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Two–matrix models, W algebras and 2D gravity *

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1 Introduction

In the last few years a lot of research has been devoted to matrix models. The idea behind such effort is that they represent sums over discretized Riemann surfaces with some structure on them, i.e. they are candidates to represent 2D gravity coupled to $c \leq 1$ conformal matter. In particular a one Hermitean matrix model with a quadratic and a cubic interaction can be thought of as a sum over all possible triangulations of compact Riemann surfaces. Consequently it is believed to be a discretized version of pure gravity in 2D. Analogously, one Hermitean matrix models with higher interaction terms are thought to represent 2D gravity coupled to conformal matter [1]. These models are known to lead in the continuum limit to the KdV hierarchy. Most of the recent research has concentrated on them as well as on Hermitean matrix models depending on a continuous parameter, which are known to provide a description of the so–called $c = 1$ string theory. Matrix models à la Kontsevitch (i.e. one–matrix models coupled to a suitable external matrix) have also been studied. As for two–matrix models, it was shown by Kazakov, [2], that a two–matrix model with two quartic interactions and a bilinear coupling represents the Ising model over discretized Riemann surfaces. This opened the way to the analysis of two–matrix models (and of multi–matrix models), as models relevant to the description of 2D gravity coupled to conformal matter. However considerably less attention has been paid to them (see anyhow [3]). The reason is understandable: although faith on solvability of these models is widespread, the technical difficulties one is faced with when trying to actually solve them look at first formidable.

The message I want to convey in these lectures is that not only can two–matrix models – here we will limit ourselves to them – be actually solved, but that they provide a unifying description, in the sense that they supply solutions relating to one–matrix models (both constant and depending on a continuous parameter), models à la Kontsevitch [4], etc., and

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possibly solutions not to be found anywhere else. The analysis of the content of two–matrix models is still under way and we are not ready for a comprehensive report. However there is already plenty of evidence of what we just said [5],[6],[7],[8],[9],[10], [11],[12],[13]. In particular here we want to show two things: how two–matrix models can be used to describe the $c = 1$ string theory and how from two–matrix models one can extract the n–KdV hierarchies and the related topological field theory models.

The idea at the basis of our treatment of two–matrix models is to transform the initial functional integral problem into a discrete integrable linear system subject to some constraints (the coupling conditions). We end up in this way with the discrete Toda lattice hierarchy. The latter (either in its usual form or in the extended form) underlies all our calculations: our aim is to compute the correlation functions (CF's) of each model, which in turn may be expressed in terms of the integrable flows of the Toda hierarchy, subject to the coupling conditions. This is the general setting for *unreduced* two–matrix models. I will also consider other models or subsystems of the initial system, in which the relevant hierarchy is not the Toda lattice hierarchy, but a reduced differential hierarchy obtained from the latter. These models will be referred to as *reduced* models.

An important part of these lectures concerns the connection with theories of interest in physics. On the one hand it will be shown how a very simple unreduced two–matrix model represents the so–called $c = 1$ string theory compactified at the self–dual point: such model reproduces not only the tachyonic but also all the discrete states and their correlators. On the other hand we make contact with $c < 1$ string theory via the reduced models, more precisely the models whose underlying integrable hierarchy are generalized KdV hierarchies. They represent well–known topological field theories coupled to topological gravity.

The paper is organized as follows. In section 2 we collect all the preliminary results on ordinary and extended two–matrix models: associated linear systems and (Toda lattice) hierarchies, coupling conditions and so on. In section 3 the W constraints are derived. Section 4 is devoted to the calculation of CF's in unreduced models with two different methods: the first is based on the direct resolution of the relevant W constraints, the second on explicitly solving the coupling conditions; the latter method turns out to be by far the most powerful. Section 5 describes the connection with $c = 1$ string theory. Section 6 is devoted to deriving integrable differential hierarchies together with their integrable reductions from the original Toda lattice hierarchy of the model. Section 7 is mostly pedagogical: it shows how to derive CF's in unreduced models by means of the differentiable hierarchies of the previous section. Section 8 does the same for completely reduced models, i.e. for models based on p–KdV hierarchies: we show that it is possible in these models to derive very general formulas for correlators. Finally in section 9 we discuss the physical connections of the results of section 8, i.e. the interpretations in terms of critical indices and topological field theories.

This review is mostly an elaboration based on the papers [5] through [12].

2 The extended two–matrix model

The model of two Hermitean $N \times N$ matrices M_1 and M_2, is introduced in terms of the partition function

$$Z_N(t,c) = \int dM_1 dM_2 e^{TrU}, \qquad U = V_1 + V_2 + gM_1M_2 \tag{2.1}$$

with potentials

$$\bar{V}_\alpha = \sum_{r=1}^{p\alpha} \bar{t}_{\alpha,r} M_\alpha^r \qquad \alpha = 1,2. \tag{2.2}$$

where p_α are finite numbers. These potentials define the model. We denote by $\mathcal{M}_{p_1,p_2}$ the corresponding two–matrix model. Models of this type will be referred to as *ordinary two–matrix models.*

However we are interested in computing all the correlation functions (CF's) of this model, i.e. the correlation functions of the operators

$$\tau_k = \mathrm{Tr} M_1^k, \qquad \sigma_k = \mathrm{Tr} M_2^k, \qquad \forall k, \qquad \chi = Tr(M_1M_2)$$

which are defined by

$$\ll \tau_{r_1} \dots \tau_{r_n} \sigma_{s_1} \dots \sigma_{s_m} \chi^l \gg = \frac{\partial^{n+m+l}}{\partial t_{1,r_1} \dots \partial t_{1,r_n} \partial t_{2,s_1} \dots \partial t_{2,s_m} \partial g^l} \ln Z_N(t,g) \tag{2.3}$$

For this reason we complete the above model by replacing (2.2) with the more general potentials

$$V_\alpha = \sum_{r=1}^{\infty} t_{\alpha,r} M_\alpha^r, \qquad \alpha = 1,2 \tag{2.4}$$

where $t_{\alpha,r} \equiv \bar{t}_{\alpha,r}$ for $r \leq p_\alpha$.

In other words we have embedded the original couplings $\bar{t}_{\alpha,r}$ into two infinite sets of couplings. Therefore we have two types of couplings. The first type consists of those couplings (the barred ones) that define the model: they represent the true *dynamical* parameters of the theory; they are kept non-vanishing throughout the calculations. The second type encompasses the remaining couplings, which are introduced only for computational purposes and are set to zero in formulas like (2.3), i.e. the partition function is but a generating function for these couplings. In terms of ordinary field theory the former are analogous to the interaction couplings, while the latter correspond to external sources (coupled to composite operators). From now on we will not make any formal distinction between them. This will allow us to obtain very general and powerful formulas – the W–constraints and the flow equations. Case by case we will specify which are the interaction couplings and which are the external ones.

After integrating the angular variables, [14], eq.(2.1) can be written as

$$Z_N(t,C) = \text{const} \int \prod_{\alpha=1}^{2} \prod_{i=1}^{N} d\lambda_{\alpha,i} \Delta(\lambda_1)\Delta(\lambda_2) \exp U, \tag{2.5}$$

$$U = \sum_{k=1}^{\infty} t_{1,k} \sum_{i=1}^{N} \lambda_{1,i}^k + \sum_{r=1}^{\infty} t_{2,r} \sum_{i=1}^{N} \lambda_{2,i}^r + g \sum_{i=1}^{N} \lambda_{1,i}\lambda_{2,i} \tag{2.6}$$

where Δ stands for the Vandermonde determinant.

It is however convenient for many purposes to enlarge the model by introducing new interaction terms as follows

$$U \to \mathcal{U} = \sum_{k=1}^{\infty} t_{1,k} \sum_{i=1}^{N} \lambda_{1,i}^k + \sum_{r=1}^{\infty} t_{2,r} \sum_{i=1}^{N} \lambda_{2,i}^r + \overline{\mathcal{V}}, \qquad \overline{\mathcal{V}} = \sum_{k,r\geq 1} g_{k,r} \sum_{i=1}^{N} \lambda_{1,i}^k \lambda_{2,i}^r$$

where $g_{1,1} \equiv g$. We notice that this corresponds, in general, to adding to the original model (2.1) not terms like $Tr(M_1^k M_2^r)$, but rather composite operators of the form $Tr(D_1^k D_2^r)$, where D_1 and D_2 are the diagonal eigenvalue matrices of M_1 and M_2, respectively.

We will call this new model the *extended two–matrix model.* It is evident that we can recover the ordinary model from the extended one by setting to zero all the extra coupling. We can generalize to the extended model everything that can be done with the original one (2.1). In particular the formula (2.3) is generalized in an obvious way to the extended model.

Henceforth all the couplings $g_{k,r}$, except g, will be referred to as *extra*. Moreover, for the sake of compactness, we will often use the convention

$$t_{1,r} \equiv g_{r,0}, \qquad t_{2,s} \equiv g_{0,s}$$

The models introduced so far will be also referred to as *unreduced models*, in opposition to the reduced ones, which will be introduced later on.

Before we proceed one remark is in order. The partition function Z_N is evidently not always well defined for generic values of the couplings. We will give later on a definition of Z_N which extends to regions of the parameter space where the integral (2.5) is not well defined.

Next we introduce the orthogonal monic polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \qquad \eta_n(\lambda_2) = \lambda_2^n + \text{lower powers}$$

and the orthogonal functions

$$\Psi_n(\lambda_1) = e^{V_1(\lambda_1)}\xi_n(\lambda_1), \qquad \Phi_n(\lambda_2) = e^{V_2(\lambda_2)}\eta_n(\lambda_2).$$

The orthogonality relations are

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1) e^{\mathcal{V}(\lambda_1,\lambda_2)} \Phi_m(\lambda_2) = \delta_{nm} h_n(t,g). \tag{2.7}$$

In these equations

$$V_\alpha(\lambda_\alpha) = \sum_{r=1}^{\infty} t_{\alpha,r}\lambda_\alpha^r, \qquad \alpha = 1,2, \qquad \mathcal{V}(\lambda_1,\lambda_2) = \sum_{k,r\geq 1} g_{k,r}\lambda_1^k\lambda_2^r$$

Using these relations and the properties of the Vandermonde determinants, one can easily calculate the partition function

$$Z_N(t,g) = \text{const } N! \prod_{i=0}^{N-1} h_i \tag{2.8}$$

Knowing the partition function means knowing the coefficients $h_n(t,g)$.

Before we proceed further we need to introduce more notations and conventions. For any matrix M we define a new matrix $\mathcal{M}$

$$\mathcal{M} = H^{-1}MH, \qquad H_{ij} = h_i\delta_{ij}, \qquad \bar{M}_{ij} = M_{ji}, \qquad M_l(j) \equiv M_{j,j-l}.$$

As usual we introduce the natural gradation

$$deg[E_{ij}] = j - i, \qquad (E_{ij})_{n,m} \equiv \delta_{in}\delta_{jm}$$

and, for any given matrix M, if all its non–zero elements have degrees in the interval $[a,b]$, then we will simply write: $M \in [a,b]$. Moreover M_+ will denote the upper triangular part of M (including the main diagonal), while $M_- = M - M_+$. We will write $\mathrm{Tr}(M) = \sum_{i=0}^{N-1} M_{ii}$.

Let us come now to the main point of this section: *the information concerning the partition functions of our matrix models can be encoded in three elements: 1) two integrable linear systems, 2) the coupling conditions, and 3) the reconstruction formulas for Z_N.*

In order to see this let us introduce the following Q– and P–type matrices

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1)\lambda_\alpha e^{\mathcal{V}(\lambda_1,\lambda_2)}\Phi_m(\lambda_2) \equiv Q_{nm}(\alpha)h_m, \quad \alpha = 1,2. \tag{2.9a}$$

$$\int d\lambda_1 d\lambda_2 \Big(\frac{\partial}{\partial\lambda_1}\Psi_n(\lambda_1)\Big)e^{\mathcal{V}(\lambda_1,\lambda_2)}\Phi_m(\lambda_2) \equiv P_{nm}(1)h_m \tag{2.9b}$$

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1)e^{\mathcal{V}(\lambda_1,\lambda_2)}\Big(\frac{\partial}{\partial\lambda_2}\Phi_m(\lambda_2)\Big) \equiv P_{mn}(2)h_n \tag{2.9c}$$

Both $Q(1)$ and $\bar{Q}(2)$ are Jacobi matrices: their pure upper triangular part is $I_+ = \sum_i E_{i,i+1}$.

Here come now the three above–mentioned basic elements. The *first* is provided by the following coupling constraints

$$P(1) + \sum_{r,s\geq 1} rg_{r,s}Q(1)^{r-1}Q(2)^s = 0, \qquad \bar{\mathcal{P}}(2) + \sum_{r,s\geq 1} sg_{r,s}Q(1)^rQ(2)^{s-1} = 0 \tag{2.10}$$

These coupling conditions lead to the $W_{1+\infty}$–constraints (see below). They are derived by remarking that the integral of the derivative of the integrand in the LHS of (2.9a) w.r.t λ_1

and λ_2, respectively, vanishes. In particular, in the ordinary two–matrix model, eq.(2.10) becomes

$$P(1) + gQ(2) = 0, \qquad gQ(1) + \bar{\mathcal{P}}(2) = 0, \tag{2.11}$$

The coupling conditions are understood to be evaluated at vanishing external couplings. Therefore from eq.(2.11) it follows at once that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \qquad \alpha = 1, 2$$

where

$$m_1 = p_2 - 1, \qquad m_2 = 1, \qquad n_1 = 1, \qquad n_2 = p_1 - 1$$

We recall that p_α is the polynomial order of V_α when only the dynamical couplings are taken into account. Therefore, the finite order of V_α entails a finite band structure for the Q matrices.

The *second element* are the discrete linear systems associated to the two–matrix model. To write them down we will denote the semi–infinite column vectors with components $\Psi_0, \Psi_1, \Psi_2, \ldots$, and $\Phi_0, \Phi_1, \Phi_2, \ldots$, by Ψ and Φ, respectively. The *first discrete linear system* is

$$\begin{cases} Q(1)\Psi(\lambda_1) = \lambda_1 \Psi(\lambda_1), \\ \frac{\partial}{\partial g_{r,0}} \Psi(\lambda_1) = Q^r_+(1)\Psi(\lambda_1), \qquad r \geq 1 \\ \frac{\partial}{\partial g_{r,s}} \Psi(\lambda_1) = -\Big(Q^r(1)Q^s(2)\Big)_- \Psi(\lambda_1), \qquad r \geq 0, \quad s \geq 1 \\ \frac{\partial}{\partial \lambda} \Psi(\lambda_1) = P(1)\Psi(\lambda_1). \end{cases} \tag{2.12}$$

The corresponding consistency conditions are

$$[Q(1), \quad P(1)] = 1, \tag{2.13a}$$

$$\frac{\partial}{\partial g_{r,s}} Q(1) = [Q(1), \Big(Q^r(1)Q^s(2)\Big)_-], \qquad r \geq 0, \; s \geq 0, \; r + s \geq 1 \tag{2.13b}$$

$$\frac{\partial}{\partial g_{r,0}} P(1) = [Q^r_+(1), P(1)], \qquad r \geq 1 \tag{2.13c}$$

$$\frac{\partial}{\partial g_{r,s}} P(1) = [P(1), \Big(Q^r(1)Q^s(2)\Big)_-], \qquad r \geq 0, \; s \geq 1 \tag{2.13d}$$

Let us see, as an example, the derivation of eqs.(2.12). One starts from the remark that, by differentiating (2.7), w.r.t. $t_{1,k}$ one gets

$$\delta_{n,m} \frac{\partial h_n}{\partial t_{1,k}} = \int \partial\lambda_1 \partial\lambda_2 \Big(\frac{\partial \xi_n}{\partial t_{1,k}} \eta_m + \xi_n \frac{\partial \eta_m}{\partial t_{1,k}} + \xi_n \lambda_1^k \eta_m \Big) e^{V_1 + V_2 + \nu} \tag{2.14}$$

From this we extract in particular

$$\frac{\partial h_n}{\partial t_{1,k}} = \int d\lambda_1 \partial\lambda_2 \xi_n e^{V_1+V_2+\mathcal{V}} \eta_n \lambda_1^k \equiv < \xi_n \lambda_1^k \eta_n > \tag{2.15}$$

where we have introduced a shortened notation for the integral. One the other hand, using the spectral equation in the form

$$\lambda_1^k \xi_n = \sum_{l=0}^{n+k} Q^k(1)_{n,l} \xi_l$$

the very definition of the monic polynomial ξ_n and eq.(2.14) with $n > m$, we find

$$\frac{\partial \xi_n}{\partial t_{1,k}} = -\sum_{l=0}^{n-1} Q^k(1)_{n,l} \xi_l \tag{2.16}$$

Now, remembering the definition of the auxiliary orthogonal functions ψ_n

$$\frac{\partial \psi_n}{\partial t_{1,k}} = \frac{\partial \xi_n}{\partial t_{1,k}} e^{V_1} + \xi_n \lambda_1^k e^{V_1} =$$
$$-\sum_{l=0}^{n-1} Q^k(1)_{n,l} \psi_l + \sum_{l=0}^{n+k} Q^k(1)_{n,l} \psi_l = \sum_{l=n}^{n+k} Q^k(1)_{n,l} \psi_l$$

which is exactly the second equation of the linear system. The other equations can be derived in a similar way.

The *second discrete linear system* is

$$\begin{cases} \bar{\mathcal{Q}}(2)\Phi(\lambda_2) = \lambda_2 \Phi(\lambda_2), \\ \frac{\partial}{\partial g_{0,s}} \Phi(\lambda_2) = \bar{\mathcal{Q}}_+^s(2)\Phi(\lambda_2), & s \geq 1 \\ \frac{\partial}{\partial g_{r,s}} \Phi(\lambda_2) = -\left(\bar{\mathcal{Q}}^s(2)\bar{\mathcal{Q}}^r(1)\right)_- \Phi(\lambda_2), & r \geq 1, s \geq 0 \\ \frac{\partial}{\partial \lambda} \Phi(\lambda_2) = P(2)\Psi(\lambda_2). \end{cases} \tag{2.17}$$

The corresponding consistency conditions are

$$[\bar{\mathcal{Q}}(2), \quad P(2)] = 1 \tag{2.18a}$$

$$\frac{\partial}{\partial g_{r,s}} Q(2) = [\left(Q^r(1)Q^s(2)\right)_+, \quad Q(2)], \qquad r \geq 0, \ s \geq 0, \ r+s \geq 1 \tag{2.18b}$$

$$\frac{\partial}{\partial g_{r,s}} P(2) = [P(2), \left(\bar{\mathcal{Q}}^s(2)\bar{\mathcal{Q}}^r(1)\right)_-], \qquad r \geq 1, s \geq 0 \tag{2.18c}$$

$$\frac{\partial}{\partial g_{0,s}} P(2) = [\bar{\mathcal{Q}}_+^s(2), P(2)], \qquad s \geq 1 \tag{2.18d}$$

The *third ingredient* we need is the link between the quantities that appear in the linear systems and in the coupling conditions with the original partition function. We have

$$\frac{\partial}{\partial g_{r,s}} \ln Z_N(t,g) = \mathrm{Tr}\left(Q^r(1)Q^s(2)\right), \tag{2.19}$$

It is evident that, by using the above flow equations, we can express all the derivatives of Z_N in terms of the elements of the Q matrices. For example

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{\alpha,r}} \ln Z_N(t,g) = \left(Q^r(\alpha)\right)_{N,N-1}, \quad \alpha = 1,2 \tag{2.20}$$

Knowing all the derivatives with respect to the coupling parameters we can reconstruct the partition function up to an overall integration constant (depending only on N).

The three elements just introduced provide a larger definition of our system. In fact we notice that the initial path integral exists only in a very restricted region of the parameter space. The above formulas provide the continuation of our system to regions of the parameter space where the path–integral is ill–defined.

We will use the following coordinatization for $Q(1)$ and $Q(2)$

$$Q(1) = I_+ + \sum_i \sum_{l=0}^{\infty} a_l(i) E_{i,i-l}, \qquad \bar{Q}(2) = I_+ + \sum_i \sum_{l=0}^{\infty} b_l(i) E_{i,i-l} \tag{2.21}$$

One can immediately see that, for example,

$$\left(Q_+(1)\right)_{ij} = \delta_{j,i+1} + a_0(i)\delta_{i,j}, \qquad \left(Q_-(2)\right)_{ij} = R_i \delta_{j,i-1} \tag{2.22}$$

where $R_{i+1} \equiv h_{i+1}/h_i$. As a consequence of this coordinatization, eq.(2.20) gives in particular the important relation

$$\frac{\partial^2}{\partial t_{1,1}^2} \ln Z_N(t,g) = a_1(N), \tag{2.23}$$

and

$$\frac{\partial^2}{\partial t_{1,1} t_{2,1}} \ln Z_N(t,g) = R_N \tag{2.24}$$

In our model, the four matrices $Q(1), Q(2)$ and $P(1), P(2)$ are not completely independent. In fact the two P matrices can be expressed in terms of the two Q matrices as follows

$$P_{nm}(1) = \sum_{r=1}^{\infty} r t_{1,r} Q_{nm}^{r-1}(1) + n Q_{nm}^{-1}(1) + \sum_{r=1}^{\infty} \frac{\partial \ln Z_n}{\partial t_{1,r}} Q_{nm}^{-r-1}(1); \tag{2.25}$$

$$\bar{\mathcal{P}}_{nm}(2) = \sum_{r=1}^{\infty} r t_{2,r} Q_{nm}^{r-1}(2) + m Q_{nm}^{-1}(2) + \sum_{r=1}^{\infty} Q_{nm}^{-r-1}(2) \frac{\partial \ln Z_m}{\partial t_{2,r}}. \tag{2.26}$$

Let us see in some detail the derivation of the first equation. From the spectral equation in the first discrete linear system, we see that polynomials $\xi_n(\lambda_1)$ satisfy the recursion relation

$$\lambda_1 \xi_n(\lambda_1) = \sum_{m=0}^{n} Q(1)_{nm} \xi_m(\lambda_1)$$

Solving this recursion relation we get

$$\xi_n(\lambda_1) = \det\Big(\lambda_1 - Q(1)\Big)_{n\times n}$$

The determinant on the RHS refers to the principal minor delimited by (and including) the n–th row and n–th column. Equivalently

$$\ln \xi_n(\lambda_1) = n \ln \lambda_1 - \sum_{r=1}^{\infty} \frac{1}{l\lambda_1^r} \frac{\partial \ln Z_n}{\partial t_{1,r}}$$

This implies

$$\ln \Psi_n(\lambda_1) = \sum_{r=1}^{\infty} r t_{1,r} \lambda_1^{r-1} + n \ln \lambda_1 - \sum_{r=1}^{\infty} \frac{1}{r\lambda_1^r} \frac{\partial \ln Z_n}{\partial t_{1,r}} \tag{2.27}$$

Taking the derivative with respect to λ_1, and using the spectral relation, we have

$$\begin{aligned} \frac{\partial \Psi_n(\lambda_1)}{\partial \lambda_1} &= \Big(\sum_{r=1}^{\infty} r t_{1,r} \lambda_1^{r-1} + n\lambda_1^{-1} + \sum_{r=1}^{\infty} \frac{\partial \ln Z_n}{\partial t_{1,r}} \lambda_1^{-r-1}\Big) \Psi_n(\lambda_1) \\ &= \Big(\sum_{r=1}^{\infty} r t_{1,r} Q_{nm}^{r-1}(1) + n Q_{nm}^{-1}(1) + \sum_{r=1}^{\infty} \frac{\partial \ln Z_n}{\partial t_{1,r}} Q_{nm}^{-r-1}(1)\Big) \Psi_m(\lambda_1) \end{aligned}$$

From the last equality we read off formula (2.25). Similarly we obtain

$$\ln \Phi_n(\lambda_2) = \sum_{r=1}^{\infty} r t_{2,r} \lambda_2^{r-1} + n \ln \lambda_2 - \sum_{r=1}^{\infty} \frac{1}{r\lambda_2^r} \frac{\partial \ln Z_n}{\partial t_{2,r}} \tag{2.28}$$

from which we can extract (2.26).

Finally we write down in explicit form the *first flows*, i.e. the $t_{1,1}$– and $t_{2,1}$–flows, which will play a very important role in what follows

$$\frac{\partial}{\partial t_{1,1}} a_l(j) = a_{l+1}(j+1) - a_{l+1}(j) + a_l(j)\Big(a_0(j) - a_0(j-l)\Big) \tag{2.29a}$$

$$\frac{\partial}{\partial t_{2,1}} a_l(j) = R_{j-l+1} a_{l-1}(j) - R_j a_{l-1}(j-1) \tag{2.29b}$$

$$\frac{\partial}{\partial t_{1,1}} b_l(j) = R_{j-l+1} b_{l-1}(j) - R_j b_{l-1}(j-1) \tag{2.29c}$$

$$\frac{\partial}{\partial t_{2,1}} b_l(j) = b_{l+1}(j+1) - b_{l+1}(j) + b_l(j)\Big(b_0(j) - b_0(j-l)\Big) \tag{2.29d}$$

2.1 The Toda lattice hierarchy

Let us discuss now the properties of the lattice hierarchies we have obtained above. We start from the first flow equations. In particular, from eq.(2.29a), we have

$$\frac{\partial}{\partial t_{2,1}} a_0(j) = R_{j+1} - R_j \tag{2.30}$$

while, from(2.19), one can show that

$$\frac{\partial}{\partial t_{1,1}} R_j = R_j(a_0(j) - a_0(j-1)) \tag{2.31}$$

Combining these two equations we obtain the following 2-dimensional lattice equation

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{2,1}} \ln R_j = R_{j+1} - 2R_j + R_{j-1} \tag{2.32}$$

In terms of the coordinates $\phi_j = \ln h_j$, the above equation takes the familiar form

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{2,1}} \phi_j = e^{\phi_{j+1}-\phi_j} - e^{\phi_j - \phi_{j-1}} \tag{2.33}$$

Eq.(2.33) is nothing but the two–dimensional Toda lattice equation. The other flows relevant to the ordinary two–matrix model, generate the complete (ordinary) two–dimensional Toda lattice hierarchy [15].

The flows relevant to the extended two–matrix model form another hierarchy which will be referred to as the extended two–dimensional Toda lattice hierarchy.

We have seen above that the linear systems corresponding to this hierarchies are restricted by the coupling conditions (2.10); therefore we can say that 2–matrix models are nothing but constrained 2–dimensional Toda lattices.

The discrete linear systems ensuing from the two-matrix models are integrable. In fact all the flows commute with one another; for example

$$\frac{\partial}{\partial g_{k,l}} \frac{\partial}{\partial g_{r,s}} Q(1) = \frac{\partial}{\partial g_{k,l}} \left[Q(1), \left(Q^r(1) Q^s(2) \right)_- \right] =$$
$$\frac{\partial}{\partial g_{r,s}} \left[Q(1), \left(Q^k(1) Q^l(2) \right)_- \right] = \frac{\partial}{\partial g_{r,s}} \frac{\partial}{\partial g_{k,l}} Q(1)$$

as can be easily proved with a simple application of the definition $M = M_+ + M_-$, for any matrix M.

3 $W_{1+\infty}$ constraints

There is a way to put together flow equations and coupling conditions that lead to an elegant algebraic structure, the $W_{1+\infty}$ constraints (or simply W–constraints). These algebraic constraints on the partition function for the extended two–matrix model were obtained in [6]. They take the form

$$\left(\mathcal{L}_n^{[r]}(1) - (-1)^r T_n^{[r]}(1) \right) Z_N(t;g) = 0, \qquad r \geq 1; \qquad n \geq -r. \tag{3.34a}$$
$$\left(\mathcal{L}_n^{[r]}(2) - (-1)^r T_n^{[r]}(2) \right) Z_N(t;g) = 0, \qquad r \geq 1, \qquad n \geq -r \tag{3.34b}$$

The generators $\mathcal{L}_n^{[r]}(1)$ are differential operators involving N and $t_{1,k}$, while $\mathcal{L}_n^{[r]}(2)$ have the same form with $t_{1,k}$ replaced by $t_{2,k}$. The $T_n^{[r]}(1)$ are differential operators involving the couplings $g_{r,s}$ (except $g_{r,0}$) and commute with the $\mathcal{L}_n^{[r]}(1)$. The $T_n^{[r]}(2)$ are differential operators involving the couplings $g_{r,s}$ (except $g_{0,s}$) and commute with the $\mathcal{L}_n^{[r]}(2)$.

The $\mathcal{L}_n^{[r]}(1)$ satisfy a $W_{1+\infty}$ algebra

$$[\mathcal{L}_n^{[r]}(1), \mathcal{L}_m^{[s]}(1)] = (sn - rm)\mathcal{L}_{n+m}^{[r+s-1]}(1) + \ldots, \tag{3.35}$$

for $r, s \geq 1$; $n \geq -r, m \geq -s$. Here dots denote lower than $r + s - 1$ rank operators. In particular

$$[\mathcal{L}_n^{[1]}(1), \mathcal{L}_m^{[1]}(1)] = (n - m)\mathcal{L}_{n+m}^{[1]}(1) \tag{3.36a}$$
$$[\mathcal{L}_n^{[2]}(1), \mathcal{L}_m^{[1]}(1)] = (n - 2m)\mathcal{L}_{n+m}^{[2]}(1) + m(m+1)\mathcal{L}_{n+m}^{[1]}(1) \tag{3.36b}$$
$$[\mathcal{L}_n^{[2]}(1), \mathcal{L}_m^{[2]}(1)] = 2(n - m)\mathcal{L}_{n+m}^{[3]}(1) - (n - m)(n + m + 3)\mathcal{L}_{n+m}^{[2]}(1) \tag{3.36c}$$

We see that once we know these generators and $\mathcal{L}_{-2}^{[2]}(1)$, the remaining ones are produced by the algebra itself. For the explicit expression of the basic generators, see below.

The algebra of the $\mathcal{L}_n^{[r]}(2)$ is just a copy of the above one, and is isomorphic to the algebra satisfied by the $T_n^{[r]}(1)$ and by the $T_n^{[r]}(2)$ separately.

The $W_{1+\infty}$ constraints for the ordinary two–matrix model are simpler. They take the form

$$W_n^{[r]} Z_N(t, g) = 0, \qquad \tilde{W}_n^{[r]} Z_N(t, g) = 0 \quad r \geq 0; \quad n \geq -r, \tag{3.37}$$

where

$$W_n^{[r]} \equiv (-c)^n \mathcal{L}_n^{[r]}(1) - \mathcal{L}_{-n}^{[r+n]}(2) \tag{3.38a}$$
$$\tilde{W}_n^{[r]} \equiv (-c)^n \mathcal{L}_n^{[r]}(2) - \mathcal{L}_{-n}^{[r+n]}(1) \tag{3.38b}$$

Here we have used also $\mathcal{L}_n^{[0]}(1)$. These are given by

$$\mathcal{L}_n^{[0]} \equiv \frac{\partial}{\partial t_{1,n}} + N\delta_{n,0}, \qquad n \geq 0$$

These generators represent an abelian extension of the $W_{1+\infty}$ algebra, of which they form an abelian subalgebra. The other generators behave tensorially with respect to $\mathcal{L}_n^{[0]}$.

3.1 Derivation of the W constraints

To prove the W constraints it is very convenient to use the extended two–matrix model. We recall all the definitions of section 2 and we introduce the additional P–type matrices

$$P_\circ(1)\xi = \frac{\partial}{\partial \lambda_1}\xi, \qquad P_\circ(2)\eta = \frac{\partial}{\partial \lambda_2}\eta.$$

where we appended the label o in order to distinguish these operators from the previously defined $P(1)$ and $P(2)$, and ξ and η are the infinite vectors $\xi = (\xi_0, \xi_1, \ldots)^t$ and $\eta = (\eta_0, \eta_1, \ldots)^t$

As in section 2 we can get the spectral equations

$$\lambda_1 \xi = Q(1)\xi, \qquad \lambda_2 \eta = \bar{Q}(2)\eta, \tag{3.39}$$

and the equations of motion of the polynomials, in the form of two coupled linear systems, which we will not write down here. One can easily see that the coupling conditions become

$$P_o(1) + \hat{V}_1'(1) + \sum_{a,b\geq 1} a g_{a,b} Q^{a-1}(1) Q^b(2) = 0, \tag{3.40a}$$

$$\bar{\mathcal{P}}_o(2) + \hat{V}_2'(2) + \sum_{a,b\geq 1} b g_{a,b} Q^{a}(1) Q^{b-1}(2) = 0. \tag{3.40b}$$

where we have introduced the notation

$$\hat{V}_1'(1) = \sum_{k=1}^{\infty} k t_{1,k} Q^{k-1}(1),$$

$$\hat{V}_1''(1) = \sum_{k=2}^{\infty} k(k-1) t_{1,k} Q^{k-2}(1),$$

etc.

3.1.1 The Virasoro constraints

The $W_{1+\infty}$–algebraic constraints contain a particularly important subset, the Virasoro constraints. We begin with them. We simply write the coupling conditions (3.40a) as an infinite series of equations

$$\mathrm{Tr}\Big(Q^{n+1}(1)\big(P_o(1) + \hat{V}'(1)+) + \sum_{a,b\geq 1} a g_{a,b} Q^{a-1}(1) Q^b(2)\big)\Big) = 0, \quad n \geq -1 \tag{3.41}$$

Now using the relation between partition function and Q matrix (2.19), these can be shown to be equivalent to

$$(\mathcal{L}_n^{[1]}(1) + T_n^{[1]}(1)) Z_N(t; g) = 0, \qquad n \geq -1. \tag{3.42}$$

where

$$\begin{aligned} \mathcal{L}_n^{[1]}(1) \;\equiv\; & \sum_{k=1}^{\infty} k t_{1,k} \frac{\partial}{\partial t_{1,k+n}} + (N + \frac{n+1}{2}) \frac{\partial}{\partial t_{1,n}} + \frac{1}{2} \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_{1,k} \partial t_{1,n-k}} \\ & + N t_{1,1} \delta_{n,-1} + \frac{1}{2} N(N+1) \delta_{n,0}, \quad n \geq -1. \end{aligned} \tag{3.43}$$

and

$$T_n^{[1]}(1) \;\equiv\; \sum_{a,b\geq 1} a g_{a,b} \frac{\partial}{\partial g_{a+n,b}}, \qquad n \geq -1 \tag{3.44}$$

These operators satisfy the Virasoro algebras

$$[\mathcal{L}_n^{[1]}(1), \mathcal{L}_m^{[1]}(1)] = (n-m)\mathcal{L}_{n+m}^{[1]}(1), \qquad n, m \geq -1, \tag{3.45a}$$

$$[\mathcal{L}_n^{[1]}(1), T_m^{[1]}(1)] = 0, \qquad n, m \geq -1, \tag{3.45b}$$

$$[T_n^{[1]}(1), T_m^{[1]}(1)] = (n-m)T_{n+m}^{[1]}(1), \qquad n, m \geq -1, \tag{3.45c}$$

In order to recover the result we are interested in for our original two–matrix model we should simply set, in the above equations, $g_{a,b} = 0$ except for $g_{1,1} = g$.

3.1.2 Higher rank constraints

The Virasoro constraints we just obtained may be referred to as rank 2 tensorial constraints. In the following we deal with higher rank constraints. In order to get the spin–3 operators, we introduce the following notations

$$\begin{aligned}
\hat{\mathcal{V}} &\equiv \sum_{a,b\geq 1} g_{a,b} Q^a(1) Q^b(2), \\
\hat{\mathcal{V}}' &\equiv \sum_{a,b\geq 1} a g_{a,b} Q^{a-1}(1) Q^b(2), \\
:\hat{\mathcal{V}}^2: &\equiv \sum_{a,b,a',b'\geq 1} g_{a,b} g_{a',b'} Q^{a+a'}(1) Q^{b+b'}(2).
\end{aligned}$$

From the trivial relation

$$\int d\lambda_1 d\lambda_2 \frac{\partial^2}{\partial \lambda_1^2}\Big(\xi_n(\lambda_1, t) e^{V_1+V_2+\mathcal{V}} \eta_m(\lambda_2, t)\Big) = 0,$$

we get the following identity

$$-P_o^2(1) + \hat{V}''(1) + (\hat{V}')^2(1) + 2\hat{V}'(1)\hat{\mathcal{V}}' + :\hat{\mathcal{V}}'^2: + \hat{\mathcal{V}}'' = 0, \tag{3.46}$$

Then multiplying (3.46) by $Q^{n+2}(1)$ from the left and taking the trace, we obtain the rank–3 constraints

$$(\mathcal{L}_n^{[2]}(1) - T_n^{[2]}(1)) Z_N(t; g) = 0, \qquad n \geq -2. \tag{3.47}$$

with the definitions

$$\begin{aligned}
\mathcal{L}_n^{[2]}(1) \equiv\ & \sum_{l_1,l_2=1}^{\infty} l_1 l_2 t_{1,l_1} t_{1,l_2} \frac{\partial}{\partial t_{1,l_1+l_2+n}} + \sum_{l=1}^{\infty} l t_{1,l} \sum_{k=1}^{l+n-1} \frac{\partial^2}{\partial t_{1,k} \partial t_{1,l+n-k}} \\
& + \frac{1}{2(n+3)} \sum_{l=1}^{n-2} \sum_{k=1}^{n-l-1} (n-l+2) \frac{\partial^3}{\partial t_{1,l} \partial t_{1,k} \partial t_{1,n-l-k}} \\
& + (2N+n+2)\Big(\sum_{l=1}^{\infty} l t_{1,l} \frac{\partial}{\partial t_{1,l+n}} + \frac{1}{2} \sum_{l=1}^{n-1} \frac{\partial^2}{\partial t_{1,l} \partial t_{1,n-l}}\Big) \\
& + (N^2 + (n+2)N + \frac{1}{3}(n+1)(n+2)) \frac{\partial}{\partial t_{1,n}} \\
& + (2N^2 t_{1,2} + N t_{1,1}^2)\delta_{n,-2} + N(N+1) t_{1,1} \delta_{n,-1} + \frac{1}{3} N(N+1)(N+2)\delta_{n,0}
\end{aligned} \tag{3.48}$$

for $n \geq -1$, and

$$T_n^{[2]}(1) \equiv \sum_{a,b,a',b' \geq 1} aa' g_{a,b} g_{a',b'} \frac{\partial}{\partial g_{a+a'+n,b+b'}} + \sum_{a,b \geq 1} a(a-1) g_{a,b} \frac{\partial}{\partial g_{a+n,b}}, \qquad n \geq -2. \tag{3.49}$$

They satisfy the following algebra

$$[\mathcal{L}_n^{[2]}(1), \mathcal{L}_m^{[1]}(1)] = (n-2m)\mathcal{L}_{n+m}^{[2]}(1) + m(m+1)\mathcal{L}_{n+m}^{[1]}(1) \tag{3.50a}$$

$$[\mathcal{L}_n^{[2]}(1), \mathcal{L}_m^{[2]}(1)] = 2(n-m)\mathcal{L}_{n+m}^{[3]}(1) - (n-m)(n+m+3)\mathcal{L}_{n+m}^{[2]}(1) \tag{3.50b}$$

$$[T_n^{[2]}(1), T_m^{[1]}(1)] = (n-2m)T_{n+m}^{[2]}(1) - m(m+1)T_{n+m}^{[1]}(1) \tag{3.50c}$$

$$[T_n^{[2]}(1), T_m^{[2]}(1)] = 2(n-m)T_{n+m}^{[3]}(1) + (n-m)(n+m+3)T_{n+m}^{[2]}(1) \tag{3.50d}$$

where we have introduced the spin–4 operators $T_n^{[3]}(1)$ and $\mathcal{L}_n^{[3]}(1)$.

It is evident that we must introduce higher and higher tensor operators in order to close the algebra generated by $\{(\mathcal{L}_n^{[1]}(1)+T_n^{[1]}(1)), n \geq -1; (\mathcal{L}_n^{[2]}(1)-T_n^{[2]}(1)), n \geq -2\}$. For example, from eqs.(3.50b) and (3.50d), we obtain the rank–4 operators $\mathcal{L}_n^{[3]}(1)$ and $T_n^{[3]}(1)$. In turn the commutators of these operators with $\mathcal{L}_n^{[2]}(1)$ and $T_n^{[2]}(1)$ generate the rank–5 operators and so on. In this way we will introduce the whole sequence of operators $T_n^{[r]}(1)$ and $\mathcal{L}_n^{[r]}(1)$. All these generators form a closed $W_{1+\infty}$–algebra

$$[T_n^{[r]}(1), T_m^{[s]}(1)] = (sn - rm)T_{n+m}^{[r+s-1]}(1) + \ldots, \tag{3.51a}$$

$$[\mathcal{L}_n^{[r]}(1), \mathcal{L}_m^{[s]}(1)] = (sn - rm)\mathcal{L}_{n+m}^{[r+s-1]}(1) + \ldots, \tag{3.51b}$$

$$[T_n^{[r]}(1), \mathcal{L}_m^{[s]}(1)] = 0, \tag{3.51c}$$

for $r, s \geq 1$; $n \geq -r, m \geq -s$. Here dots denote lower than $r+s-1$ rank operators. Since we have already shown that a combination of the rank–2 and rank–3 operators *annihilate* the partition function, eqs.(3.42) and (3.47), then, due to the above $W_{1+\infty}$ algebra, the same combination of higher rank generators will annihilate the partition function, that is to say

$$\left(\mathcal{L}_n^{[r]}(1) - (-1)^r T_n^{[r]}(1)\right) Z_N(t;g) = 0, \qquad r \geq 1; \qquad n \geq -r. \tag{3.52}$$

In the same way from the second linear system we can derive another copy of $W_{1+\infty}$ algebra as well as another version of the $W_{1+\infty}$ constraints

$$\left(\mathcal{L}_n^{[r]}(2) - (-1)^r T_n^{[r]}(2)\right) Z_N(t;g) = 0, \qquad r \geq 1, \qquad n \geq -r, \tag{3.53}$$

where $\mathcal{L}_n^{[r]}(2)$ can be obtained from $\mathcal{L}_n^{[r]}(1)$ by simply replacing the $t_{1,k}$ parameters with the $t_{2,r}$ ones. As for the $T_n^{[r]}(2)$ generators, we have, for example

$$T_n^{[1]}(2) \equiv \sum_{a,b \geq 1} b g_{a,b} \frac{\partial}{\partial g_{a,b+n}}, \qquad n \geq -1$$

$$T_n^{[2]}(2) \equiv \sum_{a,b,a',b' \geq 1} bb' g_{a,b} g_{a',b'} \frac{\partial}{\partial g_{a+a',b+b'+n}} \tag{3.54a}$$

with the following algebra, which is isomorphic to (3.51a–3.51c)

$$[T_n^{[r]}(2), T_m^{[s]}(2)] = (sn - rm)T_{n+m}^{[r+s-1]}(2) + \dots, \tag{3.55a}$$
$$[\mathcal{L}_n^{[r]}(2), \mathcal{L}_m^{[s]}(2)] = (sn - rm)\mathcal{L}_{n+m}^{[r+s-1]}(2) + \dots, \tag{3.55b}$$
$$[T_n^{[r]}(2), \mathcal{L}_m^{[s]}(2)] = 0, \tag{3.55c}$$

for $r, s \geq 1;\ n \geq -r,\ m \geq -s$.

In conclusion, we see that, in the case of the extended two–matrix model, there are two isomorphic $W_{1+\infty}$ algebra constraints .

From eq.(3.52), it is easy to see that if we return to the ordinary two–matrix model, i.e. we set all the $g_{a,b}, a, b \geq 1$ equal to zero, and only keep $g_{1,1} \equiv g \neq 0$, we have

$$T_n^{[r]}(1)Z_N(t;g) = c^r \mathrm{Tr}\Big(Q^{r+n}(1)Q^r(2)\Big)Z_N(t;g) \tag{3.56a}$$
$$T_n^{[r]}(2)Z_N(t;g) = c^r \mathrm{Tr}\Big(Q^r(1)Q^{r+n}(2)\Big)Z_N(t;g). \tag{3.56b}$$

Substituting these into eqs.(3.52) and (3.53), we obtain (3.37).

3.2 Bosonization formulae for $\mathcal{L}_n^{[r]}$ and $T_n^{[r]}$.

There is a compact way to write down the above generators. It consists of introducing the current

$$J_\alpha(z) = \sum_{r=1}^{\infty} r t_{\alpha,r} z^{r-1} + \frac{N}{z} + \sum_{r=1}^{\infty} z^{-r-1}\frac{\partial}{\partial t_{\alpha,r}}, \qquad \alpha = 1,2 \tag{3.57}$$

and defining the density

$$\mathcal{L}_\alpha^{[n]}(z) = \frac{1}{n+1} : \Big(-\partial + J_\alpha(z)\Big)^{n+1} : \cdot 1$$

Here the : : refers to the usual normal oredering between $t_{\alpha,k}$ and $\frac{\partial}{\partial t_{\alpha,k}}$. Then $\mathcal{L}_k^{[n]}(\alpha)$ can be recovered as

$$\mathcal{L}_k^{[n]}(\alpha) = \mathrm{Res}_{z=0}(\mathcal{L}_\alpha^{[n]}(z)z^{n+k}), \qquad k \geq -n \tag{3.58}$$

As for the $T_n^{[r]}$ operators we proceed as follows. We define the bilocal densities

$$H_1(z,w) = \sum_{r,s\geq 0} r g_{r,s} z^{r-1} w^{s-1}, \qquad H_2(z,w) = \sum_{r,s\geq 0} s g_{r,s} z^{r-1} w^{s-1},$$
$$K(z,w) = \sum_{r,s\geq 0} z^{-r-1} w^{-s-1} \frac{\partial}{\partial g_{r,s}}$$

where r and s are never simultaneously 0, and

$$\mathcal{W}_1^{[r]} =: K\Big(w^{-1}\partial_z + H_1\Big)^r : \cdot 1, \qquad \mathcal{W}_2^{[r]} =: K\Big(z^{-1}\partial_w + H_2\Big)^r : \cdot 1 \tag{3.59}$$

Then

$$T_n^{[r]}(1) = \mathrm{Res}_{z=0}\mathrm{Res}_{w=0}\left(\mathcal{W}_1^{[r]} z^{r+n} w^r\right) \tag{3.60a}$$

$$T_n^{[r]}(2) = \mathrm{Res}_{z=0}\mathrm{Res}_{w=0}\left(\mathcal{W}_2^{[r]} z^{r} w^{r+n}\right) \tag{3.60b}$$

4 Correlation functions in unreduced models.

As we have already remarked, calculating the CF's of a two–matrix model means calculating the partial derivatives of the free energy with respect to the couplings, i.e. is tantamount to determining the free energy up to a constant depending only on N. In this regard it will be useful to consider N itself as a coupling, $N \equiv g_{0,0}$, and, in order to represent more pictorially the dependence on it, we pretend that N, like any other coupling, couples to an operator Q. We will see later on how to calculate correlators containing Q insertions.

We have (at least) three methods to calculate CF's. The first consists of directly solving the W constraints; the second consists of determining from the coupling conditions the explicit form of $Q(1)$ and $Q(2)$ and then using the flows of the discrete Toda lattice hierarchy; the third method is based on passing from the discrete hierarchy to a purely differential one and integrating the flows of the latter. The correlators of Q have to be calculated separately (see next section).

In this section we solve some unreduced models with the above methods. The second method is extremely powerful and allows us to obtain compact expression of CF's for all genera. However, for pedagogical reasons, we start with the first.

Before we start with explicit calculations it is necessary to make an important premise.

4.1 Homogeneity and genus expansion.

The CF's we compute are genus expanded. The genus expansion is strictly connected with the homogeneity properties of the CF's. As we will see the contribution pertinent to any genus is a homogeneous function of the couplings (and N) with respect to appropriate degrees assigned to all the involved quantities. Precisely, we assign to the couplings the following degrees

$$deg(\ ) \equiv [\ ], \qquad [g_{r,s}] = y - y_1 r - y_2 s, \tag{4.1}$$

where y, y_1, y_2, are arbitrary constants. In agreement with the convention just made, it follows that, in particular, $[N] = y$.

If we rescale the couplings as follows

$$g_{r,s} \longrightarrow \lambda^{[g_{r,s}]} g_{r,s}$$

on the basis of the analysis of Bessis, Itzykson and Zuber, [14], we expect the free energy to scale like

$$F \to \sum_{h=0}^{\infty} \lambda^{2y(1-h)} F_h \tag{4.2}$$

where F_h is the genus h contribution. In other words

$$[F_h] = 2y(1-h) \tag{4.3}$$

The CF's will be expanded accordingly. Such expectation, based on a path integral analysis, remains true in our setup due to the fact that the homogeneity properties carry over to the Toda lattice hierarchy. To this end we have simply to consider a genus expansion for all the coordinate fields that appear in $Q(1)$ and $Q(2)$, see (2.21, 2.22). The Toda lattice hierarchy splits accordingly. In genus 0 the following assignments

$$[a_l] = (l+1)y_1, \qquad [b_l] = (l+1)y_2, \qquad [R] = y_1 + y_2 \tag{4.4}$$

correspond exactly to the assignments (4.1) and $[F_0] = 2y$.

4.2 First method: solving the W–constraints

In this and the following subsections we limit ourselves to ordinary matrix models. Let us simplify our notation by setting

$$t_{1,k} \equiv t_k, \qquad\qquad t_{2,k} \equiv s_k$$

and rewrite the W constraints (3.37) in terms of CF's. For example, $W_{-1}^{[1]} Z_N = 0$ and $\tilde{W}_{-1}^{[1]} Z_N = 0$ become, respectively

$$\sum_{k=2}^{\infty} k t_k \ll \tau_{k-1} \gg + N t_1 + c \ll \sigma_1 \gg = 0 \tag{4.5a}$$

$$\sum_{k=2}^{\infty} k s_k \ll \sigma_{k-1} \gg + N s_1 + c \ll \tau_1 \gg = 0 \tag{4.5b}$$

Instead $W_0^{[1]} Z_N = 0$ and $\tilde{W}_0^{[1]} Z_N = 0$ give rise to the same equation

$$\sum_{k=1}^{\infty} k t_k \ll \tau_k \gg = \sum_{k=1}^{\infty} k s_k \ll \sigma_k \gg \tag{4.6}$$

The constraint $W_1^{[1]} Z_N = 0$ becomes

$$\begin{aligned} & g\Big(\sum_{l=1}^{\infty} l t_l \ll \tau_{l+1} \gg + (N+1) \ll \tau_1 \gg \Big) + \sum_{l_1,l_2=1}^{\infty} l_1 l_2 s_{l_1} s_{l_2} \ll \sigma_{l_1+l_2-1} \gg \\ & + \sum_{l=3}^{\infty} l s_l \sum_{k=1}^{l-2} \Big(\ll \sigma_k \sigma_{l-k-1} \gg + \ll \sigma_k \gg \ll \sigma_{l-k-1} \gg \Big) \\ & + (2N+1) \sum_{l=2}^{\infty} l s_l \ll \sigma_{l-1} \gg + (N^2+N) s_1 = 0 \end{aligned} \tag{4.7a}$$

It should be clear by now that $\tilde{W}_1^{[1]} Z_N = 0$ gives an exactly symmetrical equation where t_k and τ_k are interchanged with s_k and σ_k, respectively. In a similar way we can write all the other W–constraints.

Let us consider now the problem at issue in a simplified situation, i.e. in genus 0. This will be enough to shed light on the main features. In this case the W–constraints come out simplified: to obtain their genus 0 expression we have simply to select the homogeneous part of highest degree with respect to the degrees assigned in the previous section, see (4.1, 4.3). In any CF we henceforth append a label $\ll \gg_0$ to indicate this contribution. With this convention the genus 0 version of (4.5a), (4.5b) and (4.6) remain (formally) the same, while (4.7a) takes the form

$$\begin{aligned} & g\Big(\sum_{l=1}^{\infty} l t_l \ll \tau_{l+1} \gg_0 + N \ll \tau_1 \gg_0 \Big) + \sum_{l_1,l_2=1}^{\infty} l_1 l_2 s_{l_1} s_{l_2} \ll \sigma_{l_1+l_2-1} \gg_0 \\ + & \sum_{l=3}^{\infty} l s_l \sum_{k=1}^{l-2} \ll \sigma_k \gg_0 \ll \sigma_{l-k-1} \gg_0 + 2N \sum_{l=2}^{\infty} l s_l \ll \sigma_{l-1} \gg_0 + N^2 s_1 = 0 \end{aligned} \tag{4.8}$$

and so on.

Now we have to specify the two–matrix model we want to solve. Let us consider the model specified by the conditions

$$t_k = 0, \quad k > 2, \qquad\qquad s_k = 0, \quad k > 2, \tag{4.9}$$

As explained in the introduction, we refer to this model as the $\mathcal{M}_{2,2}$ model. The CF's in $\mathcal{M}_{2,2}$ will be functions of N, g, t_1, t_2, s_1, s_2 only. This means that, apart from N the latter are the internal or interacting couplings of the model. We recall that, once the model is specified the CF's are denoted with the symbol $< \cdot >$ instead of $\ll \cdot \gg$. With these specifications the genus 0 W–constraints (4.5a), (4.5b), (4.6) and (4.8) become, respectively

$$\begin{aligned} & 2t_2 < \tau_1 > + N t_1 + g < \sigma_1 > = 0, \\ & 2s_2 < \sigma_1 > + N s_1 + g < \tau_1 > = 0, \\ & 2t_2 < \tau_2 > + t_1 < \tau_1 > = 2s_2 < \sigma_2 > + s_1 < \sigma_1 >, \\ & g(2t_2 < \tau_3 > + t_1 < \tau_2 > + N < \tau_1 >) \\ & \qquad + 4s_2^2 < \sigma_3 > + 4s_1 s_2 < \sigma_2 > + (s_1^2 + 4N s_2) < \sigma_1 > + N^2 s_1 = 0 \end{aligned}$$

Writing down the appropriate formulas for the other constraints one quickly realizes that these formulas form a recursive and overdetermined system of algebraic equations for the one–point correlation functions. For example we obtain

$$\begin{aligned} < \tau_1 > &= N \frac{2s_2 t_1 - g s_1}{g^2 - 4 s_2 t_2}, \\ < \tau_2 > &= N \Big(\frac{2s_2 t_1 - g s_1}{g^2 - 4 s_2 t_2} \Big)^2 + N^2 \frac{2 s_2}{c^2 - 4 s_2 t_2} \end{aligned} \tag{4.11}$$

etc. The expression for $< \sigma_n >$ are obtained from $< \tau_n >$ with the exchange $t_k \leftrightarrow s_k$.

One can derive multi–point correlation functions by simply differentiating an appropriate number of times the W–constraints with respect to the couplings and repeating the same procedure.

The calculation of higher genus contributions can be done by writing the W–constraints appropriate for genus 1, genus 2, etc. One realizes that CF's at genus h can be computed if we know the contributions at genus $< h$.

We can summarize the previous conclusions as follows:

The W–constraints of $\mathcal{M}_{2,2}$ can be solved and the CF's $< \tau_{k_1} \ldots \tau_{k_n} \sigma_{l_1} \ldots \sigma_{l_m} >$ can be calculated by using recursiveness both with respect to the number $k_1 + \ldots + k_n + l_1 + \ldots + l_n$ and with respect to the genus.

This method is very effective in genus 0, but it may become very cumbersome in higher genera, [11]. We do not insist with it here since we have a much more powerful one at hand.

4.3 The second method: solving the coupling conditions

This method is based on the remark that in many models we can explicitly solve the coupling constraints (2.10). It is then elementary to compute correlators by means of eq.(2.19) and the lattice Toda flow equations. In this subsection we discuss in detail the (ordinary) model $\mathcal{M}_{2,2}$.

We start with the following:

Lemma. *The matrices $Q(1)$ and $Q(2)$ relevant to the model $\mathcal{M}_{2,2}$ are specified, in reference to the coordinatization (2.21,2.22), by the following coordinates*

$$\begin{aligned} a_0(n) \equiv a_0 &= \frac{gs_1 - 2s_2t_1}{4s_2t_2 - g^2}, \qquad a_1(n) = -\frac{2s_2 n}{4s_2t_2 - g^2}, \\ b_0(n) \equiv b_0 &= \frac{gt_1 - 2t_2s_1}{4s_2t_2 - g^2}, \qquad b_1(n) = -\frac{2t_2 n}{4s_2t_2 - g^2}, \qquad R(n) = \frac{ng}{4s_2t_2 - g^2} \end{aligned} \tag{4.12}$$

while the remaining coordinates vanish.

Proof. The coupling conditions (2.10) for the model $\mathcal{M}_{2,2}$ are

$$\begin{aligned} P_\circ(1) + t_1 + 2t_2Q(1) + gQ(2) &= 0 \\ \overline{\mathcal{P}}_\circ(2) + s_1 + 2s_2Q(2) + gQ(1) &= 0 \end{aligned}$$

Using the fact that $(P_\circ)_{n,n-1}(i) = n$ for $i = 1, 2$, they can be explicitly written in terms of the coordinates as follows.

$$\begin{aligned} 2t_2R(n) + gb_1(n) &= 0 \qquad & ga_1(n) + 2s_2R(n) &= 0 \\ t_1 + 2t_2a_0(n) + gb_0(n) &= 0 \qquad & s_1 + ga_0(n) + 2s_2b_0(n) &= 0 \\ n + 2t_2a_1(n) + gR(n) &= 0 \qquad & gR(n) + 2s_2b_1(n) + n &= 0 \end{aligned}$$

These equations can be easily solved and give (4.12).

Using this lemma we can obtain the exact correlators of the model $\mathcal{M}_{2,2}$. To start with it is convenient to rewrite $Q(1)$ and $Q(2)$ as

$$Q(i) = \alpha_i \epsilon_+ + \beta_i I + \gamma_i \epsilon_-, \qquad i = 1,2 \tag{4.13}$$

where

$$\epsilon_+ \equiv I_+ = \sum_{n=0}^{\infty} E_{n,n+1}, \qquad I = \sum_{i=0}^{\infty} E_{n,n}, \qquad \epsilon_- = \sum_{n=1}^{\infty} n E_{n,n-1}$$

and

$$\alpha_1 = 1, \quad \beta_1 = a_0, \quad \gamma_1 = \frac{2s_2}{g^2 - 4s_2t_2}, \quad \alpha_2 = -\frac{2t_2}{g}, \quad \beta_2 = b_0, \quad \gamma_2 = \frac{g}{4s_2t_2 - g^2}$$

By means of the formulas

$$[\epsilon_+, \epsilon_-] = I, \qquad \mathrm{Tr}\left(\epsilon_-^k \epsilon_+^l\right) = \delta_{k,l} \sum_{n=k}^{N-1} \frac{n!}{(n-k)!} = \delta_{n,k}\, k! \begin{pmatrix} N \\ k+1 \end{pmatrix} \tag{4.14}$$

we can now make explicit computations. For example

$$\begin{aligned} \mathrm{Tr}\left(Q(1)^n\right) &= \sum_{2l=0}^{n} \begin{pmatrix} n \\ 2l \end{pmatrix} \mathrm{Tr}(\epsilon_+ + \gamma_1 \epsilon_-)^{2l} \beta_1^{n-2l} \\ &= \sum_{2l=0}^{n} \sum_{k=0}^{l} \frac{n! 2^{-k}}{(n-2l)!k!(l-k)!} \begin{pmatrix} N \\ l-k+1 \end{pmatrix} \gamma_1^{l-k} \beta_1^{n-2l} \end{aligned} \tag{4.15}$$

From this formula, using (2.19), we can immediately get the exact one–point CF's:

$$< \tau_n > \;=\; \sum_{2l=0}^{n} \sum_{k=0}^{l} \frac{n! 2^{-k}}{(n-2l)!k!(l-k)!} \begin{pmatrix} N \\ l-k+1 \end{pmatrix} \Big(\frac{2s_2}{g^2 - 4s_2t_2}\Big)^l \Big(\frac{2s_2t_1 - gs_1}{g^2 - 4s_2t_2}\Big)^{n-2l}$$

In a similar way we can derive $< \sigma_n >$.

Finally using the genus expansion (4.2) we can extract the genus by genus correlators. The genus h contributions to the one–point CF's in the model $\mathcal{M}_{2,2}$ are

$$< \tau_n >_h = \sum_{2l=0}^{n} \sum_{k=0}^{l} \frac{(-1)^{2h-k} 2^{-k} n! \beta_{2h-k}(l-k) N^{l+1-2h}}{(n-2l)!k!(l-k)!(l-k+1)!} \Big(\frac{2s_2}{g^2 - 4s_2t_2}\Big)^l \Big(\frac{2s_2t_1 - gs_1}{g^2 - 4s_2t_2}\Big)^{n-2l} \tag{4.16}$$

where

$$\beta_k(n) = \sum_{1 \le r_1 \le r_2 \dots \le r_k \le n} r_1 r_2 \dots r_k, \qquad 1 \le k \le n, \qquad \beta_0(n) = 1, \qquad \beta_k(n) = 0 \quad \text{otherwise}$$

We have, for example,

$$\beta_1(n) = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}, \qquad \beta_2(n) = \frac{3n+2}{4} \begin{pmatrix} n+1 \\ 3 \end{pmatrix},$$

In genus 0 we recover the results of the previous subsection. In a similar way we can derive exact expressions for multi–point functions, [12].

5 Discrete states and $c = 1$ string theory

In this section we will refer to the extended two–matrix model, namely we will be interested also in the extra couplings. However we will consider only one interacting coupling, $g_{1,1} \equiv g$, beside N. We refer to this model as the (extended) $\mathcal{M}_{0,0}$ model. Despite its simplicity it turns out to be very interesting. It in fact provides a description of the discrete states of the $c = 1$ string theory. The latter has been studied in a large number of papers, [16] – for simplicity, we will constantly refer to the $c = 1$ string theory compactified at the self-dual radius. One of its remarkable aspects is the appearance of discrete states organized in sl_2 representations. The states corresponding to the highest weights, $\mathcal{T}_n$, and lowest weights, $\mathcal{T}_{-n}$, are known as purely tachyonic ones. They are pretty well studied and quite a few correlation functions have been calculated. The extra discrete states are less well–known.

The purpose of this section is to show that the extended two–matrix model can accommodate *discrete states* organized in finite dimensional sl_2 representations, analogous to the $c = 1$ string theory ones, and to calculate their correlation functions. Comparing our results with existing CF's of $c = 1$ string theory, we conclude that the extended two–matrix model does provide a description of the discrete states of $c = 1$ string theory.

5.1 Definition and properties of the discrete states

In reference to the extended two–matrix model, *we call discrete states the operators* $\chi_{r,s}$ *coupled to the* $g_{r,s}$. In this definition $r = s = 0$ is excluded. However, for later use we also set $\chi_{0,0} \equiv Q$ – remember that Q is the operator coupled to $g_{0,0} \equiv N$. From the very definition it is apparent that, classically, $\chi_{r,s}$ is represented by $\sum_{i=1}^{N} \lambda_i^r \mu_i^s$, where, to simplify our notation we set $\lambda_1 = \lambda$ and $\lambda_2 = \mu$. As a consequence these states carry a built–in sl_2 structure. For let us define

$$H = \frac{1}{2}\sum_{i=0}^{N}\Big(\lambda_i \frac{\partial}{\partial \lambda_i} - \mu_i \frac{\partial}{\partial \mu_i}\Big), \qquad E_+ = \sum_{i=1}^{N} \lambda_i \frac{\partial}{\partial \mu_i}, \qquad E_- = \sum_{i=1}^{N} \mu_i \frac{\partial}{\partial \lambda_i}$$

Then

$$[H, E_\pm] = \pm E_\pm, \qquad [E_+, E_-] = 2H$$

and

$$H\chi_{r,s} = \frac{1}{2}(r-s)\chi_{r,s}, \qquad E_+\chi_{r,s} = s\chi_{r+1,s-1}, \qquad E_-\chi_{r,s} = r\chi_{r-1,s+1}$$

Therefore the set $\{\chi_{r,s} = \sum_{i=1}^{N} \lambda_i^r \mu_i^s, \quad r + s = n\}$ form an (unnormalized) representation of this algebra of dimension $n + 1$.

These states $\chi_{r,s}$ will be identified with the discrete states of the $c = 1$ string theory. In particular $T_r \equiv \chi_{r,0}$ and $T_{-s} \equiv \chi_{0,s}$ will be identified with the purely tachyonic states $\mathcal{T}_r$ and $\mathcal{T}_{-s}$.

5.2 Correlation functions of the discrete states. Generalities

The correlation functions of the extended two–matrix model are defined by

$$\ll \chi_{r_1,s_1} \cdots \chi_{r_k,s_k} \gg = \frac{\partial}{\partial g_{r_1,s_1}} \cdots \frac{\partial}{\partial g_{r_k,s_k}} \ln Z_N$$

As anticipated above, our purpose in this section is to calculate the correlation functions in $\mathcal{M}_{0,0}$, i.e. we set $g_{i,j} = 0$ except for $g_{1,1} \equiv g$, which is left undetermined. As a consequence the CF's will be functions of g and N. We denote by the symbol $< \cdot >$ the CF's calculated in $\mathcal{M}_{0,0}$.

A remark is in order concerning N. The latter will eventually be identified with the renormalized cosmological constant x of the $c = 1$ string theory. As such we had better think of it as a continuous rather than a discrete variable. There is a natural way to pass from N to a continuous variable x: it consists of rescaling the lattice $(0, 1, \ldots, N, \ldots)$, on which all the quantities are defined, by N^y and simultaneously rescaling all the quantities according to their degree (4.1), and then take $N \to \infty$. In this section it is perfectly adequate to set $y = 1$ and $y_1 = y_2 = 0$. We can follow two courses: either we do all the calculation in the discrete formalism and then take the continuum limit, or we can shift to the continuum formalism from the beginning and perform the calculations using the differential and integral calculus. In the following we give examples of both.

To start with we use again the first method, i.e. we simply write down the W constraints in terms of the CF's. We obtain a set of (overdetermined) algebraic equations which in general one can solve recursively. Let us see first of all two very general lemmas.

Lemma 1. The CF satisfy the symmetry property

$$< \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} > = < \chi_{s_1,r_1} \cdots \chi_{s_n,r_n} > \tag{5.1}$$

This is due to the symmetry of the W constraints (3.34a) and (3.34b) and of the $\mathcal{M}_{0,0}$ under the exchange $1 \leftrightarrow 2$.

Lemma 2. In $\mathcal{M}_{0,0}$ the CF's satisfy

$$(r_1 + \ldots + r_n - s_1 - \ldots - s_n) < \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} > = 0 \tag{5.2}$$

In order to prove this lemma one rewrites the W constraints (3.34a) and (3.34b) with $r = 1$ and $n = 0$ as follows

$$\sum_{r \geq 1, s \geq 0} r g_{r,s} < \chi_{r,s} > + \frac{1}{2} N(N+1) = 0$$

$$\sum_{r \geq 0, s \geq 1} s g_{r,s} < \chi_{r,s} > + \frac{1}{2} N(N+1) = 0$$

respectively. One differentiates these equations w.r.t. $g_{r_1,s_1}, \ldots, g_{r_n,s_n}$ and sets $g_{r,s} = 0$ except $g_{1,1} = g$. One gets

$$(r_1 + \ldots + r_n) < \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} > + g < \chi_{1,1} \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} > = 0$$

$$(s_1 + \ldots + s_n) < \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} > + g < \chi_{1,1} \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} > = 0$$

Subtracting the second equation from the first we obtain the result.

This lemma reflects the sl_2 structure of the discrete states since it means nothing but the conservation of the H eigenvalue (conservation of the third angular momentum component).

Next consider the W constraints

$$\left(\mathcal{L}_0^{[r]}(1) - (-1)^r T_0^{[r]}(1)\right) Z_N(t;g) = 0, \qquad r \geq 1 \tag{5.3}$$

and evaluate it in $\mathcal{M}_{0,0}$. We get

$$\mathcal{L}_0^{[r]}(1) Z_N = \frac{1}{r+1} N(N+1)\ldots(N+r) Z_N \tag{5.4}$$

$$T_0^{[r]}(1) Z_N = g^r < \chi_{r,r} > Z_N \tag{5.5}$$

This allows us to calculate $< \chi_{r,r} >$ for any genus. Remember that the genus h and genus $h-1$ contributions differ by N^2. Therefore the genus h contribution, $< \chi_{r,r} >_h$, is given by

$$< \chi_{r,r} >_h = \begin{cases} \frac{N^{r+1-2h}}{(r+1)(-g)^r} b_{2h}(r) & r > 2h-1 \\ 0 & r \leq 2h-1 \end{cases} \tag{5.6}$$

The numbers $b_k(n)$ have been introduced in the previous section.

The passage from N to x, discussed above, amounts to just substituting N with x in (5.6) and $< \chi_{r,r} >$ with the renormalized correlator. Other exact results will be obtained later on with a different method.

5.3 Correlation functions. Genus 0 results.

In genus 0 (alias dispersionless limit) one can produce very general compact formulas for the CF's. We proceed again with the first method, i.e. we find the W constraints appropriate for genus 0 and solve them. Henceforth we denote N directly by x (the cosmological constant) – with this substitution the correlators will be understood to be the renormalized ones, see above. As usual we will denote the genus zero part of every CF $< \cdot >$ by $< \cdot >_0$.

The simplest result and the first ingredient we need is $< \chi_{n,n} >_0$, which has been calculated above, eq.(5.6). We have

$$< \chi_{n,n} >_0 = \frac{x^{n+1}}{(-g)^n} \frac{1}{n+1} \tag{5.7}$$

For the remaining CF's the relevant W constraints are

$$\mathcal{L}_{-r}^{[r]}(1) Z(t;g,x) = (-1)^r T_{-r}^{[r]}(1) Z(t;g,x), \qquad r \geq 1 \tag{5.8a}$$

$$\mathcal{L}_n^{[r]}(1) Z(t;g,x) = (-1)^r T_n^{[r]}(1) Z(t;g,x), \qquad r \geq 1, \qquad n \geq 1 \tag{5.8b}$$

and the analogous ones with $1 \to 2$. But we will not have to consider the latter due to Lemma 1. Performing the degree analysis one easily extracts the dispersionless limit. In particular we have

$$T_n^{[r]}(1) = \sum_{\substack{i_1,\ldots,i_r \geq 1 \\ j_1,\ldots,j_r \geq 1}} i_1 \ldots i_r g_{i_1,j_1} \ldots g_{i_r,j_r} \frac{\partial}{\partial g_{i_1+\ldots+i_r+n, j_1+\ldots+j_r}}. \tag{5.9}$$

These generators satisfy the algebra

$$[T_n^{[r]}(1), T_m^{[s]}(1)] = (sn - rm) T_{n+m}^{[r+s-1]}(1), \qquad r,s \geq 1; \quad n \geq -r, \quad m \geq -s \tag{5.10}$$

which characterizes the area–preserving diffeomorphisms. The algebra of the $T_n^{[r]}(2)$ is just a copy of the above one. Similar simplified (but not quite as simple) expressions can be gotten for the $\mathcal{L}$–type generators.

The first step consists of differentiating (5.8a) with respect to $g_{n,n-r}$ with $n > r$ and evaluating the result for $\mathcal{M}_{0,0}$. One gets

$$g^r < \chi_{0,r}\chi_{n,n-r} >_0 + rng^{r-1} < \chi_{n-r,n-r} >_0 = 0$$

Inserting (5.7) we obtain

$$< \chi_{0,r}\chi_{n,n-r} >_0 = \frac{x^n}{(-g)^n} r = < \chi_{r,0}\chi_{n-r,n} >_0 \tag{5.11}$$

The second equality follows from Lemma 1. Now differentiate (5.8b) w.r.t. $g_{n,n+p}$ with $p > 0$, and evaluate the result in $\mathcal{M}_{0,0}$. One gets

$$\begin{aligned} x^r < \chi_{p,0}\chi_{n,n+p} >_0 \;\; = \;\; & (-1)^r \Big(rng^{r-1} < \chi_{n+p+r-1,n+p+r-1} >_0 \\ & + g^r < \chi_{p+r,r}\chi_{n,n+p} >_0 \Big) \end{aligned} \tag{5.12}$$

Finally, using (5.11) and (5.7) one gets

$$< \chi_{p+r,r}\chi_{n,n+p} >_0 = \frac{x^{n+p+r}}{(-g)^{n+p+r}} \frac{(p+r)(n+p)}{n+p+r} = < \chi_{r,p+r}\chi_{n+p,n} >_0 \tag{5.13}$$

One can write down this result in a label–independent way as follows

$$< \chi_{r_1,s_1}\chi_{r_2,s_2} >_0 = \frac{x^\Sigma}{(-g)^\Sigma} \frac{M(r_1,s_1)M(r_2,s_2)}{\Sigma} \tag{5.14}$$

where $\Sigma = r_1 + r_2 = s_1 + s_2$ and $M(r,s) = max(r,s)$. This formula also holds when the two labels of χ coincide.

The procedure just outlined for two–point functions works in general. For the generic three point functions one gets

$$< \chi_{r_1,s_1}\chi_{r_2,s_2}\chi_{r_3,s_3} >_0 = \frac{x^{\Sigma-1}}{(-g)^\Sigma} M(r_1,s_1)M(r_2,s_2)M(r_3,s_3) \tag{5.15}$$

where $\Sigma = r_1 + r_2 + r_3 = s_1 + s_2 + s_3$.

For the n–point functions with n > 3, there is more than one possibility. To explain this point we will consider for a while only n–point functions $< \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} >_0$ with $r_k \neq s_k$ for $k = 1, \ldots, n$ (the cases of coincident indices of χ are obtained as limiting cases), and define their signature to be (p, q), where p is the number of label $r_k > s_k$ and q is the number of labels $r_k < s_k$. For n–point functions with signature $(1, n-1)$ we have the general formula

$$< \chi_{r_1,s_1} \cdots \chi_{r_n,s_n} >_0 = \frac{x^{\Sigma-n+2}}{(-g)^{\Sigma}} M(r_1, s_1) \ldots M(r_n, s_n)(\Sigma - 1) \ldots (\Sigma - n + 3)$$
$$\Sigma = r_1 + \ldots + r_n = s_1 + \ldots + s_n = j_1 + \ldots + j_n \tag{5.16}$$

if $\Sigma > n-2$, and vanishes otherwise. Here we have expressed Σ in terms of the more standard sl_2 j labels.

For other signatures, although there are in principle no obstacles, explicit formulas are more elaborate to work out.

We remark that the above formulas have been derived for states $\chi_{r,s}$ with r and s not simultaneously vanishing. To obtain CF's involving p insertions of $Q \equiv \chi_{0,0}$, one has simply to differentiate p times with respect to x the corresponding CF without Q insertions. For CF's containing only Q insertions, see below.

Now, particularizing eq.(5.16) to purely tachyonic states, we get

$$< T_r T_{-s_1} \ldots T_{-s_m} >= (r-1)(r-2) \ldots (r-m+2) \frac{x^{r-m+1}}{(-g)^r}, \qquad r = s_1 + \ldots + s_m \tag{5.17}$$

Up to overall numerical factors, this formula coincides with the analogous formula for pure tachyons $\mathcal{T}_r$ of the $c = 1$ string theory, provided we identify x with the renormalized cosmological constant.

This and similar correspondences allow us to conclude:

Our results either coincide or extend previous ones about discrete states. Therefore, our conjecture that $\chi_{r,s}$ are representatives of the discrete states of the $c = 1$ string theory is confirmed.

5.4 Correlation functions of discrete states. Higher genera.

To calculate correlators in higher genus we can use the W–constraints as we did above for genus 0. However this method becomes more and more cumbersome as we pass to higher and higher genera. Therefore we shift to the second method introduced in the previous section, i.e. we explicitly solve the coupling conditions for $Q(1)$ and $Q(2)$ and use the flow equations of the Toda lattice hierarchy to obtain compact formulas for correlators at all genera.

These remarkable results are a distinctive feature of the representation of the $c = 1$ string theory by means of the two–matrix model, compared to other approaches.

The explicit solutions of the coupling conditions is already available since, from this point of view, $\mathcal{M}_{0,0}$ is nothing but a particular case of $\mathcal{M}_{2,2}$. We have simply to recall the formulas of the last section and set $t_1 = t_2 = s_1 = s_2 = 0$.

The Jacobi matrices $Q(1)$ and $Q(2)$ take a very simple form

$$Q(1) = I_+, \qquad Q(2) = \sum_{l=1}^{\infty} \frac{l}{-g} E_{l,l-1} \tag{5.18}$$

and

$$R_l = \frac{l}{-g} \tag{5.19}$$

Let us see, as an example, the 2–point correlators of pure tachyons. The calculation proceeds in three steps.

1. Evaluating the correlator on the lattice: Using eq.(2.19) and eqs.(2.13b, 2.18b), we immediately get

$$\frac{\partial^2}{\partial t_{1,m} \partial t_{2,n}} \ln Z_j(g) = \mathrm{Tr}\Big([Q^m(1), Q^n(2)]\Big)$$

Evidently it is non–vanishing only if $m = n$, and a simple calculation shows that

$$\frac{\partial^2}{\partial t_{1,n} \partial t_{2,n}} \ln Z_j(g) = \mathrm{Tr}\Big([Q^n(1), Q^n(2)]\Big) = \sum_{l=j-n+1}^{j} \prod_{i=l}^{l+n-1} R_i$$

$$= \frac{1}{(-g)^n} \sum_{l=j-n+1}^{j} l(l+1)\dots(l+n-1) \tag{5.20}$$

2. Shifting to the continuum formalism: Let us see an example of calculation with the continuum formalism. As we saw above we have to rescale the lattice by N and all the quantities according to their degree. We then define

$$x \equiv \frac{n}{N}, \qquad \epsilon \equiv \frac{1}{N}$$

and renormalized quantities as follows

$$g_{r,s}^{\mathrm{ren}} = \frac{g_{r,s}}{N}, \qquad F^{\mathrm{ren}}(x, \epsilon) = \frac{F_n}{N^2}$$

We then take $N \to \infty$, treat x as a continuous variable and operate according to the rules of the ordinary differential calculus. We stress that whenever both discrete and continuum formalisms can be used, they lead to coincident results.

The *renormalized* correlation functions are the derivatives of $F^{\mathrm{ren}}(x, \epsilon)$ with respect to the renormalized coupling constants

$$\frac{\partial^2}{\partial t_{1,n}^{\mathrm{ren}} \partial t_{2,n}^{\mathrm{ren}}} F^{\mathrm{ren}}(x, \epsilon) \equiv < T_n T_{-n} >_{\mathrm{ren}}$$

For simplicity however, from now on, we drop the label ren wherever it should appear. From the context it will be clear whether the quantities in question are unrenormalized (before the continuum limit) or renormalized (after the continuum limit).

A straightforward calculation shows

$$< T_n T_{-n} > = \frac{1}{(-g)^n} \sum_{l=1-n}^{0} (x + l\epsilon)(x + (l+1)\epsilon) \ldots (x + (l+n-1)\epsilon) \tag{5.21}$$

$$= \frac{1}{(-g)^n} \sum_{l=0}^{n-1} (x + l\epsilon)(x + (l-1)\epsilon) \ldots (x + (l-n+1)\epsilon) \tag{5.22}$$

Separating the contributions from different genera: In order to single out the contributions from different genera, we recall the topological expansion for the free energy

$$F(x, \epsilon) = \sum_{h=0}^{\infty} F_h(x) \epsilon^{2h} \tag{5.23}$$

Further we observe that eq.(5.22) can be re–expressed as

$$\begin{aligned} < T_n T_{-n} > &= \frac{1}{(-g)^n} \Big(\sum_{l=1-n}^{0} e^{-l\epsilon\partial_x} \Big) x(x+1\epsilon) \ldots (x + (n-1)\epsilon) \\ &= \frac{1}{(-g)^n} \Big(\sum_{l=0}^{n-1} e^{l\epsilon\partial_x} \Big) x(x-1\epsilon) \ldots (x - (n-1)\epsilon) \\ &= \sum_{h \geq 0} < T_n T_{-n} >_h \epsilon^{2h} \end{aligned} \tag{5.24}$$

The second equality exhibits the invariance under the exchange $\epsilon \to -\epsilon$, which amounts to the vanishing of the odd powers of ϵ in the last expression. In other words, the CF's have the same kind of expansion as the free energy.

Now let us define another series of numbers

$$\alpha_i(n) = \sum_{l=0}^{n-1} l^i, \qquad i \geq 0;$$

beside the $\beta_i(n)$ defined above. The first few $\alpha_i(n)$ are

$$\alpha_0(n) = n, \qquad \alpha_1(n) = \binom{n}{2}, \qquad \alpha_2(n) = \frac{2n-1}{3} \binom{n}{2},$$

In terms of these numbers, we have the expansion

$$x(x+\epsilon)(x+2\epsilon) \ldots (x + (n-1)\epsilon) = \sum_{i=0}^{n} \beta_i(n+1) x^{n-i} \epsilon^i,$$

$$\sum_{l=0}^{n-1} e^{-l\epsilon\partial_x} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \alpha_l(n) \epsilon^l \partial_x^l$$

Plugging these formulas into eq.(5.24), and remembering that terms with the odd powers of ϵ vanish, we obtain the general two–point function (which include the contributions from all the genera)

$$< T_n T_{-n} >= \frac{1}{(-g)^n} \sum_{0 \leq 2h \leq n} \epsilon^{2h} \sum_{l=0}^{2h} (-1)^l \begin{pmatrix} n-l \\ 2h-l \end{pmatrix} \alpha_{2h-l}(n) \beta_l(n+1) x^{n-2h} \tag{5.25}$$

where g is the renormalized coupling constant. The h–genus correlator is

$$\begin{aligned} < T_n T_{-n} >_h &= \frac{x^{n-2h}}{(-g)^n} \sum_{l=0}^{2h} (-1)^l \begin{pmatrix} n-l \\ 2h-l \end{pmatrix} \alpha_{2h-l}(n) \beta_l(n+1), \qquad 2h < n; \\ < T_n T_{-n} >_h &= 0, \qquad 2h \geq n. \end{aligned} \tag{5.26}$$

In particular we have

$$< T_n T_{-n} >_0 = \frac{1}{(-g)^n} n x^n, \tag{5.27a}$$

$$< T_n T_{-n} >_1 = \frac{1}{(-g)^n} n \begin{pmatrix} n+1 \\ 4 \end{pmatrix} x^{n-2}, \tag{5.27b}$$

$$< T_n T_{-n} >_2 = \frac{1}{(-g)^n} \frac{(3n^2 - n - 6)n}{8} \begin{pmatrix} n+1 \\ 6 \end{pmatrix} x^{n-4}. \tag{5.27c}$$

It is clear that there are no obstacles to compute more complicated correlators (see more examples in [11]), except the manageability of the formulas.

5.5 Connection with the Penner model

The model $\mathcal{M}_{0,0}$ we studied in the previous sections has a topological nature. Remarkable evidence of this is provided by its connection with the Penner model, [18]. Such connection is to be found in the famous N–dependent integration constant (not determined by the W constraints) which was mentioned after eq.(2.20).

From eq.(2.24) and eq.(2.32) we obtain

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{2,1}} \ln R_n = \frac{\partial^2}{\partial t_{1,1} \partial t_{2,1}} (F_{n+1} - 2F_n + F_{n-1}), \tag{5.28}$$

Where $F_n = \ln Z_n$. After integrating twice with respect to $t_{1,1}$ and $t_{2,1}$, we have

$$\ln R_n = F_{n+1} - 2F_n + F_{n-1} \tag{5.29}$$

Possible integration constants have been dropped since they are irrelevant for the present problem (they provide analytic contributions to the free energy).

At this point it is more convenient to shift, as we did above, to a continuum formalism (from now on the quantities we mention are understood to be the renormalized ones). We set

$$F_{n\pm 1} = e^{\pm\epsilon\partial_x} F(x,\epsilon)$$

Thus the continuum version of the equation (5.29) is

$$\ln R(x) = \frac{1}{\epsilon^2}\Big[\exp(\epsilon\partial_x) + \exp(-\epsilon\partial_x) - 2\Big] F(x,\epsilon). \tag{5.30}$$

Once we know $R(x)$, we can solve this equation to obtain the free energy.

Now let us recall the genus expansion (5.23) for the free energy. We set for simplicity $g = -1$, therefore $R(x) = x$, which corresponds to the genus 0 contribution, while higher genus contributions of R vanish. Therefore we can easily obtain the recursion relations among the free energy in different genera

$$\partial_x^2 F_0(x) = \ln R(x)$$

and

$$\sum_{l=1}^{n} \frac{1}{(2l)!}\partial_x^{2l} F_{n-l}(x) = 0, \qquad \forall n \geq 2. \tag{5.31}$$

Using these recursion relations we can obtain the free energy for any genus. For example,

$$F_1(x) = -\frac{1}{12}\ln x, \qquad F_2(x) = -\frac{1}{240}x^{-2},$$

and so on. In general we have

$$F_h(x) = \frac{B_{2h}}{2h(2h-2)}x^{2-2h}, \qquad \forall h \geq 2. \tag{5.32}$$

Eq.(5.32) is the free energy of the Penner model. We see that $F_h(x)(h \geq 2)$ indeed scales according to the power $2(1-h)$ while F_0 and F_1 exhibit logarithmic scaling violation, which is a typical feature of the $c = 1$ string theory coupled to gravity. The coefficients of the powers of x are topological numbers. We recall that the (virtual) Euler characteristics of moduli space of Riemann surfaces with n punctures in genus h was computed by Harer and Zagier [17] and rederived by Penner,[18], with Feynman graph techniques

$$\chi_h^{(n)} = \frac{(-1)^n(2h-3+n)!(2h-1)}{n!(2h)!}B_{2h}$$

Therefore

$$F_h = \chi_h^{(0)} x^{2-2h}, \qquad < Q^n >_h = n!\chi_h^{(n)} x^{2-2h-n} \tag{5.33}$$

In particular we see that Q, the operator coupled to x, is to be interpreted as the puncture operator in the present theory.

It is a common belief that the $c = 1$ string theory is a topological field theory with primaries, puncture equation, recursion relations and a Landau–Ginzburg formulation. On the basis of our calculations, [11], many features of TFT in genus 0 can actually be brought to light in this model, but the nature of the underlying TFT looks quite different from the TFT's discussed in the last section.

6 Differential Hierarchies of Two–Matrix Models.

In the two previous sections we have given several examples of *unreduced models* and of the powerful methods we have at our disposal to solve them. There is another method (the third mentioned at the beginning of section 4) and other models (the *reduced* ones), which are based on a different approach: extracting differential hierarchies of integrable equations from the Toda lattice hierarchy.

Let us return to section 2 and let us concentrate from now on on *ordinary* two–matrix models. We saw there that two–matrix models can be represented by means of coupled discrete linear systems, whose consistency conditions give rise to the Toda lattice hierarchy. Here we review the method, used in [5] and [6], to transform the discrete linear systems into equivalent differential systems whose consistency conditions are purely differential hierarchies. This is tantamount to separating the N dependence from the dependence on the couplings.

The clue to the construction are the first flows, i.e. the $t_{1,1}$ and $t_{2,1}$ flows. For the sake of simplicity let us consider the *system I* and the flow (2.29a). Let us consider the generic situation in which $Q(1)$ has $m_1 = p_2 - 1$ lower diagonal lines (see the parametrization (2.21)). To begin with let us notice that

$$\frac{\partial}{\partial t_{1,1}}\Psi_n = \Psi_{n+1} + a_0(n)\Psi_n \tag{6.1}$$

and let us adopt for any function $f(t)$ the convention $f' \equiv \frac{\partial f}{\partial t_{1,1}} \equiv \partial f$. We can rewrite

$$\Psi_n = \hat{B}_n \Psi_{n+1} \tag{6.2}$$

where

$$\hat{B}_n \equiv \frac{1}{\partial - a_0(n)} = \partial^{-1}\sum_{l=0}^{\infty}(a_0(n)\partial^{-1})^l \tag{6.3}$$

Then it is an easy exercise to prove that the discrete spectral equation

$$Q(1)\Psi(\lambda_1) = \lambda_1\Psi(\lambda_1)$$

is transformed into the pseudodifferential one

$$L_n(1)\Psi_n = \lambda_1\Psi_n \tag{6.4}$$

where

$$L_n(1) = \partial + \sum_{l=1}^{m_1} a_l(n)\hat{B}_{n-l}\hat{B}_{n-l+1}\dots\hat{B}_{n-1} \tag{6.5}$$
$$= \partial + \sum_{l=1}^{m_1} a_l(n)\frac{1}{\partial - a_0(n-l)}\cdot\frac{1}{\partial - a_0(n-l+1)}\cdots\frac{1}{\partial - a_0(n-1)}$$

Proceeding in the same way for the other equations of *system I* we obtain the new system in differential form

$$\begin{cases} L_n(1)\Psi_n = \lambda_1\Psi_n \\ \frac{\partial}{\partial t_{1,r}}\Psi_n = \left(L_n^r(1)\right)_+\Psi_n, \\ \frac{\partial}{\partial t_{2,r}}\Psi_n = -\left(L_n^r(2)\right)_-\Psi_n, \\ M_n(1)\Psi_n = \frac{\partial}{\partial\lambda_1}\Psi_n \end{cases} \tag{6.6}$$

The subscript + appended to a pseudo–differential operator represents the purely differential part of it. The subscript – represents the complementary part. In particular we have the following consistency conditions

$$\frac{\partial}{\partial t_{1,r}}L_n(1) = [(L_n^r(1))_+, L_n(1)], \tag{6.7}$$
$$\frac{\partial}{\partial t_{2,r}}L_n(1) = [L_n(1), (L_n^r(2))_-], \tag{6.8}$$

They are purely differential equations.

Let us come now to the n dependence of the above equations. The operator $L_n(1)$ in (6.5) depends on the coordinates of different lattice points. To deal with this complication, we introduce m_1 "fields" $S_1, ..., S_{m_1}$, related to the "field" a_0 in the following way

$$S_i(n) \equiv a_0(n-i) \tag{6.9}$$

Then we can rewrite $L_n(1)$ in the following way

$$L_n(1) = \partial + \sum_{l=1}^{m_1} a_l(n)\frac{1}{\partial - S_l(n)}\cdot\frac{1}{\partial - S_{l-1}(n)}\cdots\frac{1}{\partial - S_1(n)}$$

with the result that $L_n(1)$ is expressed in terms of fields evaluated at the same lattice point. Of course the fields S_i are not independent. However we will consider these fields as completely independent from one another in all the intermediate steps of our calculations and only eventually impose the condition (6.9).

To further simplify the notation we will consider henceforth the lattice label n on the same footing as the couplings and write

$$a_i(n, ...) \equiv a_i(n)(...), \qquad S_i(n, ...) \equiv S_i(n)(...)$$

where the dots denote the dependence on $t_{1,k}, t_{2,k}$ and g. So the expression of $L_1(n)$ gets further simplified to

$$L = \partial + \sum_{l=1}^{m} a_l \frac{1}{\partial - S_l} \cdot \frac{1}{\partial - S_{l-1}} \cdots \frac{1}{\partial - S_1} \tag{6.10}$$

where, for simplicity, we have dropped the label (1) too. A similar simplification has to be understood also for the other equations of the *system I* above. This simplified form is the one we constantly refer to throughout this and the following section.

A similar treatment can be applied to the second linear system as well. Therefore *the information concerning matrix models can be stored in two differential linear systems + the first flow equations (2.29a, 2.29b,2.29c,2.29d).* The former determine the dependence on the couplings, while the latter fix the dependence on N. Therefore what we have accomplished so far is the separation of the dependence on N from the dependence on the couplings.

From now on, we will concentrate on the system I. Therefore, from now on we will refer to the pseudodifferential linear system

$$L\Psi = \lambda\Psi \tag{6.11}$$

$$\frac{\partial}{\partial t_r}\Psi = (L^r)_+\Psi, \qquad r = 1, 2, 3, ... \tag{6.12}$$

whose consistency conditions are

$$\frac{\partial}{\partial t_r} L = [(L^r)_+, L] \tag{6.13}$$

where $t_r \equiv t_{1,r}$.

The problem we are faced with next is to recognize and classify all the hierarchies of the type (6.13). This is easy. L is nothing but a particular realization of the KP operator

$$L_{KP} = \partial + \sum_{l=1}^{\infty} w_l \partial^{-l}$$

where, in general, w_l are unrestricted coordinates, while in these realizations they are precise functions of the fields a_l and S_l. (6.13) are integrable hierarchies (see below) which are classified by the number $2m$ of fields. But this is not enough. For one can obtain new integrable hierarchies *via hamiltonian reduction.* This is what we are going to see next. In particular, each integrable hierarchy corresponds to a different model, i.e. to a different kind of solution of the two matrix model. In the case of a reduced hierarchy, we call the corresponding model *a reduced model.*

6.1 $2m$–field realization of the KP hierarchy and their hamiltonian reductions

The main point of this subsection can be summarized as follows: 1) *for each m in (6.10) there are m distinct integrable hierarchies which are obtained by suppressing successively the fields S_l; 2) of each such hierarchy it is possible to write down a Drinfeld–Sokolov linear system from which the Lax pair realization can be easily extracted; 3) at the end of this cascade procedure we find the $(m+1)$–th KdV hierarchy.*

To start with let us give a few definitions and useful formulas. The general differential or pseudo–differential operator we have to do with is of the form

$$A = \partial^k + a\partial^{k-2} + b\partial^{k-3} + \ldots$$

The corresponding flows then are given by

$$\frac{\partial}{\partial t_r} A = [A_+^{\frac{r}{k}}, A] \tag{6.14}$$

We define

$$< A >= \int dx a_{-1}(x), \qquad A = \ldots + a_{-1}(x)\partial^{-1} + \ldots$$

and for any differential operator X we set $f_X(A) =< AX >$. Two Poisson brackets are then given by

$$\{f_X, f_Y\}_1(A) = < A[Y,X]_R > \tag{6.15}$$

$$\{f_X, f_Y\}_2(A) = < (XA)_+ YA > - < (AX)_+ AY > + + \frac{1}{k}\int [A,Y]_{(-1)}\big(\partial^{-1}[A,X]_{(-1)}\big) \tag{6.16}$$

In the first bracket

$$[X,Y]_R = \frac{1}{2}\big([RX,Y] + [X,RY]\big), \qquad RX = X_+ - X_-, \qquad X_- = X - X_+$$

and in the second bracket the label (-1) indicates the coefficient of ∂^{-1}. We can further introduce the Hamiltonians via

$$H_r = \frac{k}{r} < A^{\frac{r}{k}} >, \qquad \forall k \geq 1$$

Integrability of the system is guaranteed since these quantities are in involution and the Poisson brackets are compatible, i.e.

$$\{H_{r+k}, f\}_1 = \{H_r, f\}_2 \qquad \text{for any function} \quad f$$

In particular the flow equations are given by

$$\frac{\partial}{\partial t_r} f = \{f, H_r\}_2 = \{f, H_{r+k}\}_1 \tag{6.17}$$

6.1.1 The NLS hierarchy and its reduction.

The simplest example of L,(6.10), corresponds to $m = 1$. It gives rise to the NLS hierarchy. We set $a_1 = a$ and $S_1 = S$. Then

$$L = \partial + a\frac{1}{\partial - S} \tag{6.18}$$

Now, following the prescription of the previous section, we can calculate anything we need. In particular the first Poisson brackets are

$$\{a(x), a(y)\}_1 = 0 \qquad \{S(x), S(y)\}_1 = 0$$
$$\{a(x), S(y)\}_1 = \partial_x \delta(x - y)$$

and the second ones are

$$\{a(x), a(y)\}_2 = (2a\partial + a')\delta(x - y) \tag{6.19}$$
$$\{S(x), S(y)\}_2 = 2\partial\delta(x - y) \tag{6.20}$$
$$\{a(x), S(y)\}_2 = (S\partial + \partial^2)\delta(x - y) \tag{6.21}$$

For later reference we call this algebra $W(1,1)$. Henceforth we will use for Poisson brackets the simplified notation of (6.21) where the fields and derivatives (denoted also by a prime) appearing in the RHS are understood to be evaluated at x.

The Hamiltonians are

$$H_1 = \int dx\ a, \qquad H_2 = \int dx\ aS, \qquad H_3 = \int dx\ (a^2 + aS^2 + a'S), \quad \text{etc.}$$

and the flow equations are easy to compute. For example the t_2 flow equations are:

$$\frac{\partial S}{\partial t_2} = -S'' + 2S'S + 2a', \qquad \frac{\partial a}{\partial t_2} = a'' + 2(aS)' \tag{6.22}$$

which is a disguised form of the non–linear Schrödinger equation.

Let us study now the hamiltonian reduction

$$S = 0$$

This constraint is second class and we have to follow Dirac's procedure. For the second Poisson brackets we introduce the corresponding Dirac bracket in the reduced system

$$\{a(x), a(y)\}_{2D} = (2a\partial + a' + \frac{1}{2}\partial^3)\delta(x - y) \tag{6.23}$$

If we use this and the constrained H_3 Hamiltonian we obtain

$$\frac{\partial}{\partial t_3}a = a''' + 6aa' \tag{6.24}$$

which is the KdV equation.

The first Poisson bracket is not reducible. So we have to look for another Poisson bracket for the reduced system. It is not difficult to find it as well as the series of Hamiltonians using the fact that the two Poisson brackets must be compatible and the Hamiltonians in involution. This is however a long procedure. There is a much quicker method, which consists of finding a Lax pair representation. For this particular case the solution is universally known: the Lax operator is (up to a rescaling of the flow parameters)

$$\partial^2 + a$$

With this and the formulas of the introduction we can calculate Poisson brackets, Hamiltonians and flows (i.e. the KdV hierarchy) of the reduced system. This goes as follows. Let us introduce

$$F_{2k}(x) = \frac{\delta H_{2k}}{\delta a(x)} \tag{6.25}$$

Then, imposing the compatibility between the two Poisson brackets (6.17) we find the recursion relation for the flows

$$\frac{\partial a}{\partial t_{2k+1}} = F'_{2k+2} = (\partial^3 + 4a\partial + 2a')F_{2k} \tag{6.26}$$

with $F_2 = a$.

The above suggests us a general method to study the reductions of the more complex systems with $m > 2$. We will impose restriction constraints which will turn out to be second class. We will compute the Dirac brackets corresponding to the second Poisson structure and find the equations of motion of the reduced system. It will then be easy to identify the appropriate Lax operator.

6.1.2 Four–field representation of the KP hierarchy and its reductions.

The simplest integrable system that appears in matrix models after the NLS system is the four–field representation of the KP hierarchy. It naturally leads, via reduction, to the Boussinesq hierarchy. Let us describe this system in some detail.

The Lax operator is

$$L = \partial + a_1 \frac{1}{\partial - S_1} + a_2 \frac{1}{\partial - S_2}\frac{1}{\partial - S_1}. \tag{6.27}$$

The first and second (non–vanishing) Poisson brackets are

$$\begin{aligned} &\{a_1(x), S_1(y)\}_1 = \delta'(x-y), \qquad \{a_1(x), S_2(y)\}_1 = \delta'(x-y), \\ &\{a_2(x), a_2(y)\}_1 = (2a_2\partial + a_2')\delta(x-y), \\ &\{a_2(x), S_2(y)\}_1 = \big(\partial^2 + (S_2 - S_1)\partial\big)\delta(x-y). \end{aligned} \tag{6.28}$$

and

$$
\begin{aligned}
&\{a_1(x), a_1(y)\}_2 = (2a_1\partial + a_1')\delta(x-y), \qquad \{a_1(x), a_2(y)\}_2 = (3a_2\partial + 2a_2')\delta(x-y),\\
&\{a_1(x), S_1(y)\}_2 = (\partial^2 + S_1\partial)\delta(x-y), \qquad \{a_1(x), S_2(y)\}_2 = (2\partial^2 + S_2\partial)\delta(x-y),\\
&\{a_2(x), a_2(y)\}_2 = [(2a_2' + 4a_2S_2 - 2a_2S_1)\partial + a_2'' + (2a_2S_2 - a_2S_1)']\delta(x-y), \qquad (6.29)\\
&\{a_2(x), S_2(y)\}_2 = \Big(a_1\partial + (\partial + S_2)(\partial + S_2 - S_1)\partial\Big)\delta(x-y),\\
&\{S_1(x), S_1(y)\}_2 = 2\delta'(x-y), \qquad \{a_2(x), S_1(y)\}_2 = 0,\\
&\{S_1(x), S_2(y)\}_2 = \delta'(x-y), \qquad \{S_2(x), S_2(y)\}_2 = 2\delta'(x-y),
\end{aligned}
$$

respectively. For later reference we call the second Poisson structure algebra $W(1,2)$. The hamiltonians in involutions are

$$
\begin{aligned}
H_1 &= \int dx a_1, \qquad H_2 = \int dx[a_1S_1 + a_2]\\
H_3 &= \int dx[a_1S_1^2 - a_1S_1' + a_2(S_1 + S_2) + a_1^2], \quad \text{etc.}
\end{aligned}
$$

The second flow equations are

$$
\begin{aligned}
\frac{\partial}{\partial t_2}a_1 &= a_1'' + 2a_2' + 2(a_1S_1)'\\
\frac{\partial}{\partial t_2}a_2 &= a_2'' + 2a_2'S_2 + 2a_2(S_1 + S_2)' \qquad (6.30)\\
\frac{\partial}{\partial t_2}S_1 &= 2a_1' + 2S_1S_1' - S_1''\\
\frac{\partial}{\partial t_2}S_2 &= 2a_1' + 2S_2S_2' - S_2'' - 2S_1''
\end{aligned}
$$

First reduction.

Let us study now the possible reductions of this integrable system. We will first impose the reduction

$$S_1 = 0, \qquad (6.31)$$

With respect to the second Poisson structure (6.29) this is a second class constraint; moreover $\{S_1(x), S_1(y)\}_2$ has an inverse. We can therefore proceed according to Dirac and define improved Poisson brackets. These are

$$
\begin{aligned}
&\{a_1(x), a_1(y)\}_{2D} = (2a_1\partial + a_1' + \frac{1}{2}\partial^3)\delta(x-y),\\
&\{a_1(x), a_2(y)\}_{2D} = (3a_2\partial + 2a_2')\delta(x-y),\\
&\{a_1(x), S_2(y)\}_{2D} = (\frac{3}{2}\partial^2 + S_2\partial)\delta(x-y),\\
&\{a_2(x), a_2(y)\}_{2D} = [(2a_2' + 4a_2S_2)\partial + a_2'' + 2(a_2S_2)']\delta(x-y),\\
&\{a_2(x), S_2(y)\}_{2D} = \Big(a_1\partial + (\partial + S_2)^2\partial\Big)\delta(x-y),\\
&\{S_2(x), S_2(y)\}_{2D} = \frac{3}{2}\delta'(x-y) \qquad (6.32)
\end{aligned}
$$

For later reference we call this algebra $W(2,1)$. Now the reduced Hamiltonian is a restriction of the original Hamiltonian H_2 to the constrained manifold, i.e.

$$H_2 \to H_2' = \int dx a_2$$

Consequently the equations of motion of the constrained system are

$$\begin{aligned} \frac{\partial}{\partial t_2} a_1 &= 2a_2' \\ \frac{\partial}{\partial t_2} a_2 &= a_2'' + 2(a_2 S_2)' \\ \frac{\partial}{\partial t_2} S_2 &= a_1' + 2S_2 S_2' - S_2'' \end{aligned} \tag{6.33}$$

The first Poisson bracket (6.28) is not reducible. One can then proceed to search for another Poisson structure, as we did with the reduced NLS model. One finds that with the following choice

$$\{a_1(x), a_1(y)\}_{1'} = 2\partial\delta(x-y), \qquad \{a_2(x), S_2(y)\}_{1'} = \partial\delta(x-y). \tag{6.34}$$

and with the Hamiltonian

$$H_3' = \int dx \Big(a_2(S_2^2 - S_2') + a_1 a_2\Big) \tag{6.35}$$

one obtains the same equations of motion (6.33) and the two Hamiltonians commute. However, as we have repeatedly noticed, there is a much quicker and powerful way to see the integrability of the reduced system. In fact we can write down a Lax pair for it. The latter is based on the following pseudo-differential operator

$$\tilde{L} = \partial^2 + a_1 + a_2 \frac{1}{\partial - S_2} \tag{6.36}$$

By means of the formulas in the introduction we can compute the corresponding Poisson brackets, Hamiltonians and flow equations. Among the latter we find all the above results. We conclude that the reduced system is integrable.

Had we imposed, instead of $S_1 = 0$, the condition $S_2 = 0$, we would have found the same result up to field redefinitions.

Second reduction.

Let us further reduce our hierarchy by suppressing the remaining S field

$$S_2 = 0 \tag{6.37}$$

With Dirac's procedure we find from (6.32)

$$\begin{aligned} \{a_1(x), a_1(y)\}_{2D'} &= (2a_1\partial + a_1' + 2\partial^3)\delta(x-y), \\ \{a_1(x), a_2(y)\}_{2D'} &= \Big(3a_2\partial + 2a_2' - \partial^2 a_1 - \partial^4\Big)\delta(x-y), \\ \{a_2(x), a_2(y)\}_{2D'} &= [2a_2'\partial + a_2'' - \frac{2}{3}(a_1 + \partial^2)(\partial a_1 + \partial^3)]\delta(x-y), \end{aligned} \tag{6.38}$$

This is nothing but the W_3 algebra. The corresponding equations of motion calculated from the doubly constrained Hamiltonian H_2 are

$$\frac{\partial}{\partial t_2}a_1 = 2a_2' - a_1'', \qquad \frac{\partial}{\partial t_2}a_2 = a_2'' - \frac{2}{3}(a_1 a_1' + a_1''') \tag{6.39}$$

This is known as the Boussinesq equation and it is the first of an integrable hierarchy of equations (the Boussinesq or 3–KdV hierarchy), which can be given a Lax pair representation by means of the operator

$$\tilde{\tilde{L}} = \partial^3 + a_1\partial + a_2 \tag{6.40}$$

Like in the KdV case, we give the recursion relations that allow us to calculate all the flows. Let us define

$$F_r(x) = \frac{\delta H_r}{\delta a_1(x)}, \qquad G_r(x) = \frac{\delta H_r}{\delta a_2(x)}, \qquad r \neq 3n$$

Then imposing the compatibility of the two relevant Poisson brackets, (6.17), we find the following recursion relations

$$\begin{aligned} \frac{\partial a_1}{\partial t_r} = 3G_{r+3}' \;&=\; 2F_r''' + 2a_1F_r' + a_1'F_r + 3a_2G_r' + 2a_2'G_r \\ &\quad -a_1G_r'' - 2a_1'G_r' - a_1''G_r - G_r'''' \end{aligned} \tag{6.41}$$

and

$$\begin{aligned} \frac{\partial a_2}{\partial t_r} = 3F_{r+3}' \;&=\; F_r'''' + 2a_1F_r'' + a_2'F_r + 3a_2F_r' + 2a_2'G_r' + a_2''G_r \\ &\quad -\frac{2}{3}\Big(a_1a_1'G_r + a_1^2G_r' + 2a_1G_r''' + a_1'''G_r + 3(a_1'G_r')' + G_r'''''\Big) \end{aligned} \tag{6.42}$$

with $F_1 = 1,\ G_1 = 0$ and $F_2 = 0,\ G_2 = 1$.

Conclusion.

As a result of successively reducing the four–field KP hierarchy we find two more distinct integrable hierarchies. The last one is the well–known Boussinesq hierarchy. This suggests that if we start from the system defined by the general Lax operator (6.10), we obtain, via reduction, m more Lax operators. This conjecture has been proven in [9]. In conclusion *for every $p = m+1$ we identify p systems or hierarchies, denoted henceforth with the symbol $\mathcal{S}_p^l$, where l counts the number of nonvanishing S fields, $0 \leq l \leq m$. In particular the case $l = m$ corresponds to the $2m$–field representation of the KP hierarchy, while $l = 0$ corresponds to the p–KdV hierarchy.*

Finally let us make a comment on the naturalness of the reduction procedure. Our reduction procedure consists of restricting the second Hamiltonian according to the chosen constraints and calculating the second Poisson brackets according to the Dirac recipe. The

rest is automatic. In the framework of the Hamiltonian systems this reduction procedure is completely natural and allows us to select consistent subsystems of a given integrable system, or, in other words, to find solutions of the initial systems in which a part of the degrees of freedom are disregarded. We stressed in section 2 that our unreduced systems are equivalent to the initial path integral and that they even provide an enlarged definition of the latter. However we do not know in general how to go back from a reduced system to the path integral formulation. We know that in the simplest case (one–matrix model giving rise to the NLS hierarchy, see [6]) the reduction is the KdV hierarchy and that the KdV hierarchy corresponds to the even potential case via double scaling limit.

6.1.3 A DS representation of the previous systems.

The integrable models and reductions we have been considering so far can be synthesized in a very compact and useful form via suitable Drinfeld–Sokolov (DS) linear systems. From them one can easily extract the corresponding Lax pair. Let us start with the linear system

$$\begin{pmatrix} \partial - S & -1 \\ a & \partial - \lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi \end{pmatrix} = 0 \tag{6.43}$$

From this it is elementary to see

$$(\partial + a\frac{1}{\partial - S})\psi = \lambda\psi \tag{6.44}$$

from which we recognize the spectral equation for the NLS Lax operator, (6.18).

Similarly

$$\begin{pmatrix} \partial & -1 \\ a - \lambda & \partial \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \end{pmatrix} = 0, \qquad \text{and} \qquad \left(\partial^2 + a\right)\psi = \lambda\psi \tag{6.45}$$

The last is nothing but the spectral equation for the KdV Lax operator.

Let us pass now to the systems with $m = 2$. We have

$$\begin{pmatrix} \partial - S_2 & -1 & 0 \\ 0 & \partial - S_1 & -1 \\ a_2 & a_1 & \partial - \lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi \end{pmatrix} = 0 \tag{6.46}$$

so that

$$\left(\partial + a_1\frac{1}{\partial - S_1} + a_2\frac{1}{\partial - S_2}\frac{1}{\partial - S_1}\right)\psi = \lambda\psi \tag{6.47}$$

Next

$$\begin{pmatrix} \partial - S_2 & -1 & 0 \\ 0 & \partial & -1 \\ a_2 & a_1 - \lambda & \partial \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi \\ \psi_2 \end{pmatrix} = 0, \qquad \text{and} \qquad \left(\partial^2 + a_1 + a_2\frac{1}{\partial - S_2}\right)\psi = \lambda\psi \tag{6.48}$$

Finally

$$\begin{pmatrix} \partial & -1 & 0 \\ 0 & \partial & -1 \\ a_2 - \lambda & a_1 & \partial \end{pmatrix} \begin{pmatrix} \psi \\ \psi_1 \\ \psi_2 \end{pmatrix} = 0, \qquad \text{and} \qquad \left(\partial^3 + a_1\partial + a_2\right)\psi = \lambda\psi \tag{6.49}$$

We recognize the Lax operators $L, \tilde{L}$ and $\tilde{\tilde{L}}$, introduced above.

On the basis of these examples it is easy to extend this to a generic m. The initial system is

$$\begin{pmatrix} \partial - S_m & -1 & 0 & \cdots & 0 & 0 \\ 0 & \partial - S_{m-2} & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \partial - S_1 & -1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 & \partial - \lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdots \\ \psi_m \\ \psi \end{pmatrix} = 0 \tag{6.50}$$

From this we obtain the equation $L\psi = \lambda\psi$ where L is the same Lax operator as in eq.(6.10). The other systems can be obtained by: 1) moving $-\lambda$ by one place to the left in the last row, 2) suppressing the surviving S field with lowest index, 3) moving ψ one place up in the vector, and 4) extracting the equation for ψ. We obtain in such a way m more Lax operators. We proved in [12] that these represent the m distinct integrable reductions of the system represented by the Lax operator (6.10), which we called $\mathcal{S}_p^l$, $0 \leq l \leq m$.

6.2 W algebras.

One of the most fascinating aspects of matrix models is the emergence of structures characteristic of continuous field theories. One example are the W algebras. Here we use the term W algebras in a comprehensive sense, it includes also linear algebras, not only quadratic ones.

Leaving aside, for the sake of brevity, the lattice W algebras, which would be worth a discussion of their own, let us concentrate on the W algebras that accompany the $\mathcal{S}_p^l$ models. In such models we have first of all the W algebras of the generators connected with the W constraints (section 3). These are quantum W algebras. Next we have the classical W algebras originated by the Poisson structures. Each Poisson structure gives rise to one W_∞ algebra and one finite W algebra. This goes as follows.

The model $\mathcal{S}_p^l$ is defined by the Lax operator

$$L_{(n,l)} = \partial^n + \sum_{k=1}^{n-1} a_k \partial^{n-k-1} + \sum_{k=1}^{l} a_{n+k-1} \frac{1}{\partial - S_k} \frac{1}{\partial - S_{k-1}} \cdots \frac{1}{\partial - S_1}, \quad n \geq 1, \quad l \geq 0 \tag{6.51}$$

where $n = p - l$, which involves $n + 2l - 1$ independent fields. We have seen above a few examples of such Lax operator. Each of them gives rise to an integrable hierarchy and

via successive reductions we arrive at the $(n+l)$–KdV hierarchy. The operator $L_{(n,l)}$ is a particular case of

$$L_{gen} = \partial^n + \sum_{k=1}^{n-1} a_k \partial^{n-k-1} + \sum_{r=0}^{\infty} u_r \partial^{-r-1}, \tag{6.52}$$

The operator L_{gen} is also integrable and is endowed with two Hamiltonian structures. The corresponding Poisson brackets form two W_∞ algebras [9]. The coordinates u_r can be expressed as functions of the coordinates a_k and S_k of (6.51) – for this reason we say that the integrable hierarchy corresponding to $L_{(n,l)}$ is a $(n+2l-1)$–field representation of the hierarchy corresponding to L_{gen}. If we calculate the Poisson brackets of a_k and u_r using their expressions in terms of a_k and S_k and the Poisson brackets of $\mathcal{S}_p^l$, we exactly find the two W_∞ algebras just mentioned, [9].

By far the most important structure is the second Poisson structure. In the case of $L_{(n,l)}$ we call it $W(n,l)$. We saw some examples above. A very remarkable theorem, [10], asserts that $W(n,l)$ is isomorphic to the direct sum of the algebra W_n, the algebra W_{n+l} and a U(1) factor, i.e. a current J with Poisson bracket: $\{J(x), J(y)\} \sim \partial_x \delta(x,y)$. The isomorphism is realized via a Miura transformation, i.e. via a non–local transformation which is well–known in 2D field theories to lead to a vertex operator or free field representation.

It is remarkable that these algebras $W(n,l)$ appear in classical 2D field theories. For example, the W_3 algebra is characteristic of the sl_3 Toda field theory; the $W(1,1)$ algebra appears in the conformal affine Liouville theory and in the SL(2,R) WZNW model; the algebra $W(2,1)$ can be shown to be isomorphic (via the same kind of non–local transformation mentioned above) to the so–called Berdshadsky–Polyakov algebra; and so on. No doubt such link between two–matrix models and 2D conformal field theories is one of the most fascinating and mysterious aspects of two–matrix models.

7 Third method: more unreduced models.

After the previous long introduction, we are ready to calculate correlation functions via the third method, i.e. by making use of the differential flow equations (plus the W constraints). Contrary to the previous examples, here we limit ourselves to one sector of the ordinary two–matrix models, say sector 1. In other words we consider a model defined by an integrable system $\mathcal{S}_p^l$ and disregard the dependence on the $t_{2,k}$'s. In practice we set all the couplings to 0 except a few among the $t_{1,k}$, which define the model. As for the coupling g, we set it to zero too, but in some cases this recipe is too simplistic and one must proceed differently. We will make a comment on this point later on.

It is very convenient to simplify our notations. From now on let us denote $t_{1,k}$ by t_k and N by t_0. We recall that we introduced above an operator $\tau_0 \equiv Q$ coupled to t_0, namely we attributed the dependence on N to the interaction of τ_0. Finally we use the symbol $\mathcal{M}_p^l$, $0 \le l \le p-1$, to denote the model based on the the integrable hierarchy corresponding to $\mathcal{S}_p^l$.

One should always bear in mind, anyhow, that $\mathcal{M}_p^l$ is either a particular case or a reduced subsystem of a suitable $\mathcal{M}_{p_1,p_2}$.

7.1 The NLS hierarchy and the corresponding $\mathcal{M}_2^1$ model.

The first model we want to study is the one based on the NLS hierarchy, which was introduced in section 4.1.1, and is therefore still an unreduced model. We recall that such model is characterized not only by the NLS differential hierarchy, but also by the first flow equations, which are

$$\frac{\partial}{\partial t_1} S_n = a_{n+1} - a_n, \qquad \frac{\partial}{\partial t_1} a_n = a_n(S_n - S_{n-1}) \tag{7.1}$$

and that eq.(2.23) takes the form

$$\frac{\partial^2}{\partial t_1^2} \ln Z_N(t) = a_N \tag{7.2}$$

Let us set $2t_2 = -1$, $t_k = 0$ for $k \geq 3$. These conditions fix the so-called *first critical point.* The meaning of the notion *critical point* will be explained later on: it is clear from now however that specifying the critical point is tantamount to specifying the interacting couplings, i.e. the model.

In this model, therefore, we have an explicit dependence only on $N \equiv t_0$ and on t_1. It has become customary to call the set of variable left unspecified the *small phase space* of the model. We denote by $< \cdot >$ the CF's in this small phase space.

In order to study the dependence on N, for any function f_N on the lattice we define the derivative ∂_0 by means of

$$\left(e^{\partial_0} - 1\right) f_N \equiv f_{N+1} - f_N \tag{7.3}$$

One can easily verify that ∂_0 can be identified with the derivative with respect to $N \equiv t_0$, at least as long as f_N is a polynomial in N.

In particular, using the first flow equations, (7.1), we obtain

$$\left(e^{\partial_0} - 1\right) a = S' \tag{7.4}$$

$$\left(e^{\partial_0} - 1\right) S = (\ln a)' \tag{7.5}$$

Next, a is connected to the partition function through eq.(7.2). Therefore we have

$$a =< \tau_1 \tau_1 > \tag{7.6}$$

From this and (7.4) we find

$$S =< (e^Q - 1)\tau_1 > \tag{7.7}$$

and (7.5) becomes the compatibility condition

$$< \left(e^Q - 1\right)^2 >= \ln\left(< \tau_1\tau_1 >\right) + f(t_0) \tag{7.8}$$

where $f(t_0)$ is an integration constant (independent of t_1). The above equations are obtained via formal integration. Formal integration is an operation we will often use in the following. It is the formal inverse of ∂ (no integration constant involved) and is a proper operation in the context of integrable hierarchies whenever the pseudodifferential calculus applies.

The starting point to compute correlation functions is provided by the W-constraints. Equation (4.5a), at $g = 0$, gives

$$< \tau_1 >= t_0 t_1 \tag{7.9}$$

Therefore

$$< \tau_1 Q >= t_1, \qquad < \tau_1\tau_1 >= t_0, \qquad < \tau_1\tau_1 Q >= 1 \tag{7.10}$$

while all the other correlation functions containing at least one τ_1 insertion vanish. Therefore, in particular, we have

$$a = t_0, \qquad S = t_1 \tag{7.11}$$

Therefore a very simple use of the simplest W constraint and the first flow equations has allowed us to calculate a and S at the first critical point. In the following we will make use of the NLS integrable hierarchy to calculate the remaining one-point CF's. For simplicity here we limit ourselves to genus 0 CF's. To extract the relevant part of the flows we do a homogeneity analysis as in section 3. Since we have fixed $2t_2 = -1$, we can preserve homogeneity only if $x = 2y_1$ in the assignments (4.1). We make an even simpler choice, namely we assign the following degrees

$$[t_k] = 2 - k, \qquad [F_0] = 4, \qquad [a_0] = 2, \qquad [S_0] = 1$$

where F_0, a_0 and S_0 are the genus 0 part of $F = \ln Z$, a and S, respectively. We then extract the leading part of the flow equations w.r.t. such degrees. The resulting flows form the dispersionless hierarchy, i.e.

$$\frac{\partial S_0}{\partial t_r} = \sum_{\substack{k \\ 0\le 2k\le r}} \binom{r}{2k}\binom{2k}{k}\left(a_0^k S_0^{r-2k}\right)' \tag{7.12}$$

$$\frac{\partial a_0}{\partial t_r} = \sum_{\substack{k \\ 2\le 2k\le r+1}} \binom{r}{2k-1}\binom{2k-1}{k}\left(a_0^k S_0^{r-2k+1}\right)' \tag{7.13}$$

From (7.12) we obtain

$$< \tau_r Q >_0 = \sum_{\substack{k \\ 0\le 2k\le r}} \frac{r!}{(r-2k)!(k!)^2} a_0^k S_0^{r-2k} \tag{7.14}$$

while from (7.13) we get

$$< \tau_r \tau_1 >_0 = \sum_{\substack{k \\ 2 \leq 2k \leq r+1}} \frac{r!}{(r-2k+1)!(k-1)!k!} a_0^k S_0^{r-2k+1} \tag{7.15}$$

At the first critical point $a_0 = a$ and $S_0 = S$. If we insert (7.11) into (7.14) and (7.15), integrate over t_0 and t_1, respectively, and compare the results in order to fix the integration constants, we obtain in the small phase space

$$< \tau_r >_0 = \sum_{\substack{k \\ 2 \leq 2k \leq r+2}} \frac{r!}{(r-2k+2)!(k-1)!k!} t_0^k t_1^{r-2k+2} \tag{7.16}$$

This result is nothing but a particular case of eq.(4.16), obtained by setting $s_1 = 0, c = 0$ and $2t_2 = 2s_2 = -1$.

We remark that, in order to obtain this result, we have made use mostly of the flows, while we have made very little use of the W constraints. As we already pointed out, the solution of our problems (the CF's) are not only solutions of the Toda hierarchy, but also of the coupling conditions. The W constraints contain, in general, both the information about the hierarchy and the information about the coupling conditions. This is the reason why they are needed to fix some indeterminacy left by the flow equations.

Higher genus contributions and multipoint correlation functions can be calculated along the same lines, by taking into account non-dispersive terms and by considering higher order derivatives, respectively. We find a confirmation of the results of section 3. However, in this way, we cannot compute CF's containing only Q insertions. Such information is not contained in the W constraints. In order to compute $< Q^n >$ one must proceed as in the section 5.5. The results are the same for $n > 2$, but, for example, we find

$$< QQ >_0 = \ln t_0 - \ln(g^{-1} - g) \tag{7.17}$$

By the way, this result fixes the integration constant $f(t_0)$ introduced above.

We understood above that we should set $g = 0$ at the end of our calculations, but of course the RHS of (7.17) only makes sense if $g \neq 0$. It is true that higher genus and higher n correlation functions of Q depend only on t_0 and not on g, but it would seem that our recipe of setting $g = 0$ may lead to unacceptable results in the case of eq.(7.17). In fact one could proceed another way, keeping $g \neq 0$ but setting $s = 0$ in the results of section 3. The CF's one obtains for τ_k are the same as those obtained setting $g = 0$ modulo some rescalings of t_0, t_1 and F. This procedure is certainly more laborious. From these considerations we can conclude the following: setting $g = 0$ is an expedient way to arrive at the results, however, when pure Q correlators are involved, this recipe has to be used with some care.

7.2 The $\mathcal{M}_3^2$ model

The model $\mathcal{M}_3^2$ is specified by the Lax operator (6.27) which arises in the context of the previous section when we consider a potential V_2 of highest order 3. We have therefore altogether four fields a_1, a_2, S_1, S_2. The Poisson brackets, the hamiltonians and some of the flow equations have been given in [9].

The calculus of CF's via the hierarchy is similar to the previous example, but a bit more complicated. We subdivide it in several steps.

Step 1. Meaning of the fields. We specify the connection of the fields characterizing the model with the correlation functions. Due to equation (2.23) we have

$$a_1 = \ll \tau_1 \tau_1 \gg \tag{7.18}$$

The interpretation for a_2 can be inferred from the second flow equation for a_1, [9]:

$$\frac{\partial a_1}{\partial t_2} = \left(a_1' + 2a_2 + 2a_1 S_1\right)' \tag{7.19}$$

Therefore we have

$$a_2 = \frac{1}{2} \ll \tau_2 \tau_1 \gg - \frac{1}{2} \ll \tau_1 \tau_1 \tau_1 \gg - \ll \tau_1 \tau_1 \gg S_1 \tag{7.20}$$

via formal integration of eq.(7.19).

Looking at eq.(7.20), we see that the meaning of a_2 in terms of correlation functions is clear once the meaning of S_1 is clear. An interpretation for the latter and for S_2 can be derived from the first flows. In $\mathcal{M}_3^2$ the first flows are

$$\begin{aligned}
a_0'(N) &= a_1(N+1) - a_1(N) \\
a_1'(N) &= a_2(N+1) - a_2(N) + a_1(N)\big(a_0(N) - a_0(N-1)\big) \\
a_2'(N) &= a_2(N)\big(a_0(N) - a_0(N-2)\big)
\end{aligned}$$

Let us recall the operator $D_0 = e^{\partial_0} - 1$ defined above. Using the definition of the S fields, the first flows can be rewritten as

$$S_1' + D_0 S_1' = D_0 a_1 \tag{7.22a}$$
$$a_1' = D_0 a_2 + a_1 D_0 S_1 \tag{7.22b}$$
$$a_2' = a_2\big(D_0 S_1 + D_0 S_2\big) \tag{7.22c}$$

Once again we set $N \equiv t_0$ and recall the operator Q coupled to it. Keeping track of the operator Q is tantamount to studying the N dependence in the CF's. Then (7.22a) implies

$$S_1 = \ll \left(1 - e^{-Q}\right)\tau_1 \gg, \qquad S_2 = \ll e^{-Q}\left(1 - e^{-Q}\right)\tau_1 \gg \tag{7.23}$$

Similarly, eqs.(7.22c) and (7.22b) become

$$a_2 = \exp\Big(\ll (e^Q - 1)(1 - e^{-2Q}) \gg \Big) f \tag{7.24a}$$

$$D_0 a_2 = \ll \tau_1\tau_1\tau_1 \gg - \ll \tau_1\tau_1 \gg \ll (e^Q - 1)(1 - e^{-Q})\tau_1 \gg \tag{7.24b}$$

where f is an integration constant independent of t_1. These two are compatibility conditions which come from the request that ∂ and ∂_0 commute.

Eqs.(7.18, 7.20) and (7.23) provide the connection between fields and CF's we were looking for.

Step 2. The genus 0 contribution. Here we are interested in the genus 0 part of the CF's, which corresponds to considering the dispersionless limit of the hierarchy, i.e. the limit in which only the first derivative terms (in t_1 and t_0) are retained, and keeping the leading order terms in every equation. In this limit the two fields $S_{1,0}$ and $S_{2,0}$ collapse to a single field S_0 for

$$S_{1,0} = \ll \tau_1 Q \gg = S_{2,0}$$

The t_2 dispersionless flow equations are

$$\frac{\partial a_1}{\partial t_2} = \Big(2a_2 + 2a_1 S\Big)' \tag{7.25}$$

$$\frac{\partial a_2}{\partial t_2} = 2a_2' S + 4a_2 S' \tag{7.26}$$

$$\frac{\partial S}{\partial t_2} = \Big(S^2 + 2a_1\Big)' \tag{7.27}$$

Here and in the forthcoming formulas of this subsection, for simplicity, a_1, a_2 and S denotes the genus 0 part of the corresponding fields.

The compatibility equations (7.24a) and (7.24b) become

$$a_2 = e^{2\ll QQ\gg_0} f_0(t_0) \tag{7.28a}$$

$$\ll \tau_1\tau_1\tau_1 \gg_0 = \partial_0 a_2 + \ll \tau_1\tau_1 \gg_0 \ll \tau_1 QQ \gg_0 \tag{7.28b}$$

Moreover, from (7.20),

$$a_2 = \frac{1}{2} \ll \tau_2\tau_1 \gg_0 - \ll \tau_1\tau_1 \gg_0 \ll \tau_1 Q \gg_0 \tag{7.29}$$

Step 3. The first critical point. In the previous examples the first critical point was characterized by the fact that the correlation functions contain only nonnegative integral powers of the couplings. In this model this circumstance is implemented if

$$3t_3 = -1, \qquad t_k = 0 \quad k > 3 \tag{7.30}$$

We want to preserve homogeneity, therefore in the above degree assignment we set $x = 3y_1$. Here too it is enough to put

$$[t_k] = 3 - k \quad k \geq 0, \qquad [a_1] = 2, \qquad [a_2] = 3, \qquad [S] = 1, \qquad [F_0] = 6 \tag{7.31}$$

For a_1, a_2, S these are the genus 0 degrees. The CF's will therefore be homogeneous functions of N, t_1, t_2. These parameters form the small phase space.

This choice of the critical point corresponds, in the path integral language, to considering the potential

$$V_1 = t_1\tau_1 + t_2\tau_2 - \frac{1}{3}\tau_3 \tag{7.32}$$

as the interacting potential, while the other τ_k are regarded as external couplings.

Step 4. Integrating the flow equations. Now, instead of proceeding as in the NLS case, we make a small detour just to show how tiny is the role of the W constraints in this method. We make an ansatz for the fields a_1, a_2 and S, based on homogeneity

$$S = at_2, \qquad a_1 = bt_1 + ct_2^2, \qquad a_2 = dt_0 + et_1t_2 + ft_2^3$$

where a, b, c, d, e, f are numerical constants to be determined. If we plug these expressions in the t_2 flow above, we find

$$a = 2b, \qquad c = 2b^2, \qquad e = 0, \qquad f = 0$$

Then we can write

$$\begin{aligned} &< \tau_1\tau_1 >_0 = \frac{1}{2}(at_1 + a^2t_2^2) \\ &< \tau_2\tau_1 >_0 = 2dt_0 + a^2t_1t_2 + a^3t_2^3 \\ &< \tau_1 Q >_0 = at_2 \end{aligned}$$

where, as above, the symbol $< \cdot >$ denotes the correlation function evaluated in the small phase space and $< \cdot >_0$ its genus 0 part. The second equation is obtained from (7.25) via formal integration. Integrating these equations with respect to t_1, t_2 and t_0 respectively, and comparing the results we find $2d = a$ and

$$< \tau_1 >_0 = \frac{1}{4}at_1^2 + at_0t_2 + \frac{1}{2}a^2t_1t_2^2 + \frac{1}{4}a^3t_2^4 \tag{7.33}$$

After this example it should be clear how to proceed in order to calculate $< \tau_2 >_0$. We need to know $< \tau_2\tau_1 >_0$, $< \tau_2 Q >_0$ and $< \tau_2\tau_2 >_0$. The first two are obtained by formally integrating the t_2 flows of a_1 and S. The third is obtained in the following way. We differentiate eq.(7.29) with respect to t_2 and get

$$\frac{\partial a_2}{\partial t_2} = \frac{1}{2} \ll \tau_2\tau_2\tau_1 \gg_0 - a_1\frac{\partial S}{\partial t_2} - \frac{\partial a_1}{\partial t_2}S$$

Then we plug eqs.(7.25) and (7.27) into this equation and obtain

$$\frac{1}{2} \ll \tau_2\tau_2\tau_1 \gg_0 = \left(4a_2S + 2a_1S^2 + a_1^2\right)' \tag{7.34}$$

This can be formally integrated. So altogether we get

$$\begin{aligned} \ll \tau_2\tau_2 \gg_0 &= 8a_2S + 4a_1S^2 + 2a_1^2 \\ \ll \tau_2\tau_1 \gg_0 &= 2a_2 + 2a_1S \\ \ll \tau_2 Q \gg_0 &= S^2 + 2a_1 \end{aligned}$$

Evaluating the RHS's at the first critical point, integrating and comparing as before, we find

$$< \tau_2 >_0 = at_0t_1 + 2a^2t_0t_2^2 + a^3t_1t_2^3 + \frac{1}{2}a^2t_1^2t_2 + \frac{1}{2}a^4t_2^5 \tag{7.35}$$

We can continue in this way and calculate the other one–point CF's, the pattern for the derivation does not change. They all depend on the constant a. This constant cannot be determined on the basis of the flow equations alone. Before we pass to the determination of a, let us remark that while the RHS of eq.(7.26) is not a derivative w.r.t. t_1, the RHS of (7.34) is. This circumstance, crucial for us to be able to calculate the CF (otherwise we would be left with an undetermined integration constant), is typical of all the flows and is rooted in the integrability of the hierarchy.

Step 5. The W–constraints. The undetermined constant a is what is left for us to compute and it is the only thing the flow equations cannot tell us. In order to compute its value we have to use the W constraints. Precisely, we only need the $W_{-1}^{[1]}Z_N = 0$ constraint at $g = 0$. It takes the simple form

$$- < \tau_2 > +2t_2 < \tau_1 > +t_0t_1 = 0$$

Inserting the above expressions for $< \tau_2 >_0$ and $< \tau_1 >_0$ we find that the constant a must be equal to 1. This is a general fact: provided $a = 1$, the correlation functions calculated by integrating the flow equations satisfy all the W–constraints if $g = 0$.

In conclusion we find

$$S = t_2, \qquad a_1 = \frac{1}{2}(t_1 + t_2^2), \qquad a_2 = \frac{1}{2}t_0 \tag{7.36}$$

Replacing these values into the flow equations and integrating we find all the one–point correlators. Examples:

$$\begin{aligned} < \tau_1 >_0 &= \frac{1}{4}t_1^2 + t_0t_2 + \frac{1}{2}t_1t_2^2 + \frac{1}{4}t_2^4, \\ < \tau_2 >_0 &= t_0t_1 + 2t_0t_2^2 + t_1t_2^3 + \frac{1}{2}t_1^2t_2 + \frac{1}{2}t_2^5, \\ < \tau_3 >_0 &= \frac{1}{4}t_1^3 + 3t_0t_1t_2 + \frac{3}{2}t_1^2t_2^2 + \frac{9}{4}t_1t_2^4 + \frac{3}{4}t_0^2 + 4t_0t_2^3 + t_2^6, \end{aligned}$$

and so on. One can in fact show that the expressions for a_1, a_2, S and τ_1, τ_2 are valid to every genus.

A remark is in order concerning the constant a determined in this section. The value of this constant $a = 1$ is directly connected with the chosen fixed point $3t_3 = -1$. The correlation function with generic a are appropriate to the choice $3t_3 = -\frac{1}{a}$.

We stop here with the analysis of the model $\mathcal{M}_3^2$. We could of course compute more complicated correlators, but our purpose here was simply to show the third method at work and to collect an important piece of information: the differential flow equations are almost entirely what we need in order to calculate CF's.

8 Correlators in reduced models

In this section we concentrate on reduced models of the type $\mathcal{M}_p^0$ which are based on the p–KdV hierarchy. The calculation of CF's is based on the flow equations, which in general we know in the form of recursive relations. These allow us to calculate all the genus 0 correlators up to a normalization constant. The examples of the previous section are clear enough from this point of view. The point is that in any genus we have new indeterminate constants. The W–constraints play the only role of determining such constants. However, in the case of the reduced models, a new problem arises: the W–constraints stem from the coupling conditions, but we do not completely know the latter. Most of the information concerning the coupling conditions is in fact stored in the differential system of the model: the Lax operator inherits such information via the number of non–vanishing diagonal lines of the original $Q(1)$ matrix. Therefore it is not surprising that the flow equations are almost enough to determine the CF's. However not all the information concerning the coupling conditions is contained in the differential hierarchy characterizing the model.

In order to retrieve all the information we have to use the W algebra structure. Let us proceed heuristically. In the reduced models we are interested in the solutions that do not depend on the second sector (i.e. on $t_{2,k}$). If we look at eq.(3.34a), we see that such solutions should therefore satisfy to W–constraints of the form

$$\mathcal{L}_n^{[r]}(1) Z_N = 0, \qquad r \geq 1; \qquad n \geq -r \tag{8.37}$$

However the form of the generators might change, since the generators $\mathcal{L}_n^{[r]}$ are appropriate for the original Toda hierarchy, not necessarily for the reduced hierarchy. This is actually what happens: we have to change the form of the generators in such a way that they be compatible with the reduced flows. This will be done case by case below.

8.1 The KdV hierarchy and the associated $\mathcal{M}_2^0$ model

We showed that we are allowed to impose the constraint $S = 0$ on the NLS system while preserving integrability. In other words there is a consistent subsystem of the NLS system, of

which we can easily compute the flows. These are the KdV flows. We just explained that, in addition, we have to determine W constraints that match the KdV flows. In the present case we recall that only the odd flows survive the reduction. Therefore the t_{2n} are disregarded. It is therefore natural to forget t_0 as well.

The appropriate W constraints take the form

$$L_n\sqrt{Z} = 0, \qquad n \geq -1 \tag{8.38}$$

where

$$\begin{aligned} L_{-1} &= \sum_{k=1}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k-1}} + \frac{t_1^2}{16} \\ L_0 &= \sum_{k=0}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k+1}} + \frac{1}{16} \\ L_n &= \sum_{k=0}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k+2n+1}} + \sum_{k=0}^{n-1}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2n-2k-1}}, \qquad n >_0 \end{aligned} \tag{8.39}$$

These generators satisfy the commutation relations of the Virasoro algebra.

Here we limit ourselves to proving that (8.38) are in agreement with the KdV flows. Let us differentiate (8.38) with $n > 0$ with respect to $t_1 \equiv x$. Using F_{2k} defined in subsection 6.1.1, we can write

$$\begin{aligned} &\sum_{k=0}^{\infty}(2k+1)t_{2k+1}F_{2k+2n+2} + \partial^{-1}F_{2n+2} \\ &\qquad + \sum_{k=0}^{n-1}\left(2\frac{\partial}{\partial t_{2k+1}}F_{2n-2k} + F_{2k+2}\partial^{-1}F_{2n-2k} + \partial^{-1}F_{2k+2}F_{2n-2k}\right) = 0 \end{aligned}$$

Now we apply to it the recursion operator $\partial^3+4a\partial+2a'$. What we obtain, by using eq.(6.26), is nothing but the constraint $L_{n+1}\sqrt{Z} = 0$ differentiated twice with respect to x. To see this we have to apply the remarkable formula

$$\begin{aligned} F_{2n+4} &= F''_{2n+2} + 3F_2F_{2n+2} + \sum_{k=0}^{n-1}\Big(2F_{2k+2}F''_{2n-2k} - F'_{2k+2}F'_{2n-2k} \\ &\quad +4F_2F_{2k+2}F_{2n-2k} - F_{2k+2}F_{2n-2k+2}\Big) \end{aligned}$$

which can be obtained once again from the recursion relation (6.26). As for the cases $n = 0$ and $n = -1$, which are excluded by the above argument, they can be verified straightforwardly.

Let us come now to the correlators. We proceed as in the NLS case. The first critical point is defined by $6t_3 = -1$ and $t_{2k+1} = 0$ for $k > 1$. From the $n = -1$ Virasoro constraint we get immediately

$$< \tau_1 >= \frac{t_1^2}{2}, \qquad a =< \tau_1\tau_1 >= t_1, \qquad < \tau_1\tau_1\tau_1 >= 1$$

Knowing a we can now proceed as above using the KdV flows. In particular in genus 0 we have the dispersionless flows (we do not have to use the symbol a_0 for the genus 0 part of a since $a = t_1$ is the exact all genus expression)

$$\frac{\partial}{\partial t_{2k+1}} a = 2^k \frac{(2k+1)!!}{k!} a^k a' \tag{8.40}$$

We easily find

$$< \tau_{2k+1} >_0 = \frac{2^k (2k+1)!!}{(k+2)!} t_1^{k+2} \tag{8.41}$$

In these formulae $(2k+1)!! = (2k+1)(2k-1)\ldots 1$.

If we use, instead, the complete hierarchy, it is not very difficult to find the all genus expression for the one–point function, see [12]. At genus h we find

$$< \tau_{2k+1} >_h = \frac{2^k}{12^h h!} \frac{(2k+1)!!}{(k-3h+2)!} t_1^{k-3h+2} \tag{8.42}$$

8.2 The Boussinesq hierarchy and the associated $\mathcal{M}_3^0$ model

The 3–KdV or Boussinesq hierarchy was obtained in 6.1.2 as a reduction of the $\mathcal{M}_3^2$ model. It is described by two fields a_1 and a_2 and is specified by the Lax operator

$$L = \partial^3 + a_1 \partial + a_2 \tag{8.43}$$

In the Boussinesq hierarchy the t_{3k} flows with $k = 1, 2, 3...$ do not appear. It is therefore natural to ignore $t_0 \equiv N$ as well.

The first few flows of the Boussinesq hierarchy are

$$\frac{\partial a_1}{\partial t_2} = 2a_2' - a_1'', \qquad \frac{\partial a_2}{\partial t_2} = a_2'' - \frac{2}{3} a_1''' - \frac{2}{3} a_1 a_1' \tag{8.44}$$

$$\frac{\partial a_1}{\partial t_4} = \Big(-\frac{1}{3} a_1''' + \frac{2}{3} a_2'' - \frac{2}{3} a_1 a_1' + \frac{4}{3} a_1 a_2 \Big)' \tag{8.45}$$

$$\frac{\partial a_2}{\partial t_4} = \Big(\frac{1}{3} a_2''' - \frac{2}{9} a_1'''' + \frac{2}{3} a_1 a_2' + \frac{2}{3} a_2^2 - \frac{2}{3} a_1 a_1'' - \frac{1}{2} a_1' a_1' - \frac{4}{27} a_1^3 \Big)' \tag{8.46}$$

The correlation function interpretation of the fields a_1 and a_2 is given by eq.(2.23) and the first of (8.44):

$$a_1 = < \tau_1 \tau_1 >, \qquad 2a_2 = < \tau_1 \tau_2 > + < \tau_1 \tau_1 \tau_1 >, \tag{8.47}$$

As in the KdV case the difficulty with this model is the coupling conditions. In this case too we get around it by finding effective W constraints, which are determined by imposing consistency with the flow equations. One can prove that the effective W constraints are

$$L_n^{[r]}\, Z^{1/3} = 0, \qquad r = 1,2, \quad n \geq -r \tag{8.48}$$

where

$$L_n^{[1]} = \frac{1}{3}\sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+3n}} + \frac{1}{6}\sum_{\substack{k,l\\ k+l=3n}} \frac{\partial^2}{\partial t_k \partial t_l} + \frac{1}{6}\sum_{\substack{k,l\\ k+l=-3n}} k l t_k t_l + \frac{1}{9}\delta_{n,0}, \quad \forall n$$

$$\begin{aligned}\sqrt{3}L_n^{[2]} &= \frac{1}{9}\sum_{l_1,l_2=1}^{\infty} l_1 l_2 t_{l_1} t_{l_2} \frac{\partial}{\partial t_{l_1+l_2+3n}} + \frac{1}{9}\sum_{\substack{l,k,j\\ l-k-j=-3n}} l t_l \frac{\partial^2}{\partial t_k \partial t_j} \\ &+ \frac{1}{27}\sum_{\substack{l,k,j\\ l+k+j=3n}} \frac{\partial^3}{\partial t_l \partial t_k \partial t_j} + \frac{1}{27}\sum_{\substack{l,k,j\\ l+k+j=-3n}} l k j t_l t_k t_j, \quad \forall n\end{aligned}$$

In these expressions summations are limited to the terms such that no index involved is either negative or multiple of 3. The above two sets of generators form a closed algebra, the W_3 algebra,

$$\begin{aligned}[L_n^{[1]}, L_m^{[1]}] &= (n-m)L_{n+m}^{[1]} + \frac{1}{6}(n^3-n)\delta_{n+m,0} \\ [L_n^{[1]}, L_m^{[2]}] &= (2n-m)L_{n+m}^{[2]} \\ [L_n^{[2]}, L_m^{[2]}] &= -\frac{1}{54}(n-m)\Big((n^2+m^2+4nm)+3(n+m)+2\Big)L_{n+m}^{[1]} \\ &+ \frac{1}{9}(n-m)\Lambda_{n+m} + \frac{1}{810}n(n^2-1)(n^2-4)\delta_{n+m,0}\end{aligned}$$

where

$$\Lambda_n = \sum_{k\leq -1} L_k^{[1]} L_{n-k}^{[1]} + \sum_{k\geq 0} L_{n-k}^{[1]} L_k^{[1]}$$

This corresponds to the quantum W_3 algebra with central charge 2.

From now on it is very easy to proceed. First we define the first critical point by

$$4t_4 = -1, \qquad t_k = 0 \quad k > 4$$

so that the small phase space is formed by t_1, t_2. From the $L_{-1}^{[1]} Z^{1/3} = 0$ constraint we find immediately $< \tau_1 >= 6t_1 t_2$. This is an exact result, valid for all genera. It follows that

$$a_1 = 6t_2, \qquad a_2 = 3t_1 \tag{8.49}$$

is also an exact result.

In the following we limit ourselves to genus 0 results. We extract the genus 0 part of the constraints by assigning the degree

$$[t_k] = 4 - k \quad k \geq 0, \qquad [F_0] = 8$$

where F_0 is the genus 0 part of F, and keep the leading order terms in the W constraints. Similarly we can extract the genus 0 or dispersionless flow equations by simply dropping in the RHS's the terms containing more than one first order t_1 derivative.

Inserting these expressions into the dispersionless flows and integrating, we find immediately the results

$$\begin{aligned} < \tau_1 > &= 6t_1t_2, \qquad < \tau_2 > = 3t_1^2 - 8t_2^3 \\ < \tau_4 >_0 &= 12t_1^2t_2 - 16t_2^4, \qquad < \tau_5 >_0 = 5t_1^3 - 40t_1t_2^3 \end{aligned} \tag{8.50}$$

These are actually particular cases of a compact formula which one can derive from the recursion relations (6.41) and (6.42), [12]:

$$< \tau_{3n-2+\epsilon} >_0 = \sum_{2l+3j=3n+2+\epsilon} \frac{2^l(-1)^{\frac{l-1-\epsilon}{3}}}{3^{n-j-2+\epsilon}} \frac{(3n-3+2\epsilon)!!!\,(3n-2+2\epsilon)!!!}{(3j)!!!\; l!!!\;(l-1)!!!\;(n-1+\epsilon)!!!}\, t_1^j\, t_2^l \tag{8.51}$$

where $\epsilon = 0, 1$. $n!!!$ is the 3 by 3 factorial, i.e.$n!!! = n(n-3)(n-6)\ldots$, and $0!!! = (-1)!!! = 1$ by convention.

For more results on this model, see [12].

8.3 Other models

Let us generalize what we have just done for the Boussinesq hierarchy to the $\mathcal{M}_p^0$ (or p–th KdV) models (see the W generators in Appendix). The general recipe is as follows. One must first of all disregard all the t_k with k a multiple of p; the first critical point is

$$(p+1)t_{p+1} = -1, \qquad t_k = 0 \qquad k > p+1 \tag{8.52}$$

and the degree assignment is

$$[t_k] = p + 1 - k, \qquad [F_0] = 2p + 2, \qquad [a_{1,0}] = 2, \ldots, [a_{p-1,0}] = p \tag{8.53}$$

where $a_{i,0}$ is the genus 0 part of a_i. The CF's will be homogeneous functions of $t_1, \ldots, t_{p-1}$.

In all the cases the method consists of fixing the form of the fields by means of effective W–constraints, which in turn are determined imposing compatibility with the relevant flow equations. Once this is done the CF's are directly obtained from the flow equations.

To give one more example, in the model $\mathcal{M}_4^0$ we have

$$a_1 = 12t_3, \qquad a_2 = 8t_2, \qquad a_3 = 4t_1 + 18t_3^2 \tag{8.54}$$

and

$$\begin{aligned} < \tau_1 > &= 8t_2^2 + 12t_1t_3, \qquad < \tau_2 > = 16t_1t_2 - 72t_2t_3^2 \\ < \tau_3 > &= 6t_1^2 - 72t_2^2t_3 + 81t_3^4 \end{aligned} \tag{8.55}$$

8.4 Higher critical points

The origin of the term *critical* point will be discussed in the next subsection. In all the previous examples the first critical point has been characterized by a dependence of the basic fields on the couplings specified by homogeneous polynomials with non–negative integer powers. Higher critical points are characterized still by a homogeneous dependence, but with rational and/or negative powers of the couplings.

For the models of type $\mathcal{M}_p^0$ (p–th KdV hierarchy) characterized by a small phase space with parameters $t_1, ..., t_{p-1}$, higher critical points are specified by

$$kt_k = b, \qquad t_l \neq 0 \qquad \text{for } l \neq 1, \ldots, p-1, \qquad k \neq np \tag{8.56}$$

while all the other couplings vanish. Here b is a pure number whose actual value is a matter of convention. For example, the choice $b = 1$ or $b = -1$ is made in such a way as to avoid complex coefficients in the small phase space expressions of the CF's of the basic fields. The t_k's $\neq 0$ are the true self–interacting couplings of the model.

The degree assignment is

$$[t_l] = k - l, \qquad [F_0] = 2k, \qquad [a_{1,0}] = 2, \ldots, [a_{p-1,0}] = p \tag{8.57}$$

Here are a few examples. For the model $\mathcal{M}_3^0$ the second critical point is fixed by

$$5t_5 = 1, \qquad t_4 = 0, \qquad t_l = 0 \quad l > 5$$

As usual let us impose the constraint $L_{-1}^{[1]}\ Z^{1/3} = 0$. It implies

$$< \tau_2 >= -6t_1t_2 \tag{8.58}$$

This is an exact result and gives $a_2 = -3t_2$. Using again the second flow equations we also find (in genus 0)

$$a_{1,0} = 3t_1^{\frac{1}{2}} \tag{8.59}$$

Now it is easy to extract the CF's from the flow equations. Here are the simplest ones which we will need in a subsequent discussion.

$$\begin{aligned} &< \tau_1 >_0 = 2t_1^{\frac{3}{2}} - 3t_2^2, \qquad < \tau_2 > = -6t_1t_2 \\ &< \tau_4 >_0 = -8t_1^{\frac{3}{2}}t_2 + 4t_2^3, \qquad < \tau_5 >_0 = -2t_1^{\frac{5}{2}} + 15t_1t_2^2 \end{aligned} \tag{8.60}$$

Another example is the second critical point of the model $\mathcal{M}_4^0$:

$$6t_6 = -1, \qquad t_5 = 0, \qquad t_l = 0 \quad l > 6$$

The constraint $L_{-1}^{[1]}\ Z^{1/4} = 0$ tells us that

$$< \tau_2 >= 8t_2^2 + 12t_1t_3 \tag{8.61}$$

From this and the second flow equations we can derive

$$a_{1,0} = -\frac{4}{3}t_1t_3^{-1}, \qquad a_2 = 6t_3, \qquad a_{3,0} = \frac{4}{9}t_1^2t_3^{-2} + 4t_2 \tag{8.62}$$

from which we can easily derive the CF's.

9 Critical points and topological field theories

In the last section we discuss the connection of the $\mathcal{M}_p^0$ models with 2D gravity and topological field theories.

On the basis of the continuum approach of 2D gravity coupled to conformal matter we would expect a series of critical points near which the 'chemical potential' behaves like

$$t^{\frac{2}{p+q-1}} \tag{9.63}$$

where t is the renormalized cosmological constant and p and q are relatively prime positive integers. Different values of the latter correspond to different models. For example, the case $p = 2$ and $q = 3$ corresponds to pure gravity. The question is whether our models can account for such behaviour. In our models the chemical potential is a_1, and it is natural to identify t_1 with t. The answer to the above question is completely clear in the case $\mathcal{M}_2^0$. At the critical point defined by $kt_k = b$, we have $a_1 \equiv a \sim t_1^{\frac{2}{k-1}}$, therefore we have simply to put $p = 2$ and $q = k - 2$ to reproduce (9.63).

An obvious generalization for $p = 3$ would be to consider $\mathcal{M}_3^0$, and set $q = k - 3$ at the critical point determined by $kt_k = b$. However, out of the examples of the previous section, there is a perfect matching only at the second critical point ($k = 5$), not at the first ($k = 4$). In the latter case, nevertheless, the degrees match exactly: $[a_1] = 2$, $[t_1] = 3$. This means that we can reproduce the desired critical behaviour (9.63) in a definite scaling region

$$t_2 \sim t_1^{\frac{2}{3}}$$

which is allowed by the degree analysis.

This is general. For generic p in (9.63), select the model $\mathcal{M}_p^0$, and set $q = k - p$ at the critical point defined by $kt_k = b$. Then choose a suitable scaling region in the coupling space – *which always exists without violating homogeneity* – and the behaviour (9.63) of a_1 will always be reproduced.

Let us come now to the topological field theory properties of the models $\mathcal{M}_p^0$. The topological field theory (TFT) properties of such models is well-known, [19], [20]. They are the models corresponding to the A series, in the ADE classification, coupled to topological gravity. What we want to discuss here is how this TFT properties reproduce themselves at higher genus and at higher critical points.

We recall that a TFT is defined by the set of primary fields ϕ_i, $i = 1, \ldots, n$. Among them one, say ϕ_1, plays a special role. The n–th point CF's are pure numbers and, in particular, the 3–point CF's (in genus 0) $C_{ijk} =< \phi_i \phi_j \phi_k >_0$ are crucial in the definition of TFT. The metric η_{ij} coincides by definition with C_{1ij} and is required to be invertible. The inverse metric is denoted η^{ij}. A defining property of the topological field theory is the associativity condition

$$C_{ijp}\eta^{pq}C_{qkl} = C_{ikp}\eta^{pq}C_{qjl} \tag{9.64}$$

Here and in the following repeated upper and lower indices are understood to be summed over.

When we couple such models to topological gravity, primaries gets dressed by gravity and give rise to a series of descendants $\sigma_k(\phi)$, where $k = 0, 1, 2, \ldots$ and $\sigma_0(\phi) = \phi$. Moreover ϕ_1 plays the role of the puncture operator P. The structure of the resulting theory is governed by two sets of equations, the puncture equations

$$< P\sigma_{k_1}(\phi_1)\ldots\sigma_{k_n}(\phi_n) >_0 = \sum_{i=1}^{n} a_{k_i,i} < \sigma_{k_1}(\phi_1)\ldots\sigma_{k_i-1}(\phi_i)\ldots\sigma_{k_n}(\phi_n) >_0 \tag{9.65}$$

and the recursion relations

$$< \sigma_{k_i}(\phi_i)XY >_0 = a_{k_i,i} < \sigma_{k_i-1}(\phi_i)\phi_p > \eta^{pq} < \phi_q XY > \tag{9.66}$$

where $a_{k_i,i}$ are numbers depending on k_i and i (as well as on the normalizations we choose), and X, Y are any two primaries or descendants. Notice that eq.(9.64) easily follows from (9.66) as a particular case. For take $k_i = 1$, $X = \phi_k$ and $Y = \phi_l$, then differentiate both sides w.r.t. the coupling of ϕ_j. If the metric is constant, eq.(9.64) follows from symmetry under the exchange $\phi_k \leftrightarrow \phi_l$.

Let us see this formalism at work in a simple but significant case, $\mathcal{M}_3^0$ (the case $\mathcal{M}_2^0$ is too simple and many interesting aspects are obscured). In this case the primary fields are τ_1 and τ_2; moreover P has to be identified with τ_1. The metric η_{ij} , $i, j = 1, 2$, is given by $\eta_{11} = \eta_{22} = 0$ and $\eta_{12} = \eta_{21} = 6$. It is easy to verify many cases of eqs.(9.65) and (9.66) directly from the explicit formulas of the previous section. Our normalization requires $a_{k_i,i} = 3k_i + i$.

It is however important to realize that eqs.(9.65) and (9.66) are nothing but particular cases of equations we know very well. For let us take the first Virasoro constraint for $\mathcal{M}_3^0$, $L_{-1}^{[1]}\, Z^{1/3} = 0$, differentiate it w.r.t. $t_{k_1}, \ldots t_{k_n}$, with $k_1, \ldots k_n \neq 1, 2, 3l$, and evaluate it at the first critical point. We obtain

$$< P\tau_{k_1}\ldots\tau_{k_n} > = \sum_{i=1}^{n} k_i < \tau_{k_1}\ldots\tau_{k_i-3}\ldots\tau_{k_n} > \tag{9.67}$$

This is exactly eq.(9.65). Similarly the recursion relations are nothing but the dispersionless flow equations evaluated at the critical point. For example, the recursion relation

$$< \tau_4 PP >_0 = 4 < PP >_0 \frac{1}{6} < \tau_2 PP >$$

is nothing but the flow $\frac{\partial a_1}{\partial t_4} = \frac{4}{3}(a_1 a_2)'$, and so on.

Therefore eqs.(9.65) and (9.66), which were tailored [19],[20] to couple a TFT model to topological gravity, are nothing but particular cases of very general equations in two–matrix models. Looking at eqs.(9.65) and (9.66) as they are, it is far from clear how to generalize them to higher genus and higher critical points, while in the context of matrix models this

generalization is given for free. In higher genus (9.65) is derived from the first Virasoro constraint (which is valid at every genus), (9.66) are replaced by the full flows.

Let us see now what happens at the second critical point of $\mathcal{M}_3^0$. A simple repetition of what we have just done for the first critical point, leads to disappointing results. The reason is that, at the second critical point, the role of the puncture operator is not played by τ_1 anymore, but by the operator τ_2, i.e. $P \equiv \tau_2$. Once we realize this, everything works. The metric is the same as at the first critical point (up to a sign, this depends on our having chosen b=+1) and the associative condition is satisfied. The puncture equation has the form (9.67), which is derived from the first Virasoro constraint, provided of course $P \equiv \tau_2$. The recursion relation (9.66) holds too and, once again, can be replaced by the dispersionless flows evaluated at the critical point.

The same can be said about the second critical point of $\mathcal{M}_4^0$. In fact the above can be generalized not only to the higher critical points of $\mathcal{M}_3^0$ and $\mathcal{M}_4^0$, but to all the critical points of $\mathcal{M}_p^0$. In general, if the critical point is determined by the condition (8.56), the puncture operator P has to be identified with τ_{k-p}. The metric is constant and (up to normalization) the same as at the first critical point. The entire theory is regulated by equations which can be given the form of the W–constraints and the flow equations. This conclusion sounds like a triviality in our two–matrix model framework, but it is not so if we start from a TFT and couple it to topological gravity.

Appendix

The W generators appropriate to the $\mathcal{M}_p^0$ model are given in terms of

$$L_n^{[1]} = \frac{1}{p}\sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+pn}} + \frac{1}{2p}\sum_{k+l=pn} \frac{\partial^2}{\partial t_k \partial t_l} + \frac{1}{2p}\sum_{k+l=-np} klt_k t_l + \frac{p^2-1}{24p}\delta_{n,0}$$

$$\sqrt{p}L_2^{[2]} = \frac{1}{p^2}\sum_{k,l=1}^{\infty} klt_k t_l \frac{\partial}{\partial t_{k+l-2p}} + \frac{1}{p^2}\sum_{\substack{k,j,l \\ l=j+k+2p}} lt_l \frac{\partial^2}{\partial t_k \partial t_j} + \frac{1}{3p^2}\sum_{\substack{k,j,l \\ k+j+l=2p}} kjlt_k t_j t_l$$

From these we can generate all the W_p algebra. In the above formulas n is any integer and multiples of p are excluded among the summation indices.

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