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On the gravitational field of thin disks

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1. ABSTRACT

The main aim of this note is to discuss an exact solution of Einstein's vacuum equations with a rotating black hole and a thin disk which is appropriate as a test bed for codes that we recently applied in calculating profiles of spectral lines from self-gravitating disks. The solution has been obtained by the "cut and identify" method from the Kerr spacetime and it satisfies the dominant energy condition. We also briefly summarize recent analytic calculations of various disk models, both within the Newtonian and the general relativistic framework.

2. INTRODUCTION

In a widely accepted model of active galactic nuclei (AGN) a supermassive black hole is surrounded by an accretion disk or a torus (Begelman, Blandford & Rees 1984; Blandford, Netzer & Woltjer 1990). Most of the observed radiation is generated by viscous forces in the region close to the black hole horizon. This model meets most of the observational properties of AGN. An observational support for it could be strengthened if the system contains a solar-mass star is captured in an orbit near the black hole. Collisions of the star with the accretion disk can periodically modify the radiation flux from the source and give us information about the evolution of the orbit. Its orientation with respect to the observer will be affected by general relativistic effects (periastron shift, Lense-Thirring precession) and corresponding modulation with specific periodicities may provide us with the evidence of the black hole in the core (Karas & Vokrouhlický 1994). The model of a star orbiting a supermassive black hole has been considered by several authors in different situations. A mechanism of the tidal capture of a star by a black hole has been discussed by Hills (1988), Rees (1988), and Novikov, Pethick & Polnarev (1992). Star-disk collisions offer another possibility (Syer, Clarke & Rees 1991; Pineault & Landry 1994). Ostriker (1983) proposes that the collisions could result in a viscous drag on the accretion disk. Zentsova (1983) calculates a temperature profile of a bright spot which is created in the place were the star crashes through the disk. She estimates that the maximum local intensity is in the UV band. The collisions are highly supersonic (Mach number $10^2 - 10^3$) and they still need to be better understood. Assuming that the core of AGN is embedded in a dense star cluster, Zurek, Siemiginowska & Colgate (1992) estimate that the amount of the disk material swept out of the disk by stars' passages is enough to form the gas clouds of the broad line region. Syer et al. (1991) calculate time-scales for the evolution of the orbital parameters in the Newtonian approximation: star-disk collisions result in circularization of the orbit, the inclination is reduced and the star becomes a part of the disk. Vokrouhlický & Karas (1993) obtained similar results for a star moving in the field of a Kerr black hole. In contrast to the Newtonian case, the probability that the star will be captured by the black hole is now higher because subsequent collisions can set the star in an unstable orbit which ends in the black hole. Finally, the gravitational field of the central object and the disk modify the shape of spectral features coming from the disk (Laor

1991). Karas, Lanza & Vokrouhlický (1995) calculated predicted profiles of spectral lines from self-gravitating disks within the framework of exact general relativistic solutions.

Most of the works mentioned above rely substantially on numerical computations. For example, in our recent work (Karas et al. 1995) we employed numerical solutions of Einstein's equations describing a disk around a rotating black hole constructed by Lanza (1992) and computed observed spectra by solving (numerically, again) geodesic equations. Therefore, it appears desirable to understand how the gravitational field near a disk around a central object is formed and to have relatively simple non-trivial exact solutions for testing numerical codes. In this contribution we first discuss analytic models of the disk gravitational field in the Newtonian gravity. Then we present a general relativistic solution which appears convenient for our purposes. Although we refer to related works of other authors, it should be stressed that we did not attemt to write a consistent review.

3. NEWTONIAN DISKS

Unlike the gravitomagnetic precession and the pericenter shift which are specifically non-Newtonian effects, the effects induced by the disk attraction do not disappear in the Newtonian limit. We therefore started with the Newtonian description of a combined gravitational potential of the central mass and the equatorial, axially symmetric disk, $V = V_c + V_d$. At this moment, we assumed V_c to be spherically symmetric so that the precession effects due to the disk are clearly visible and do not interfere with those which could arise from a non-sphericity of V_c. Test particle motion in the field of a disk has important applications in celestial mechanics. Unfortunately, standard series expansions (Ward 1981) do not converge if the particle crosses the disk. Linear methods which have been adopted to treat the trajectories intersecting the disk (Lemaitre & Dubru 1991) have unacceptably bad features at small inclinations. In that case, we must resort to a numerical approach. It should be emphasized that the orbits intersecting the disk at moderate inclinations are of particular interest to us for two reasons: (i) they are an outcome of different circularization processes acting on the star. Initially highly eccentric and inclined orbits get circularized and ground to the disk plane on relatively short time-scales of the order $10^3 - 10^5$ orbital periods; (ii) the effects of the disk gravity on the orbital evolution become the most pronounced at small inclinations.

The problem of determining the gravitational field of an infinitesimally thin disk with an arbitrary radial distribution of the disk material $\sigma(R)$ (or, alternatively, the electrostatic field of a charged disk) attracts longstanding interest in physics and astrophysics. First we give expressions for the potential and its gradient which are to be solved. Because we did not want to restrict ourselves to any specific $\sigma(R)$ *a priori*, we evaluated the gravitational field numerically. We employed direct evaluation of the disk potential and its gradient. Disadvantage of this approach consists in occurence of non-algebraic (elliptic) functions in the integrand of final formulae [see Eqs. (1) and (3)–(4)]. Analytic solutions are rare. In the case of infinite disks, formulae containing

only algebraic functions in the integrands and several analytical results have been obtained recently by Evans & de Zeeuw (1992).

We introduced the system of polar coordinates $\{R, \Phi\}$ adjusted to the disk and the system of cylindric coordinates $\{z, \rho, \phi\}$ with the origin in the disk center ($z = 0$ is the disk plane). The potential of the disk with the radius a evaluated at an arbitrary location (ρ, z) is given by

$$V_{\rm d}(\rho, z) = -4\int_0^a \frac{\sigma(R)R}{B(R)} K[k(R)]\, dR\,, \tag{1}$$

where

$$k^2(R) = \frac{4\rho R}{B^2(R)}\,, \qquad B^2(R) = z^2 + (\rho + R)^2\,.$$

Here, $K(k)$ is the elliptic integral of the first kind. No apparent reduction of the integral (1) is possible until the radial distribution of the density $\sigma(R)$ is specified. An important example is a uniform disk, $\sigma(R) \equiv \sigma_0$. The potential (1) in terms of elliptic integrals K, E and Π reads (Lass & Blitzer 1983) is

$$V_{\rm d}(\rho, z)_{|\sigma=\sigma_0} \equiv V_{\rm u}(\rho, z; \sigma_0) = 2\sigma_0\Big[\pi|z| - B(a)\,E[k(a)] - \tfrac{a^2-\rho^2}{B(a)} K[k(a)] - \tfrac{a-\rho}{a+\rho}\tfrac{z^2}{B(a)}\,\Pi\left[\alpha^2; k(a)\right]\Big] \tag{2}$$

with

$$\alpha^2 = \frac{4a\rho}{(a+\rho)^2}\,.$$

Though derived by a straightforward integration this result does not seem to be generally known (cf. Appendix B in Farouki et al. 1992). The expression for the potential inside the disk ($z = 0$, $\rho < a$) can be simplified by applying the Gauss transformation of elliptic functions (Byrd & Friedman 1971). One obtains

$$V_{\rm u}(\rho, 0; \sigma_0) = -4a\sigma_0 E(\rho/a)\,,$$

a more compact formula than the one given by Lass & Blitzer (1983).

The components of the gravitational force are given by the gradient of the potential (1):

$$\frac{\partial V_{\rm d}}{\partial \rho} = -\frac{2}{\rho}\int_0^a \frac{\sigma(R)R}{B(R)}\left\{E[k(R)]\frac{R^2-\rho^2+z^2}{A^2(R)} - K[k(R)]\right\} dR\,, \tag{3}$$

$$\frac{\partial V}{\partial z} = -4z\int_0^a \frac{\sigma(R)\,R\,E[k(R)]}{B(R)\,A^2(R)}\, dR\,, \tag{4}$$

where we denoted

$$A^2(R) = z^2 + (\rho - R)^2\,.$$

For the uniform disk

$$\frac{\partial V_{\rm u}}{\partial \rho} = -\frac{2\sigma_0}{\rho B(a)}\left[B^2(a)\,E[k(a)] - \left(z^2 + a^2 + \rho^2\right) K[k(a)]\right]\,, \tag{5}$$

$$\frac{\partial V_{\rm u}}{\partial z} = 2\sigma_0 \left[\pm\pi - \frac{z}{B(a)} \left(K\left[k(a)\right] + \frac{a-\rho}{a+\rho}\, \Pi\left[\alpha^2(a)\,; k(a)\right] \right) \right] . \tag{6}$$

The integrands in Eqs. (1) and (3)–(4) diverge at the disk plane, $z \to 0$, although the result of the integration is certainly finite because the potential must be continuous across the disk. Taking into account the relation (Byrd & Friedman 1971)

$$K(k) \propto \ln\left(1-k^2\right) \tag{7}$$

for $k \approx 1$, one concludes that the divergence is proportional to $\ln z$. To get rid of numerical errors in evaluating these integrals we divided diverging expressions into two parts by setting $\sigma(R) \equiv [\sigma(R) - \sigma(\rho)] + \sigma(\rho)$. The potential is then

$$V_{\rm d}(\rho, z) = -\,4 \int_0^a \frac{[\sigma(R) - \sigma(\rho)]\, R}{B(R)}\, K\left[k(R)\right]\, dR + V_{\rm u}\left[\rho, z; \sigma(\rho)\right] , \tag{8}$$

where the second term corresponds to the potential of a disk with a constant density $\sigma(\rho)$ given by Eq. (2). Now the integrand in (8) is well behaved. A sharp decrease of the integrand near $R \approx \rho$, $z \approx 0$ can be treated by appropriate numerical methods (Pérez-Jordá, San-Fabián & Moscardó 1992). The same approach can be successfully applied to the force components (3)–(4) as well.

In view of the properties of the integrands described above it appears rather important to discuss special cases which can be treated analytically. We considered both finite and infinite disks. Different strategies can be applied to express the gravitational field of a Newtonian axisymmetric disk analytically, at least in a formal way. Here, we briefly summarize the four important approaches.

The disks of a finite radius a. Lynden-Bell & Pinneault (1978a, b) expose a general direct algorithm. They employ integral quadratures which follow from classical formulae for the gravitational field of concentric spheroids in the limit $e \to 1$. These integrals are only rarely carried out in simple functions. The limit of homogeneous spheroids (MacLaurin series of spheroids) is such an example in this family. It results in the disk with the surface density

$$\sigma(R) = \frac{3M_{\rm d}}{2\pi a^2} \sqrt{1 - \left(\frac{R}{a}\right)^2} .$$

(Brandt, 1960) and the potential in the disk plane ($z = 0; \rho < a$)

$$V_{\rm d}(\rho, 0) = \frac{3\pi M}{8a^3} \left(5a^2 - 4\rho^2\right) .$$

Mestel's disk (Mestel 1963, Lynden-Bell & Pinneault 1978a) is another example. It is characterized by a constant orbital velocity v of the disk elements. The density distribution is given by

$$\sigma(R) = \frac{v^2}{\pi^2 R} \arccos(R/a)$$

(see also Lemaitre & Dubru 1991) and the potential in the disk is

$$V_{\rm d}(\rho,0) = -v^2 \ln\left(\frac{\rho}{2a}\right) .$$

We do not repeat extensive expressions for the potential $V(\rho,z)$ of Mestel's disk [Eq. (2.18) of Lynden-Bell & Pineault 1978a]. Let us only remind that, besides the uniform disk (2), this is one of the rare cases when the analytic expression for the potential valid at an arbitrary point (ρ,z) is known. Only very recently, Brada & Milgrom (1994) discovered an interesting family of finite elliptical disks.

Another approach employs expansions in the fundamental set of the solutions of the Laplace equation (Hunter 1963, Morse & Feshbach 1953). It is well known that the Laplace equation is separable in spheroidal coordinates and the Legendre polynomials form a fundamental set of solutions. The crucial problem, however, is to join the expansions of the potential valid far from the disk to those near the disk because they are based on different modes of the Legendre polynomials (Hunter 1963). In general, one even does not know a spatial surface where the joining condition is to be imposed. As above, the only simple case occurs for the region inside the disk.

Infinite disks, $a \to \infty$. Considerable progress has been achieved recently by Evans & de Zeeuw (1992), followed by relativistic generalisations by Bičák, Lynden-Bell & Katz (1993) and Bičák, Lynden-Bell & Pichon (1993). The method, conceptually unifying several classes of solutions (Toomre 1963; Mestel 1963), is based on a trick due to Kuzmin (1956). We recontructed numerically several solutions which belong to the Toomre and the Kalnajs-Mestel disks as presented by Evans & de Zeeuw (1992) by setting the disk edge sufficiently large.

The last method to be reminded here evaluates integrals (1) and (3)–(4) directly. This method is however restricted to the disk plane ($z = 0$). First, let us define quantities

$$\psi\,[\alpha,\beta; f(\xi)] = \int_\alpha^\beta d\xi\, \sigma(\xi)\, \xi\, f(\xi) \tag{9}$$

and

$$\varphi\,[\alpha,\beta; g(\eta)] = \int_\alpha^\beta d\eta\, \frac{\sigma(\eta)}{\eta^2}\, g(\eta) . \tag{10}$$

One employs the Gauss transformation of elliptic functions to obtain the following form of the potential and its gradient in the equatorial plane (the disk is supposed to have the inner edge a and the outer edge b):

$$V_{\rm d}(\rho,0) = \rho\, \varphi\,[\rho/b, \rho/a; K(\eta)] \tag{11}$$

for the region below the inner disk edge ($\rho < a$),

$$V_{\rm d}(\rho,0) = \rho\, \{\varphi\,[\rho/b, 1; K(\eta)] + \psi\,[a/\rho, 1; K(\xi)]\} \tag{12}$$

for the region inside the disk ($a < \rho < b$), and

$$V_{\rm d}(\rho,0) = \rho\, \psi\,[a/\rho, b/\rho; K(\xi)] \tag{13}$$

outside the disk ($\rho > b$). Similarly, one obtains

$$\frac{\partial V_{\rm d}(\rho,0)}{\partial \rho} = 2\rho\, \varphi\left[\rho/b, \rho/a; \eta\frac{d}{d\eta}K(\eta)\right] \tag{14}$$

for the region below the inner disk edge ($\rho < a$),

$$\frac{\partial V_{\rm d}(\rho,0)}{\partial \rho} = 2\rho\, \left\{\varphi\left[\rho/b, 1; \eta\frac{d}{d\eta}K(\eta)\right] - \psi\left[a/\rho, 1; \frac{d}{d\xi}\xi K(\xi)\right]\right\} \tag{15}$$

for the region inside the disk ($a < \rho < b$), and

$$\frac{\partial V_{\rm d}(\rho,0)}{\partial \rho} = -2\rho\, \psi\left[a/\rho, b/\rho; \frac{d}{d\xi}\xi K(\xi)\right] \tag{16}$$

outside the disk ($\rho > b$). Here, we denoted $\xi = R/\rho$ and $\eta = \rho/R$. Note that the difference between (11)–(13) and (14)–(16) is essentially given by the functional kernel in ψ and φ functions. The right-hand-side of (12) and (15) is evaluated as a principal value.

Previous integrals can be performed for several profiles of the surface density of the type $\sigma_n(R) = \sigma_0 R^{-n}$ (Figure 1). As an example,

$$\frac{\partial V_{\rm d}}{\partial \rho}(\rho,0) = -\frac{4\sigma_0}{\rho}\left[K\left(\frac{\rho}{a}\right) - K\left(\frac{\rho}{b}\right)\right] \tag{17}$$

for ($\rho < a$),

$$\frac{\partial V_{\rm d}}{\partial \rho}(\rho,0) = -\frac{4\sigma_0}{\rho}\left[\frac{a}{\rho}K\left(\frac{a}{\rho}\right) - K\left(\frac{\rho}{b}\right)\right] \tag{18}$$

for ($a < \rho < b$), and

$$\frac{\partial V_{\rm d}}{\partial \rho}(\rho,0) = \frac{4\sigma_0}{\rho}\left[bK\left(\frac{b}{\rho}\right) - aK\left(\frac{a}{\rho}\right)\right] \tag{19}$$

for ($\rho > b$) in the case of the exponent $n = 1$.

4. GENERAL RELATIVISTIC DISKS

Following the work by Bardeen & Wagoner (1971) many authors have studied the structure of relativistic rotating fluid bodies—either within the context of neutron stars, self-gravitating disks or toroids (for references see Nishida, Eriguchi & Lanza 1992). Exact solutions, in particular those describing a disk around a black hole, have been recently summarized by Lemos & Letelier (1994). These authors found also new families of exact black hole–thin disk solutions. It appears, however, that analytic solutions which have been discovered so far cannot be applied to astrophysically realistic situations. Mainly for this reason, Lanza (1992) has solved *numerically* Einstein's equations for the stationary axisymmetric spacetimes with a thin self-gravitating disk and a rotating black hole. Lanza's work describes Keplerian pressure-less disks but his code which employs the multigrid method can be generalized to deal with a more complicated equation of state of the disk material.

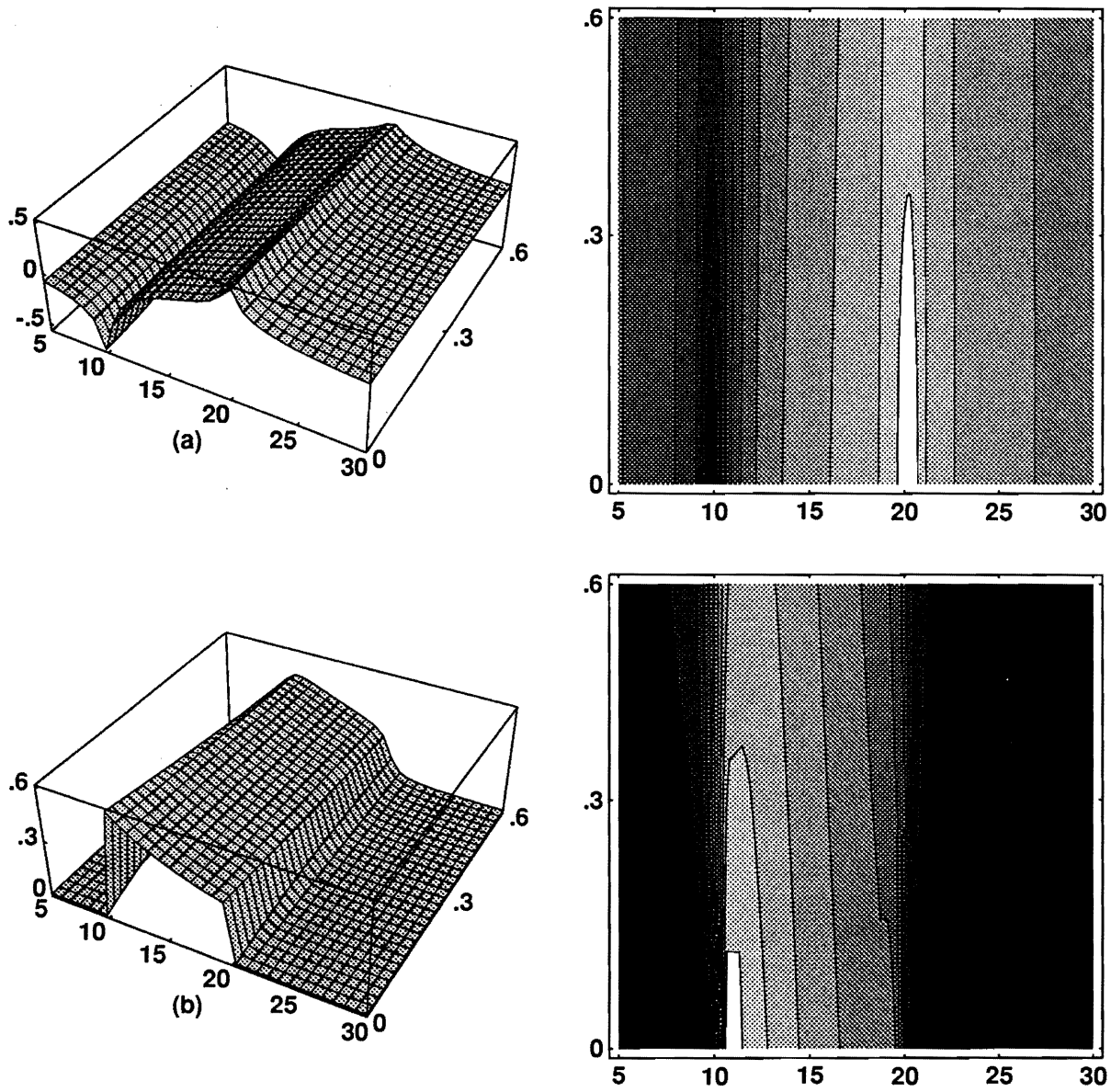

Fig. 1.— Three-dimensional plots and contour plots of the radial (a) and vertical (b) components of the force ($\partial V/\partial \rho$ and $\partial V/\partial z$, respectively) (arbitrary units) due to the Newtonian disk. The disk extends in the region $10 < R < 20$ in dimensionless units and it has the surface density distribution $\propto R^{-1}$. In this figure, $5 < \rho < 30$, $0 < z < 0.6$.

Even thick disks can be studied after modifications. We were interested in constructing an exact solution which is not "too much" unrealistic as far as the source term is concerned, and which could be useful in testing our numerical codes. For this purpose we required a solution describing a thin disk around a rotating black hole. We started with the Kerr spacetime and employed a well-known method of constructing external gravitational field of thin shells (Israel 1966; Barrabès & Israel 1991). Analogous approach have been used by Abramowicz, Arkuszewski & Muchotrzeb (1976), Turakulov (1989; 1990), and Pichon & Lynden-Bell (1994) in a different context of exact solutions. Bičák & Ledvinka (1993) have constructed pressure-less counter-rotating disks in the Kerr geometry. We adopt their notation.

The Kerr black hole (dimensionless angular momentum parameter a) has nonvanishing metric components in canonical coordinates $\{t, \phi, \rho, z\}$

$$g_{tt} = -\frac{L}{E}, \qquad g_{\phi t} = -\frac{B}{E},$$

$$g_{\phi\phi} = \rho^2\frac{E}{L} - \frac{B^2}{LE}, \qquad g_{zz} = g_{\rho\rho} = \frac{E}{F}$$

with

$$L = p^2x^2 + a^2y^2 - 1, \qquad E = (px+1)^2 + a^2y^2,$$
$$F = p^2(x^2 - y^2), \qquad B = 2a(1-y^2)(px+1),$$
$$p^2 = 1 - a^2.$$

Here, x and y are prolate coordinates which are related to ρ, z by

$$z = pxy, \qquad \rho^2 = p^2(x^2 + y^2 - x^2y^2 - 1).$$

A disk around a Kerr black hole is now constructed by cutting the region $|z| < \mathcal{Z}(\rho)$ out of the original manifold and by identifying $z = \mathcal{Z}(\rho)$ with $z = -\mathcal{Z}(\rho)$. We wanted to construct the disk with an inner edge above the black hole horizon and, therefore, we assumed that $\mathcal{Z}(\rho)$ is a space-like surface and $\mathcal{Z}(\rho) = 0$ for $\rho < \rho_{\rm inn}$, $\rho_{\rm inn} > 1 + \sqrt{(1-a^2)}$.

Following Israel's formalism, we introduced an orthonormal tetrad

$$e^\alpha_{(t)} = \delta^\alpha_{(t)}, \qquad e^\alpha_{(\rho)\pm} = \delta^\alpha_{(\rho)} \pm z'\delta^\alpha_{(z)},$$

$$e^\alpha_{(\phi)} = \delta^\alpha_{(\phi)}, \qquad n^\pm_\alpha = \frac{1}{\sqrt{g^{\rho\rho}(z')^2 + g^{zz}}}\left(\delta^{(z)}_\alpha \mp z'\delta^{(\rho)}_\alpha\right);$$

[$z' \equiv d\mathcal{Z}/d\rho$; "$\pm$" corresponds to $z = \pm\mathcal{Z}(\rho)$]. The stress-energy tensor S_{ab} in the disk is related to jumps in the extrinsic curvature

$$[K_{ab}] \equiv \left[n_\alpha e^\beta_{(b)}\nabla_\beta e^\alpha_{(a)}\right]^{z=-\mathcal{Z}(\rho)}_{z=\mathcal{Z}(\rho)} \tag{20}$$

by Einstein's equations

$$-8\pi\left(S_{ab}-\tfrac{1}{2}g_{ab}S\right)=[K_{ab}]\,. \tag{21}$$

Associated four-dimensional tensor in the disk is

$$S^{\alpha\beta}=e^{\alpha}_{(a)}e^{\beta}_{(b)}S^{ab} \tag{22}$$

and it vanishes outside the disk. Finally, we introduced an orthonormal basis of vectors $\{V^{\mu}_{(\alpha)}\}$ of an observer orbiting around the black hole in the disk:

$$V^{\mu}_{(t)}=N(1,\Omega,0,0),\qquad V^{\mu}_{(\phi)}=J(\beta,1,0,0),$$

$$V^{\mu}_{(\rho)}=L(0,0,1,\delta),\qquad V^{\mu}_{(z)}=M(0,0,\epsilon,1),$$

and we required that the stress-energy tensor in the observer frame, $S^{\star}_{(\alpha)(\beta)}=V^{\mu}_{(\alpha)}V^{\nu}_{(\beta)}S_{\mu\nu}$, is diagonal. Now $S^{\mu\nu}$ can be written in the form

$$S^{\mu\nu}=\sigma V^{\mu}_{(t)}V^{\nu}_{(t)}+P_{\phi}V^{\mu}_{(\phi)}V^{\nu}_{(\phi)}+P_{\rho}V^{\mu}_{(\rho)}V^{\nu}_{(\rho)} \tag{23}$$

with

$$\sigma=S^{\star}_{(t)(t)},\qquad P_{\phi}=S^{\star}_{(\phi)(\phi)},\qquad P_{\rho}=S^{\star}_{(\rho)(\rho)}.$$

This shows that the disk material has both radial and azimuthal pressure. With a special choice of the identification surface, $\mathcal{Z}(\rho)=const$, one obtains $P_{\rho}=0$ and our results are reduced to those presented by Bičák & Ledvinka (1993). If we further set $a=0$, eq. (3.50) of Bičák, Lynden-Bell & Katz (1993) is reproduced. As an example, Figure 2 corresponds to the disk with an inner edge at $\rho_{\rm inn}=20$ around a black hole $a=0.8$ and the identification taken at $\mathcal{Z}(\rho)=(\rho-\rho_{\rm inn})^3$, $\rho>\rho_{\rm inn}$. It can be seen that this solution satisfies the dominant energy condition (Hawking & Ellis 1973). however, there is always a region with $P_{\rho}<0$. The metric outside the disk remains identical to the Kerr metric. This is useful in testing codes for computing spectra of accretion disks, because the solution of the geodesic equation—otherwise requiring computation of numerical derivatives of a numerically given metric—is known analytically in the Kerr metric.

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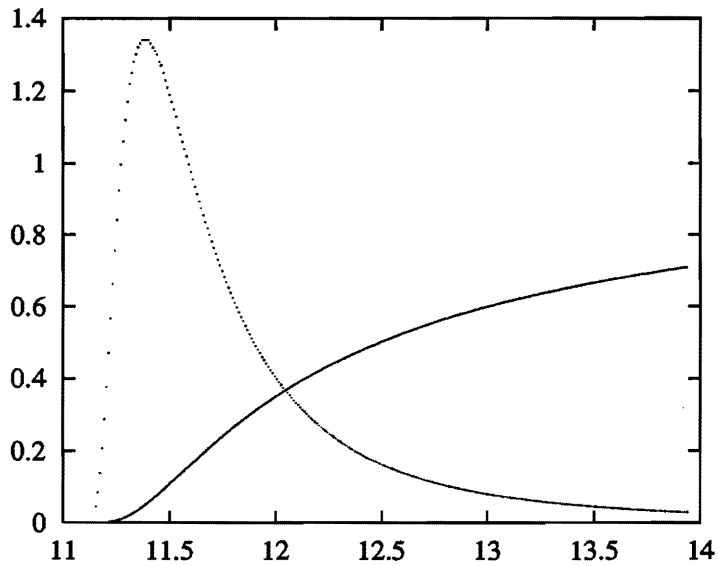

Fig. 2.— σ (solid line) and $\sigma - |P_\rho| - |P_z|$ (dotted line) in arbitrary units, as a function of the circumferential radius $\mathcal{R} = \sqrt{g_{\phi\phi}}$ starting from the inner edge of the disk at $\mathcal{R} = 11.2$.

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