, then one can argue that on larger scales these travelling effects have no time to propagate. Should however the assumption $H_{ab} = 0$ turn out to be too restrictive on physical grounds, either on scales larger than H^{-1}, or on smaller scales, it is nevertheless worth studying these models, as this is the most general class of inhomogeneous solutions of Einstein equations considered so far (see below). Since there are no sound (because of $p = 0$) or gravity (because of $H_{ab} = 0$) waves in these models, they were dubbed silent universes[8]. At this point, we can state the following:

- Claim 1: for $\Lambda \neq 0$ the attractor for expanding regions in a silent universe is the de Sitter model, i.e. these regions inflate and tend to a de Sitter vacuum phase (cosmic no–hair theorem).
- Claim 2: for $\Lambda = 0$, the attractor for the expanding regions in a silent universe is the Milne model, i.e. these regions tend to locally spherical voids.
- Claim 3: for $\Lambda = 0$, in the appropriate phase space for silent universe (see below) there is a 1-dimensional attracting set for collapsing regions, given by a closed curve, whose points represent Kasner models. Since all but a set of measure zero of the Kasner models have a spindle–like filamentary singularity, and because of the nature of the attracting set, the final fate of generic collapsing regions in a silent universe is to fall into a triaxial spindle singularity.
- Claim 4: for recollapsing regions in a $\Lambda \neq 0$ silent universe, Λ becomes soon negligible, and everything goes as in Claim 3.

Before going ahead in justifying these claims, further remarks on the irrotational perfect fluids models with $H_{ab} = 0$ are in order. These spacetimes where first considered in [11], where it was shown that they admit an orthonormal tetrad associated with the matter 4–velocity [12] u^a which is the simultaneous eigenframe for the shear σ_{ab} of the matter flow and for the electric Weyl tensor E_{ab}: in all but two special cases the vectors of this tetrad are hypersurface orthogonal, so that a coordinate system exists in which the metric g_{ab}, σ_{ab} and E_{ab} are all diagonal. The first cosmological implementation of these models was however given in [13], where it was shown that, if the flow is taken to be geodesic, $\dot{u}^a = 0$ (as it is the case for dust), the evolution equations simplify a great deal, reducing to only six ordinary differential equations for ρ (matter density), Θ (expansion scalar), σ_1 and σ_2 (two independent eigenvalues of the traceless shear tensor σ_{ab}), and E_1 and E_2 (eigenvalues of the traceless electric Weyl tensor E_{ab}). Mathematically, this fact corresponds to the lack of travelling information between neighboring fluid elements, and justifies the name "silent universes" for these models. Thus, all the environmental effects on each fluid element are coded in the initial data, which must satisfy the appropriate constraint equations[11, 14], while the subsequent evolution of the physical quantities characterizing each fluid element is purely local and given by the following evolution equations:

$$\dot{\rho} = -\Theta \rho , \tag{1}$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 2\,\sigma^2 - \frac{1}{2}\rho + \Lambda\,, \tag{2}$$

$$\dot{\sigma}_+ = {\sigma_+}^2 - \frac{1}{3}{\sigma_-}^2 - \frac{2}{3}\,\Theta\,\sigma_+ - E_+\,, \tag{3}$$

$$\dot{\sigma}_- = -2\,\sigma_+\,\sigma_- - \frac{2}{3}\,\Theta\,\sigma_- - E_-\,, \tag{4}$$

$$\dot{E}_+ = \sigma_-\,E_- - 3\,E_+\,\sigma_+ - \Theta\,E_+ - \frac{1}{2}\rho\,\sigma_+\,, \tag{5}$$

$$\dot{E}_- = 3\,\sigma_-\,E_+ + 3\,E_-\,\sigma_+ - \Theta\,E_- - \frac{1}{2}\rho\,\sigma_-\,, \tag{6}$$

where we have found convenient to define $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$ and $E_\pm = \frac{1}{2}(E_1 \pm E_2)$, $\sigma^2 = 3\,{\sigma_+}^2 + {\sigma_-}^2$ being the shear magnitude. Setting $\sigma_- = E_- = 0$ one obtains the dynamics of the Szekeres models[15] for $\Lambda = 0$, and that of their generalization[16] for $\Lambda > 0$[17]. As shown in [11], these degenerate models are the only ones of Petrov type D, while the general models with $\omega_{ab} = H_{ab} = 0$ are of the general Petrov type I. For dust the metric is[11, 2]

$$ds^2 = -dt^2 + \sum_{\alpha=1}^{3} {\ell_\alpha}^2(\vec{x},t)(dx^\alpha)^2\,, \tag{7}$$

and the local evolution of the three scale factors ℓ_α and that of their local average ℓ is given by

$$\frac{\dot{\ell}}{\ell} = \frac{1}{3}\Theta\,, \quad \frac{\dot{\ell}_\alpha}{\ell_\alpha} = \sigma_\alpha + \frac{1}{3}\Theta\,, \tag{8}$$

i.e. it will be determined (up to a factor which is constant along each flow line) upon integrating system (1)–(6). Of course, it will be possible to recover the whole metric (with its spatial dependence) only after the constraint equations will be integrated, but this is not only beyond our purpouses, but not necessary at all in order to disclose the local dynamics of our models. Instead this can be given by a qualitative analysis of the dynamical system (1)–(6) and (8), and of the related system (9)–(15) given below[1, 2].

For $\Lambda = 0$, system (1)–(6) has a single stationary point[18], given by the origin in phase-space, $\rho = \Theta = \sigma_\pm = E_\pm = 0$, corresponding to Minkowski spacetime. For $\Lambda > 0$ this point bifurcates in 10 new points. Of these, 3 are static ($\Theta = 0$), with only one physically meaningful (the other 2 have $\rho < 0$). and representing the Einstein universe, with $\rho = 2\Lambda$ and $\sigma_\pm = E_\pm = 0$. Another point represents the spatially flat de Sitter spacetime, with $\Theta = \sqrt{3\Lambda}$ (assuming expansion), and $\rho = \sigma_\pm = E_\pm = 0$. Two other points have $\sigma_- = E_- = 0$, thus are degenerate and represent an oblate and a prolate configuration expanding exponentially with $\Theta = \sqrt{\Lambda/3}$ and $\Theta = \sqrt{\Lambda}$ respectively. Finally, the two other pairs of stationary points are physically equivalent replicas of these two latter. Now, the linearized stability analysis of these stationary points shows that all of them are saddle points, thus unstable, except the one representing de Sitter, which is asymptotically stable if $\Theta > 0$, i.e. de Sitter *is the unique attractor for expansion.* It then follows that those patches of the $H_{ab} = 0$ models that do not recollapse expand toward a de Sitter phase and isotropize, i.e. $\sigma_\pm \to 0$ and $E_\pm \to 0$. In terms of the usual density parameter $\Omega = \Omega_M + \Omega_V$, with the matter and vacuum density

parameters defined as usual, $\Omega_M = 3\rho/\Theta^2$ and $\Omega_V = 3\Lambda/\Theta^2$, it follows that $\Omega \to 1$, with the 3-curvature ${}^{(3)}R$ of the slices orthogonal to u^a vanishing, ${}^{(3)}R \to 0$. This justifies our Claim 1. Even in case the portion of the universe undergoing inflation was initially negligibly small, in a timescale of order $\alpha = \sqrt{3/\Lambda}$ most of the universe volume will be isotropized by a phase of de Sitter-like inflation: a picture which is in the spirit of chaotic inflation [19]. According to the inflationary scenario, this de Sitter period will be ended by a suitable reheating process, after which a standard FRW phase will occur with Ω close to unity.

Another view of the phase-space for the H_{ab} models is achieved using Ω_M and Ω_V, together with the other dimensionless variables[20] $\Sigma_\pm = \sigma_\pm/\Theta$ and $\varepsilon_\pm = E_\pm/\Theta^2$. Obviously these variables diverge for $\Theta = 0$, so that one has to separately consider the $\Theta > 0$ and $\Theta < 0$ cases. To this end, it is convenient to introduce a new "time" $\tau = \pm 3 \ln \ell$, using the minus (plus) for the $\Theta < 0$ ($\Theta > 0$) case, so that $d\tau/dt > 0$ in both cases.

Denoting by a prime the derivative with respect to τ, by $\Sigma^2 = 3\,{\Sigma_+}^2 - {\Sigma_-}^2$ the magnitude of the dimensionless shear, and by $\Omega_G = \Omega_M - 2\Omega_V$, a sort of effective gravitational density parameter, the evolution equations for our variables for $\Theta < 0$ read

$$\Theta' = \frac{\Theta}{6}(2 + 12\,\Sigma^2 + \Omega_G)\,, \tag{9}$$

$$\Omega_V' = -\frac{\Omega_V}{3}(2 + 12\,\Sigma^2 + \Omega_G)\,, \tag{10}$$

$$\Omega_M' = -\frac{\Omega_M}{3}(12\,\Sigma^2 - 1 + \Omega_G)\,, \tag{11}$$

$$\Sigma_+' = \frac{\Sigma_+}{6}(2 - 12\,\Sigma^2 - 6\,\Sigma_+ - \Omega_G) + \frac{1}{3}{\Sigma_-}^2 + \varepsilon_+\,, \tag{12}$$

$$\Sigma_-' = \frac{\Sigma_-}{6}(2 - 12\,\Sigma^2 + 12\,\Sigma_+ - \Omega_G) + \varepsilon_-\,, \tag{13}$$

$$\varepsilon_+' = \frac{\varepsilon_+}{3}(1 - 12\,\Sigma^2 + 9\,\Sigma_+ - \Omega_G) - \Sigma_-\,\varepsilon_- + \frac{1}{6}\Sigma_+\,\Omega_M\,. \tag{14}$$

$$\varepsilon_-' = \frac{\varepsilon_-}{3}(1 - 12\,\Sigma^2 - 9\,\Sigma_+ - \Omega_G) - 3\,\Sigma_-\,\varepsilon_+ + \frac{1}{6}\Sigma_-\,\Omega_M\,. \tag{15}$$

We see that now the Raychaudhuri equation (9) is decoupled from the rest of the system (10)–(15), i.e. all other equations do not depend on Θ. The advantage we get in introducing system (10)–(15) is that, in addition to the non-static ($\Theta \neq 0$) stationary points of system (1)–(6), we also have a set of stationary points representing models for which at least one of the variables in (1)–(6) diverges, while those in (10)–(15) obviously have finite constant values. More precisely, system (10)–(15) admits many isolated stationary points (3 of which are unphysical, $\Omega_M = -3$), with many degenerate equivalent triplets, and 2 physically equivalent sets of points parametrized by Σ_+.

The physically significant and distinct points are listed in Table 1; points L I, L II and L III have $\Omega_V \neq 0$ and are the same non-static points of system (1)–(6) described above. Point D I-D VI are degenerate points with $\Omega_V = 0$ (i.e. for finite Θ they represent $\Lambda = 0$ models), and the set T III and its conjugate $\overline{\text{T III}}$ (given by the sign exchange $\Sigma_- \to -\Sigma_-$ and

Table 1: Stationary points of system (10)–(15) (only the physically interesting points are listed). For expansion ($\Theta > 0$) the de Sitter point L I is asymptotically stable, and for collapse ($\Theta < 0$) the attractor is given by the points of the Kasner set, i.e. the T III family and its conjugate $\overline{\text{T III}}$. All other points are saddles. Models marked with M are equivalent to Minkowski. The functions in the T III family read: $\Sigma_- = \frac{1}{\sqrt{3}}\sqrt{1-9{\Sigma_+}^2}$; $\varepsilon_+ = \frac{1}{3}\Sigma_+(6\Sigma_+ + 1) - \frac{1}{9}$; $\varepsilon_- = -\frac{\sqrt{3}}{9}(6\Sigma_+ - 1)\sqrt{1-9{\Sigma_+}^2}$.

Point	Ω_M	Ω_V	Σ_+	Σ_-	ε_+	ε_-	Model
L I	0	1	0	0	0	0	*de Sitter*
L II	0	3	$-1/3$	0	$1/3$	0	*prolate*
L III	0	9	$2/3$	0	0	0	*oblate*
D I	1	0	0	0	0	0	*Flat FRW*
D II	0	0	0	0	0	0	*Milne* (M)
D III	0	0	$1/6$	0	0	0	*Szekeres* (M)
D IV	0	0	$-1/3$	0	0	0	*Kasner* (M)
D V	0	0	$1/3$	0	$2/9$	0	*Kasner*
D VI	0	0	$-1/12$	0	$1/32$	0	*Szekeres*
T III	0	0	Σ_+	$\Sigma_-(\Sigma_+)$	$\varepsilon_+(\Sigma_+)$	$\varepsilon_-(\Sigma_+)$	*Kasner*

$\varepsilon_- \to -\varepsilon_-$ in the functions given in the caption of Table 1) represent triaxial configurations parametrized by Σ_+, and form a closed curve in phase space.

The linearized stability analysis shows again that for expansion ($\Theta > 0$) with $\Lambda \neq 0$ point L I, locally representing a flat de Sitter universe, is the unique attractor. If we set $\Lambda = \Omega_V = 0$, we decrease by one the dimension of phase space, i.e. system (11)–(15) then gives the dynamics of silent universes in the matter dominated era. In this case, point D II – representing a Milne universe – is the unique attractor for expansion ($\Theta > 0$), which justifies our Claim 2. For collapsing phases ($\Theta < 0$) the attractor is given by the curve T III–$\overline{\text{T III}}$. It can be shown[2], by looking at the local behavior of the metric (7), that each stationary point of the sets T III and $\overline{\text{T III}}$ is locally equivalent to a Kasner model, generically tending to a triaxial spindle singularity, with pancakes arising only as special degenerate cases, equivalent to Minkowski (see e.g. [21]). More specifically, each point on this attracting curve is stationary in itself, with the Jacobian of system (10)–(15) having a vanishing eigenvalue, the others being negative for collapse ($\Theta < 0$). Thus this attracting curve is normal hyperbolic [22], and therefore only a set of measure zero of initial conditions can give rise to trajectories falling in the isolated pancake points. Thus if collapse occurs $\Theta \to -\infty$ and $\Omega_V \to 0$, while $\Sigma_\pm$ and $\varepsilon_\pm$ tend to finite values, and $\Omega_M \to 0$ as well, and the outcome of the stability analysis for system (10)–(15) is that collapsing configurations generically fall in spindle–like Kasner singularities, with matter and Λ having no effects in the final stage. This justifies our Claims 3 and 4.

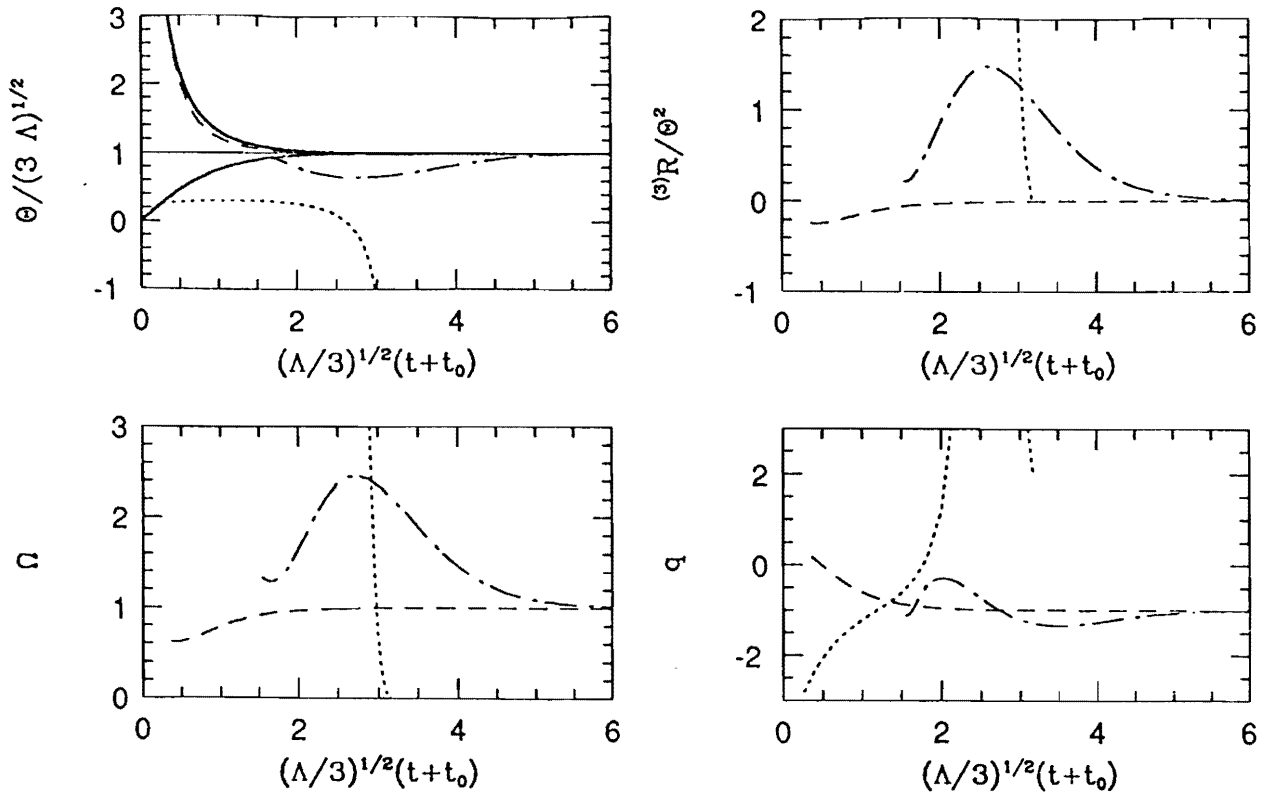

Figure 1: For three inflationary cases, with one finally re-collapsing, we plot four significant dimensionless variables. Top left: $\Theta/\sqrt{3\Lambda}$ against $\sqrt{\Lambda/3}(t+t_0)$, where t_0 is an appropiately choosen time constant, depending on the initial value of the expansion. The $\Theta = \sqrt{3\Lambda}$ thick line represents flat de Sitter, the upper bound open de Sitter, and the bound closed de Sitter. In the strip $|\Theta| \leq \sqrt{3\Lambda}$ we have ${}^{(3)}R > 0$. Top right: the dynamically normalized 3-curvature scalar ${}^{(3)}R/\Theta^2$. Bottom left: $\Omega = \Omega_M + \Omega_V$. Bottom right: the deceleration parameter $q = -3\dot{\Theta}/\Theta^2 - 1$ ($q < 0$ during inflation, $q < -1$ during super-inflation, and $q \to 2$ during collapse). Note that for the case with ${}^{(3)}R > 0$, Ω diverges from unity until super–inflation occurs, as depicted by q. For the re-collapsing case, before diverging at turn-around ($\Theta = 0$), ${}^{(3)}R/\Theta^2$ and Ω are out of scale.

To get a clue of what goes on locally in an inhomogeneous universe with an effective cosmological constant we plot in Fig. 1 three significant inflationary cases of several numerical integrations of system (1)–(6), with one finally recollapsing, and two expanding, one with ${}^{(3)}R < 0$, and the other with ${}^{(3)}R > 0$. The de Sitter phase is asymptotically approached with $\Theta/\sqrt{3\Lambda} \to 1$, ${}^{(3)}R \to 0$, $\Omega \to 1$ and $q \to -1$. We have found that in many cases $\Theta/\sqrt{3\Lambda}$ gets below the flat de Sitter line, then approaching it from below, asymptotically going as a closed de Sitter universe, with ${}^{(3)}R \to 0^+$. Related to this, we have observed in our numerical results that ${}^{(3)}R$ does not change sign during the evolution, although we cannot formulate this as a rule; also, note that although ${}^{(3)}R \to \infty$ in the collapse, ${}^{(3)}R/\Theta^2 \to 0$: the approach to the singularity is Weyl dominated. Finally, we observe that, for the case with ${}^{(3)}R > 0$, Ω does not approach unity even during the inflationary phase, unless super-inflation [23] occurs, with $\dot{\Theta} > 0$ and $q < -1$.

In summary, we have presented here an analysis of $H_{ab} = \omega_{ab} = p = 0$ models with and without a cosmological constant Λ. A positive Λ is taken to represent the local value of the vacuum energy density in the early universe, possibly causing the occurrence of an inflationary phase in certain patches of the universe with suitable initial conditions. Of course in a realistic scenario such a cosmological constant will ultimately decay into radiation, reheating the universe to a standard FRW phase. In the matter dominated era, our models with $\Lambda = 0$ seem to give a fair coarse grained view of the possible non–linear evolution of the small fluctuations present at decoupling, with expanding regions tending to locally spherical voids, and collapsing ones tending to spindle–like singularities, rather then to pancakes as generally predicted in the Newtonian approximation. The picture that emerges is that of a generally inhomogeneous universe, with large patches of it where, thanks to an early inflationary evolution, the local properties are close to that of a flat FRW model. In this picture, however, the observable parameters H_0, Ω_0 and q_0 should be thought as local values, not to be interpreted as giving the global properties of the universe. On sub–horizon scales, the question which is left open is if collapsing configurations will still tend to a spindle once H_{ab} will be switched on (at least perturbatively), thus definitively contradicting the Zel'dovich pancaking of Newtonian theory, or if pancaking will be somehow be reintroduced by H_{ab}: this will be the subject of a future investigation.

MB would like to thank British PPARC (grant GR/J 36440) and Università degli Studi di Trieste for financial support, and SISSA for hospitality. SM and OP acknowledge Italian MURST for financial support.

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