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MONODROMY OF CERTAIN PAINLEVE'–VI TRANSCENDENTS AND REFLECTION GROUPS

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Abstract. We study the global analytic properties of the solutions of a particular family of Painlevé VI equations with the parameters $\beta = \gamma = 0$, $\delta = \frac{1}{2}$ and α arbitrary. We introduce a class of solutions having critical behaviour of algebraic type and completely compute the structure of analytic continuation of these solutions in terms of an auxiliary reflection group in the three dimensional space. The analytic continuation is given in terms of an action of the braid group on the triples of generators of the reflection group. This result is used to classify all the algebraic solutions of the Painlevé VI equation.

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INTRODUCTION

In this paper we will study the structure of analytic continuation of the solutions of the following differential equation

$$\begin{aligned} y_{xx} =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y_x^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y_x \\ &+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2+\frac{x(x-1)}{(y-x)^2}\right], \end{aligned} \qquad PVI\mu$$

in the complex plane, μ is an arbitrary complex parameter satisfying the condition $2\mu \notin \mathbb{Z}$.

This is a particular case of the general Painlevé VI equation (see [Ince]) PVI$(\alpha,\beta,\gamma,\delta)$, depending on four parameters $\alpha,\beta,\gamma,\delta$. The equation PVI$\mu$ is specified by the following choice of the parameters

$$\alpha=\frac{(2\mu-1)^2}{2}, \qquad \beta=\gamma=0 \qquad \delta=\frac{1}{2}.$$

The general solution $y(x;c_1,c_2)$ of PVI$(\alpha,\beta,\gamma,\delta)$ satisfies the following two important properties (see [Pain]):

1) The solution $y(x;c_1,c_2)$ can be analytically continued to a meromorphic function on the universal covering of $\overline{\mathbb{C}}\backslash\{0,1,\infty\}$.
2) For generic values of the integration constants c_1,c_2 and of the parameters $\alpha,\beta,\gamma,\delta$, the solution $y(x;c_1,c_2)$ can not be expressed via elementary or classical transcendental functions.

The first claim is the so-called *Painlevé property* (see below) of the equation PVI$(\alpha,\beta,\gamma,\delta)$. The points $0,1,\infty$ are called *critical points* of the equation. The solutions $y(x;c_1,c_2)$ have complicated singularities at the critical points. The position of these complicated singularities does not depend on the choice of the particular solution (the so-called *fixed singularities*). All the other singularities are poles, whose position depends on the integration constants (the so-called *movable singularities*).

All the second order ordinary differential equations of the type:

$$y_{xx}=\mathcal{R}(x,y,y_x),$$

where $\mathcal{R}$ is rational in y_x and meromorphic in x and y and satisfies the Painlevé property of absence of movable critical singularities, were classified by Painlevé and Gambier (see [Pain], and [Gamb]). Only six of these equations, which are given in the *Painlevé-Gambier list,* satisfy the property 2), i.e. they can not be reduced to known differential equations for elementary and classical special functions. The PVI$(\alpha,\beta,\gamma,\delta)$ is the most general equation of these six: all the others can be obtained from PVI$(\alpha,\beta,\gamma,\delta)$ by a confluence procedure (see [Ince] §14.4).

We do not discuss here the various physical applications of particular solutions of the Painlevé equations. We mention only the paper [Tod] where our PVIμ appears in the problem of the construction of self-dual Bianchi-type IX Einstein metrics, and the paper [Dub] where the same equation was used to classify the solutions of WDVV equation in $2D$-topological field theories.

The solutions of the six Painlevé equations are called *Painlevé transcendents.* This name could be misleading; indeed, for some particular values of $(c_1,c_2,\alpha,\beta,\gamma,\delta)$, the solution $y(x;c_1,c_2)$ can be expressed via classical functions. For example Picard (see in [Ok]) showed that the general solution of PVI$(0,0,0,\frac{1}{2})$ can be expressed via elliptic functions, and, more recently, Hitchin [Hit] obtained the general solution of PVI$(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{3}{8})$ in terms of the Jacobi theta-functions (see also [Man]). Particular examples of classical solutions, that can be expressed via hypergeometric functions, of PVI were first constructed by Lukashevich [Luka]. A general approach to study the classical solutions of PVI was proposed by Okamoto (see [Ok1][Ok2]). One of the main tools of this approach is the symmetry group of PVI: the particular solutions are those being invariant with respect to some symmetry of PVI. The symmetries act in a non trivial way on the space of the parameters $(\alpha,\beta,\gamma,\delta)$. Okamoto described the fundamental region of the action of this symmetry group and showed that all the classical solutions known at that moment, fit into the boundary of this fundamental region.

The theory of the classical solutions of the Painlevé equations was developed by Umemura and Watanabe ([Um], [Um1], [Um2], [Um3], [Wat]); in particular, all the one-parameter families of classical solutions of PVI were classified. Watanabe also proved that, loosely speaking, all the other classical solutions of PVI (i.e. not belonging to the one-parameter families) could only be given by algebraic functions.

Examples of algebraic solutions were found in [Hit1], for PVI$(\frac{1}{8}, -\frac{1}{8}, \frac{1}{2k^2}, \frac{1}{2} - \frac{1}{2k^2})$, for an arbitrary integer k. Other examples for PVIμ were constructed in [Dub]. They turn out to be related to the group of symmetries of the regular polyhedra in the three dimensional space.

The main aim of our work is to elaborate a tool to classify all the algebraic solutions of the Painlevé VI equation (for the other five Painlevé equations, all algebraic solutions have already been classified (see [Kit], [Wat1], [Mur] and [Mur1]). Our idea is very close to the main idea of the classical paper of Schwartz (see [Schw]) devoted to the classification of the algebraic solutions of the Gauss hypergeometric equation. Let $y(x; c_1, c_2)$ be a branch of a solution of PVI. The analytic continuation of this branch along any closed path γ avoiding the singularities, is a new branch $y(x; c_1^\gamma, c_2^\gamma)$, with new integration constants c_1^γ, c_2^γ. Since all the singularities of the solution on $\overline{\mathbb{C}}\backslash\{0, 1, \infty\}$ are poles, the result of the analytic continuation depends only on the homotopy class of the loop γ on the Riemann sphere with three punctures. As a consequence the structure of the analytic continuation is described by an action of the fundamental group:

$$\gamma \in \overline{\mathbb{C}}\backslash\{0, 1, \infty\}, \quad \gamma : (c_1, c_2) \to (c_1^\gamma, c_2^\gamma). \tag{0.1}$$

To classify all the algebraic solutions of Painlevé VI, all the finite orbits of this action must be classified.

Our problem is more involved than the Schwartz's linear analogue of it, because (0.1) is not a linear representation, but a non-linear action of the fundamental group. Moreover our problem is also more involved than the problem of the classification of the algebraic solutions of the other five Painlevé equations; in fact, the PVI is the only equation on the list having non-abelian fundamental group of complement of the critical locus.

Although the main idea seems to work for the general PVI$(\alpha, \beta, \gamma, \delta)$, we managed to completely describe the action (0.1), and to solve the problem of the classification of the algebraic solutions, only for the particular one-parameter family PVIμ. Nevertheless, we have decided to publish these results separately, postponing the investigation of the general case to another paper (in an effort to keep the paper within a reasonable size, we also postpone the study of the resonant case $2\mu \in \mathbb{Z}$). One of the motivations for the present publication is a nice geometrical interpretation of the structure of the analytic continuation (0.1), that seems to disappear in the general PVI equation.

We now outline the main results and describe the structure of the paper. Let us introduce a class of solutions of PVIμ a-priori containing all the algebraic solutions. We say that a branch of a solution $y(x; c_1, c_2)$ has *critical behaviour of algebraic type*, if there exist three real numbers l_0, l_1, l_∞ and three non-zero complex numbers a_0, a_1, a_∞, such that

$$y(x) = \begin{cases} a_0 x^{l_0} \left(1 + \mathcal{O}(x^\varepsilon)\right), & \text{as} \quad x \to 0, \\ 1 - a_1 (1-x)^{l_1} \left(1 + \mathcal{O}((1-x)^\varepsilon)\right), & \text{as} \quad x \to 1, \\ a_\infty x^{1-l_\infty} \left(1 + \mathcal{O}(x^{-\varepsilon})\right), & \text{as} \quad x \to \infty, \end{cases} \tag{0.2}$$

where $\varepsilon > 0$ is small enough. We show that there exists a three-parameter family of solutions of PVIμ with critical behaviour of algebraic type, where μ itself is a function of l_0, l_1, l_∞. Of course, for an algebraic solution, the indices l_0, l_1, l_∞ must be rational.

It turns out that the three-parameter family of solutions (0.2) is closed under the analytic continuation (0.1), if and only if μ is real. One of our main results is the parameterisation of the solutions (0.2) by ordered triples of planes in the three dimensional *Euclidean space*(see section 1.4). In particular, the indices l_0, l_1, l_∞ are related to the angles $\pi r_0, \pi r_1, \pi r_\infty$ between the planes:

$$l_i = \begin{cases} 2r_i & \text{if} \quad 0 < r_i \le \dfrac{1}{2} \\ 2 - 2r_i & \text{if} \quad \dfrac{1}{2} \le r_i < 1 \end{cases}, \quad i = 0, 1, \infty$$

The parameter μ is determined within the ambiguity $\mu \mapsto \pm\mu + n$, $n \in \mathbb{Z}$, by the equation:

$$\sin^2 \pi\mu = \cos^2 \pi r_0 + \cos^2 \pi r_1 + \cos^2 \pi r_\infty + 2 \cos \pi r_0 \cos \pi r_1 \cos \pi r_\infty .$$

This ambiguity and the one due to the reordering of the planes can be absorbed by the symmetries of PVIμ described in section 1.2.

We compute the analytic continuation (0.1) in terms of some elementary operations on the planes. This computation leads to prove that, for an algebraic solution of PVIμ, the reflections in the planes must generate the symmetry group of a regular polyhedron in $\mathbb{R}^3$. A main result of this paper in the classification theorem of all algebraic solutions of PVIμ. They are in one-to-one correspondence, modulo the symmetries of the equations described in section 1.2, with the reciprocal pairs of the three-dimensional regular polyhedra and star-polyhedra (the description of the star-polyhedra can be found in [Cox]). The solutions corresponding to the regular tetrahedron, cube and icosahedron are the ones obtained in [Dub]. The solutions corresponding to the regular great icosahedron, and regular great dodecahedron are new. Our method allows not only to classify the solutions, but also to obtain the explicit formulae, as we have done in section 2.4.

The main tool to obtain these results is the isomonodromy deformation method (see [Fuchs], [Sch] and [JMU], [ItN], [FlN]). The Painlevé VI is represented as the equation of isomonodromy deformation of the auxiliary Fuchsian system

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_x}{z-x} \right) Y. \tag{0.3}$$

For PVIμ the 2×2 matrices A_0, A_1, A_x are nilpotent and

$$A_0 + A_1 + A_x = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix}.$$

The entries of the matrices A_i are complicated expressions of x, y, y_x and of some quadrature $\int R(x,y)\mathrm{d}x$. The monodromy of (0.3) remains constant if and only if $y = y(x)$ satisfies PVI. Thus the solutions of PVIμ are parameterised by the monodromy data of the Fuchsian system (0.3) (see section 1.1). In section 1.2, we compute the structure of the analytic continuation in terms of the monodromy data. On this basis, in section 1.3, we classify all the monodromy data of the algebraic solutions of PVIμ. To this end we classify all the rational solutions of certain trigonometric equations using the method of an old paper by Gordan (see [Gor]). In section 1.4, we parameterise the monodromy data of PVIμ, by ordered triples of planes in the three-dimensional space. The structure of the analytic continuation of the solutions of PVIμ is reformulated in terms of a certain action of the braid group B_3 on the triples of planes. The group G generated by the reflections with respect to the planes remains unchanged. For the algebraic solutions, the group G turns out to coincide with the symmetry group of one of the regular polyhedra in the three-dimensional Euclidean space. We give also another proof, suggested by E. Vinberg, of this result. Here, we also establish that the class of solutions of PVIμ parameterised by triples of planes in the three dimensional Euclidean space is invariant with respect to the analytic continuation.

In the second part of the paper, we identify this class of solutions of PVIμ with the class of solutions having critical behaviour of algebraic type (0.2). In section 2.1, we prove that the solution $y(x)$ of the form (0.2), is uniquely determined by its asymptotic behaviour near one of the critical points, i.e. by any of the pairs (a_0, l_0), (a_1, l_1),(a_∞, l_∞) defining the critical behaviour. In particular, we prove that, for an algebraic solution of PVIμ, the indices l_0, l_1, l_∞ must satisfy:

$$0 < l_i \leq 1, \qquad i = 0, 1, \infty.$$

To derive the *connection formulae* establishing the relations between these pairs, we use (see section 2.3) the properly adapted method of Jimbo (see [Jim]). This method allows to express the monodromy data of the auxiliary Fuchsian system (0.3), in terms of the parameters (a_0, l_0), (a_1, l_1) or (a_∞, l_∞). For convenience of the reader, and also due to some differences between the assumptions of Jimbo's work and ours, we give a complete derivation of the connection formulae in section 2.2. Using the results of the sections 1.3 and 1.4, we complete the computation of the critical behaviour (0.2) for all the branches of the analytic continuation

of the solution. The result of this computation is used in section 2.4 to obtain the explicit formulae for the algebraic solutions of PVIμ.

Remark. The resulting classification of the algebraic solutions of PVIμ is in striking similarity to the Schwartz's classification (see [Schw]) of the algebraic solutions of the hypergeometric equation. According to Schwartz, the algebraic solutions of the hypergeometric equation, considered modulo contiguity transformations, are of fifteen types (the first type consists of an infinite sequence of solutions). The rows $(2-15)$ of the Schwartz's list (see, for example, the table in section 2.7.2 of [Schw]) correspond to the triples of generating reflections of the symmetry groups of regular polyhedra in the three-dimensional Euclidean space (we are grateful to E. Vinberg for bringing our attention to this point). The parameter λ, μ, ν of the hypergeometric equation shown in the table are just the angles between the mirrors of the reflections, divided by π.

According to our classification, the algebraic solutions of PVIμ, considered modulo symmetries, are in one-to-one correspondence to the classes of equivalence of the triples of generating reflections in the symmetry groups of regular polyhedra. the equivalence being defined by an action of the braid group B_3 on the triples. We find that in the groups $G = W(A_3)$ and $G = W(B_3)$, respectively the symmetry groups of the regular tetrahedron and of the cube or regular octahedron, there is only one equivalence class of triples of generating reflections; these are given respectively by the rows $(2,3)$ and by $(4,5)$ of the Schwartz's table. In the group $W(H_3)$ of symmetries of regular icosahedron or regular dodecahedron, there are three equivalence classes of triples of reflections which are given respectively by the rows $(6,8,13)$, $(11,14,15)$ and $(7,9,10,12)$ of the Schwartz's table and correspond to icosahedron, great icosahedron and great dodecahedron (or to their reciprocal pairs, see [Cox]). To establish the correspondence, we associate to a regular polyhedron a *standard* system of generating reflections in the following way: let H be the center of the polyhedron, O the center of a face, P a vertex of this face and Q the center of an edge of the same face through the vertex P. Then the reflections with respect to the planes HOP, HOQ and HPQ are the standard system of generators (see figure 7).

Summarizing, we see that the list of all the algebraic solutions of PVIμ is obtained by folding of the list of Schwartz modulo the action of the braid group. This relation between the algebraic solutions of PVIμ and the algebraic hypergeometric functions looks to be surprising also from the point of view of the results of Watanabe (see [Wat]) who classified all the one-parameter families of classical solutions of PVIμ (essentially, all of them are given by hypergeometric functions). Using these results, one can easily check that our algebraic solutions do not belong to any of the one-parameter families of classical solutions of PVIμ.

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1. STRUCTURE OF ANALYTIC CONTINUATION AND ALGEBRAIC SOLUTIONS OF PVIμ

1.1. Painlevé VI equation as isomonodromy deformation equation.

In this section we show how the PVIμ equation can be reduced to the isomonodromy deformation equation of an auxiliary Fuchsian system (see [Sch], [JMU]); moreover we describe the parameterisation, essentially due to Schlesinger, of the solutions of the PVIμ equation by the monodromy data of such Fuchsian system.

An auxiliary Fuchsian system and its monodromy data. Let us consider the following Fuchsian system with four regular singularities at $z = u_1, u_2, u_3, \infty$:

$$\frac{\mathrm{d}}{\mathrm{d}z} Y = \mathcal{A}(z) Y, \qquad z \in \overline{\mathbb{C}} \backslash \{u_1, u_2, u_3, \infty\} \tag{1.1}$$

where

$$\mathcal{A}(z) = \frac{\mathcal{A}_1}{z-u_1} + \frac{\mathcal{A}_2}{z-u_2} + \frac{\mathcal{A}_3}{z-u_3},$$

$\mathcal{A}_i$ being 2×2 matrices, and u_1, u_2, u_3 being pairwise distinct complex numbers. We will see in the latter part of this section that for the particular case PVIμ of the Painlevé VI equation, the matrices $\mathcal{A}_i$ must satisfy the following conditions:

$$\mathcal{A}_i^2 = 0 \quad \text{and} \quad \mathcal{A}_\infty := -\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3 = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}. \tag{1.2}$$

In this paper we consider the *non-resonant* case $2\mu \notin \mathbb{Z}$.

The solution $Y(z)$ of the system (1.1) is a multi-valued analytic function in $\mathbb{C}\backslash\{u_1, u_2, u_3\}$, and its multivaluedness is described by the so-called *monodromy matrices.*

Let us briefly recall the definition of the monodromy matrices of the Fuchsian system (1.1). We have to fix a basis $\gamma_1, \gamma_2, \gamma_3$ of loops in the fundamental group, with base point at $\hat{x} \to \infty$, of the punctured Riemann sphere

$$\pi_1\left(\overline{\mathbb{C}}\backslash\{u_1, u_2, u_3, \infty\}, \hat{x}\right),$$

and to fix a fundamental matrix for the system (1.1). To fix the basis of the loops, we first perform some cuts between the singularities, namely we cut three parallel segments π_i between the point at infinity and each u_i; the segments π_i are ordered according to the order of the points u_1, u_2, u_3, as in the figure 1. Take γ_i to be a simple closed curve starting and finishing at infinity, going around u_i in positive direction (γ_i is oriented counter-clockwise, u_i lies inside, while the other singular points lie outside) and not crossing the cuts π_i. Near ∞ we take every loop γ_i close to the cut π_i as in the figure 1.

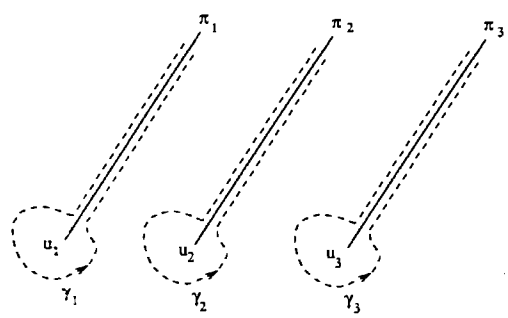

Fig.1. The cuts π_i between the singularities u_i and the oriented loops γ_i.

Now, we fix the fundamental matrix $Y_\infty(z)$ of the system (1.1) in such a way that

$$Y_\infty = \left(1 + O(\frac{1}{z})\right)\begin{pmatrix} z^{-\mu} & 0 \\ 0 & z^{\mu} \end{pmatrix}, \quad \text{as} \quad z \to \infty, \tag{1.3}$$

where $z^\mu := e^{\mu \log z}$, with the choice of the principal branch of the logarithm, with the branch-cut along the common direction of π_1, π_2, π_3. Such a fundamental matrix $Y_\infty(z)$ exists, due to the non-resonance condition, and it is uniquely determined. It can be analytically continued to a multivalued analytic function on the universal covering of $\overline{\mathbb{C}}\backslash\{u_1, u_2, u_3, \infty\}$. For any element $\gamma \in \pi_1\left(\overline{\mathbb{C}}\backslash\{u_1, u_2, u_3, \infty\}, \hat{x}\right)$ we denote the result of the analytic continuation of $Y_\infty(z)$ along the loop γ by $\gamma[Y_\infty(z)]$. Then, $\gamma[Y_\infty(z)]$ and $Y_\infty(z)$ are two fundamental matrices in the neighborhood of infinity, so they must be related by the following relation:

$$\gamma[Y_\infty(z)] = Y_\infty(z) M_\gamma$$

for some constant invertible 2×2 matrix M_γ depending only on the homotopy class of γ. Particularly, the matrix $M_\infty := M_{\gamma_\infty}$ is given by:

$$M_\infty = \begin{pmatrix} \exp(2i\pi\mu) & 0 \\ 0 & \exp(-2i\pi\mu) \end{pmatrix}, \tag{1.4}$$

here γ_∞ is a loop around infinity in the clock-wise direction.

The *monodromy representation* is an anti-homomorphism:

$$\begin{array}{rcl} \pi_1\left(\overline{\mathbb{C}}\backslash\{u_1,u_2,u_3,\infty\},\hat{x}\right) & \to & SL_2(\mathbb{C}) \\ \gamma & \mapsto & M_\gamma \end{array}$$

$$M_{\gamma\tilde{\gamma}} = M_{\tilde{\gamma}} M_\gamma. \tag{1.5}$$

The matrices $M_i := M_{\gamma_i}$, where the γ_i, $i = 1,2,3$, are the generators of the fundamental group, are called *the monodromy matrices* of the Fuchsian system (1.1), and generate the *monodromy group of the system.* Moreover, due to the fact that, in our particular case, the $\mathcal{A}_i$ are nilpotent, they satisfy the following relations:

$$\det(M_i) = 1, \quad \mathrm{Tr}(M_i) = 2, \quad \text{for} \quad i = 1,2,3, \tag{1.6}$$

with $M_i = \mathbf{1}$ if and only if $\mathcal{A}_i = 0$. Since the loop $(\gamma_1\gamma_2\gamma_3)^{-1}$ is homo-topic to γ_∞, the following relation holds:

$$M_\infty M_3 M_2 M_1 = \mathbf{1}. \tag{1.7}$$

A simultaneous conjugation $\mathcal{A}_i \mapsto D^{-1}\mathcal{A}_i D$, $i = 1,2,3$ of the coefficients $\mathcal{A}_i$ of the Fuchsian system (1.1) by a diagonal matrix D, implies the same conjugation of the monodromy matrices $M_\gamma \mapsto D^{-1}M_\gamma D$, for any $\gamma \in \pi_1\left(\overline{\mathbb{C}}\backslash\{u_1,u_2,u_3,\infty\},\hat{x}\right)$.

We now recall the definition of the *connection matrices.* Let us assume that $M_i \neq \mathbf{1}$, or equivalently $\mathcal{A}_i \neq 0$, for every $i = 1,2,3$. We choose the fundamental matrices $Y_i(z)$ of the system (1.1), such that:

$$Y_i = G_i\left(\mathbf{1} + \mathcal{O}(z-u_i)\right)(z-u_i)^J, \quad \text{as} \quad z \to u_i, \tag{1.8}$$

where J is the Jordan normal form of $\mathcal{A}_i$, namely $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the invertible matrix G_i is defined by $\mathcal{A}_i = G_i J G_i^{-1}$, and the choice of the branch of $\log(z-u_i)$ needed in the definition of

$$(z-u_i)^J = \begin{pmatrix} 1 & \log(z-u_i) \\ 0 & 1 \end{pmatrix}$$

is similar to the one above. The fundamental matrix $Y_i(z)$ is uniquely determined up to the ambiguity:

$$Y_i(z) \mapsto Y_i(z)R_i,$$

where R_i is any matrix commuting with J.

Continuing, along, say, the right-hand-side of the cut π_i, the solution Y_∞ to a neighborhood of u_i, we obtain another fundamental matrix around u_i, that must be related to $Y_i(z)$ by:

$$Y_\infty(z) = Y_i(z)C_i, \tag{1.9}$$

for some invertible matrix C_i. The matrices C_1, C_2, C_3 are called *connection matrices,* and are related to the monodromy matrices as follows:

$$M_i = C_i^{-1}\exp(2\pi i J)C_i, \qquad i = 1,2,3. \tag{1.10}$$

Lemma 1.1. *Given three matrices M_1, M_2, M_3, $M_i \neq \mathbf{1}$ for every $i = 1,2,3$, satisfying the relation (1.6) and (1.7), then*

i) there exist three matrices C_1, C_2, C_3 satisfying the (1.10). Moreover they are uniquely determined by the matrices M_1, M_2, M_3, up to the ambiguity $C_i \mapsto R_i^{-1}C_i$.

ii) If the matrices M_1, M_2, M_3, are the monodromy matrices of a Fuchsian system of the form (1.1), then any triple satisfying (1.10) can be realized as the connection matrices of the Fuchsian system itself.

Proof of the lemma 1.1. i) By the (1.6), the monodromy matrices have all the eigenvalues equal to one; moreover they can be reduced to the Jordan normal form because $M_i \neq \mathbf{1}$; namely there exists a matrix $\tilde{C}_i$ such that:

$$M_i = \tilde{C}_i^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\tilde{C}_i;$$

taking

$$C_i = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} \tilde{C}_i$$

we obtain the needed matrix. Two such matrices C_i and C_i' give the same matrix M_i if and only if $C_i^{-1}C_i'$ commutes with J, namely if and only if they are related by $C_i = R_i^{-1}C_i'$. ii) Let us now assume that C_1', C_2', C_3' are the connection matrices of a Fuchsian system of the form (1.1), with monodromy matrices M_1, M_2, M_3; id est $Y_\infty(z) = Y_i'(z)C_i'$, $i = 1, 2, 3$, for some choice of the solutions Y_1', Y_2' and Y_3' of the form (1.8). We have

$$M_i = (C_i')^{-1}\exp(2\pi i J)C_i' = C_i^{-1}\exp(2\pi i J)C_i, \quad i = 1, 2, 3.$$

So the matrices $R_i = C_i'C_i^{-1}$ must commute with J and, with respect to the new solutions $Y_i(z) = Y_i'(z)R_i$, the connection matrices are C_1, C_2 and C_3. QDE

Now we state the result about the correspondence between monodromy data and coefficients of the Fuchsian system, for a given set of poles:

Lemma 1.2. *Two Fuchsian systems (1.1) with the same poles u_1, u_2 and u_3, and the same value of μ, coincide if and only if they have the same monodromy matrices M_1, M_2, M_3, with respect to the same basis of the loops γ_1, γ_2 and γ_3.*

Proof of the Lemma 1.2. Let $Y_\infty^{(1)}(z)$ and $Y_\infty^{(2)}(z)$ be the fundamental matrices of the form (1.3) of the two Fuchsian systems. Let us consider the following matrix:

$$Y(z) := Y_\infty^{(2)}(z)Y_\infty^{(1)}(z)^{-1}.$$

$Y(z)$ is an analytic function around infinity:

$$Y(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty.$$

Since the monodromy matrices coincide, $Y(z)$ is a single valued function on $\overline{\mathbb{C}}\backslash\{u_1, u_2, u_3\}$. Let us prove that $Y(z)$ is analytic also at the points u_i. Due to the lemma 1.1, we can choose the fundamental matrices $Y_i^{(1)}(z)$ and $Y_i^{(2)}(z)$ in such a way that

$$Y_\infty^{(1),(2)}(z) = Y_i^{(1),(2)}(z)C_i \quad i = 1, 2, 3.$$

with the same connection matrices C_i. Then near the point u_i,

$$Y(z) = G_i^{(2)}\left(\mathbf{1} + \mathcal{O}(z - u_i)\right)\left[G_i^{(1)}\left(\mathbf{1} + \mathcal{O}(z - u_i)\right)\right]^{-1}.$$

This proves that $Y(z)$ is an analytic function on all $\overline{\mathbb{C}}$ and then, by the Liouville theorem

$$Y(z) = \mathbf{1},$$

and the two Fuchsian systems must coincide.

Corollary 1.1. *Two Fuchsian systems (1.1) with the same poles u_1, u_2 and u_3, and the same value of μ, are conjugated*

$$\mathcal{A}_i^{(1)} = D^{-1}\mathcal{A}_i^{(2)}D, \qquad i = 1, 2, 3,$$

with a diagonal matrix D, if and only if their monodromy matrices $M_i^{(1)}$ and $M_i^{(2)}$, with respect to the same basis of the loops γ_1, γ_2 and γ_3, are conjugated:

$$M_i^{(1)} = D^{-1}M_i^{(2)}D, \qquad i = 1, 2, 3.$$

The isomonodromy deformations of the Fuchsian system (1.1) and the Painlevé equation PVIμ. We now want to deform the poles of the Fuchsian system keeping the monodromy fixed. The theory of these deformations is described by the following two results:

Theorem 1.1. *Let M_1, M_2, M_3 be the monodromy matrices of the Fuchsian system:*

$$\frac{\mathrm{d}}{\mathrm{d}z}Y^0 = \left(\frac{A_1^0}{z-u_1^0} + \frac{A_2^0}{z-u_2^0} + \frac{A_3^0}{z-u_3^0}\right)Y^0, \tag{1.11}$$

of the above form (1.2), with pair-wise distinct poles, and with respect to some basis $\gamma_1, \gamma_2, \gamma_3$ of the loops in $\pi_1\left(\overline{\mathbb{C}}\backslash\{u_1^0, u_2^0, u_3^0, \infty\}, \hat{x}\right)$. Then there exists a neighborhood $U \subset \mathbb{C}^3$ of the point $u^0 = (u_1^0, u_2^0, u_3^0)$ such that, for any $u = (u_1, u_2, u_3) \in U$, there exists a unique triple $\mathcal{A}_1(u)$, $\mathcal{A}_2(u)$, $\mathcal{A}_3(u)$ of analytic matrix valued functions such that:

$$\mathcal{A}_i(u^0) = \mathcal{A}_i^0, \quad i = 1, 2, 3,$$

and the monodromy matrices of the Fuchsian system

$$\frac{\mathrm{d}}{\mathrm{d}z}Y = A(z;u)Y = \left(\frac{\mathcal{A}_1(u)}{z-u_1} + \frac{\mathcal{A}_2(u)}{z-u_2} + \frac{\mathcal{A}_3(u)}{z-u_3}\right)Y, \tag{1.12}$$

with respect to the same basis[1] $\gamma_1, \gamma_2, \gamma_3$ of the loops, coincide with the given M_1, M_2, M_3. The matrices $\mathcal{A}_i(u)$ are the solutions of the Cauchy problem with the initial data $\mathcal{A}_i^0$ for the following Schlesinger equations:

$$\frac{\partial}{\partial u_j}\mathcal{A}_i = \frac{[\mathcal{A}_i, \mathcal{A}_j]}{u_i - u_j}, \quad \frac{\partial}{\partial u_i}\mathcal{A}_i = -\sum_{j\neq i}\frac{[\mathcal{A}_i, \mathcal{A}_j]}{u_i - u_j}. \tag{1.13}$$

The solution $Y_\infty^0(z)$ of (1.11), with the form (1.3), can be, uniquely continued, for $z \notin U$, to an analytic function

$$Y_\infty(z, u), \quad u \in U,$$

such that

$$Y_\infty(z, u^0) = Y_\infty^0(z).$$

This continuation is the local solution of the Cauchy problem with the initial data Y_∞^0 for the following system, that is compatible to the system (1.12):

$$\frac{\partial}{\partial u_i}Y = -\frac{\mathcal{A}_i(u)}{z-u_i}Y.$$

Moreover the functions $\mathcal{A}_i(u)$ and $Y_\infty(z, u)$ can be continued analytically to global meromorphic functions on the universal coverings of

$$\mathbb{C}^3\backslash\{diags\} := \left\{(u_1, u_2, u_3) \in \mathbb{C}^3 \,|\, u_i \neq u_j \text{ for } i \neq j\right\},$$

and

$$\left\{(z, u_1, u_2, u_3) \in \mathbb{C}^4 \,|\, u_i \neq u_j \text{ for } i \neq j \text{ and } z \neq u_i,\ i = 1, 2, 3\right\},$$

respectively.

The proof of this theorem can be found, for example, in [Mal], [Miwa], [Sib]. We recall the theorem of solvability of the inverse problem of the monodromy (see [Dek]):

Theorem 1.2. *Given three arbitrary matrices, satisfying (1.6) and (1.7), with M_∞ of the form (1.4), and given a point $u^0 = (u_1^0, u_2^0, u_3^0) \in \mathbb{C}^3\backslash\{diags\}$, for any neighborhood U of u^0, there exist $(u_1, u_2, u_3) \in U$ and a Fuchsian system of the form (1.1), with the given monodromy matrices, the given μ and with poles in u_1, u_2, u_3.*

Remark. Fuchsian systems of the form (1.1), with coefficients $\mathcal{A}_i$ satisfying (1.2), depend on four parameters, one of them being μ. The triples of the monodromy matrices satisfying (1.6) and (1.7) depend on four

[1] *Observe that the basis $\gamma_1, \gamma_2, \gamma_3$ of $\pi_1\left(\overline{\mathbb{C}}\backslash\{u_1, u_2, u_3, \infty\}, \hat{x}\right)$ varies continuously with small variations of u_1, u_2, u_3. This new basis is homo-topic to the initial one, so we can identify them.*

parameters too. Loosely speaking, the theorems N1 and N2 claim that, not only the monodromy matrices are first integrals for the equations of isomonodromy deformation (1.13), but they provide a full system of first integrals for such equations.

Remark. We observe that the isomonodromy deformations equations preserve the connection matrices C_i too. This follows from the lemma 1.1.

Reduction to the PVIμ equation. Let us now explain how to reduce the Schlesinger equations (1.13) to the PVIμ equation. The Schlesinger equations are invariant with respect to the gauge transformations of the form:

$$\mathcal{A}_i \mapsto D^{-1}\mathcal{A}_i D, \quad i = 1,2,3, \quad \text{for any } D \text{ diagonal matrix.}$$

First of all we have to factor out such gauge transformations; to this aim, we introduce two coordinates (p,q) on the quotient of the space of the matrices satisfying (1.2) with respect to the equivalence relation

$$\mathcal{A}_i \sim D^{-1}\mathcal{A}_i D, \quad i = 1,2,3, \quad \text{for any } D \text{ diagonal matrix.} \tag{1.14}$$

The coordinates (p,q) are defined as follows: q is the root of the following linear equation:

$$[\mathcal{A}(q;u_1,u_2,u_3)]_{12} = 0,$$

and p is giver by:

$$p = [\mathcal{A}(q;u_1,u_2,u_3)]_{11},$$

where $\mathcal{A}(z;u_1,u_2,u_3)$ is given in (1.12). The matrices $\mathcal{A}_i$ are expressed rationally in terms of the coordinates (p,q) and an auxiliary coordinate k, coming from the gauge freedom (1.14):

$$\begin{aligned}
(\mathcal{A}_i)_{11} &= -(\mathcal{A}_i)_{22} = \frac{q-u_i}{2\mu P'(u_i)}\left[P(q)p^2 + 2\mu\frac{P(q)}{q-u_i} + \mu^2(q+2u_i-\sum_j u_j)\right],\\
(\mathcal{A}_i)_{12} &= -\mu k\frac{q-u_i}{P'(u_i)},\\
(\mathcal{A}_i)_{21} &= k^{-1}\frac{q-u_i}{4\mu^3 P'(u_i)}\left[P(q)p^2 + 2\mu\frac{P(q)}{q-u_i} + \mu^2(q+2u_i-\sum_j u_j)\right]^2,
\end{aligned} \tag{1.15}$$

for $i = 1,2,3$, where $P(z) = (z-u_1)(z-u_2)(z-u_3)$ and $P'(z) = \frac{dP}{dz}$. The Schlesinger equations in these coordinates reduce to:

$$\begin{aligned}
\frac{\partial q}{\partial u_i} &= \frac{P(q)}{P'(u_i)}\left[2p + \frac{1}{q-u_i}\right]\\
\frac{\partial p}{\partial u_i} &= -\frac{P'(q)p^2 + (2q+u_i-\sum_j u_j)p + \mu(1-\mu)}{P'(u_i)},
\end{aligned} \tag{1.16}$$

for $i = 1,2,3$. The system of the *reduced Schlesinger equations* (1.16) is invariant under the transformations of the form

$$u_i \mapsto au_i + b, \qquad q \mapsto aq + b, \qquad p \mapsto \frac{p}{a}.$$

We introduce the following new invariant variables:

$$\begin{aligned}
x &= \frac{u_2-u_1}{u_3-u_1},\\
y &= \frac{q-u_1}{u_3-u_1};
\end{aligned} \tag{1.17}$$

the system (1.16), expressed in the these new variables, reduces to the PVIμ equation for $y(x)$.

Remark. The system (1.16) admits the following *degenerate solutions* (see [Ok1] and [Wat]):

$$q \equiv u_i \quad \text{for some } i,$$

and p, in the variable x, can be reduced to a Gauss hypergeometric function. Moreover the monodromy group of the system (1.1) reduces to the monodromy group of the Gauss hypergeometric equation, namely the following lemma holds true:

Lemma 1.3. *The solutions of the full Schlesinger equations, corresponding to the solution $q \equiv u_i$, for some i, have the form:*

$$A_i(u) \equiv 0, \quad \text{and for } j \neq i \quad A_j(u) = D(u)^{-1} A_j^0 D(u),$$

where $D(u)$ is a diagonal matrix depending on u, and A_j^0 is a constant matrix. The monodromy matrix M_i of the corresponding Fuchsian system, turns out to be the identity. Conversely, if one of the monodromy matrices M_i is the identity, $M_i = \mathbf{1}$, then the solution of (1.16) is degenerate.

Proof of the lemma 1.3. The matrix A_i, for $q \equiv u_i$, is identically 0, thanks to (1.15). Having $A_i \equiv 0$, M_i is $\mathbf{1}$. Conversely, if $M_i = \mathbf{1}$, then $A_i \equiv 0$. Solving the Schlesinger equations (1.12), we obtain $q \equiv u_i$, and p reducible to a Gauss hypergeometric function. QDE

The degenerate solutions do not give any solution of the PVIμ equation. All the other solutions do, via (1.17). Conversely, starting from any solution $y(x)$ of PVIμ, we arrive at the solution:

$$q = (u_3 - u_1) y\left(\frac{u_2 - u_1}{u_3 - u_1}\right) + u_1$$
$$p = \frac{P'(u_2)}{2P(q)} y'\left(\frac{u_2 - u_1}{u_3 - u_1}\right) - \frac{1}{2}\frac{1}{q - u_1}$$

of the reduced Schlesinger equations (1.16). To obtain a solution of the full Schlesinger equations, the function k must be given by a quadrature:

$$\frac{\partial k}{\partial u_i} = (2\mu - 1)\frac{q - u_i}{P'(u_i)}.$$

We conclude this section summarizing all the above results in the following:

Theorem 1.3. *The branches of solutions of the PVIμ equation near a given point $x_0 \in \overline{\mathbb{C}}\backslash\{0, 1, \infty\}$, are in one-to-one correspondence with the triples of the monodromy matrices M_1, M_2, M_3 satisfying (1.6) and (1.7), none of them being equal to $\mathbf{1}$, considered modulo diagonal conjugations.*

1.2. The structure of the analytic continuation.

We parameterised branches of the solutions of PVIμ by triples of monodromy matrices. Now we show how do these parameters change with a change of the branch in the process of analytic continuation of the solutions along a path in $\overline{\mathbb{C}}\backslash\{0, 1, \infty\}$. Recall that, as it follows from the theorem 1.1, the solutions of PVIμ, defined in a neighborhood of a given point $x_0 \in \overline{\mathbb{C}}\backslash\{0, 1, \infty\}$, can be analytically continued to a meromorphic function on the universal covering of $\overline{\mathbb{C}}\backslash\{0, 1, \infty\}$. Observe that the fundamental group $\pi_1\left(\overline{\mathbb{C}}\backslash\{0, 1, \infty\}\right)$ is non-abelian. As a consequence, the global structure of the analytic continuation of the solutions of PVI is more involved than that of the other Painlevé equations. In fact for PI,..., PV the solutions have only two critical singularities and the corresponding fundamental group in abelian.

As a first step we introduce a parameterisation of the monodromy matrices.

The parameterisation of the monodromy data Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be three linear operators $\mathcal{M}_i : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying (1.6). We introduce, for them, a parameterisation which will be useful for studying the analytic continuation of the solutions of the PVIμ equation.

Lemma 1.4. *If $\mathcal{M}_1$, $\mathcal{M}_2$ are such that*

$$\mathrm{Tr}(\mathcal{M}_1 \mathcal{M}_2) \neq 2,$$

then there exists a basis in $\mathbb{C}^2$ such that, in this basis, the matrices of $\mathcal{M}_1$, $\mathcal{M}_2$ have the form:

$$M_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \tag{1.18}$$

where $x_1 = \sqrt{2 - \mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2)}$; *when* $\mathcal{M}_1$, $\mathcal{M}_2$ *are such that* $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) = 2$, *they have a common eigenvector, and then there exists a basis in* $\mathbb{C}^2$ *such that, in this basis, the matrices* M_1, M_2 *are both upper-triangular.*

Proof of the lemma 1.4. Due to the (1.6), there exist two vectors e_1 and e_2 such that

$$\mathcal{M}_1 e_1 = e_1, \qquad \mathcal{M}_2 e_2 = e_2.$$

We now prove that these two vectors are linearly dependent if and only if $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) = 2$; in fact if the two vectors are linearly dependent, then we can find a linear independent vector e_2' such that, in the basis (e_1, e_2') the matrices of M_1, M_2 have the form:

$$M_1 = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & \lambda_2 \\ 0 & 1 \end{pmatrix},$$

then $\mathrm{Tr}(M_1M_2) = 2$; conversely, in the basis (e_1, e_2') the matrix M_1 has the form $M_1 = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix}$, and requiring that

$$\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) = 2, \quad \mathrm{eigenv}(M_2) = 1,$$

also the matrix M_2 must have the above form $M_2 = \begin{pmatrix} 1 & \lambda_2 \\ 0 & 1 \end{pmatrix}$; then, the two vectors e_1 and e_2 are linearly dependent. As a consequence, if $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) \neq 2$, the two vectors e_1 and e_2 are linearly independent, and in the basis (e_1, e_2) the matrices of M_1, M_2 have the form:

$$M_1 = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 \\ \lambda_2 & 1 \end{pmatrix},$$

with $\mathrm{Tr}(M_1M_2) = 2 + \lambda_1\lambda_2$; rescaling the basic vectors (e_1, e_2), we obtain the (1.18). QDE

Lemma 1.5. *Let* $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ *satisfy also the condition (1.7), with* $\mathcal{M}_\infty$ *given by (1.4), with* $2\mu \notin \mathbb{Z}$. *Then the following statements are true:*

i) If two of the following numbers

$$\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2), \quad \mathrm{Tr}(\mathcal{M}_1\mathcal{M}_3), \quad \mathrm{Tr}(\mathcal{M}_3\mathcal{M}_2)$$

are equal to 2, then one of the matrices of M_i *is equal to one.*

ii) If $\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) \neq 2$, *then there exists a basis in* $\mathbb{C}^2$ *such that, in this basis, the matrices* M_1, M_2 *and* M_3 *have the form*

$$M_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 + \frac{x_2x_3}{x_1} & -\frac{x_2^2}{x_1} \\ \frac{x_3^2}{x_1} & 1 - \frac{x_2x_3}{x_1} \end{pmatrix}, \tag{1.19}$$

where

$$\mathrm{Tr}(M_1M_2) = 2 - x_1^2, \quad \mathrm{Tr}(M_3M_2) = 2 - x_2^2, \quad \mathrm{Tr}(M_1M_3) = 2 - x_3^2,$$

and

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 4\sin^2 \pi\mu. \tag{1.20}$$

iii) If two triples of matrices M_1, M_2, M_3 *and* M_1', M_2', M_3', *satisfying (1.7), with none of them equal to* **1**, *have the form (1.19) with parameters* (x_1, x_2, x_3) *and* (x_1', x_2', x_3') *respectively, then these triples are conjugated*

$$M_i = T^{-1}M_i'T$$

with some invertible matrix T, *if and only if the triple* (x_1', x_2', x_3') *is equal to the triple* (x_1, x_2, x_3), *up to the change of the sign of two of the coordinates.*

Proof of the lemma 1.5.

i) Let us assume that

$$\mathrm{Tr}(\mathcal{M}_1\mathcal{M}_2) = 2, \quad \mathrm{Tr}(\mathcal{M}_1\mathcal{M}_3) = 2;$$

let e_1 and e_3 be the common eigenvectors of $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_1$, $\mathcal{M}_3$ respectively, according to the lemma 1.4. If $\mathcal{M}_i \neq \mathbf{1}$, then the eigenvectors e_1 and e_3 coincide. In fact, for any $\mathcal{M}_i \neq \mathbf{1}$ there exists a unique eigenvector; being e_1 and e_3 eigenvectors of the same operator $\mathcal{M}_1 \neq \mathbf{1}$, they coincide. Then we can find a linear independent vector e_2' such that, in the basis (e_1, e_2') the matrices of M_1, M_2, M_3 all have the form:

$$M_i = \begin{pmatrix} 1 & \lambda_i \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, 3.$$

Then

$$\mathrm{Tr}(M_3 M_2 M_1) = 2.$$

This contradicts the assumption $2\mu \notin \mathbb{Z}$.

ii) Let us choose the basis such that, according to the lemma 1.4. the matrices of M_1, M_2 have the form (1.18). Solving the equations:

$$\mathrm{Tr}(M_3 M_2) = 2 - x_2^2, \quad \mathrm{Tr}(M_2 M_3) = 2 - x_3^2,$$

we arrive at the formula (1.19). The (1.20) is obtained by straightforward computations from:

$$\mathrm{Tr}(M_3 M_2 M_1) = \cos 2\pi\mu.$$

iii) The two triples of matrices M_1, M_2, M_3 and M_1', M_2', M_3' are conjugated

$$M_i = T^{-1} M_i' T$$

with some invertible matrix T, if and only if they are the matrices of the same operators $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, written in different bases. Since the traces do not depend on the choice of the basis, then

$$x_i^2 = x_i'^2, \quad i = 1, 2, 3.$$

According to the proof of the lemma 1.4, the basis (e_1, e_2) is uniquely determined up to changes of sign. A change of sign $e_1 \mapsto -e_1$ correspond to the change of sign $x_1 \mapsto -x_1$; then the form of the matrix M_3 is preserved if and only if we change one of the signs of x_2 or x_3. QDE

Remark. The matrices (1.19) have a simple geometrical meaning. Let us consider the three-dimensional linear space with a basis (e_1, e_2, e_3) and with a skew-symmetric bilinear form $\{\cdot, \cdot\}$ such that

$$\{e_1, e_2\} = x_1, \quad \{e_1, e_3\} = x_3, \quad \{e_2, e_3\} = x_2.$$

Let us consider the reflections R_1, R_2, R_3 in this space, with respect to the hyper-planes skew-orthogonal to the basic vectors:

$$R_i(x) = x - \{e_i, x\} e_i, \qquad i = 1, 2, 3.$$

The reflections have a one-dimensional invariant subspace, namely the kernel of the bilinear form. The matrices of the reflections acting on the quotient are the (1.19).

Definition: a triple (x_1, x_2, x_3) is called *admissible* if it has at most one coordinate equal to zero. Two such triples are called *equivalent* if they are equal up to the change of two signs of the coordinates.

Observe that for an admissible triple (x_1, x_2, x_3), none of the matrices (1.19) is equal to the identity. So the admissible triples correspond to the non-degenerate solutions of the reduced Schlesinger equations (1.16). Moreover two equivalent triples generate the same solution. We can summarize the above results in the following

Theorem 1.4. *The branches of solutions of the PVIμ equation, near a given point $x_0 \in \overline{\mathbb{C}}\backslash\{0,1,\infty\}$, are in one-to-one correspondence with the equivalence classes of the admissible triples satisfying (1.20).*

Proof of the Theorem 1.4. Starting from a solution of PVIμ we obtain the monodromy matrices satisfying (1.6). None of them is equal to the identity. So the canonical form (1.19) of M_1, M_2, M_3 is determined uniquely up to a choice of the admissible triple (x_1, x_2, x_3) within the equivalence class. Conversely, given an admissible triple (x_1, x_2, x_3) satisfying (1.20), we obtain the matrices M_1, M_2, M_3 of the form (1.19). The matrix $M_3M_2M_1$ is diagonalizable with the eigenvalues $\exp(\pm 2\pi i\mu)$ (here we use the non-resonance condition $2\mu \notin \mathbb{Z}$). Reducing this matrix to the diagonal form

$$M_1M_2M_3 = T^{-1}\begin{pmatrix} \exp(2\pi i\mu) & 0 \\ 0 & \exp(-2\pi i\mu) \end{pmatrix} T$$

we obtain the monodromy matrices TM_iT^{-1} satisfying (1.6) and thus specifying a branch of the solution of PVIμ.

Monodromy data and symmetries of PVIμ. The Painlevé VI equation possesses a rich family of symmetries, i. e. transformations of the dependent and independent variables (y, x), and also of the parameters, that preserve the shape of the equation. The theory of these symmetries, and its applications to the construction of particular solutions, was developed in [Ok]. Here we enlist the symmetries which preserve our PVIμ and compute their action on the monodromy data.

First of all the permutations of the poles u_1, u_2, u_3 generate the action of the symmetric group S_3 on the solutions $y(x)$. In particular the involution

$$i_1 : u_2 \leftrightarrow u_3,$$

produces the transformation

$$x \mapsto \frac{1}{x}, \qquad y \mapsto \frac{1}{y},$$

and

$$i_2 : u_1 \leftrightarrow u_3,$$

produces the transformation

$$x \mapsto 1 - x, \qquad y \mapsto 1 - y.$$

Both these transformations clearly preserve the equation PVIμ.

Let us compute the action of these symmetries on the monodromy data. The only thing that changes is the basis in the fundamental group $\pi_1(\overline{\mathbb{C}}\backslash\{u_1, u_2, u_3, \infty\})$. In fact, the cuts π_1, π_2, π_3 along which we take our basis $\gamma_1, \gamma_2, \gamma_3$, are ordered according to the order of the poles. Applying the transformation i_1 we then arrive to the new basis $\gamma_1', \gamma_2', \gamma_3'$ shown in figure 2.

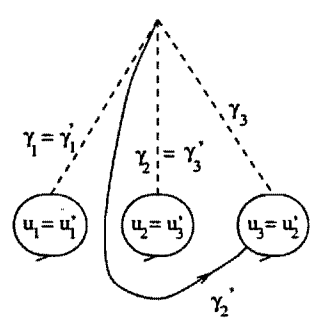

Fig.2. The new basis $\gamma_1', \gamma_2', \gamma_3'$ obtained by the action of i_1.

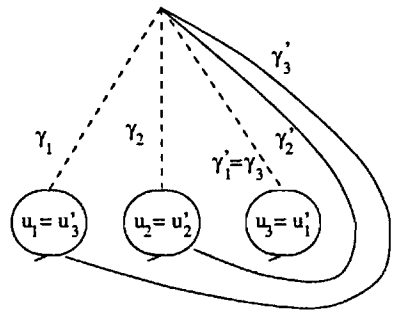

Fig.3. The new basis $\gamma_1', \gamma_2', \gamma_3'$ obtained by the action of i_2.

This new basis has the following form:

$$\gamma_1' = \gamma_1, \quad \gamma_2' = \gamma_2\gamma_3\gamma_2^{-1}, \quad \gamma_3' = \gamma_2.$$

As a consequence the new monodromy matrices are:

$$M_1' = M_1, \quad M_2' = M_2^{-1}M_3M_2, \quad M_3' = M_2.$$

For the second transformation i_2, the basis of the new loops is shown in figure 3.
It has the following form:

$$\gamma_1' = \gamma_3, \quad \gamma_2' = \gamma_3^{-1}\gamma_2\gamma_3, \quad \gamma_3' = \gamma_3^{-1}\gamma_2^{-1}\gamma_1\gamma_2\gamma_3.$$

The new monodromy matrices are:

$$M_1' = M_3, \quad M_2' = M_3M_2^{-1}M_3, \quad M_3' = M_3M_2M_1M_2^{-1}M_3^{-1}.$$

Lemma 1.6. *In the coordinates (x_1, x_2, x_3) on the space of the monodromy matrices, the action of the symmetries i_1, i_2 is given by the formulae:*

$$i_1 : (x_1, x_2, x_3) \mapsto (x_3 - x_1x_2, -x_2, x_1), \qquad i_2 : (x_1, x_2, x_3) \mapsto (-x_2, -x_1, x_1x_2 - x_3).$$

The proof is straightforward.

There are more complicated symmetries which change the value of the parameter μ. The first one comes from the following simultaneous conjugation of the coefficients of the Fuchsian system:

$$A_i \to \Sigma A_i \Sigma,$$

where

$$\Sigma = \Sigma^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Indeed,

$$\Sigma A_\infty \Sigma = -A_\infty.$$

Using the parameterisation (1.15) of the matrices A_1, A_2, A_3 by the coordinates (p, q), we arrive to the following

Lemma 1.7. *The formula:*

$$\tilde{y} = y\frac{p_0(y')^2 + p_1y' + p_2}{q_0(y')^4 + q_1(y')^3 + q_2(y')^2 + q_3y' + q_4}, \tag{1.21}$$

where:

$$
\begin{aligned}
p_0 &= x^2(x-1)^2,\\
p_1 &= 2x(x-1)(y-1)[2\mu(y-x)-y]\\
p_2 &= y(y-1)[y(y-1)-4\mu(y-1)(y-x)+4\mu^2(y-x)(y-x-1)]\\
q_0 &= x^4(x-1)^4\\
q_1 &= -4x^3(x-1)^3y(y-1)\\
q_2 &= 2x^2(x-1)^2y(y-1)[3y(y-1)+4\mu^2(y-x)(1+x-3y)]\\
q_3 &= 4x(x-1)y^2(y-1)^2[-y(y-1)-16\mu^3(y-x)^2+4\mu^2(y-x)(3y-x-1)]\\
q_4 &= y^2(y-1)^2\{y^2(y-1)^2+64\mu^3y(y-1)(y-x)^2-8\mu^2y(y-1)(y-x)(3y-x-1)+\\
&\quad +16\mu^4(y-x)^2[(x-1)^2+y(2+2x-3y)]\}
\end{aligned}
\qquad , \qquad (1.22)
$$

transform solutions of PVIμ to solutions of PVI$(-\mu)$. The class of equivalence of the monodromy data (x_1, x_2, x_3) does not change under such symmetry.

Proof of the lemma 1.7. The new monodromy matrices M_1', M_2', M_3' have the form

$$M_i' = \Sigma M_i \Sigma, \quad i = 1, 2, 3.$$

Then, the canonical form (1.19) of the monodromy operators does not change. QDE

The last symmetry is the so-called Schlesinger transformation (see [JMU]). It comes from a gauge transform

$$Y(z) \mapsto G(z)Y(z),$$

of the Fuchsian system (1.1), with an appropriate gauge $G(z)$. Omitting the details we formulate the following

Lemma 1.8. *The formula:*

$$\tilde{y} = y\frac{p_0(y')^4+p_1(y')^3+p_2(y')^2+p_3y'+p_4}{q_0(y')^4+q_1(y')^3+q_2(y')^2+q_3y'+q_4}, \tag{1.23}$$

where the p_i are

$$
\begin{aligned}
p_0 &= x^4(x-1)^4y,\\
p_1 &= 4x^3(x-1)^3y(y-1)[2\mu(y-x)-y],\\
p_2 &= 2x^2(x-1)^2y(y-1)[-12\mu\, y(y-1)(y-x)+3y^2(y-1)+\\
&\quad +4\mu(y-x)(3y^2-3xy-3y+2x)],\\
p_3 &= 4x(x-1)y^2(y-1)^2[y-2\mu(y-x)][-y(y-1)-4\mu^2(y-x)(y-x-1)+\\
&\quad +4\mu(y-1)(y-x)],\\
p_4 &= y^3(y-1)^2[-4\mu(y-x)(y-1)+y(y-1)+4\mu^2(y-x-1)(y-x)]^2,
\end{aligned}
\tag{1.24}
$$

and the q_i are the same as in (1.23), transforms a solution of PVIμ to a solution of PVI$(1+\mu)$. The class of equivalence of the monodromy data (x_1, x_2, x_3) does not change under such symmetry.

Remark. One can show that the above symmetries, and their superpositions, exhaust all the birational transformations preserving our one-parameter family of PVI equations. We will not do it here (see [Ok]). It is important, however, that these symmetries preserve the class of algebraic solutions of PVIμ. We will classify all the algebraic solutions, modulo the above symmetries.

Remark. It is not difficult to show that the denominator of the formulae (1.21) and (1.23) does not vanish identically for any solution of PVIμ, with $2\mu \notin \mathbb{Z}$. Indeed, eliminating y_{xx} and y_x form the system

$$
\begin{aligned}
y_{xx} =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y_x^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y_x\\
&+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2+\frac{x(x-1)}{(y-x)^2}\right],
\end{aligned}
$$

$$Q(y_x, y, x, \mu) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}x} Q(y_x, y, x, \mu) = 0,$$

where Q is the denominator, the resultant equation

$$(2\mu+1)^4 \mu^{16} \left[x(x-1)^2\right]^4 \left[y(y-1)(y-x)\right]^4$$

never vanishes.

The analytic continuation of the solutions of PVIμ and the braid group B_3. According to the theorem 1.1 any solution of the Schlesinger equations can be continued analytically from a point (u_1^0, u_2^0, u_3^0) to another point (u_1^1, u_2^1, u_3^1) along a path

$$(u_1(t), u_2(t), u_3(t)) \in \mathbb{C}^3 \backslash \{diags\}, \qquad 0 \le t \le 1,$$

with

$$u_i(0) = u_i^0, \quad \text{and} \quad u_i(1) = u_i^1,$$

provided that the end-points are not the poles of the solution. The result of the analytic continuation depends only on the homotopy class of the path in $\mathbb{C}^3 \backslash \{diags\}$. Particularly, to find all the branches of a solution near a given point $u^0 = (u_1^0, u_2^0, u_3^0)$, one has to compute the results of the analytic continuation along any homotopy class of closed loops in $\mathbb{C}^3 \backslash \{diags\}$, with the beginning and the end at the point $u^0 = (u_1^0, u_2^0, u_3^0)$. Let

$$\beta \in \pi_1 \left(\mathbb{C}^3 \backslash \{diags\};\, u^0\right)$$

be an arbitrary loop. Any solution of the Schlesinger equations near the point $u^0 = (u_1^0, u_2^0, u_3^0)$, is uniquely determined by the monodromy matrices M_1, M_2 and M_3, computed in the basis γ_1, γ_2, γ_3, which was chosen as above, see figure 1. Continuing analytically this solution along the loop β, we arrive to another branch of the same solution, near u^0. This new branch is specified. according to the theorem 1.3, by some new monodromy matrices M_1^β, M_2^β and M_3^β, computed in the same basis γ_1, γ_2, γ_3. Our nearest goal is to compute these new matrices for any loop $\beta \in \pi_1 \left(\mathbb{C}^3 \backslash \{diags\};\, u^0\right)$.

The fundamental group $\pi_1 \left(\mathbb{C}^3 \backslash \{diags\};\, u^0\right)$ is isomorphic to the pure (or unpermuted) braid group, P_3 with three strings (see [Bir]); this is a subgroup of the full braid group B_3. The full braid group is isomorphic to the fundamental group of the same space where the permutations are allowed:

$$B_3 \sim \pi_1 \left(\mathbb{C}^3 \backslash \{diags\}/S_3;\, u^0\right),$$

S_3 being the group of the permutations of the coordinates (u_1, u_2, u_3). Any loop in B_3 has the form

$$(u_1(t), u_2(t), u_3(t)) \in \mathbb{C}^3 \backslash \{diags\}, \qquad 0 \le t \le 1,$$

with

$$u_i(0) = u_i^0, \quad u_i(1) = u_{p(i)}^0,$$

where p is a permutation of $\{1, 2, 3\}$, id est $p : (1, 2, 3) \mapsto (p(1), p(2), p(3))$. The elements of the subgroup P_3 of the pure braids are specified by the condition $p = \mathrm{id}$.

To simplify the computations we extend the procedure of the analytic continuation to the full braid group

$$M_1,\, M_2,\, M_3 \mapsto M_1^\beta,\, M_2^\beta,\, M_3^\beta, \quad \beta \in B_3 = \pi_1 \left(\mathbb{C}^3 \backslash \{diags\}/S_3;\, u^0\right).$$

For a generic braid $\beta \in B_3$, the new monodromy matrices describe the superposition of the analytic continuation and of the permutation

$$u_i \mapsto u_{p(i)}, \quad \mathcal{A}_i \mapsto \mathcal{A}_{p(i)}. \tag{1.25}$$

The braid group B_3 admits a presentation with generators β_1 and β_2 and defining relations

$$\beta_1 \beta_2 \beta_1 = \beta_2 \beta_1 \beta_2.$$

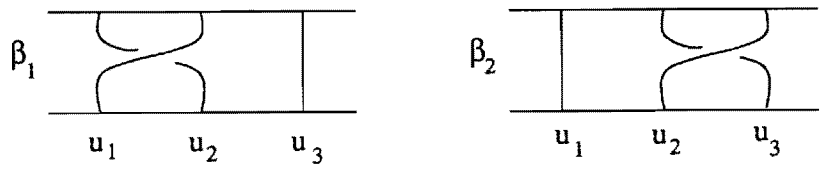

Fig.4. The geometric representation of the generators of the braid group B_3.

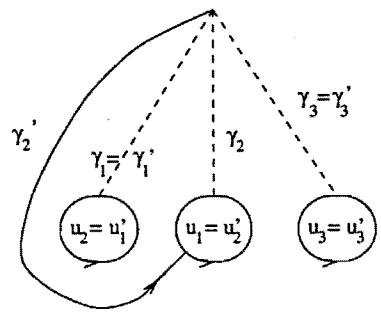

Fig.5. The new loops γ_i' obtained under the action of the braid β_1.

The generators β_1 and β_2 are shown in the figure 4.

Lemma 1.9. *For the generators β_1, β_2 shown in the figure 4, the matrices M_i^β have the following form:*

$$M_1^{\beta_1} = M_2, \quad M_2^{\beta_1} = M_2 M_1 M_2^{-1}, \quad M_3^{\beta_1} = M_3, \tag{1.26}$$

$$M_1^{\beta_2} = M_1, \quad M_2^{\beta_2} = M_3, \quad M_3^{\beta_2} = M_3 M_2 M_3^{-1}. \tag{1.27}$$

Proof of the lemma 1.9. Changing the positions of the points u_1 and u_2 by the braid β_1, the basis of the loops will be deformed into the new basis $\gamma_1', \gamma_2', \gamma_3'$ shown in the figure 5.

Thanks to the fact that we deal with isomonodromy deformations, the monodromy matrices M_i' of the system (1.1) with respect to the new basis $\gamma_1', \gamma_2', \gamma_3'$ are the same M_i, up to the reordering:

$$M_1' = M_2, \quad M_2' = M_1 \quad M_3' = M_3, \tag{1.28}$$

due to the permutation $(1,2,3) \mapsto (2,1,3)$. We want to compute the monodromy matrices with respect to the old basis $\gamma_1, \gamma_2, \gamma_3$. To this aim we notice the following obvious relation in the fundamental group:

$$\gamma_1 = \gamma_1', \quad \gamma_2 = (\gamma_1')^{-1}\gamma_2'\gamma_1' \quad \gamma_3 = \gamma_3'.$$

Using this relations and the (1.28), we immediately obtain the (1.26). Similarly the deformation of the basis of the fundamental group corresponding to the braid β_2 is shown in the figure 6.

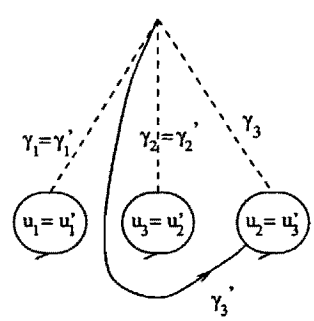

Fig.6. The new loops γ_i' obtained under the action of the braid β_2.

Here we have the permutation

$$M_1' = M_1 \quad M_2' = M_3', \quad M_3' = M_2,$$

and the relations in the fundamental group:

$$\gamma_1 = \gamma_1', \quad \gamma_2 = \gamma_2' \quad \gamma_3 = (\gamma_2')^{-1}\gamma_3'\gamma_2'.$$

From this we obtain the (1.27) and the lemma is proved. QDE

The action (1.26), (1.27) of the braid group on the triples of monodromy matrices commutes with the diagonal conjugation of them. As a consequence this action not only describes the structure of the analytic continuation of the solutions of the Schlesinger equations (1.13), but also of the reduced ones (1.16). Moreover the class of the degenerate solutions is closed under this analytic continuation. In fact if some of the matrices M_i is equal to $\mathbf{1}$, then for any β there is a j such that $M_j^\beta = \mathbf{1}$. As a consequence the following lemma holds true:

Lemma 1.10. *The structure of the analytic continuation of the solutions of the PVIμ equation is determined by the action (1.26), (1.27) of the braid group on the triples of monodromy matrices.*

Our next step is to rewrite the action (1.26), (1.27) of the braid group in the coordinates (x_1, x_2, x_3) in the space of the monodromy data. This is given by the following

Lemma 1.11. *In the coordinates (x_1, x_2, x_3), the action (1.26), (1.27) of the braid group is given by the formulae:*

$$\begin{aligned} \beta_1 &: (x_1, x_2, x_3) \mapsto (-x_1, x_3 - x_1x_2, x_2), \\ \beta_2 &: (x_1, x_2, x_3) \mapsto (x_3, -x_2, x_1 - x_2x_3). \end{aligned} \tag{1.29}$$

Proof of the lemma 1.11. The above formulae are obtained by straightforward computations from the (1.26), (1.27) by means of the parameterisation of the monodromy matrices (1.19).

We can summarize the results of this section in the following:

Theorem 1.5. *The structure of the analytic continuation of the solutions of the PVIμ equation is determined by the action (1.29) of the braid group on the triples (x_1, x_2, x_3).*

1.3. Monodromy data and algebraic solutions of the PVIμ equation.

A preliminary discussion on the algebraic solutions of the PVIμ equation and their monodromy data. Here we state some necessary condition for the triples (x_1, x_2, x_3) to generate the algebraic solutions. First of all we have to define the algebraic solutions.

Definition. A solution $y(x)$ is called *algebraic* if there exists a polynomial in two variables such that

$$F(x, y(x)) \equiv 0.$$

If $y(x)$ is an algebraic solution, then the correspondent solution $p(u_1, u_2, u_3)$, $q(u_1, u_2, u_3)$, of the reduced Schlesinger equations (1.16), is also algebraic. According to the theorem 1.1, the ramification points of the reduced Schlesinger equations (1.16) can lie only on the diagonals $u_1 = u_2$, $u_1 = u_3$, $u_3 = u_2$. Analogously the ramification points of $y(x)$ are allowed to lie only at 0, 1, ∞.

We now characterize the monodromy data such that the correspondent solution of the PVIμ equation is algebraic.

Lemma 1.12. *A necessary and sufficient condition for a solution of PVIμ to be algebraic is that the correspondent monodromy matrices, defined modulo diagonal conjugations, have a finite orbit under the action of the braid group (1.26), (1.27).*

Proof of the lemma 1.12. By definition, any algebraic function has a finite number of branches. Allowing also the permutations (1.25), we still obtain a finite number of values for M_1^β, M_2^β and M_3^β, $\beta \in B_3$ up to diagonal conjugations. QDE

Corollary 1.2. *An admissible triple (x_1, x_2, x_3) specifies an algebraic solution of PVIμ if and only if it satisfies (1.20) and its orbit, under the action (1.29) of the braid group, is finite.*

Remark. We stress that the action (1.29) preserves the relation (1.20).

In this way, the problem of the classification of all the algebraic solutions of the PVIμ reduces to the problem of the classification of all the finite orbits of the action (1.29) under the braid group in the three dimensional space (see [Dub], appendix F). Here we give a simple necessary condition for a triple (x_1, x_2, x_3) to belong to a finite orbit.

Lemma 1.13. *Let (x_1, x_2, x_3) be a triple belonging to a finite orbit. Then:*

$$x_i = -2\cos \pi r_i, \quad r_i \in \mathbf{Q}, \quad 0 \le r_i \le 1. \quad i = 1, 2, 3, \tag{1.30}$$

here $\mathbf{Q}$ is the set of the rational numbers.

Proof of the lemma 1.13. Let us prove the statement for, say, the coordinate x_1. Consider the transformation

$$\beta_1^2 : (x_1, x_2, x_3) \mapsto (x_1, x_2 + x_1x_3 - x_1^2x_2, x_3 - x_1x_2),$$

as a linear map on the plane (x_2, x_3). This linear map preserves the quadratic form

$$x_2^2 + x_3^2 - x_1x_2x_3.$$

If $x_1 = 2$, we put $r_1 = 1$; otherwise we reduce the quadratic form to the principal axes, introducing the new coordinates

$$\tilde{x}_2 = \frac{\sqrt{2 + x_1}}{2}(x_2 - x_3), \qquad \tilde{x}_3 = \frac{\sqrt{2 + x_1}}{2}(x_2 + x_3).$$

In these new coordinates the preserved quadratic form becomes a sum of squares and the transformation β_1^2 is a rotation of angle $\pi + 2\alpha$, where α is such that $x_1 = -2\cos\alpha$. To have a finite orbit of $(\tilde{x}_2, \tilde{x}_2)$, under the iterations of β_1^2, the angle α must be a rational multiple of π. In this way the statement for x_1 is proved. To prove it for x_2 and x_3, we have to consider the iterations of β_2^2 and $\beta_2^{-1}\beta_1^2\beta_2$ respectively. QDE

Remark. Thanks to the above lemma, for the finite orbits of the braid group, it is equivalent to deal with the triples (x_1, x_2, x_3), or with the *triangles* with angles $(\pi r_1, \pi r_2, \pi r_3)$, with $x_i = -2\cos \pi r_i$ and $0 \le r_i \le 1$ (we may assume, changing, if necessary, two of the signs, that at most one of the x_i is positive). We stress that the quantity

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 - 4$$

is greater than 0 if and only if the triangle (r_1, r_2, r_3) is hyperbolic, namely $\sum r_i < 1$; it is equal to 0, if and only if the triangle (r_1, r_2, r_3) is flat, namely $\sum r_i = 1$, and it is less than 0 if and only if the triangle (r_1, r_2, r_3) is spherical, namely $\sum r_i > 1$. Thanks to the (1.20), a flat triangle gives a resonant value of μ, and it is thus forbidden.

Classification of the triples (x_1, x_2, x_3) corresponding to the algebraic solutions. We deal with the classification of all the finite orbits of the triples (x_1, x_2, x_3) of the form (1.30), with at most[3] one r_i being equal to $\frac{1}{2}$. According to the lemma 1.13, any point of these B_3-orbits must have the same form (1.30). This condition is crucial in the classification.

Definition. We say that an admissible triple (x_1, x_2, x_3) is *good* if, for any braid $\beta \in B_3$ one has

$$\beta(x_1, x_2, x_3) = (-2\cos \pi r_1^\beta, -2\cos \pi r_2^\beta, -2\cos \pi r_3^\beta),$$

with some rational numbers $0 \le r_i^\beta \le 1$.

[3] This corresponds to the fact that we deal only with admissible triples.

Theorem 1.6. *Any good triple is equivalent, up to permutations, to*

$$(-2\cos\pi r_1, -2\cos\pi r_2, -2\cos\pi r_3),$$

where the rational numbers (r_1, r_2, r_3) *are in the following list:*

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{3}\right) \qquad \left(\frac{1}{3},\frac{1}{3},\frac{2}{3}\right) \tag{1.31}$$

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{4}\right) \qquad \left(\frac{2}{3},\frac{1}{4},\frac{1}{4}\right) \tag{1.32}$$

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{5}\right) \qquad \left(\frac{2}{3},\frac{1}{5},\frac{1}{5}\right) \qquad \left(\frac{4}{5},\frac{4}{5},\frac{4}{5}\right) \tag{1.33}$$

$$\left(\frac{1}{2},\frac{1}{3},\frac{2}{5}\right) \qquad \left(\frac{2}{3},\frac{2}{5},\frac{2}{5}\right) \qquad \left(\frac{2}{5},\frac{2}{5},\frac{2}{5}\right) \tag{1.34}$$

$$\left(\frac{1}{2},\frac{2}{5},\frac{1}{5}\right) \qquad \left(\frac{3}{5},\frac{1}{3},\frac{1}{5}\right) \qquad \left(\frac{2}{5},\frac{1}{3},\frac{1}{3}\right) \qquad \left(\frac{2}{3},\frac{1}{3},\frac{1}{5}\right). \tag{1.35}$$

In particular the above sets of triangles (1.31), (1.32), (1.33), (1.34) and (1.35) define five different finite orbits under the action of the braid group.

Remark. We observe that the above orbits (1.31), (1.32), (1.33), (1.34) and (1.35), contain also every permutation of the given triples. If we consider the action of the pure braid group P_3 on each triangle of the above orbits, except the (1.32), they again define the same orbits, with the same number of points. Concerning the triangles of (1.32), they define three different orbits of three points. So the orbit (1.31) has four points, the three orbits (1.32) have three points each, (1.33) and (1.34) have ten points each and (1.35) has eighteen points. These orbits give rise to all the algebraic solutions of the PVIμ equation, for μ is given by (1.20). The number of the points of each orbit with respect to the action of P_3 coincides with the number of the branches of the correspondent algebraic solution. We stress that this theorem claims that the good triples have necessarily a finite orbit.

Proof of the theorem 1.6. The braid group acting on the classes of triples (x_1, x_2, x_3), is generated by the braid β_1 and by the cyclic permutation:

$$(x_1, x_2, x_3) \mapsto (x_3, x_1, x_2).$$

As a consequence it is enough to study the operator:

$$(x_i, x_j, x_k) \mapsto (-x_i, x_j, x_k - x_i x_j),$$

up to cyclic permutations. This transformation works on the triangles as follows:

$$(r_i, r_j, r_k) \mapsto (1 - r_i, r_j, r'_k), \tag{1.36}$$

where r'_k is such that:

$$\cos\pi r'_k = \cos\pi r_k + 2\cos\pi r_i \cos\pi r_j. \tag{1.37}$$

As a first consequence we have immediately that the points (1.31), (1.32), (1.33), (1.34) and (1.35) define five different finite orbits.

The first step is to classify all the rational triples (r_i, r_j, r_k) such that r'_k, defined by (1.37) is a rational number, $1 > r'_k > 0$, for every choice of $i \neq j \neq k \neq i$, $i, j, k = 1, 2, 3$. Equivalently we want to classify all the rational solutions of the following equation:

$$\cos\pi r_k + \cos\pi(r_i + r_j) + \cos\pi(r_i - r_j) + \cos\pi(1 - r'_k) = 0,$$

or all the rational quadruples $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ such that:

$$\cos 2\pi\varphi_1 + \cos 2\pi\varphi_2 + \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4 = 0, \tag{1.38}$$

where the φ_i are related with the r_i by the following relations:

$$\varphi_1 = r_k/2, \quad \varphi_2 = \frac{r_i + r_j}{2}, \quad \varphi_3 = \frac{|r_i - r_j|}{2}, \quad \varphi_4 = \frac{|1 - r'_k|}{2}. \tag{1.39}$$

Such a classification is given by the following:

Lemma 1.14. *The only rational solutions $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, $0 \le \varphi_i < 1$, considered up to permutations, of the equation (1.38) are the following non-trivial ones:*

$$\left(\frac{1}{30}, \frac{11}{30}, \frac{2}{5}, \frac{1}{6}\right) \tag{a}$$

$$\left(\frac{7}{30}, \frac{17}{30}, \frac{1}{5}, \frac{1}{6}\right) \tag{b}$$

$$\left(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{1}{6}\right) \tag{c}$$

and the trivial ones, namely for $\cos 2\pi\varphi_4 = 0$:

$$\varphi_4 = \frac{1}{4}, \qquad \cos 2\pi\varphi_1 + \cos 2\pi\varphi_2 + \cos 2\pi\varphi_3 = 0 \tag{d}$$

which has the following solutions, obtained in [Cro]:

$$(d.1) : \left(\frac{1}{3}, \frac{1}{10}, \frac{3}{10}, \frac{1}{4}\right), \qquad (d.2) : \left(\varphi, \varphi + \frac{1}{3}, \varphi + \frac{2}{3}, \frac{1}{4}\right), \qquad (d.3) : \left(\frac{1}{4}, \varphi, |\varphi - \frac{1}{2}|, \frac{1}{4}\right),$$

where φ is any rational number $0 \le \varphi < 1$; for $\cos 2\pi\varphi_4 = 1$:

$$\varphi_4 = 0, \qquad \cos 2\pi\varphi_1 + \cos 2\pi\varphi_2 + \cos 2\pi\varphi_3 + 1 = 0, \tag{e}$$

which has the following solutions, obtained in [Gor]:

$$(e.1) : \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, 0\right), \qquad (e.2) : \left(\frac{1}{2}, \varphi, |\varphi - \frac{1}{2}|, 0\right), \qquad (e.3) : \left(\frac{1}{3}, \frac{1}{5}, \frac{2}{5}, 0\right),$$

where φ is any rational number $0 \le \varphi < 1$; for:

$$\cos 2\pi\varphi_1 + \cos 2\pi\varphi_2 = 0, \qquad \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4 = 0, \tag{f}$$

which has the following solutions:

$$\varphi_2 = |1/2 - \varphi_1|, \qquad \varphi_4 = |1/2 - \varphi_3|,$$

where φ_1, φ_3 are some rational numbers $0 \le \varphi_i < 1$.

Proof of the lemma 1.14. We follow the same procedure explained in [Gor] and [Cro]. In this proof we use the same notations as in [Cro], except for the φ_i which there are called r_i. Let us recall the notations. Let $\varphi_k = \frac{n_k}{d_k}$ where r_k, n_k are either positive coprime integers, $r_k > n_k$, or $n_k = 0$. Let p the largest prime which is a divisor of d_1, d_2, d_3, or d_4 and let δ_k, l_k, c_k, ν_k be the integers such that

$$d_k = \delta_k p^{l_k} \quad \text{and} \quad n_k = c_k\delta_k + \nu_k p^{l_k},$$

where δ_k is prime to p, $0 \le c_k < p^{l_k}$, $c_k = 0$ if $l_k = 0$, but otherwise c_k is prime to p. So

$$\varphi_k = \frac{\nu_k}{\delta_k} + \frac{c_k}{p^{l_k}} = f_k + \frac{c_k}{p^{l_k}}.$$

We assume that $l_1 \ge l_2 \ge l_3 \ge l_4$ and define, for the real variable x the function:

$$g_k(x) = \begin{cases} \frac{1}{2}\left[e^{2\pi i f_k} x^{c_k p^{l_1 - l_k}} + e^{-2\pi i f_k} x^{p^{l_1} - c_k p^{l_1 - l_k}}\right] & \text{if} \quad c_k \neq 0 \\ \cos 2\pi\varphi_k & \text{if} \quad c_k = 0 \end{cases}$$

and, in our case:

$$U(x) = \sum_1^4 g_k(x).$$

As in [Cro], $g_k(e^{\frac{2\pi i}{p^{l_1}}}) = \cos 2\pi\varphi_k$ and $U(e^{\frac{2\pi i}{p^{l_1}}}) = 0$. Let us introduce the polynomial

$$P(x) = 1 + x^{p^{l_1 - 1}} + x^{2p^{l_1 - 1}} \cdots x^{(p-1)p^{l_1 - 1}}.$$

This is the minimal polynomial of $e^{\frac{2\pi i}{p^{l_1}}}$ with coefficients in $\mathbf{Q}$, such that i) $P\left(e^{\frac{2\pi i}{p^{l_1}}}\right) = 0$ and ii) $P(x)$ is irreducible in the ring of the polynomials with rational coefficients. A stronger result was proved by Kronecker (see [Kr]): the polynomial $P(x)$ remains irreducible over any extension of the form $\mathbf{Q}(\zeta_1, \cdots, \zeta_n)$, where ζ_i is a root of the unity of the order coprime with p. As a consequence, the following lemma holds true (see [Cro])

Lemma 1.15. *If we express the polynomial $U(x)$ as a sum of polynomials $U_t(x)$,*

$$U(x) = \sum_{t=0}^{p^{l_1-1}-1} U_t(x),$$

where $U_t(x)$ contains those terms of $U(x)$ of the form bx^c with $c = t \bmod (p^{l_1-1})$, then $U_t(x)$ is divisible by $P(x)$.

We now apply this lemma in our case. The indices of the powers of x are:

$$c_1, \quad p^{l_1} - c_1, \quad c_2 p^{l_1-l_2}, \quad p^{l_1} - c_2 p^{l_1-l_2}, \quad c_3 p^{l_1-l_3}, \quad p^{l_1} - c_3 p^{l_1-l_3}, \quad c_4 p^{l_1-l_4}, \quad p^{l_1} - c_4 p^{l_1-l_4};$$

if all the following conditions are satisfied:

$$l_1, l_2, l_3 > 1, \quad l_1 > l_2, l_3, l_4, \quad l_2 > l_3, l_4, \quad l_3 > l_4, \quad l_4 > 0,$$

then there are no indices equal to each other $\bmod (p^{l_1-1})$ and there is no solution of (1.38). So we have to study the cases in which one of them is violated.

1): $l_1 = 1 \geq l_2 \geq l_3 \geq l_4$. In this case, since the degree of $U(x)$ is less than p, and the degree of $P(x)$ is $p-1$, being $U(x)$ divisible by $P(x)$, we must have $U(x) = mP(x)$, for some constant m. There are four possibilities:

1.1) : $l_1 = l_2 = l_3 = l_4 = 1$ then $U(0) = 0$ and P(0)=1. Then $m = 0$ and $U(x) \equiv 0$; moreover if the sum of two (three) terms representing two (three) of the functions g_k vanishes, then the sum of the two (three) functions vanishes. As a consequence there are only the following possibilities:

1.1.1) : $g_i = -g_j$ and $g_k = -g_l$ for some distinct $i, j, k, l = 1, \cdots 4$. This gives rise to the trivial case (f).

1.1.2) : $g_l = 0$ for some $l = 1, \cdots 4$; this is the trivial case (d).

1.1.3) : $U(x)$ contains only two powers of x. If $b_1, \cdots, b_4$ are the coefficients if one of the powers x^c, then:

$$b_1 + b_2 + b_3 + b_4 = 0, \quad \text{and} \quad \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} = 0,$$

namely $b_1, \cdots b_4$ are the solutions of the following biquadratic equation:

$$z^4 + (b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4)z^2 + b_1b_2b_3b_4 = 0;$$

as a consequence $b_i + b_j = 0$, $b_l + b_k = 0$, $\frac{1}{b_i} + \frac{1}{b_j} = 0$ and $\frac{1}{b_l} + \frac{1}{b_k} = 0$, for some distinct $i, j, k, l = 1, \cdots 4$. Then this case reduces to the trivial case (f).

1.2) $l_1 = l_2 = l_3 = 1$, $l_4 = 0$; then $U(0) = \cos 2\pi\varphi_4$ and then $U(x) = \cos 2\pi\varphi_4 P(x)$, where $P(x)$ is a polynomial with p powers of x. Since in U we have at most 7 powers and p must be prime, then p can only be equal to $2, 3, 5, 7$.

1.2.1) Case $p = 2$. Since p is the largest prime in $d_1, \cdots, d_4$, we must have $d_1 = d_2 = d_3 = d_4 = 2$ and $\delta_k = 1$. Then $\nu_k = 0$, $c_k = 1$ and this provides no solution.

1.2.2) Case $p = 3$. In this case there are the two following possibilities:

$$\frac{1}{2}e^{2\pi i f_1} + \frac{1}{2}e^{2\pi i f_2} + \frac{1}{2}e^{2\pi i f_3} = \cos 2\pi\varphi_4 = \frac{1}{2}e^{-2\pi i f_1} + \frac{1}{2}e^{-2\pi i f_2} + \frac{1}{2}e^{-2\pi i f_3}$$

or

$$\frac{1}{2}e^{-2\pi i f_1} + \frac{1}{2}e^{2\pi i f_2} + \frac{1}{2}e^{2\pi i f_3} = \cos 2\pi\varphi_4 = \frac{1}{2}e^{2\pi i f_1} + \frac{1}{2}e^{-2\pi i f_2} + \frac{1}{2}e^{-2\pi i f_3}.$$

In both the case one can show that there are no solutions. In fact, for example, in the first case, one has to solve the following equations:

$$2\cos 2\pi\varphi_4 = \cos 2\pi f_1 + \cos 2\pi f_2 + \cos 2\pi f_3, \qquad \sin 2\pi f_1 + \sin 2\pi f_2 + \sin 2\pi f_3 = 0.$$

Using the classification of all the possible rational solution of the case (d) given in [Cro], one can show that there are no solutions.

1.2.3) Case $p = 5$. In this case we have:

$$\frac{1}{2}e^{2\pi i f_k} = \frac{1}{2}e^{-2\pi i f_k} = \frac{1}{2}e^{2\pi i f_i} + \frac{1}{2}e^{\pm 2\pi i f_j} = \cos 2\pi\varphi_4,$$

for some distinct $i, j, k = 1, 2, 3$. Then f_k is 0 or $\frac{1}{2}$ and $\varphi_4 = \frac{1}{6}$ or $\varphi_4 = \frac{1}{3}$ respectively. Following the same computations of [Cro] we obtain the two solutions (a) and (b).

1.2.4) Case $p = 7$. In this case we have:

$$\frac{1}{2}e^{2\pi i f_1} = \frac{1}{2}e^{-2\pi i f_1} = \frac{1}{2}e^{2\pi i f_2} = \frac{1}{2}e^{-2\pi i f_2} = \frac{1}{2}e^{2\pi i f_3} = \frac{1}{2}e^{-2\pi i f_3} = \cos 2\pi\varphi_4,$$

which has the following solutions:

$$f_1 = f_2 = f_3 = 0 \quad \text{and} \quad \varphi_4 = \frac{1}{6} \quad \text{or} \quad f_1 = f_2 = f_3 = \frac{1}{2} \quad \text{and} \quad \varphi_4 = \frac{1}{3}.$$

Following the same computations of [Cro], we obtain the solution (c).

1.3) $l_1 = l_2 = 1$ and $l_3 = l_4 = 0$. Then $U(x) = (\cos 2\pi\varphi_3 + \cos 2\pi\varphi_4)P(x)$; again in U we have at most 5 powers and then $p = 2, 3, 5$. The case $p = 2$ is treated as in [Cro];

1.3.1) : In the case $p = 3$ either

$$\frac{1}{2}e^{2\pi i f_1} + \frac{1}{2}e^{2\pi i f_2} = \frac{1}{2}e^{-2\pi i f_1} + \frac{1}{2}e^{-2\pi i f_2} = \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4,$$

or:

$$\frac{1}{2}e^{2\pi i f_1} + \frac{1}{2}e^{-2\pi i f_2} = \frac{1}{2}e^{-2\pi i f_1} + \frac{1}{2}e^{2\pi i f_2} = \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4.$$

In the former case, for $f_1 = f_2$, with $\cos 2\pi f_2 = \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4$ and this gives again the solution (b). The latter case is equivalent.

1.3.2) : In the case $p = 5$ one has:

$$\frac{1}{2}e^{2\pi i f_1} = \frac{1}{2}e^{-2\pi i f_1} = \frac{1}{2}e^{2\pi i f_2} = \frac{1}{2}e^{-2\pi i f_2} = \cos 2\pi\varphi_3 + \cos 2\pi\varphi_4,$$

which gives $f_1 = f_2 = 0$ or $f_1 = f_2 = \frac{1}{2}$. We treat the former case (the latter is equivalent); then $\cos 2\pi\varphi_3 + \cos 2\pi\varphi_4 = \frac{1}{2}$ and we can show that this case reduces to the trivial solutions (d) and (e).

1.4) $l_1 = 1$ and $l_2 = l_3 = l_4 = 0$. In this case, as in [Cro], there is no solution, but the trivial one (d).

2) $l_1 \geq 2$, $l_1 \geq l_2, l_3, l_4$. This case can be treated as the analogous one in [Cro]. This concludes the proof of the lemma 1.14. QDE

We now use the above lemma to classify all the triangles which can give rise to the algebraic solutions. Every quadruple generates twelve triangles. In fact given a solution $(\varphi_1, \cdots \varphi_4)$ we have six ways to choose the pair (φ_i, φ_j) such that

$$\cos 2\pi\varphi_i + \cos 2\pi\varphi_j = 2\cos\pi(\varphi_i + \varphi_j)\cos\pi(\varphi_i - \varphi_j);$$

chosen the pair (φ_i, φ_j), we have two ways for choosing φ_k, in order to have the triangle

$$(\varphi_k, \varphi_i + \varphi_j, |\varphi_i - \varphi_j|).$$

The remaining φ_l is, by definition, such that the above triangle is mapped, by the braid (1.36), to:

$$(|\varphi_i - \varphi_j|, |1 - \varphi_i - \varphi_j|, \varphi_l).$$

Let us analyze all the triangles generated by the solutions of the equation (1.38), and keep the *good* ones, namely the ones for which the new r'_k, given by (1.37), is rational for every i, j, k, cyclic permutation of $1, 2, 3$.

We observe that, a necessary and sufficient condition for (r_1, r_2, r_3) to give a good triple, is that every permutation of it, not only cyclic, must give again a good triple. In fact let us suppose that (r_1, r_2, r_3) gives a bad triple. Then there is not any rational r_4 such that

$$\cos \pi r_4 = \cos \pi r_1 + 2 \cos \pi r_2 \cos \pi r_3,$$

or equivalently, for the correspondent $\varphi_1, \varphi_2, \varphi_3$

$$\varphi_1 = \frac{r_1}{2}, \ \varphi_2 = \frac{r_2 + r_3}{2}, \ \varphi_3 = \frac{|r_2 - r_3|}{2},$$

there is not any rational φ_4 such that $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ satisfy (1.38). We now show that the permuted triple (r_1, r_3, r_2) is again bad. In fact, if it was good, then the correspondent $\varphi'_1, \varphi'_2, \varphi'_3$ admit a rational φ'_4 such that $\varphi'_1, \varphi'_2, \varphi'_3, \varphi'_4$ satisfy (1.38). But this is absurd because the new $\varphi'_1, \varphi'_2, \varphi'_3$ coincide with the old ones $\varphi_1, \varphi_2, \varphi_3$. So the triple (r_1, r_3, r_2) is again bad. As a consequence if (r_1, r_2, r_3) is a good triple, then the permuted triple (r_1, r_3, r_2) is again good (otherwise, by the above argument (r_1, r_2, r_3) is bad). This proves our claim; in fact the good triples are defined up to cyclic permutations, then, allowing the permutation $(r_1, r_2, r_3) \mapsto (r_1, r_3, r_2)$, all the permutations are allowed.

So we will exclude all the triangles (r_1, r_2, r_3) for which there exists at least a permutation that gives rise to values of $(\varphi_1, \varphi_2, \varphi_3)$ for which there is no rational φ_4 such that $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ is a solution of (1.38).

Solution (a).

$$\left(\frac{1}{15}, \frac{1}{30}, \frac{23}{30}\right), \quad \left(\frac{1}{15}, \frac{1}{5}, \frac{8}{15}\right), \quad \left(\frac{1}{15}, \frac{7}{30}, \frac{17}{30}\right), \quad \left(\frac{4}{15}, \frac{7}{30}, \frac{13}{30}\right), \quad \left(\frac{11}{15}, \frac{11}{30}, \frac{13}{30}\right), \quad \left(\frac{11}{15}, \frac{2}{15}, \frac{1}{5}\right),$$

$$\left(\frac{4}{5}, \frac{2}{15}, \frac{1}{5}\right), \quad \left(\frac{1}{5}, \frac{1}{5}, \frac{7}{15}\right), \quad \left(\frac{1}{3}, \frac{11}{30}, \frac{13}{30}\right), \quad \left(\frac{2}{3}, \frac{1}{30}, \frac{7}{30}\right), \quad \left(\frac{1}{3}, \frac{1}{5}, \frac{3}{5}\right), \quad \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right).$$

The last two points

$$\left(\frac{1}{3}, \frac{1}{5}, \frac{3}{5}\right) \qquad \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right) \tag{1.40}$$

belong to the orbit (1.35). The above values, except the (1.40), give rise, for example, to the following values of $(\varphi_1, \varphi_2, \varphi_3)$, given by (1.39), (written in the same order as the correspondent generating triangles):

$$\left(\frac{1}{60}, \frac{5}{12}, \frac{7}{20}\right), \quad \left(\frac{1}{10}, \frac{3}{10}, \frac{7}{30}\right), \quad \left(\frac{7}{60}, \frac{19}{60}, \frac{1}{4}\right), \quad \left(\frac{7}{60}, \frac{7}{20}, \frac{1}{12}\right), \quad \left(\frac{11}{60}, \frac{7}{12}, \frac{3}{20}\right),$$

$$\left(\frac{1}{10}, \frac{13}{30}, \frac{3}{10}\right), \quad \left(\frac{1}{15}, \frac{1}{2}, \frac{3}{10}\right), \quad \left(0, \frac{1}{5}, \frac{7}{30}\right), \quad \left(\frac{1}{20}, \frac{11}{60}, \frac{23}{60}\right), \quad \left(\frac{1}{60}, \frac{9}{20}, \frac{13}{60}\right).$$

there isn't any rational number φ_4 such that any of the quadruples build with these triples and φ_4 is in the class described by the lemma 1.14.

Solution (b).

$$\left(\frac{7}{15}, \frac{11}{30}, \frac{23}{30}\right), \quad \left(\frac{7}{15}, \frac{2}{5}, \frac{11}{15}\right), \quad \left(\frac{7}{15}, \frac{1}{30}, \frac{11}{30}\right), \quad \left(\frac{2}{15}, \frac{1}{30}, \frac{19}{30}\right), \quad \left(\frac{2}{15}, \frac{1}{30}, \frac{17}{30}\right), \quad \left(\frac{2}{15}, \frac{1}{15}, \frac{3}{5}\right),$$

$$\left(\frac{2}{5}, \frac{1}{15}, \frac{2}{5}\right), \quad \left(\frac{2}{5}, \frac{11}{15}, \frac{2}{5}\right), \quad \left(\frac{1}{3}, \frac{1}{30}, \frac{13}{30}\right), \quad \left(\frac{1}{3}, \frac{11}{30}, \frac{23}{30}\right), \quad \left(\frac{2}{5}, \frac{1}{3}, \frac{4}{5}\right), \quad \left(\frac{1}{3}, \frac{1}{3}, \frac{4}{5}\right).$$

The last two points are equivalent to:

$$\left(\frac{1}{3},\frac{1}{5},\frac{3}{5}\right) \qquad \left(\frac{2}{3},\frac{1}{3},\frac{1}{5}\right) \tag{1.41}$$

of the orbit (1.35). As before one can show that if (r_1, r_2, r_3) is one of the above values, except the (1.41), then there exists a permutation such that the r'_k defined by (1.37) is no-more rational. In fact we obtain for example the following values of $(\varphi_1, \varphi_2, \varphi_3)$, which don't fall in the values obtained in the lemma 1.14:

$$\left(\frac{11}{60},\frac{37}{60},\frac{3}{20}\right), \quad \left(\frac{1}{5},\frac{3}{5},\frac{2}{15}\right), \quad \left(\frac{1}{60},\frac{5}{12},\frac{1}{20}\right), \quad \left(\frac{1}{60},\frac{3}{4},\frac{23}{60}\right), \quad \left(\frac{1}{60},\frac{47}{60},\frac{7}{20}\right),$$

$$\left(\frac{1}{30},\frac{23}{30},\frac{11}{30}\right), \quad \left(\frac{1}{30},\frac{2}{5},0\right), \quad \left(\frac{11}{30},\frac{2}{5},0\right), \quad \left(\frac{1}{60},\frac{23}{60},\frac{1}{20}\right), \quad \left(\frac{11}{60},\frac{11}{20},\frac{13}{60}\right).$$

Solution (c).

$$\left(\frac{2}{7},\frac{1}{7},\frac{5}{7}\right), \quad \left(\frac{2}{7},\frac{5}{42},\frac{19}{42}\right), \quad \left(\frac{2}{7},\frac{11}{42},\frac{25}{42}\right), \quad \left(\frac{4}{7},\frac{11}{42},\frac{25}{42}\right), \quad \left(\frac{4}{7},\frac{2}{7},\frac{4}{7}\right), \quad \left(\frac{4}{7},\frac{1}{42},\frac{13}{42}\right),$$

$$\left(\frac{1}{7},\frac{1}{42},\frac{29}{42}\right), \quad \left(\frac{1}{7},\frac{5}{42},\frac{23}{42}\right), \quad \left(\frac{1}{7},\frac{1}{7},\frac{4}{7}\right), \quad \left(\frac{1}{3},\frac{1}{7},\frac{3}{7}\right), \quad \left(\frac{1}{3},\frac{2}{7},\frac{4}{7}\right), \quad \left(\frac{1}{3},\frac{1}{7},\frac{5}{7}\right).$$

As before one can show that if (r_1, r_2, r_3) is one of the above values then there exists a permutation such that the r'_k defined by (1.37) is no-more rational. In fact we obtain for example the following values of $(\varphi_1, \varphi_2, \varphi_3)$, which are not included in the values described by the lemma 1.14:

$$\left(\frac{1}{14},\frac{1}{2},\frac{3}{14}\right), \quad \left(\frac{5}{84},\frac{31}{84},\frac{1}{12}\right), \quad \left(\frac{11}{84},\frac{37}{84},\frac{13}{84}\right), \quad \left(\frac{11}{84},\frac{7}{12},\frac{1}{84}\right), \quad \left(\frac{1}{7},\frac{4}{7},0\right), \quad \left(\frac{1}{84},\frac{37}{84},\frac{11}{84}\right),$$

$$\left(\frac{1}{84},\frac{5}{12},\frac{23}{84}\right), \quad \left(\frac{5}{84},\frac{29}{84},\frac{17}{84}\right), \quad \left(\frac{1}{7},\frac{2}{7},0\right), \quad \left(\frac{1}{14},\frac{8}{21},\frac{1}{21}\right), \quad \left(\frac{1}{7},\frac{19}{42},\frac{5}{42}\right), \quad \left(\frac{1}{14},\frac{11}{21},\frac{4}{21}\right).$$

Solution (d.1).

$$\left(\frac{2}{3},\frac{3}{20},\frac{3}{20}\right), \quad \left(\frac{1}{3},\frac{1}{20},\frac{9}{20}\right), \quad \left(\frac{1}{5},\frac{1}{20},\frac{1}{20}\right), \quad \left(\frac{1}{5},\frac{1}{30},\frac{19}{30}\right), \quad \left(\frac{1}{5},\frac{1}{12},\frac{7}{12}\right), \quad \left(\frac{2}{5},\frac{1}{12},\frac{5}{12}\right),$$

$$\left(\frac{3}{5},\frac{3}{20},\frac{7}{20}\right), \quad \left(\frac{3}{5},\frac{7}{30},\frac{13}{30}\right), \quad \left(\frac{1}{2},\frac{7}{30},\frac{13}{30}\right), \quad \left(\frac{1}{2},\frac{1}{30},\frac{11}{30}\right), \quad \left(\frac{2}{3},\frac{1}{5},\frac{2}{5}\right), \quad \left(\frac{1}{2},\frac{1}{5},\frac{2}{5}\right).$$

The last two points are equivalent to:

$$\left(\frac{1}{3},\frac{1}{5},\frac{3}{5}\right) \qquad \left(\frac{1}{2},\frac{2}{5},\frac{1}{5}\right) \tag{1.42}$$

of the orbit (1.35), which is now complete. We can again exclude all the other values of (r_1, r_2, r_3), with the same trick as above.

Solution (d.2). In this case we have that any of the triangles generated is equivalent to one of the following:

$$\left(|1-2\varphi|,\frac{|1-4\varphi|}{4},\frac{1+4\varphi}{4}\right), \quad \left(\frac{1}{2},\frac{|1-4\varphi|}{4},\frac{|3-4\varphi|}{4}\right), \quad \left(\frac{1}{2},\frac{|1-4\varphi|}{4},\frac{1+4\varphi}{4}\right),$$

$$\left(2\varphi,\frac{|1-4\varphi|}{4},\frac{|3-4\varphi|}{4}\right), \quad \left(\frac{1}{2},\frac{1}{2},\frac{|1-4\varphi|}{2}\right),$$

where φ is an arbitrary rational number. The last triangle is forbidden because it has two right angles, and the first four ones are all equivalent to a flat triangle, so they are again forbidden because they give rise to a resonant value of μ.

Solution (d.3). The generated triangles are the following:

$$\left(\frac{1}{2},\frac{1}{3},2\varphi+\frac{1}{3}\right),\quad \left(|\frac{2}{3}-2\varphi|,\frac{1}{3},2\varphi+\frac{1}{3}\right),\quad \left(\frac{1}{2},\frac{2}{3},\frac{2}{3}+2\varphi\right),\quad \left(\frac{2}{3}+2\varphi,\frac{2}{3},\frac{2}{3}+2\varphi\right),$$

$$\left(\frac{2}{3}+2\varphi,\frac{|1-4\varphi|}{4},\frac{1+4\varphi}{4}\right),\quad \left(|\frac{2}{3}-2\varphi|,\frac{|1-4\varphi|}{4},\frac{1+4\varphi}{4}\right),\quad \left(\frac{1}{2},\frac{1}{3},1+2\varphi\right),\quad \left(2\varphi,\frac{1}{3},1+2\varphi\right),$$

$$\left(|\frac{2}{3}-2\varphi|,\varphi+\frac{1}{12},\varphi+\frac{7}{12}\right),\left(2\varphi,\varphi+\frac{1}{12},\varphi+\frac{7}{12}\right),\left(\frac{2}{3}+2\varphi,\varphi+\frac{5}{12},\varphi+\frac{11}{12}\right),\left(2\varphi,\varphi+\frac{5}{12},\varphi+\frac{11}{12}\right).$$

This case must be studied carefully because we have to classify the allowed values of the rational variable φ in order that, applying the transformation (1.36), we obtain always rational values. Let us analyze the first triangle. It is mapped by (1.36) to a triangle equivalent to the second:

$$\left(\frac{1}{2},\frac{1}{3},2\varphi+\frac{1}{3}\right)\mapsto\left(\frac{1}{3}+2\varphi,\frac{2}{3},2\varphi+\frac{1}{3}\right)\sim\left(|\frac{2}{3}-2\varphi|,\frac{1}{3},2\varphi+\frac{1}{3}\right). \tag{1.43}$$

If we apply the braid (1.36), with $r_i=\frac{2\pi}{3}$, $r_j=r_k=2\varphi+\frac{1}{3}$, we have to solve:

$$\cos\frac{2\pi}{3}+2\cos^2\pi(2\varphi+\frac{1}{3})=\cos\pi r'_k,$$

or, equivalently, for φ and for the new φ_k:

$$\cos\frac{2\pi}{3}+\cos 2\pi(2\varphi+\frac{1}{3})+1+\cos 2\pi\varphi_k=0.$$

We classify the values of the allowed φ using the lemma 1.14 in the case (e). We have six possibilities for φ:

i) if $\varphi_k=\frac{1}{2}$, then $\varphi=\frac{11}{24}$. In this case we obtain, from (1.43), all the points of the orbit (1.32).

ii) if $\varphi_k=\frac{1}{4}$, then $\varphi=0$. In this case we obtain, from (1.43), all the points of the orbit (1.31).

iii) if $\varphi_k=\frac{1}{2}$, then $\varphi=\frac{1}{4}$. In this case we obtain, from (1.43), the following points:

$$\left(\frac{1}{2},\frac{1}{3},\frac{5}{6}\right),\qquad \left(\frac{1}{6},\frac{2}{3},\frac{1}{6}\right).$$

We exclude them because there exists a permutation such that the r'_k defined by (1.37) is no-more rational.

iv) if φ_k is free to vary, then $2\varphi+\frac{1}{3}=\frac{1}{2}$. In this case we obtain, from (1.43), the forbidden point:

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{2}\right),$$

so we have to exclude it.

v) if $\varphi_k=\frac{1}{5}$, then $\varphi=\frac{1}{30}$, and we obtain, from (1.43), the following two points of the orbit (1.34):

$$\left(\frac{1}{2},\frac{1}{3},\frac{2}{5}\right),\qquad \left(\frac{2}{5},\frac{2}{3},\frac{2}{5}\right).$$

vi) if $\varphi_k=\frac{2}{5}$, then $\varphi=\frac{13}{30}$, and we obtain the following two points of the orbit (1.33):

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{5}\right),\qquad \left(\frac{1}{5},\frac{2}{3},\frac{1}{5}\right).$$

In the same way we can study all the other triangles and show that we don't obtain any other value but the ones described in the theorem.

Solution (e.1). The generated triangles are the following:

$$\left(0,0,\frac{2}{3}\right),\ \left(0,\frac{1}{12},\frac{7}{12}\right),\ \left(0,\frac{1}{2},\frac{2}{3}\right),\ \left(\frac{2}{3},\frac{1}{12},\frac{7}{12}\right)\ \left(\frac{2}{3},\frac{1}{3},\frac{1}{3}\right),\ \left(\frac{1}{2},\frac{1}{3},\frac{1}{3}\right),\ \left(\frac{2}{3},\frac{1}{4},\frac{1}{4}\right);$$

the first four are forbidden because there exists a permutation such that the r'_k defined by (1.37) is no-more rational. So we have to exclude them. The fifth and the sixth are points of the orbit (1.31) and the last of (1.32).

Solution (e.2). The generated triangles are the following:

$$\left(2\varphi,\frac{1}{2},\frac{1}{2}\right),\quad \left(|1-2\varphi|,\frac{1}{2},\frac{1}{2}\right),\quad \left(\frac{1}{2},\frac{1}{2},\frac{|1+4\varphi|}{2}\right),\quad (1,\varphi,\varphi),$$

$$\left(1,\frac{|1-2\varphi|}{2},\frac{|1-2\varphi|}{2}\right),\quad \left(0,\frac{1}{2},\frac{|1-4\varphi|}{2}\right),\quad (|1-2\varphi|,\varphi,\varphi),\left(0,\frac{|1-2\varphi|}{2},\frac{1+2\varphi}{2}\right),$$

$$(0,\varphi,1-\varphi),\quad \left(2\varphi,\frac{|1-2\varphi|}{2},\frac{|1-2\varphi|}{2}\right),\quad \left(|1-2\varphi|,\frac{|1-2\varphi|}{2},\frac{1+2\varphi}{2}\right),\quad (2\varphi,\varphi,1-\varphi),$$

They are all forbidden, the first three because they have two right angles, the next three ones, because we can prove that necessarily $\varphi=\frac{1}{2}$, then the first has two right angles, the second gives $|\cos\pi r'_k|=3$ and the last gives $|\cos\pi r'_k|=2$; all the others because they are equivalent to a flat triangle.

Solution (e.3). The generated triangles are the following:

$$\left(0,\frac{2}{15},\frac{8}{15}\right),\quad \left(0,\frac{1}{15},\frac{11}{15}\right),\quad \left(0,\frac{1}{5},\frac{3}{5}\right),\quad \left(\frac{2}{3},\frac{1}{5},\frac{3}{5}\right),\quad \left(\frac{2}{5},\frac{1}{15},\frac{11}{15}\right),\quad \left(\frac{4}{5},\frac{2}{15},\frac{8}{15}\right),$$

$$\left(\frac{4}{5},\frac{1}{5},\frac{1}{5}\right),\quad \left(\frac{2}{3},\frac{1}{5},\frac{1}{5}\right),\quad \left(\frac{2}{3},\frac{2}{5},\frac{2}{5}\right),\quad \left(\frac{2}{5},\frac{2}{5},\frac{2}{5}\right),\quad \left(\frac{2}{5},\frac{1}{3},\frac{1}{3}\right),\quad \left(\frac{1}{5},\frac{2}{3},\frac{1}{3}\right).$$

We exclude the first six because we can show that there exists a permutation such that the r'_k defined by (1.37) is no-more rational. The seventh and the eighth give two points of the orbit (1.34), the ninth and tenth two points of the orbit (1.37) and the last two, two points of (1.37).

Solution (f) We have obtained all the points of all the orbits of the theorem. To show that there are no other points we still have to examine the case (f). In this case all the obtained triangles are equivalent to the following:

$$\left(\frac{\varphi_1}{2},\frac{|2-\varphi_1+\varphi_3|}{4},\frac{|2-\varphi_1-\varphi_3|}{4}\right),\quad \left(\frac{\varphi_1}{2},\frac{1}{2},\frac{|1-\varphi_1|}{2}\right),\quad \left(\frac{\varphi_1}{2},\frac{|4-\varphi_1-\varphi_3|}{4},\frac{|\varphi_1-\varphi_3|}{4}\right). \tag{1.44}$$

Applying the transformation (1.36) to the above triangles, we find that we have to solve for φ_1, φ_3 and for the new φ obtained from the (1.37), the following three equations respectively:

$$\cos\pi\frac{|2-\varphi_3|}{2}+\cos\pi\varphi_1+\cos 2\pi\varphi+1=0,$$

$$\cos\pi\frac{|1+\varphi_1-\varphi_3|}{2}+\cos\pi\frac{|1-\varphi_1-\varphi_3|}{2}+\cos 2\pi\varphi=0,$$

$$2\cos\pi\frac{\varphi_1+\varphi_3}{4}+\cos\pi\frac{|3\varphi_1-\varphi_3|}{4}+\cos 2\pi\varphi=0.$$

We again can use the lemma 1.14 to prove that we don't obtain any new point. Let us show this in for the first triangle

$$\left(\frac{\varphi_1}{2}, \frac{|2-\varphi_1+\varphi_3|}{4}, \frac{|2-\varphi_1-\varphi_3|}{4}\right). \tag{1.45}$$

We have to solve the equation

$$\cos \pi \frac{|2-\varphi_3|}{2} + \cos \pi\varphi_1 + \cos 2\pi\varphi + 1 = 0. \tag{1.46}$$

Using the lemma 1.14, the possible values for $(\frac{|2-\varphi_3|}{2}, \varphi_1, \varphi)$ are the (e.1), (e,2) and (e.3). Consider the case (e.1), then the possible solutions for the pair (φ_1, φ_3), are

$$(\varphi_1, \varphi_3) = (\frac{1}{2}, \frac{2}{3}), \quad (\frac{2}{3}, 1), \quad (\frac{2}{3}, \frac{2}{3}).$$

Let us substitute these solutions in (1.45); we obtain the triangles

$$(\frac{1}{4}, \frac{13}{24}, \frac{5}{24}), \quad (\frac{1}{3}, \frac{7}{12}, \frac{1}{12}), \quad (\frac{1}{3}, \frac{1}{2}, \frac{1}{6}),$$

which are all flat, and then forbidden. Let us now consider the case (e.2). In this case we obtain two possibilities: either $\varphi_3 = 0$ and φ_1 is a free parameter, or $\varphi_1 = 1$ and φ_1 is a free parameter. In both the cases the triangle (1.45) is flat, and then forbidden. Let us now consider the last case (e.3). The possible solutions for the pair (φ_1, φ_3), are

$$(\varphi_1, \varphi_3) = (\frac{2}{5}, \frac{2}{3}), \quad (\frac{2}{3}, \frac{6}{5}), \quad (\frac{2}{5}, \frac{2}{5}), \quad (\frac{4}{5}, \frac{6}{5}), \quad (\frac{4}{5}, \frac{2}{3}), \quad (\frac{2}{3}, \frac{2}{5}).$$

Substituting these values in (1.45), we obtain all flat angles. We can repeat the same proof for the other two triangles in (1.44). In this way we conclude the proof of the theorem. QDE

1.4. Monodromy data and reflection groups.

We reformulate here the above parameterisation of the monodromy data by classes of equivalence of triples (x_1, x_2, x_3) in a more geometric way. Let us consider a three-dimensional space V with a basis (e_1, e_2, e_3) and with a symmetric bilinear form $(\cdot, \cdot)$ given, in this basis, by the matrix

$$g := \begin{pmatrix} 2 & x_1 & x_3 \\ x_1 & 2 & x_2 \\ x_3 & x_2 & 2 \end{pmatrix} \tag{1.47}$$

namely

$$(e_i, e_i) = 2, \quad \text{for} \quad i = 1, 2, 3, \quad \text{and} \quad (e_1, e_2) = x_1, \quad (e_2, e_3) = x_2, \quad (e_1, e_3) = x_3.$$

Observe that the bilinear form (1.47) does not degenerate. Indeed,

$$\det g = 8 - 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3) = 8\cos^2 \pi\mu \neq 0,$$

due to the non-resonance assumption $2\mu \notin \mathbb{Z}$. The three planes p_1, p_2, p_3 orthogonal to the basic vectors (e_1, e_2, e_3) posses the following properties:

1) The normal vectors to these planes are non-isotropic (i.e. $(e_i, e_i) \neq 0$).
2) None of the planes is orthogonal to the other two.

Conversely, a real three-dimensional space V with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ and with an ordered triple of planes, satisfying the above conditions, uniquely determines the matrix g of the form (1.47), and then the monodromy data of a solution of PVIμ.

We define three reflections R_1, R_2, R_3 with respect to the three planes (p_1, p_2, p_3):

$$R_i : \begin{array}{l} V \to V \\ x \mapsto x - (e_i, x)e_i \end{array} \qquad i = 1, 2, 3.$$

These reflection have the following matrix representation in the basis (e_1, e_2, e_3):

$$R_1 = \begin{pmatrix} 1 & -x_1 & -x_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -x_1 & 1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & -x_2 & 1 \end{pmatrix}. \tag{1.48}$$

Let us consider the group $G \subset \mathcal{O}(V, (\cdot, \cdot))$ of the linear transformations of V, generated by the three reflections R_1, R_2, R_3. The matrix g is the Gram matrix of the reflection group G. We observe that, for an admissible triple, the group G is irreducible, namely there are no non-trivial subspaces of V which are invariant with respect to all the transformations of G.

We conclude that the branches of the solutions of PVIμ can be parameterised by three-dimensional groups with a marked ordered system of generating reflections R_1, R_2, R_3. Let us describe what happens with the triples of generators under the analytic continuation of the solution.

We define an action of the braid group B_3 on the systems of generators R_1, R_2, R_3 of the reflection group G:

$$\begin{aligned} \beta_1 &: (R_1, R_2, R_3) \mapsto (R_1, R_2, R_3)^{\beta_1} := (R_2, R_2R_1R_2, R_3), \\ \beta_2 &: (R_1, R_2, R_3) \mapsto (R_1, R_2, R_3)^{\beta_2} := (R_1, R_3, R_3R_2R_3), \end{aligned} \tag{1.49}$$

where $\beta_{1,2}$ are the standard generators of the braid group. Observe that the groups generated by the reflections (R_1, R_2, R_3) and $(R_1, R_2, R_3)^\beta$ coincide for any $\beta \in B_3$. In particular the following lemma holds true:

Lemma 1.16. *For any braid $\beta \in B_3$, the transformations $\beta(R_1, R_2, R_3)$ are reflections with respect to some planes orthogonal to some new basic vectors $(e_1^\beta, e_2^\beta, e_3^\beta)$. The Gram matrix with respect to the basis $(e_1^\beta, e_2^\beta, e_3^\beta)$ has the form:*

$$(e_i^\beta, e_i^\beta) = 2, \quad i = 1, 2, 3, \quad (e_1^\beta, e_2^\beta) = x_1^\beta, \quad (e_2^\beta, e_3^\beta) = x_2^\beta, \quad (e_1^\beta, e_3^\beta) = x_3^\beta,$$

where $(x_1^\beta, x_2^\beta, x_3^\beta) = \beta(x_1, x_2, x_3)$.

Proof of the lemma 1.16. It is sufficient to check the statement for the generators $\beta_{1,2}$. For $\beta = \beta_1$:

$$e_1^{\beta_1} = e_2, \quad e_2^{\beta_1} = e_1 - x_1 e_2, \quad e_3^{\beta_1} = e_3,$$

for $\beta = \beta_2$:

$$e_1^{\beta_2} = e_2, \quad e_2^{\beta_2} = e_3, \quad e_3^{\beta_2} = e_2 - x_2 e_3.$$

Computing the Gram matrix we prove the lemma. QDE

Reflection groups and algebraic solutions We observe that all the triangles, obtained in the classification theorem 1.6, are spherical. For them, the reflection group G, defined above, acts in the Euclidean space (in fact the correspondent Gram matrix g is positive definite), and it is finite. These groups coincide with the Coxeter groups A_3, B_3 and H_3. They describe respectively the symmetries of the tetrahedron, of the cube and of the icosahedron.

The classes of the algebraic solutions, described by the triangles in the list given by the theorem 1.6, are in one-to-one correspondence with the pairs of the reciprocal regular polyhedra or star-polyhedra in the three-dimensional Euclidean space. Namely (1.31) corresponds to the regular tetrahedron, (1.32) to the pair regular octahedron-cube, (1.33) to the pair regular icosahedron-regular dodecahedron, (1.34) to the pair regular great icosahedron-great star dodecahedron, (1.35) to the pair regular great dodecahedron-small star dodecahedron. To establish the correspondence we associate a triple of planes in the Euclidean three-dimensional space to a polyhedron on the list, using the following construction (see figure 7):

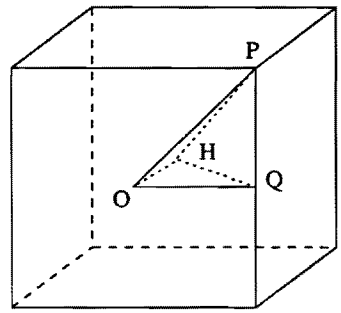

Fig.7. Here we show how to obtain a triple of planes, with a common intersection point, starting from a cube.

1) Take a face of the polyhedron.
2) Connect the center O of the face with a vertex P of it.
3) Take an edge of the face passing through the vertex P.
4) Connect the center Q of this edge with O.
5) Consider the hyper-planes through the center H of the polyhedron and the sides OP, OQ and PQ.
6) Intersect such hyper-planes with a sphere of center in the center of the polyhedron.

Classification of the monodromy data, second proof. We present here another proof of the above classification result, based on an idea suggested by E. B. Vinberg. We start with the following:

Algebraic Lemma. *Let (x, y, z) be an admissible triple of real numbers, satisfying the inequalities:*

$$x^2 + y^2 + z^2 - xyz > 4, \tag{1.50}$$

and

$$|x|,\ |y|,\ |z| \leq 2. \tag{1.51}$$

Then there exists a braid $\beta \in B_3$ such that the absolute value of some of the coordinates of $\beta(x, y, z)$ is strictly greater than 2.

Before proving the lemma, we observe that we can assume, without loss of generality, that all the coordinates of (x, y, z) are non-zero; in fact, for any admissible triple, there exists a braid $\beta \in B_3$ such that all the coordinates of $\beta(x, y, z)$ are non-zero. Let us denote by b_x, b_y and b_z the following braids:

$$\begin{array}{lll} b_x := & \beta_2, & b_x(x,y,z) = (z, -x, x - yz), \\ b_y := & \beta_2^{-1}\beta_1\beta_2, & b_y(x,y,z) = (-y + xz, -x, -z), \\ b_z := & \beta_1, & b_z(x,y,z) = (-x, z - xy, y). \end{array}$$

Lemma 1.17. *Let (x, y, z) be a triple of non-zero real numbers, satisfying*

$$0 < |z|,\ |x|,\ |y| \leq 2 \tag{1.52}$$

and

$$x^2 + y^2 + z^2 - xyz = 4 + c^2, \quad c > 0. \tag{1.53}$$

Denote $(x', y', z') := \beta(x, y, z)$, where

$$\beta = \begin{cases} b_x & \text{if} \quad |x| \leq |y|,\ |z|, \\ b_y & \text{if} \quad |y| \leq |x|,\ |z|, \\ b_z & \text{if} \quad |z| \leq |x|,\ |y|. \end{cases}$$

Then:

$$\min\{|x'|,\ |y'|,\ |z'|\} \geq \min\{|x|,\ |y|,\ |z|\} \tag{1.54}$$

and

$$|x'| + |y'| + |z'| \geq |x| + |y| + |z| + \min\{z^2, 2c\}. \tag{1.55}$$

Proof of the lemma 1.17. Let us prove the lemma in the case where $|z| \leq |x|, |y|$ and $\beta = b_z$. The other cases can be proved in the same way. If the signs of z and of xy are opposite then:

$$|y'| = |z| + |xy| \geq |z| + z^2, \quad |x'| = |x|, \quad |z'| = |z|$$

and the (1.54), (1.55) are proved. Let us suppose that the signs of z and of xy are the same. Changing the triple (x, y, z) to an equivalent one, we can assume that all the coordinates are positive. If we prove now that:

$$2z + 2c \leq xy, \tag{1.56}$$

where c is given in (1.53), we have that $|y'| = |xy - z| \geq z + 2c$ and the lemma is proved. To prove (1.56) we find the constrained minimum of the function xy on the domain D defined by the conditions (1.52) and (1.53). The Lagrange function:

$$F(x, y, z) := xy - \lambda \left(x^2 + y^2 + z^2 - xyz\right),$$

has the local maximum at:

$$x = y = \sqrt{\frac{4 + c^2 - z^2}{2 - z}},$$

and no minimum in the interior of D. It remains to study the values of the function xy on the boundary of D. If, say $z = y$, then the positive root x of the equation

$$x^2 + 2z^2 - xz^2 = 4 + c^2$$

is greater than 2. So the boundaries $z = y$ and $z = x$ are not reached for $(x, y, z) \in D$, and then $|z| < |x|, |y|$. It remains the last boundary to be studied. If, say $y = 2$, we find $x = z \pm c$. Since $x \geq z$, then $x = z + c$ and $xy = 2(z + c)$; this is the minimum of the function xy. QDE

Proof of the algebraic lemma. As observed above we can always reduce to the case where all the coordinates (x, y, z) are non-zero. Put:

$$\Delta(x, y, z) := \min\left\{x^2,\ y^2,\ z^2,\ 2\sqrt{x^2 + y^2 + z^2 - xyz - 4}\right\}.$$

Using the lemma 1.17, we can build a braid b_1 such that the coordinates:

$$(x_1, y_1, z_1) := b_1(x, y, z)$$

satisfy the inequalities

$$\min\{|x_1|, |y_1|, |z_1|\} \geq \min\{|x|, |y|, |z|\} \quad |x_1| + |y_1| + |z_1| \geq |x| + |y| + |z| + \Delta(x, y, z). \tag{1.57}$$

Since the quantity $x^2 + y^2 + z^2 - xyz - 4$ is preserved by the action of the braid group, we obtain:

$$\Delta(x_1, y_1, z_1) \geq \Delta(x, y, z).$$

If the absolute value of some of the coordinates (x_1, y_1, z_1) is greater than 2, the lemma in proved. Otherwise we apply again the construction of the lemma 1.17 to the triple (x_1, y_1, z_1). In this way we obtain a sequence of braids b_1, b_2, $b_3 \cdots$ such that the corresponding triples

$$(x_{k+1}, y_{k+1}, z_{k+1}) := b_{k+1}(x_k, y_k, z_k)$$

satisfy

$$\Delta(x_{k+1}, y_{k+1}, z_{k+1}) \geq \Delta(x_k, y_k, z_k).$$

Iterating the inequality (1.54), we obtain that

$$|x_k| + |y_k| + |z_k| \geq |x| + |y| + |z| + k\Delta(x, y, z).$$

Hence, in a finite number of steps we build a triple such that the absolute value of at least one of the coordinates in greater than 2. This concludes the proof of the algebraic lemma. QDE

Corollary 1.3. *For an algebraic solution of PVIμ, specified by an admissible triple $x_i = -2\cos 2\pi r_i$, the value of μ must be real, the strict inequalities*

$$|x_i| < 2, \quad i = 1, 2, 3, \tag{1.58}$$

hold true and the matrix g defined in (1.47) is positive definite.

Proof of the corollary 1.3. Let us prove that, for an algebraic solution, the triple (x_1, x_2, x_3) must satisfy the inequality:

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 < 4. \tag{1.59}$$

In fact, if $x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 > 4$, then, according to the algebraic lemma the triple is not a good one, and this contradicts the assumption that the solution is algebraic. If $x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 4$, then, form (1.20), $\mu = \frac{1}{2} + k$ with $k \in \mathbb{Z}$. This contradicts the basic assumption $2\mu \notin \mathbb{Z}$. Then (1.59) is satisfied and μ is a real number. Let us now prove (1.58). If, by absurd, one of the coordinates, say x_1 is such that $x_1 = \pm 2$, then

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 4 + (x_2 \mp x_3)^2,$$

and, being x_2 x_3 real numbers, (1.59) is violated. Then $x_i \neq \pm 2$, for every i. Finally, applying the Sylvester criterion to the matrix g, we prove that g is positive definite. In fact

$$\det G = 8 - 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3) > 0,$$

and, for any minor:

$$\det \begin{pmatrix} 2 & x_i \\ x_i & 2 \end{pmatrix} = 4 - x_i^2 > 0.$$

QDE

Lemma 1.18. *For an algebraic solution of PVIμ the reflection group G acts in the Euclidean space.*

The proof immediately follows from the fact that the correspondent Gram matrix is positive definite.

Corollary 1.4. *For a good triple (x_1, x_2, x_3) and for any braid $\beta \in B_3$, there exists three integer positive numbers n_{12}^β, n_{13}^β and n_{23}^β such that:*

$$\left(R_i^\beta R_j^\beta\right)^{n_{ij}^\beta} = 1, \quad \text{for} \quad i \neq j, \quad i, j = 1, 2, 3. \tag{1.60}$$

Proof of the corollary 1.4. If $(e_1 . e_2) = x_1 = -2\cos \pi r$ with $r = \frac{m}{n}$, $m, n \in \mathbb{Z}$, then R_1R_2 is a rotation by the angle $2\pi\frac{m}{n}$, hence:

$$(R_1R_2)^n = 1.$$

This holds true for any pair R_i and R_j. Moreover, for any braid $\beta \in B_3$, the triple $\beta(x_1, x_2, x_3)$ is again good, then (1.60) is proved. QDE

Corollary 1.5. *The set of the solutions of the PVIμ equation, with a real value of μ and real parameters (x_1, x_2, x_3) satisfying*

$$|x_i| < 2, \quad i = 1, 2, 3,$$

is invariant with respect to the analytic continuation.

Proof of the corollary 1.5. Applying the Sylvester criterion to the matrix g defined in (1.47), we obtain that g is positive definite. So the reflections R_1, R_2, R_3 can be realized in the Euclidean space. After a

transformation $(x_1, x_2, x_3) \mapsto (x_1^\beta, x_2^\beta, x_3^\beta) = \beta(x_1, x_2, x_3)$, the new numbers $({x_1}^\beta, {x_2}^\beta, {x_3}^\beta)$ are the entries of the Gram matrix:

$$g^\beta := \begin{pmatrix} 2 & x_1^\beta & x_3^\beta \\ x_1^\beta & 2 & x_2^\beta \\ x_3^\beta & x_2^\beta & 2 \end{pmatrix},$$

of the basis $(e_1^\beta, e_2^\beta, e_3^\beta)$, in the same Euclidean space. Then this matrix must be positive definite, namely $x_i^2 < 4$ as we wanted to prove. QDE

In the second part of this paper, we will identify the set described in the corollary 1.5 the class of solutions of PVIμ having asymptotic behaviour of algebraic type. This identification will be crucial in the computation of the five algebraic solutions of PVIμ we obtain.

As we have just shown, a good triple

$$(x_1, x_2, x_3) = \left(-2\cos\pi\frac{m_1}{n_1}, -2\cos\pi\frac{m_2}{n_2}, -2\cos\pi\frac{m_3}{n_3}\right),$$

corresponds to a representation of the Coxeter group generated by three reflections R_1, R_2, R_3 satisfying

$$R_1^2 = R_2^2 = R_3^2 = \mathbf{1}, \quad (R_1R_2)^{n_1} = (R_2R_3)^{n_2} = (R_1R_3)^{n_3} = \mathbf{1}, \tag{1.61}$$

in the three-dimensional Euclidean space. We denoted by G the image of this representation. Moreover, for any braid $\beta \in B_3$, the matrices

$$(R_1^\beta, R_2^\beta, R_3^\beta) := \beta(R_1, R_2, R_3),$$

satisfy the same identities (1.61), with some new integers $n_1^\beta, n_2^\beta, n_3^\beta$. We stress that the reflections $R_1^\beta, R_2^\beta, R_3^\beta$ generate the same group G.

Theorem 1.7. *It follows from the above property that G is an irreducible finite Coxeter group.*

Before proving the theorem, we recall some algebra. Let n be the least common multiple of n_1, n_2 and n_3. Put:

$$\zeta = 2\cos\frac{\pi}{n}.$$

Lemma 1.19. *The numbers*

$$x_i = -2\cos\pi\frac{m_i}{n_i}, \quad i = 1, 2, 3,$$

belong to the ring $\mathcal{K}_0$ of the integers of the field $\mathcal{K} := \mathbf{Q}[\zeta]$.

Recall (see [Wey]) that $\mathcal{K}$ is the normal extension of $\mathbf{Q}$ generated by ζ and $\mathcal{K}_0$ is the ring of all the algebraic integer numbers of $\mathcal{K}$, namely it consists of all the elements $x \in \mathcal{K}$ satisfying an algebraic equation of the form

$$x^k + a_1x^{k-1} + \cdots + a_k = 0, \quad \text{with} \quad a_i \in \mathbf{Z}.$$

Proof of the lemma 1.19. Let $n = n_i m_i'$, then

$$\cos\pi\frac{m_i}{n_i} = T_{m_i m_i'}\left(\cos\frac{\pi}{n}\right),$$

where

$$T_k(x) = \cos(k\arccos x) = 2^{k-1}x^k + \sum_{s=0}^{k-1} 2^{s-1}a_{ks}x^s, \tag{1.62}$$

are the Tchebyscheff polynomials of the first kind (see [Bat]). Recall that all the coefficients a_{ks} are integers, so $\cos\pi\frac{m_i}{n_i} =$ is a polynomial of ζ with integer coefficients. Moreover ζ is a root of the algebraic equation

$$2T_n(\zeta) + 2 = 0,$$

so it is a root of a monic polynomial in $\mathcal{K}$ with integer coefficients. Hence $\zeta \in \mathcal{K}_0$ and $x_i = -2T_{m_i m_i'}\left(\frac{\zeta}{2}\right) \in \mathcal{K}_0$ being a polynomial of ζ with integer coefficients, as we wanted to prove. QDE

Proof of the theorem 1.7. From the formulae (1.48) it follows that the matrices R_1, R_2 and R_3 are all defined over the same ring $\mathcal{K}_0$ of the integer algebraic numbers of $\mathcal{K}$:

$$R_i \in \mathrm{Mat}(\mathcal{K}_0, 3).$$

Moreover these matrices are orthogonal with respect to g:

$$R_i^T g R_i = g, \tag{1.63}$$

where g is defined in (1.47). Let

$$\Gamma := Gal(\mathcal{K}, \mathbf{Q})$$

the Galois group of $\mathcal{K}$ over $\mathbf{Q}$, namely the group of all the auto-morphisms

$$\phi : \mathcal{K} \to \mathcal{K},$$

such that they are constant on $\mathbf{Q}$.

For any $\phi \in \Gamma$ we denote by $\phi(R_i)$ and $\phi(g)$ the matrices obtained from R_i and g by the action:

$$(x_1, x_2, x_3) \mapsto (\phi(x_1), \phi(x_2), \phi(x_3)). \tag{1.64}$$

Lemma 1.20. *For any $\phi \in \Gamma$ the following statements hold true:*

i) $\det \phi(g) \neq 0$,

ii) The matrices $\phi(R_i)$ are orthogonal with respect to $\phi(g)$.

iii) For any $\beta \in B_3$ the matrices $\phi(R_i)^3$ satisfy the Coxeter relation (1.60).

The proof is obvious, due to the fact that any auto-morphism preserves all the algebraic relations..

From the above lemma, and from the algebraic lemma, it follows that for any $\phi \in \Gamma$, the real symmetric matrix $\phi(g)$ must be positive definite. We will show that this implies that the group G is finite. Let N be the order of the Galois group Γ. We construct the block-diagonal matrices

$$\mathcal{R}_i \in \mathrm{Mat}(\mathcal{K}_0, 3N), \quad i = 1, 2, 3,$$

as the matrices formed by 3×3 blocks on the diagonal, such that the j-th block is $\phi_j(R_i)$, for $\phi_j \in \Gamma$, $j = 1, 2, \cdots, N$. the matrices $\mathcal{R}_i$ are orthogonal with respect to $\mathcal{G}$, that is the block-diagonal matrix having $\phi_j(G)$, for $\phi_j \in \Gamma$, $j = 1, 2, \cdots, N$, on the diagonal blocks. We can apply the lemma 1.20 to the matrices $\mathcal{R}_i$ to show that they satisfy the Coxeter relation (1.60). As a consequence we obtain a representation of our reflection group G into the orthogonal group

$$\begin{aligned} G &\to \mathcal{O}\left(\mathcal{K}^{3N}, \mathcal{G}\right) \\ R_i &\mapsto \mathcal{R}_i. \end{aligned} \tag{1.65}$$

By construction the matrices $\mathcal{R}_i$ preserve the sub-lattice

$$\mathcal{K}_0^{3N} \subset \mathcal{K}^{3N}$$

of the vectors whose components are algebraic integers of the field $\mathcal{K}$. We recall (see [Wey] that the ring $\mathcal{K}_0$ of the algebraic integers of the field $\mathcal{K}$, is a finite-dimensional lattice. As a consequence the image of the representation (1.65) is a discrete subgroup of the orthogonal group. Since $\mathcal{G}$ is positive definite, this subgroup is compact and, hence, G must be finite. The theorem is proved. QDE

To complete the classification of the monodromy data related to the algebraic solutions it remains to classify the objects

$$(G, R_1, R_2, R_3),$$

where G is one of the Coxeter groups A_3, B_3 and H_3 and (R_1, R_2, R_3) is a triple of generating reflections considered modulo the action (1.49) of the braid group. This can be done by a straightforward computation of all the orbits of the triples of generating reflections. All of them were described and classified by Schwartz (see the introduction). We arrive again at the list of the theorem 1.6, where the triples (1.31) generate the group A_3 of the symmetries of the tetrahedron, (1.32) generate the group B_3 of the symmetries of the cube, while (1.33), (1.34) and (1.35) correspond to three inequivalent triples of the generating reflections of the group H_3.

2. GLOBAL STRUCTURE OF THE SOLUTIONS OF PAINLEVE' VIμ HAVING CRITICAL BEHAVIOUR OF ALGEBRAIC TYPE

In the first part of this paper, we found a class of solutions of PVIμ invariant with respect to the analytic continuation. For them, the reflection group G acts in the three-dimensional *Euclidean* space. Recall that the parameter μ must be real, the coordinates of the admissible triples (x_1, x_2, x_3) must be real and satisfy the inequality

$$-2 < x_i < 2, \quad i = 1, 2, 3.$$

In this second part, we prove that this class of solutions coincides with the class of the solutions of PVIμ having critical behaviour of the algebraic type:

$$y(x) = \begin{cases} a_0 x^{l_0} \left(1 + \mathcal{O}(x^{\varepsilon})\right), & \text{as} \quad x \to 0, \\ 1 - a_1 (1-x)^{l_1} \left(1 + \mathcal{O}((1-x)^{\varepsilon})\right), & \text{as} \quad x \to 1, \\ a_\infty x^{1-l_\infty} \left(1 + \mathcal{O}(x^{-\varepsilon})\right), & \text{as} \quad x \to \infty, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$ is small enough, and the indices l_0, l_1, l_∞ are real and the coefficients a_0, a_1, a_∞ are some complex numbers. We compute the behaviour of any branch of these solutions near the critical points. These results will be used to compute explicitly all the algebraic solutions classified in the first part.

First of all, we fix the notations. Let us choose:

$$u_1 = 0, \quad u_2 = x, \quad u_3 = 1.$$

Then the Fuchsian system (1.1) reads:

$$\frac{\mathrm{d}}{\mathrm{d}z} Y = \mathcal{A}(z, x) Y = \left(\frac{\mathcal{A}_1}{z} + \frac{\mathcal{A}_2}{z - x} + \frac{\mathcal{A}_3}{z - 1} \right) Y,$$

and putting:

$$\mathcal{A}_1 := A_0, \quad \mathcal{A}_2 := A_x, \quad \mathcal{A}_3 := A_1, \quad \mathcal{A}_\infty = A_\infty,$$

we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}z} Y = \left(\frac{A_0}{z} + \frac{A_1}{z - 1} + \frac{A_x}{z - x} \right) Y. \tag{2.2}$$

The branch cuts in $\overline{\mathbb{C}}$ are the same as in the section 1.1. We call now the basic loops $\gamma_0, \gamma_x, \gamma_1$; they are fixed as before, namely $\gamma_0, \gamma_x, \gamma_1$ play the role of the preceding $\gamma_1, \gamma_2, \gamma_3$ (see picture N1). The Schlesinger equations read:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} A_0(x) &= -\frac{[A_0, A_x]}{x}, \\ \frac{\mathrm{d}}{\mathrm{d}x} A_1(x) &= -\frac{[A_1, A_x]}{x - 1}, \\ \frac{\mathrm{d}}{\mathrm{d}x} A_x(x) &= \frac{[A_0, A_x]}{x} + \frac{[A_1, A_x]}{x - 1}. \end{aligned} \tag{2.3}$$

The correspondent monodromy matrices are

$$M_0, \quad M_x, \quad M_1,$$

which play the role of the preceding M_1, M_2, M_3 respectively. We recall that they satisfy

$$M_\infty M_1 M_x M_0 = \mathbf{1} \quad \det(M_i) = 1, \quad \mathrm{Tr}(M_i) = 2, \quad \text{for} \quad i = 0, 1, x \tag{2.4}$$

with

$$M_\infty = \begin{pmatrix} \exp(2i\pi\mu) & 0 \\ 0 & \exp(-2i\pi\mu) \end{pmatrix}. \tag{2.5}$$

With the above choice of A_∞, A_1, A_x and A_0, satisfying

$$\det A_i = 0, \quad \mathrm{Tr}(A_i) = 0, \quad i = 0, 1, \infty, \tag{2.6}$$

the non-degenerate solution $A(z, x)$ of the Schlesinger equations turns out to be related to the solution of PVI, with parameter μ in the following way (see [JMU]):

$$[A(y, x)]_{12} = 0, \quad \text{iff} \quad y(x) \text{ solves PVI}\mu, \tag{2.7}$$

with y not identically equal to $0, 1, x$.

We now state the first main theorem of this second part:

Theorem 2.1. *For any admissible triple (x_0, x_1, x_∞), $x_i \in \mathbb{R}$, $|x_i| < 2$ for $i = 0, 1, \infty$, there exists a unique branch $y(x; x_0, x_1, x_\infty)$ of the solution of PVIμ, with the parameter μ satisfying the equation:*

$$4 \sin^2 \pi\mu = x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty, \tag{2.8}$$

with the asymptotic behaviour (2.1) around the critical points $0, 1, \infty$. The indices are given by

$$l_i = \frac{1}{\pi} \arccos(\cos 2\pi r_i) = \begin{cases} 2r_i & \text{if } 0 < r_i \leq \frac{1}{2} \\ 2 - 2r_i & \text{if } \frac{1}{2} \leq r_i < 1 \end{cases} \quad i = 0, 1, \infty, \tag{2.9}$$

with

$$x_i = -2 \cos \pi r_i, \qquad i = 0, 1, \infty,$$

and the leading coefficients a_0, a_1, a_∞ are single-valued functions of the equivalence class of x_0, x_1, x_∞ and of μ, namely the coefficient a_0, for $x_0 \neq 0$, is given by:

$$a_0 = \frac{\exp(-i\pi\phi)}{4(2\mu + l_0 - 1)^2} \frac{\Gamma^2(1 - l_0)\Gamma^2(\frac{1+l_0}{2})\Gamma(\frac{1+l_0}{2} + \mu)\Gamma(\frac{1+l_0}{2} - \mu)}{\Gamma^2(l_0)\Gamma^2(\frac{1-l_0}{2})\Gamma(\frac{1-l_0}{2} + \mu)\Gamma(\frac{1-l_0}{2} - \mu)} \tag{2.10}$$

where

$$\exp(i\pi\phi) = \frac{x_0^2 x_1^2 - 2x_1^2 - 2x_0 x_1 x_2 + 2x_\infty^2 + i x_1 sign(x_0)\sqrt{4 - x_0^2}(2x_\infty - x_0 x_1)}{2(x_1^2 - x_0 x_1 x_2 + x_\infty^2)} \tag{2.11}$$

and for $x_0 = 0$

$$a_0 = \frac{x_\infty^2}{x_1^2 + x_\infty^2}. \tag{2.12}$$

The coefficient a_1 is given by the same formula with the substitution $x_0 \leftrightarrow x_1$, $l_0 \mapsto l_1$; a_∞ is given by the same formula too, after the substitution $(x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty)$ and $l_0 \mapsto l_\infty$. Conversely any solution of the PVIμ equation, with a real value of μ, having critical behaviour of algebraic type, can be obtained by the above construction.

Remark. The relation (2.8) determines μ up to the transformations

$$\mu \mapsto \pm\mu + n, \quad n \in \mathbb{Z}.$$

From the results of the section 1.2, it follows that a change $\mu \to -\mu$ corresponds to the action of the symmetry (1.21) on the solution $y(x; x_1, x_2, x_3)$, and a change $\mu \to \mu + 1$ corresponds to the action of the symmetry (1.23) on the same solution; these transformations preserve the class of the algebraic solutions.

The theorem 2.1 will be proved in the section 2.4.

2.1. Local theory of the solutions of PVIμ having critical behaviour of algebraic type

Local asymptotic behaviour around 0. In this section we characterize the local asymptotic behaviour of the solutions of PVIμ near the singular point $x = 0$. First of all let us characterize the type of asymptotic behaviour that can be related to the algebraic solutions:

Lemma 2.1. *Let us suppose that $y(x)$ is an algebraic solution of PVIμ; then the first term of its Puiseaux series is*

$$y(x) \sim a_0 x^{1-\sigma_0} \quad \text{as} \quad x \to 0. \tag{2.13}$$

for some constant $a_0 \neq 0$ and, the rational number σ_0 must satisfy $0 \leq \sigma_0 < 1$, with $a_0 \neq 1$ if $\sigma_0 = 0$.

Proof of the lemma 2.1. If $y(x)$ is an algebraic function, as $x \to 0$, then it admits an expansion in Puiseaux series around 0:

$$y(x) = \sum_{k_0}^{\infty} a_k x^{\frac{k}{n}}, \qquad k_0 \in \mathbf{Z},$$

where n is some natural number. As a consequence, for $k_0 \neq 0$, we have the following relation between the orders of the first and second derivative of y:

$$\mathcal{O}(x^2 y'') = \mathcal{O}(xy') = \mathcal{O}(y). \tag{2.14}$$

We now reduce to the common denominator the PVIμ equation and collect together all the terms of the same order, using the rule (2.14). The first term of the Puiseaux series must be chosen in order to kill the lowest term in the numerator of the PVIμ equation. If $k_0 < 0$, the lowest term is

$$-y^6 + 4\mu y^6 - 4\mu^2 y^6$$

which, for $2\mu \notin \mathbf{Z}$ cannot be zero for any choice of k_0. Then k_0 cannot be negative. If $n \geq k_0 > 0$, the lowest term is:

$$2x^2 {y'}^2 y - 2xy' y^2 - 2x^2 y'' y^2,$$

which is zero for $y = a_{k_0} x^{\frac{k_0}{n}}$. For $k_0 > n$:

$$-x^3 {y'}^2 + 2x^3 y'' y + y^2.$$

which cannot be zero. Moreover for $k_0 = 0$, the lowest term in the numerator of the PVIμ equation is

$$-a_0^4 (a_0 - 1)^2 (2\mu - 1)^2$$

and, since by assumption $2\mu \notin \mathbf{Z}$ and $a_0 \neq 0$, the only possible value of a_0 is 1. Putting $y = 1 + a_1 x^{\frac{k_1}{n}}$, we obtain, in the same way that $a_1 = 0$. This is the forbidden solution $y(x) \equiv 1$. Then k_0 can not be zero, and $y(x)$ satisfies (2.13) with $0 < l = \frac{k_0}{n} \leq 1$, namely $0 \leq \sigma_0 < 1$. QDE

In the above lemma we have seen the expected asymptotic behaviour of the algebraic solutions. We now state the main result of this section, which is more general, namely it holds also for non algebraic solutions.

Theorem 2.2. *For any pair of values (a_0, σ_0), $0 \leq \sigma_0 < 1$, there exists a unique branch of the solution of PVIμ, for a fixed μ, with the asymptotic behaviour*

$$y(x) = a_0 x^{1-\sigma_0}(1 + x^{\varepsilon} f(x)) \quad \text{as} \quad x \to 0, \tag{2.15}$$

for some $\varepsilon > 0$ and $f(x)$ smooth function such that $\lim_{x \to 0} f(x) = \text{const}$.

In order that $x^{1-\sigma_0}$ is well defined, we have to make some cut in the complex plane, namely, from now on, we cut along the line $\arg x = \varphi$ for some φ.

Proof of the existence. First of all we state the existence of solutions of the Schlesinger equations with a particular asymptotic behaviour. The following result will play an important role also in the section 2.3.

Lemma 2.2 (Sato-Miwa-Jimbo). *Given three constant matrices A_i^0, $i = 0, 1, x$ with zero eigenvalues such that $\Lambda = A_0^0 + A_x^0$ has eigenvalues $\pm\frac{\sigma}{2}$, $0 \le \sigma < 1$, and $A_1^0 = -\Lambda - A_\infty$, in any sector of $\overline{\mathbb{C}}$ containing none of the branch cuts, and sufficiently close to 0, there exists a solution of the Schlesinger equations that satisfy*

$$|A_1(x) - A_1^0| \le K|x|^{1-\sigma'} \quad \left|x^{-\Lambda}(A_1(x) - A_1^0)x^{\Lambda}\right| \le K|x|^{1-\sigma'} \tag{2.16}$$

$$\left|x^{-\Lambda}A_0(x)x^{\Lambda} - A_0^0\right| \le K|x|^{1-\sigma'} \quad \left|x^{-\Lambda}A_x(x)x^{\Lambda} - A_x^0\right| \le K|x|^{1-\sigma'}, \tag{2.17}$$

where K is some positive constant and $1 > \sigma' > \sigma$.

We want to show that it is possible to choose $A_{0,1,x}$ and Λ such that the corresponding solution $y(x)$ of the Painlevé VI equation, obtained via the equivalence (2.7), has the asymptotic behaviour (2.15). Let us consider an arbitrary constant matrix Λ with eigenvalues $\pm\frac{\sigma}{2}$; let T be the diagonalizing matrix of Λ, namely:

$$\Lambda = T\begin{pmatrix} \frac{\sigma}{2} & 0 \\ 0 & -\frac{\sigma}{2} \end{pmatrix} T^{-1}.$$

The choice of T fixes Λ and vice-versa. Now we choose $A_1^0 = -A_\infty - \Lambda$ and $A_{0,x}^0$ such that $A_0^0 + A_x^0 = \Lambda$, namely

$$A_0^0 = \frac{1}{2}\Lambda + F, \qquad A_x^0 = \frac{1}{2}\Lambda - F$$

for some constant matrix F. Then:

$$T^{-1}A_0^0T = \begin{pmatrix} \frac{\sigma}{4} & 0 \\ 0 & -\frac{\sigma}{4} \end{pmatrix} + E, \qquad T^{-1}A_x^0T = \begin{pmatrix} \frac{\sigma}{4} & 0 \\ 0 & -\frac{\sigma}{4} \end{pmatrix} - E,$$

where we can choose $E = \begin{pmatrix} 0 & \frac{b\sigma}{4} \\ \frac{-\sigma}{4b} & 0 \end{pmatrix}$, for some non-zero constant b. With this choice of E, $A_{0,x}$ have zero eigenvalues. Using the lemma 2.2, we obtain that, as $x \to 0$:

$$A_{0,x} \to T\begin{pmatrix} x^{\frac{\sigma}{2}} & 0 \\ 0 & x^{\frac{-\sigma}{2}} \end{pmatrix}\left[\begin{pmatrix} \frac{\sigma}{4} & 0 \\ 0 & -\frac{\sigma}{4} \end{pmatrix} \pm \begin{pmatrix} 0 & \frac{b\sigma x^{\sigma}}{4} \\ \frac{-\sigma x^{-\sigma}}{4b} & 0 \end{pmatrix}\right]\begin{pmatrix} x^{\frac{-\sigma}{2}} & 0 \\ 0 & x^{\frac{\sigma}{2}} \end{pmatrix}T^{-1}, \quad \text{and} \quad A_1 \to -A_\infty - \Lambda.$$

Substituting such asymptotic behaviors in the relation (2.7), taking T_{12}, $T_{11} \neq 0$ we obtain:

$$y(x) \sim -\frac{T_{12}x^{1-\sigma}}{4bT_{11}}; \tag{2.18}$$

we are now free to choose the arbitrary constants b, T_{11}, T_{12}, σ in such a way that $-\frac{T_{12}}{4bT_{11}} = a_0$, $\sigma = \sigma_0$, for any fixed a_0 and σ_0.

Proof of the uniqueness. Now we prove that the solution $y(x)$, $x \in \mathcal{B}(0, r)$, of Painlevé VI equation such that it satisfies (2.15) for some given constants a_0 and $\sigma_0 \in [0, 1)$, is uniquely determined by a and σ. Here $\mathcal{B}(0, r) = \{x \big| |x| \le r, \arg x \neq \varphi, x \neq 0\}$.
The proof is based on the fact that Painlevé VI is equivalent to the following reduced Schlesinger equations (1.16):

$$\dot{q} = \frac{(q-1)q + 2p(q-1)q(q-x)}{(x-1)x},$$

$$\dot{p} = \frac{-p^2(x - 2q - 2xq + 3q^2) - p(2q-1) - (1-\mu)\mu}{(x-1)x},$$

where:

$$q = y, \quad p = \frac{x(x-1)\dot{y} - y(y-1)}{2(y-x)y(y-1)}, \tag{2.19}$$

and the dot means the derivative $\frac{d}{dx}$ (see the system (1.16) above). We shall prove the local uniqueness of the solutions of the Hamiltonian system with the following asymptotic behaviour:

$$q(x) \sim ax^l + x^{l+\varepsilon} f(x) \qquad p(x) \sim \frac{l-1}{2a}\frac{1}{x^l} + \frac{x^\varepsilon g(x)}{x^l} \tag{2.20}$$

where $l = 1 - \sigma_0$, $a = a_0$, $\varepsilon > 0$ and $f(x)$ and $g(x)$ are some smooth functions in $\mathcal{B}(0,r)$ which tend to zero as $x \to 0$.

This is equivalent to show the main theorem. In fact from the uniqueness of q follows trivially the uniqueness of y and the following lemma holds true:

Lemma 2.3. *The estimates (2.20) on the asymptotic behaviour of $(q(x), p(x))$ are a consequence of (2.15).*

Proof of the lemma 2.3. Since $q = y$, the assertion on y is obvious due to the hypothesis (2.15); concerning p, we use its definition:

$$p = \frac{x(x-1)\dot{y} - y(y-1)}{2(y-x)y(y-1)}$$

and by a straightforward computation we show the ansatz for p. QDE

We now distinguish two cases: $0 < l < 1$, and $l = 1$. Let us consider the former case; it is convenient to introduce the new variables $(\tilde{q}, \tilde{p})$:

$$\tilde{q} = \frac{y}{x^l} \quad \tilde{p} = x^l p;$$

which have a similar asymptotic behaviour:

$$\tilde{q}(x) = a + x^\varepsilon f(x) \qquad \tilde{p}(x) = \frac{l-1}{2a} + x^\varepsilon g(x), \tag{2.21}$$

and the equations of the motion become:

$$\begin{aligned} \dot{\tilde{q}} &= f_q(\tilde{p}, \tilde{q}, x, x^l), \\ \dot{\tilde{p}} &= f_p(\tilde{p}, \tilde{q}, x, x^l), \end{aligned} \tag{2.22}$$

with:

$$f_q = \frac{-\tilde{q}(l-1-2\tilde{p}\tilde{q})}{x} - \frac{\tilde{q}(1+2\tilde{p}\tilde{q})}{x^{1-l}} + \frac{\tilde{q}(1+4\tilde{p}\tilde{q}) - 2x^{-l}\tilde{p}\tilde{q} - x^l\tilde{q}^2(1+2\tilde{p}\tilde{q})}{1-x},$$

and

$$f_p = \frac{\tilde{p}(l-1-2\tilde{p}\tilde{q})}{x} + \frac{\mu - \mu^2 + 2\tilde{p}\tilde{q} + 3\tilde{p}^2\tilde{q}^2}{x^{1-l}} + \frac{\frac{\tilde{p}^2}{x^l} - \tilde{p}(1+4\tilde{p}\tilde{q}) + x^l(\mu - \mu^2 + 2\tilde{p}\tilde{q} + 3\tilde{p}^2\tilde{q}^2)}{1-x}.$$

We want to prove the uniqueness of the solution $(\tilde{q}, \tilde{p})$ of (2.22), satisfying the ansatzs (2.21) for $t \in \mathcal{B}(0,r)$, in the ball $||\tilde{p} - \frac{l-1}{2a}||$, $||\tilde{q} - a|| \leq C_r$, where C_r is a constant that vanishes when the radius $r \to 0$, and $||f|| = \sup_{\mathcal{B}(0,r)} |f(x)|$. Let us suppose that there are two solutions $(\tilde{q}_1, \tilde{p}_1)$ and $(\tilde{q}_2, \tilde{p}_2)$ of the system (2.22), satisfying (2.21). Then, if we define $\mathrm{X} = \begin{pmatrix} \tilde{q}_1 - \tilde{q}_2 \\ \tilde{p}_1 - \tilde{p}_2 \end{pmatrix}$, we obtain, as a consequence of (2.21), that the following limits exist:

$$\lim_{|x|\to 0,\, \arg(x)=\vartheta} \frac{\mathrm{X}^{(i)}(x)}{|x|^\varepsilon} = 0, \quad i = 1, 2, \tag{2.23}$$

for some $0 < \varepsilon$, $\mathrm{X}^{(i)}$ being the i-th component of X. Moreover X satisfies the following:

$$\mathrm{X}' = \begin{pmatrix} \frac{[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)}{x} + \frac{\Delta \mathrm{Q}_1}{x^{1-l}} + \frac{\Delta \mathrm{Q}_2}{x^l} + \Delta \mathrm{Q}_3 \\ \frac{[l-1-2\tilde{q}_2(\tilde{p}_1+\tilde{p}_2)](\tilde{p}_1-\tilde{p}_2)-2\tilde{p}_1^2(\tilde{q}_1-\tilde{q}_2)}{x} + \frac{\Delta \mathrm{P}_1}{x^{1-l}} + \frac{\Delta \mathrm{P}_2}{x^l} + \Delta \mathrm{P}_3 \end{pmatrix},$$

where

$$\Delta Q_i = Q_i(\tilde{q}_1, \tilde{p}_1, x) - Q_i(\tilde{q}_2, \tilde{p}_2, x), \quad \text{and} \quad \Delta P_i = P_i(\tilde{q}_1, \tilde{p}_1, x) - P_i(\tilde{q}_2, \tilde{p}_2, x),$$

$Q_1 = \tilde{q}(1 + 2\tilde{p}\tilde{q})$, $Q_2 = -2\tilde{p}\tilde{q}$, $Q_3 = \frac{\tilde{q}(1+4\tilde{p}\tilde{q}) - 2x^{1-l}\tilde{p}\tilde{q} - x^l\tilde{q}^2(1+2\tilde{p}\tilde{q})}{1-x}$, $P_1 = \mu - \mu^2 + 2\tilde{p}\tilde{q} + 3\tilde{p}^2\tilde{q}^2$, $P_2 = \tilde{p}^2$, $P_3 = \frac{\tilde{p}^2 x^{1-l} - \tilde{p}(1+4\tilde{p}\tilde{q}) + x^l(\mu - \mu^2 + 2\tilde{p}\tilde{q} + 3\tilde{p}^2\tilde{q}^2)}{1-x}$.

We want to prove that, under the hypothesis (2.23), $X \equiv 0$ (this is equivalent to prove our theorem). Performing the constant linear transformation $X = T\,Z$, where

$$T = \begin{pmatrix} 1 & 0 \\ \frac{1-l}{2a^2} & \frac{1}{2a^2} \end{pmatrix},$$

we obtain

$$Z' = \begin{pmatrix} \frac{[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)}{x} + \frac{\Delta Q_1}{x^{1-l}} + \frac{\Delta Q_2}{x^l} + \Delta Q_3 \\ \frac{2a^2 G(\tilde{p}_1,\tilde{p}_2,\tilde{q}_1,\tilde{q}_2)}{x} + \frac{2a^2\Delta P_1 + (l-1)\Delta Q_1}{x^{1-l}} + \frac{2a^2\Delta P_2 + (l-1)\Delta Q_2}{x^l} + 2a^2\Delta P_3 + (l-1)\Delta Q_3 \end{pmatrix} \tag{2.24}$$

where

$$G(\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2) = [l-1-2\tilde{q}_2(\tilde{p}_1+\tilde{p}_2)](\tilde{p}_1-\tilde{p}_2) - 2\tilde{p}_1^2(\tilde{q}_1-\tilde{q}_2)\} + (l-1)\{[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2) + 2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2),$$

and, from (2.23):

$$\lim_{|x|\to 0,\, \arg(x)=\vartheta} \frac{Z^{(i)}(x)}{|x|^\varepsilon} = 0, \quad i = 1, 2. \tag{2.25}$$

In order to prove that $Z \equiv 0$, we fix any direction in the complex plane, namely we take x such that $\arg(x) = \vartheta$, for some fixed ϑ and we consider the real variable $t = |x|$. Then we define:

$$V^{(i)}(t) := |Z^{(i)}(x)|.$$

We want to prove that the assertion $V^{(i)}(t_0) \neq 0$ for some $t_0 > 0$ leads to an absurd. To this aim we prove a differential inequality for the right derivative $D_+V^{(i)}$ of $V^{(i)}(t)$. Since $D_+V^{(i)} \leq |Z^{(i)\prime}|$, to obtain such a differential inequality is enough to estimate from above the modulus of the components of the right-hand-side of (2.24). To this aim we notice that all the polynomials Q_i, P_i have the form:

$$Q_i = \sum_{k,n=0}^{3} a^i_{k,n}\tilde{p}^n\tilde{q}^k, \qquad P_i = \sum_{k,n=0}^{3} b^i_{k,n}\tilde{p}^n\tilde{q}^k,$$

with $a_{k,n}(x)$, $b_{k,n}(x)$ regular functions $x \in \mathcal{B}(0,r)$. As a consequence, we obtain, in the ball $||\tilde{p} - \frac{l-1}{2a}||$, $||\tilde{q} - a|| \leq C_r$, the estimates:

$$|\Delta Q_i|, \quad |\Delta P_i| \leq c^i_1|Z^{(1)}| + c^i_2|Z^{(2)}| \tag{2.26}$$

for some positive constants c^i_1, c^i_2. In-fact

$$|\Delta Q_i| = |\sum_{k,n} a_{k,n}[\tilde{q}_1^k(\tilde{p}_1^n - \tilde{p}_2^n) + \tilde{p}_2^n(\tilde{q}_1^k - \tilde{q}_2^k)]| \leq \left\{ \sum_{k=0,1,2} C_r^{(1)^k}(||a_{k,1}|| + 2C_r^{(2)}||a_{k,2}||) \right\}$$
$$\cdot |\tilde{p}_1 - \tilde{p}_2| + \left\{ \sum_{n=0,1,2,3} C_r^{(2)^n}(||a_{1,n}|| + 2C_r^{(1)}||a_{2,n}|| + 3C_r^{(1)^2}||a_{3,n}||) \right\} \cdot |\tilde{q}_1 - \tilde{q}_2|,$$

where $C_r^{(1)} = C_r + 2|a|$ and $C_r^{(2)} = C_r + \frac{1-l}{|a|}$. We obtain the (2.26) observing that $|\tilde{q}_1 - \tilde{q}_2|$, $|\tilde{p}_1 - \tilde{p}_2|$ are related to $|Z^{(1)}|$, $|Z^{(2)}|$ by the constant linear transformation T.

For the terms of order $\mathcal{O}(\frac{1}{x})$ in (2.24) we have:

$$\frac{|[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)|}{|x|} \leq \frac{|-(1-l)(\tilde{q}_1-\tilde{q}_2)+2a^2(\tilde{p}_1-\tilde{p}_2)|}{|x|}+ \\ + \frac{C_r^{(3)}|\tilde{q}_1-\tilde{q}_2|}{|x^{1-\varepsilon}|}+\frac{C_r^{(4)}|\tilde{p}_1-\tilde{p}_2|}{|x^{1-\varepsilon}|}, \tag{2.27}$$

and

$$\frac{1}{x}|2a^2\{[l-1-2\tilde{q}_2(\tilde{p}_1+\tilde{p}_2)](\tilde{p}_1-\tilde{p}_2)-2\tilde{p}_1^2(\tilde{q}_1-\tilde{q}_2)\}+(l-1)\{[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)]\cdot \\ (\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)\}| \leq \frac{C_r^{(5)}}{|x^{1-\varepsilon}|}|\tilde{q}_1-\tilde{q}_2|+\frac{C_r^{(6)}}{|x^{1-\varepsilon}|}|\tilde{p}_1-\tilde{p}_2|, \tag{2.28}$$

for some positive constants $C_r^{(3)},...,C_r^{(6)}$. Let us prove (2.27):

$$\frac{|[1-l+2\tilde{p}_1(\tilde{q}_1+\tilde{q}_2)](\tilde{q}_1-\tilde{q}_2)+2\tilde{q}_2^2(\tilde{p}_1-\tilde{p}_2)|}{|x|} \leq \frac{|-(1-l)(\tilde{q}_1-\tilde{q}_2)+2a^2(\tilde{p}_1-\tilde{p}_2)|}{|x|}+ \\ + \frac{|2ag_1(x)+\frac{l-1}{a}(f_1(x)+f_2(x))+x^\varepsilon g_1(x)(f_1(x)+f_2(x))|}{|x^{1-\varepsilon}|}|\tilde{q}_1-\tilde{q}_2|+ \\ + \frac{|2f_2^2(x)x^\varepsilon+4af_2(x)|}{|x^{1-\varepsilon}|}|\tilde{p}_1-\tilde{p}_2| \leq \frac{|-(1-l)(\tilde{q}_1-\tilde{q}_2)+2a^2(\tilde{p}_1-\tilde{p}_2)|}{|x|}+ \\ + \frac{C_r^{(3)}}{|x^{1-\varepsilon}|}|\tilde{q}_1-\tilde{q}_2|+\frac{C_r^{(4)}}{|x^{1-\varepsilon}|}|\tilde{p}_1-\tilde{p}_2|,$$

for some positive constants $C_r^{(3)}$ and $C_r^{(4)}$. The proof of (2.28) is analogous. From the estimates (2.26), (2.27), (2.28), we obtain:

$$\begin{pmatrix} |Z^{(1)\prime}| \\ |Z^{(2)\prime}| \end{pmatrix} \leq \left(\frac{1}{|x|}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}+\frac{A_1}{|x^{1-l}|}+\frac{A_2}{|x^l|}+\frac{A_3}{|x^{1-\varepsilon}|}+A_4\right)\begin{pmatrix} |Z^{(1)}| \\ |Z^{(2)}| \end{pmatrix} \tag{2.29}$$

for some constant matrices A_1, A_2, A_3 and A_4 (Here we mean $\leq$ component by component). Finally, choosing $\tilde{l}=\max\{1-\varepsilon,1-l,l\}$, we obtain from (2.29):

$$\begin{pmatrix} D_+V^{(1)} \\ D_+V^{(2)} \end{pmatrix} \leq \left(\frac{1}{t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}+\frac{\tilde{A}}{t^{\tilde{l}}}+A_4\right)\begin{pmatrix} V^{(1)} \\ V^{(2)} \end{pmatrix}, \tag{2.30}$$

for $\tilde{A}=A_1+A_2+A_3$. To show that $Z=0$ we use the following:

Comparison Theorem. *Let us consider the following systems of n first order ODEs in the real variable $t\in(0,a]$, for some $a>0$:*

$$D_+V^{(i)} \leq F^{(i)}(t,V), \quad V^{(i)}(x_0)=V_0^{(i)}, \quad i=1,..,n \tag{2.31}$$

$$\frac{dU^{(i)}}{dt}=F^{(i)}(t,U), \quad U^{(i)}(x_0)=U_0^{(i)}, \quad i=1,..,n \tag{2.32}$$

where $F^{(i)}(t,U)$ are continuous functions in $t\in(0,a]$, $||U-U_0||<b$, non-decreasing in $U^{(i)}$. If $V_0^{(i)}\geq U_0^{(i)}$, for $i=1,..n$, then $V^{(i)}(t)\geq U^{(i)}(t)$, for every $0<t\leq t_0$, $i=1,..n$.

For the proof see [Lak].

We now apply this lemma to show that the hypothesis $Z(t_0) \neq 0$ for some $t_0 > 0$ leads to an absurd. In fact our V satisfies (U9) with $V_0^{(i)} > 0$ and F linear in V given by:

$$F(t, V) = \left(\frac{1}{t} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{\tilde{A}}{t^{\tilde{l}}} + A_4 \right) V.$$

By the comparison theorem, for any solution of (2.32) with $U^{(i)}(t_0) = V_0^{(i)}$, we have $V^{(i)}(t) \geq U^{(i)}(t)$, for every $0 < t \leq t_0$, $i = 1, 2$. Moreover by standard arguments it is possible to take U in such a way that $U^{(i)}(t) \geq 0$ and to continue the functions U, V to $t = 0$ preserving the relation:

$$0 \leq U^{(i)}(t) \leq V^{(i)}(t);$$

then, by the (2.25) we obtain that

$$\lim_{t \to 0} \frac{U^{(i)}(t)}{t^{\varepsilon}} = 0, \quad i = 1, 2. \tag{2.33}$$

Now, we use the following lemma:

Lemma 2.4. *The only solution* U *of (2.31) satisfying (2.33) is* $U(t) := 0$,

Proof of the lemma 2.4. We perform the following change of variable $t^{1-\tilde{l}} = z$, the differential equation for U in the new variable z is:

$$\frac{dU}{dz} = \left(\frac{1}{z} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{1-\tilde{l}} + A(z) \right) U, \quad \text{with} \quad A(z) = \frac{\tilde{A}}{1-\tilde{l}} + \frac{A_4}{1-\tilde{l}} z^{\frac{\tilde{l}}{1-\tilde{l}}},$$

namely the matrix $A(z)$ is a rational function of $0 \leq z \leq r$. Defining $\tilde{U}$ such that $U = \begin{pmatrix} 0 & \log(z^{1/\tilde{l}}) \\ 0 & 0 \end{pmatrix} \tilde{U}$ we obtain:

$$\frac{d\tilde{U}}{dz} = \begin{pmatrix} -a_{21}(z) \log(z^{1/\tilde{l}}) & -a_{22}(z) \log(z^{1/\tilde{l}}) \\ 0 & 0 \end{pmatrix} \tilde{U} = \tilde{A}(z)\, \tilde{U}, \tag{2.34}$$

where the functions $a_{ij}(z)$ are the matrix elements of $A(z)$. The matrix $\tilde{A}(z)$ is a diagonalizable matrix function of $0 \leq z \leq r$ with integrable eigenvalues $0, -\log(z^{1/\tilde{l}}) a_{21}(z)$, and regular invertible diagonalizing matrix

$$\tilde{T}(z) = \begin{pmatrix} 1 & 1 \\ -\frac{a_{21}(z)}{a_{22}(z)} & 0 \end{pmatrix}.$$

Then the general solution of (2.34) is

$$\tilde{U} = \tilde{T}(z) \exp\left(\int_0^z \begin{pmatrix} 0 & 0 \\ 0 & -\log(\tilde{z}^{1/l}) a_{21}(\tilde{z}) \end{pmatrix} d\tilde{z} \right)$$

which gives:

$$U = U_0 \begin{pmatrix} 0 & \log(z^{1/\tilde{l}}) \\ 0 & 0 \end{pmatrix} \tilde{T}(z) \exp\left(\int_0^z \begin{pmatrix} 0 & 0 \\ 0 & -\log(\tilde{z}^{1/l}) a_{21}(\tilde{z}) \end{pmatrix} d\tilde{z} \right), \tag{2.35}$$

for some constant matrix U_0. Now it is obvious that (2.35) satisfies (2.33) if and only if $U_0 = 0$, namely $U \equiv 0$, as we wanted to prove. QDE

Using the above lemma, we obtain $U_0^{(i)} = V_0^{(i)} = 0$, and this is absurd. This concludes the proof of the uniqueness in the case $0 < l < 1$.

Let us briefly explain how to prove the uniqueness in the case $l = 1$; since the procedure is essentially the same as before, we shall skip the details. First of all we introduce some new variables $(\tilde{q}, \tilde{p})$:

$$\tilde{q}(x) = \frac{q(x)}{x} \sim a + x^{\varepsilon} f(x) \qquad \tilde{p}(x) = p(x) - \mu(1-\mu) \sim x^{\varepsilon} g(x)$$

which satisfy the equations of the motion:

$$\dot{\tilde{q}} = Q_1(\tilde{q},\tilde{p}) + \frac{1}{x-1}Q_2(\tilde{q},\tilde{p})$$
$$\dot{\tilde{p}} = -\frac{\tilde{p}}{x} + P_1(\tilde{q},\tilde{p}) + \frac{1}{x-1}P_2(\tilde{q},\tilde{p})$$

where $Q_1(\tilde{q},\tilde{p}) = 2(\mu-\mu^2+\tilde{p})(\tilde{q}-1)\tilde{q}^2$, $Q_2(\tilde{q},\tilde{p}) = \tilde{q}(\tilde{q}-1)[1+(2\mu(1-\mu)+2\tilde{p})(\tilde{q}-1)]$, $P_1(\tilde{q},\tilde{p}) = (\mu-\mu^2+\tilde{p})^2(2-3\tilde{q})\tilde{q}-\mu(1-\mu)$, and $P_2(\tilde{q},\tilde{p}) = \tilde{p}+(\mu-\mu^2+\tilde{p})^2(4\tilde{q}-3\tilde{q}^2-1)-2(\mu-\mu^2+\tilde{p})\tilde{q}$. Then, if we define X as before, we obtain

$$\mathrm{X}' = \begin{pmatrix} \Delta Q_1 + \frac{\Delta Q_2}{x-1} \\ -\frac{\tilde{p}_1-\tilde{p}_2}{x} + \Delta P_1 + \frac{\Delta P_2}{x-1} \end{pmatrix}$$

that gives rise to the differential inequality:

$$|\mathrm{X}'| \le \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{|x|} + \mathrm{A}_1 + \frac{\mathrm{A}_2}{|x-1|}\right)|\mathrm{X}|$$

for some constant matrices A_1 and A_2. Obviously X satisfies (2.23) with any $0<\varepsilon<1$. Again we apply the comparison theorem to $\mathrm{V} := \begin{pmatrix} |\mathrm{X}^{(1)}| \\ |\mathrm{X}^{(2)}| \end{pmatrix}$, along any fixed direction on the complex plane, namely we take x such that $\arg(t) = \vartheta$, for some fixed ϑ and define $t = |x|$. V satisfies (2.31) with:

$$F(t,\mathrm{V}) = \left(\frac{1}{t}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{A}_1 + \frac{\mathrm{A}_2}{t-1}\right)\mathrm{V}.$$

If, $\mathrm{V}_0^{(i)} > 0$, then, thanks to the comparison theorem, it is possible to take a solution U of (U10), with $\mathrm{U}_0^{(i)} = \mathrm{V}_0^{(i)}$, $i=1,2$, in such a way that:

$$0 \le \mathrm{U}^{(i)}(t) \le \mathrm{V}^{(i)}(t),$$

namely $\mathrm{U}^{(i)}$ satisfies (2.33). The general solution of (2.32) is

$$\mathrm{U} = \mathrm{U}_0\left(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} + \mathcal{O}(t^2)\right)$$

that satisfies (2.33) iff $\mathrm{U}_0 = 0$, namely $\mathrm{U} \equiv 0$ that is absurd. This concludes the proof of the uniqueness.

Asymptotic behaviour of the solutions of the Schlesinger equations. An important corollary to the theorem 2.2 is the following:

Theorem 2.3. *The solutions of the Schlesinger equations $A_{0,1,x}(x)$ corresponding to the solution of Painlevé VI equation with asymptotic behaviour (2.15) must satisfy the relations (2.16) and (2.17).*

Proof of the theorem 2.3. Let us consider the solution $y(x)$ of Painlevé VI equation with asymptotic behaviour (2.15) and let us suppose by absurd that the corresponding solutions of the Schlesinger equations $A_{0,1,x}(x)$ do not satisfy the relations (2.16) and (2.17). As shown in the lemma 2.2, for any constant matrices $A^0_{0,1,x}$, Λ such that $\Lambda = A^0_0 + A^0_x$ has eigenvalues $\pm\frac{\sigma}{2}$, $\sigma \in [0,1[$, and $A^0_1 = -\Lambda - A_\infty$, there exists a solution $\hat{A}_{0,1,x}(x)$ of the Schlesinger equations that satisfy the relations (2.16) and (2.17). Now, as shown in the section 2.2, we can choose $A^0_{0,1,x}$ in order that the corresponding solution $\hat{y}(x)$ of Painlevé VI equation has exactly the asymptotic behaviour (2.15). But for the uniqueness proved in theorem 2.2, we have that $y(x) = \hat{y}(x)$, namely $A_{0,1,x} = \hat{A}_{0,1,x}$, which is absurd. QDE

Asymptotic behaviour of the PVIμ transcendent around 1 and ∞. We now state the analogues of the theorem 2.2 for the local asymptotic behaviour of the solutions of (PVI) near the singular points $x = 1, \infty$:

Theorem 2.2'. *For any pair of values (a_1, σ_1), $\sigma_1 \in [0,1[$, there exists a unique branch of the solution of (PVI) with the asymptotic behaviour*

$$y(x) \sim 1 - a_1(1-x)^{1-\sigma_1} \quad as \quad x \to 1. \tag{2.36}$$

The proof of this theorem is analogous to the proof of theorem 1, namely one can state the analogous of the lemma 2.2 replacing $x \mapsto 1-x$, and then choose suitably Λ, $A^0_{0,1,x}$; the uniqueness is proved in the same way as the case $x \mapsto 0$.

Theorem 2.2". *For any pair of values $(a_\infty, \sigma_\infty)$, $\sigma_\infty \in [0,1[$, there exists a unique branch of the solution of (PVI) with the asymptotic behaviour*

$$y(x) \sim a_\infty x^{\sigma_\infty} \quad as \quad x \to \infty. \tag{2.37}$$

The proof of uniqueness is analogous to the one of theorem 2.2. The proof of existence follows the same strategy as the one of theorem 2.2, but with a different formulation of the lemma 2.2:

Lemma 2.2'. *Given some constant matrices A^0_i, $i = 0,1,x$ with zero eigenvalues such that $\Lambda = A^0_0 + A^0_t$ has eigenvalues $\pm\frac{\sigma}{2}$, $0 \le \sigma < 1$, in any sector of $\overline{\mathbb{C}}$ containing none of the branch cuts, and sufficiently close to ∞, there exists a solution of the Schlesinger which satisfy:*

$$|x^{A_\infty} A_x(x) x^{-A_\infty} - A^0_1| \le K|x|^{\sigma'-1} \quad \left|x^{\Lambda}\left(x^{A_\infty} A_x(x) x^{-A_\infty} - A^0_1\right) x^{-\Lambda}\right| \le K|x|^{\sigma'-1} \tag{2.38}$$

$$\left|x^{\Lambda} x^{A_\infty} A_{0,1}(x) x^{-A_\infty} x^{-\Lambda} - A^0_x\right| \le K|x|^{\sigma'-1}, \tag{2.39}$$

where K is some positive constant and $1 > \sigma' > \sigma$.

Proof of the lemma 2.2'. Let us consider the Schlesinger equations (2.3) and perform the change of variable $x = \frac{1}{\hat{x}}$. Moreover we put:

$$A_i(x) := x^{-A_\infty} \hat{A}_i(x) x^{A_\infty};$$

Then we can apply the lemma 2.2 to the system:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\hat{x}} \hat{A}_0(\hat{x}) &= -\frac{[\hat{A}_0, \hat{A}_1]}{\hat{x}}, \\
\frac{\mathrm{d}}{\mathrm{d}\hat{x}} \hat{A}_x(\hat{x}) &= -\frac{[\hat{A}_x, \hat{A}_1]}{\hat{x}-1}, \\
\frac{\mathrm{d}}{\mathrm{d}\hat{x}} \hat{A}_1(\hat{x}) &= \frac{[\hat{A}_0, \hat{A}_1]}{\hat{x}} + \frac{[\hat{A}_x, \hat{A}_1]}{\hat{x}-1},
\end{aligned}$$

and obtain the estimates (2.38) and (2.39). QDE

2.2. The local asymptotic behaviour and the monodromy group of the Fuchsian system

The asymptotic behaviour of the PVIμ transcendent around 0 and the monodromy matrices. In this section we relate the local asymptotic behaviour of the solution $y(x)$ of (PVI) to the monodromy data of the associated Fuchsian system (2.2). We essentially follow the same strategy of [Jim], even if we have to introduce some more tricks (see sections 4.2, 4.3, 4.4) due to the fact that our matrices $A^0_{0,1,x}$ are strongly resonant, namely their eigenvalues are all zero. The main result of this section is the following:

Theorem 2.4. *For the solution $y(x)$ of PVIμ, such that $y(x) \sim a_0 x^{1-\sigma_0}(1 + \mathcal{O}(x^\varepsilon))$, $0 < \sigma_0 < 1$, the monodromy matrices of the Fuchsian system (2.2) are parameterised as follows:*

$$\begin{aligned}
M_1 = &\frac{-i}{\sin \pi\vartheta_\infty} \cdot \\
&\cdot \begin{pmatrix} \cos \pi\sigma_0 - e^{-i\pi\vartheta_\infty} & -2e^{-i\pi\vartheta_\infty} \sin\frac{\pi(\vartheta_\infty+\sigma_0)}{2} \sin\frac{\pi(\vartheta_\infty-\sigma_0)}{2} \\ 2e^{i\pi\vartheta_\infty} \sin\frac{\pi(\vartheta_\infty+\sigma_0)}{2} \sin\frac{\pi(\vartheta_\infty-\sigma_0)}{2} & -\cos \pi\sigma_0 + e^{i\pi\vartheta_\infty} \end{pmatrix}
\end{aligned} \tag{2.40}$$

$$CM_xC^{-1} = \frac{-i}{\sin \pi\sigma_0} \begin{pmatrix} e^{i\pi\sigma_0} - 1 & 2se^{i\pi\sigma_0} \sin^2 \frac{\pi\sigma_0}{2} \\ -\frac{2}{s} e^{-i\pi\sigma_0} \sin^2 \frac{\pi\sigma_0}{2} & 1 - e^{-i\pi\sigma_0} \end{pmatrix} \tag{2.41}$$

$$CM_0C^{-1} = \frac{-i}{\sin \pi\sigma_0} \begin{pmatrix} e^{i\pi\sigma_0} - 1 & -2s \sin^2 \frac{\pi\sigma_0}{2} \\ \frac{2}{s} \sin^2 \frac{\pi\sigma_0}{2} & 1 - e^{-i\pi\sigma_0} \end{pmatrix} \tag{2.42}$$

where $\vartheta_\infty = 2\mu$ and:

$$\frac{s}{r} = \frac{1}{4a_0} \frac{2\mu + \sigma_0}{2\mu - \sigma_0} \frac{\Gamma^2(1+\sigma_0)\Gamma^2(1-\frac{\sigma_0}{2})\Gamma(1+\mu-\frac{\sigma_0}{2})\Gamma(1-\mu-\frac{\sigma_0}{2})}{\Gamma^2(1-\sigma_0)\Gamma^2(1+\frac{\sigma_0}{2})\Gamma(1+\mu+\frac{\sigma_0}{2})\Gamma(1-\mu+\frac{\sigma_0}{2})} \tag{2.43}$$

with an arbitrary complex number r and the matrix C is:

$$C = \begin{pmatrix} \sin \frac{\pi(\vartheta_\infty - \sigma_0)}{2} & r \sin \frac{\pi(\vartheta_\infty + \sigma_0)}{2} \\ \frac{1}{r} \sin \frac{\pi(\vartheta_\infty + \sigma_0)}{2} & \sin \frac{\pi(\vartheta_\infty - \sigma_0)}{2} \end{pmatrix}. \tag{2.44}$$

In the case where $\sigma_0 = 0$ the monodromy matrices of the Fuchsian system (2.2) are parameterised as follows:

$$M_1 = \frac{1}{\cos \frac{\pi\vartheta_\infty}{2}} \begin{pmatrix} e^{-i\pi\frac{\vartheta_\infty}{2}} & i\pi e^{-i\pi\frac{\vartheta_\infty}{2}} \\ -\frac{i \sin^2 \frac{\pi\vartheta_\infty}{2} e^{i\pi\frac{\vartheta_\infty}{2}}}{\pi} & e^{i\pi\frac{\vartheta_\infty}{2}} \end{pmatrix}, \tag{2.45}$$

$$M_0 = \begin{pmatrix} 1 - is \tan \frac{\pi\vartheta_\infty}{2} & \frac{-is\pi \exp(i\pi\frac{\vartheta_\infty}{2})}{\cos \frac{\pi\vartheta_\infty}{2}} \\ \frac{is \sin^2 \frac{\pi\vartheta_\infty}{2} \exp(-i\pi\frac{\vartheta_\infty}{2})}{\pi \cos \frac{\pi\vartheta_\infty}{2}} & 1 + is \tan \frac{\pi\vartheta_\infty}{2} \end{pmatrix}, \tag{2.46}$$

$$M_x = \begin{pmatrix} 1 - i(1-s) \tan \frac{\pi\vartheta_\infty}{2} & \frac{-i(1-s)\pi \exp(i\pi\frac{\vartheta_\infty}{2})}{\cos \frac{\pi\vartheta_\infty}{2}} \\ \frac{i(1-s) \sin^2 \frac{\pi\vartheta_\infty}{2} \exp(-i\pi\frac{\vartheta_\infty}{2})}{\pi \cos \frac{\pi\vartheta_\infty}{2}} & 1 + i(1-s) \tan \frac{\pi\vartheta_\infty}{2} \end{pmatrix}, \tag{2.47}$$

where $s = a_0$.

The main idea to prove this theorem is that, due to the theorem 2.3, the solutions of the Schlesinger equations corresponding to the PVIμ transcendent with the asymptotic behaviour (2.15) must satisfy the relations (2.16) and (2.17); using these relations we relate the monodromy matrices of the Fuchsian system (2.2), to the ones of two simpler systems, which are given in the following two lemmas (see [SMJ] and [Jim]):

Lemma 2.5. *Under the hypotheses (2.16), (2.17), the limit of the fundamental solution of the system (2.2), normalized at infinity, $\lim_{x\to 0} Y_\infty(z,x) = \hat{Y}(z)$, exists, for $z \in \overline{\mathbb{C}} \backslash \{B_0 \cup B_x \cup B_1 \cup B_\infty\}$, B_0 B_x, B_1 and B_∞ being balls around 0, x, 1 and ∞ respectively. Moreover this limit $\hat{Y}$ satisfies the differential equation:*

$$\frac{\mathrm{d}}{\mathrm{d}z}\hat{Y} = \left(\frac{A_1^0}{z-1} + \frac{\Lambda}{z}\right)\hat{Y}; \tag{$\hat{\Sigma}$}$$

and it has the following behaviour near the singularities of $(\hat{\Sigma})$

$$\begin{aligned} \hat{Y}(z) &= \left(1 + \mathcal{O}(\frac{1}{z})\right) z^{-A_\infty} && z \to \infty \\ &= (1 + \mathcal{O}(z))\, z^{\Lambda} \hat{C}_0 && z \to 0 \\ &= \hat{G}_1 (1 + \mathcal{O}(z-1))\, (z-1)^{J_1} \hat{C}_1 && z \to 1 \end{aligned} \tag{2.48}$$

where J_1 is the Jordan normal forms of A_1^0, $\hat{G}_1 J_1 \hat{G}_1^{-1} = A_1^0$, $A_\infty = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$ and $\hat{C}_0$, $\hat{C}_1$ are the connection matrices of the system $(\hat{\Sigma})$.

Remark. Observe that the matrix $\hat{C}_0$ is uniquely determined by the conditions (2.48).

Lemma 2.6. *Under the hypotheses (2.16), (2.17), the limit of the fundamental solution of the system (2.2), normalized around* 0, $\lim_{x\to 0} x^{-\Lambda}Y_0(xz,x) = \tilde{Y}(x)\hat{C}_0$ *exists for* $z \in \overline{\mathbb{C}}\backslash\{B_0 \cup B_x \cup B_1 \cup B_\infty\}$. *It satisfies the differential system:*

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{Y} = \left(\frac{A_x^0}{z-1} + \frac{A_0^0}{z}\right)\tilde{Y}; \tag{$\tilde{\Sigma}$}$$

and it has the following behaviour near the singularities of $(\tilde{\Sigma})$

$$\begin{aligned}\tilde{Y}(z) &= \left(1 + \mathcal{O}(\frac{1}{z})\right) z^{\Lambda} && z \to \infty\\ &= \tilde{G}_0\,(1+\mathcal{O}(z))\, z^{J_0}\tilde{C}_0 && z \to 0\\ &= \tilde{G}_x\,(1+\mathcal{O}(z-1))\,(z-1)^{J_x}\tilde{C}_x && z \to 1\end{aligned}$$

where $J_{0,x}$ *are the Jordan normal forms of* $A_{0,x}^0$, $G_{0,x}$ *are such that* $G_{0,x}J_{0,x}G_{0,x}^{-1} = A_{0,x}^0$ *and* $\tilde{C}_{0,1}$ *are the connection matrices of the system* $(\tilde{\Sigma})$.

As we have seen above, the matrices of the two systems have the following form:

$$A_0^0 = \frac{1}{2}\Lambda + F, \qquad A_x^0 = \frac{1}{2}\Lambda - F, \quad A_1^0 = -A_\infty - \Lambda,$$

for some constant matrix F, and for Λ and T such that

$$\Lambda = T\begin{pmatrix}\frac{\sigma}{2} & 0\\ 0 & -\frac{\sigma}{2}\end{pmatrix}T^{-1}. \tag{2.49}$$

Using the relations (2.6), we have that

$$F = T\begin{pmatrix}0 & \frac{b\sigma}{4}\\ \frac{-\sigma}{4b} & 0\end{pmatrix}T^{-1}, \tag{2.50}$$

for some parameter b. As a consequence the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ are determined, up to diagonal conjugation, by the four entries of the matrix T and by b.

Now, we explain how to compute the monodromy matrices of the original system (2.2), knowing the ones of the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$. Later we will show how to compute the matrices $A_{0,x,1}^0$ and the monodromy matrices of $(\hat{\Sigma})$ and $(\tilde{\Sigma})$.

Lemma 2.7. *Let* $\hat{M}_0, \hat{M}_1, \hat{M}_\infty = M_\infty$ *be the monodromy matrices of the system* $(\hat{\Sigma})$ *with respect to the fundamental matrix* $\hat{Y}$ *and the basis* $\hat{\gamma}_0 = \gamma_0\gamma_x$, γ_1 *in* $\pi_1\left(\overline{\mathbb{C}}\backslash\{0,1,\infty\}\right)$; *let* $\tilde{M}_0, \tilde{M}_1, \tilde{M}_\infty = \exp(-2\pi I\Lambda)$ *be the monodromy matrices of the system* $(\tilde{\Sigma})$ *with respect to the fundamental matrix* $\tilde{Y}$ *and the basis* $\tilde{\gamma}_0, \tilde{\gamma}_1 = \gamma_x$. *The the the monodromy matrices of the original system (2.2) are given by the formulae:*

$$M_0 = \hat{C}_0^{-1}\tilde{M}_0\hat{C}_0, \quad M_x = \hat{C}_0^{-1}\tilde{M}_1\hat{C}_0, \quad M_1 = \hat{M}_1, \tag{2.51}$$

where $\hat{C}_0$ *is defined by (2.48).*

Proof of the lemma 2.7. By the definition of $\hat{Y}$, the system $(\hat{\Sigma})$ is obtained by the merging of the singularities 0 and x of the system (2.2). We can then consider the path $\hat{\gamma}_0$ to be homotopic to $\gamma_0\gamma_x$, with $\hat{\gamma}_0$ not crossing a ball B_0 around) (see figure 8).

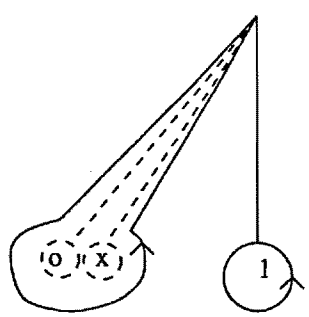

Fig.8. The paths γ_x and γ_0 merge together, as $x \to 0$.

As a consequence we obtain a relation between the monodromy matrices of the system (2.2) and the ones of the system $(\hat{\Sigma})$:

$$\begin{aligned}\hat{M}_\infty &= M_\infty,\\ \hat{M}_1 &= M_1,\\ \hat{M}_0 &= M_x M_0.\end{aligned}$$

Similarly, by the definition of $\tilde{Y}$ the system $(\tilde{\Sigma})$ is obtained by the merging (see figure 9) of the singularities $\frac{1}{x}$ and ∞ of the system for $Y'(xz') := Y(z)$:

$$\frac{\mathrm{d}}{\mathrm{d}z'}Y' = \left(\frac{A_0}{z'} + \frac{A_1}{z' - \frac{1}{x}} + \frac{A_x}{z'-1}\right)Y'.$$

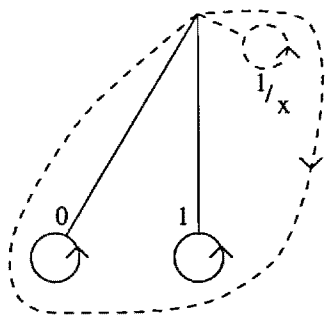

Fig.9. The paths $\gamma_{\frac{1}{x}}$ and γ_∞ merge together, as $x \to 0$, in the space of the z' variable.

So, in the basis $\hat{Y}$, the monodromy matrices of $(\tilde{\Sigma})$ have the following form:

$$\begin{aligned}\tilde{M}_\infty &= \hat{C}_0^{-1} M_\infty M_1 \hat{C}_0\\ \tilde{M}_1 &= \hat{C}_0^{-1} M_x \hat{C}_0,\\ \tilde{M}_0 &= \hat{C}_0^{-1} M_0 \hat{C}_0.\end{aligned}$$

The lemma is proved. QDE

Now we want to compute the monodromy matrices $\hat{M}_i$ and $\tilde{M}_i$ and the connection matrix $\hat{C}_0$. To this aim we have to solve the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$, namely we have to determine T and b. For $\sigma_0 \neq 0$, this can be done introducing a suitable gauge transformation of $\hat{Y}$ and $\tilde{Y}$ such that the differential systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ are equivalent to a Gauss equation. The case σ_0 will be treated later.

Reduction to the Gauss equation. First of all let us notice that both the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ have similar form; we want to reduce them, via a suitable gauge transformation, and a appropriate choice of the parameters α, β, γ, to systems of the form:

$$\frac{\mathrm{d}}{\mathrm{d}z}Y(z,\alpha,\beta,\gamma) = \left(\frac{B_0}{z} + \frac{B_1}{z-1}\right)Y(z,\alpha,\beta,\gamma) \tag{2.52}$$

where B_0, B_1 are some constant matrices with eigenvalues $1-\gamma$, 0 and $\gamma-\alpha-\beta-1$, 0 respectively and $B_0 + B_1 = -\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$.

Lemma 2.8. *For $\alpha \neq \beta$, the system (2.52) is uniquely determined, up to a diagonal conjugation*

$$B_0 \to T^{-1}B_0T, \quad B_1 \to T^{-1}B_1T, \quad \textit{with} \quad T = \begin{pmatrix}1 & 0\\ 0 & r\end{pmatrix}, r \neq 0. \tag{2.53}$$

The entries b^0_{ij} and b^1_{ij} of the matrices B_0 and B_1 respectively, are given by the formulae

$$b^0_{11} = \frac{\alpha(\gamma-1-\beta)}{\beta-\alpha}, \quad b^0_{22} = \frac{-\beta(\gamma-1-\alpha)}{\beta-\alpha}, \quad b^1_{11} = \frac{-\alpha(\gamma-1-\alpha)}{\beta-\alpha}, \tag{2.54}$$

$$b^1_{22} = \frac{\beta(\gamma-1-\beta)}{\beta-\alpha}, \quad b^0_{12}b^0_{21} = b^1_{12}b^1_{21} = \frac{-\alpha\beta(\gamma-1-\beta)(\gamma-1-\alpha)}{(\beta-\alpha)^2}. \tag{2.55}$$

The system (2.52) can be solved using the Gauss hypergeometric function. So, we can compute its connection matrices via the Kummer relations (see [Luke]) of the hypergeometric functions.

Lemma 2.9. *The solutions of (2.52) have the form* $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, *with y_1 being an arbitrary solution of the following Gauss equation:*

$$z(1-z)y_1'' + [c-(a+b+1)z]y_1' - aby_1 = 0 \tag{2.56}$$

where $a = \alpha+1$, $b = \beta$, $c = \gamma$ and y_2 given by:

$$y_2(z) = r\frac{\beta-\alpha}{\beta(\gamma-\beta-1)}\left\{z(z-1)\frac{\mathrm{d}}{\mathrm{d}z}y_1(z) + \left[\alpha z + \frac{\alpha(\gamma-\beta-1)}{\beta-\alpha}\right]y_1(z)\right\} \tag{2.57}$$

where r is the arbitrary parameter due to the ambiguity (2.53).

Proof of the lemma 2.9. Let us consider the system (2.52); performing the gauge:

$$Y(z,\alpha,\beta,\gamma) = z^{b^0_{11}}(1-z)^{b^1_{11}}U(z,\alpha,\beta,\gamma)$$

it is easy to show that u_1 satisfies the following Riemann equation:

$$u_1'' + \left[\frac{1+b^0_{11}-b^0_{22}}{z} + \frac{1+b^1_{11}-b^1_{22}+1}{z-1}\right]u_1' - \frac{b^0_{11}b^0_{22}}{z^2(z-1)^2}u_1 = 0.$$

Now u_1 is related with the solution y_G of the Gauss equation (2.56), with $a = -b^0_{11} - b^1_{11}$, $b = 1 - b^0_{22} - b^1_{22}$, $c = 1 - b^0_{11} - b^0 11$, via the relation $u_1 = z^{-b^0_{11}}(1-z)^{-b^1_{11}}y_G$. As a consequence, thanks to (2.54), (2.55), we obtain that $y_1 = y_G$ and $a = \alpha+1$, $b = \beta$, $c = \gamma$. y_2 it is given by:

$$\left(\frac{b^0_{12}}{z} + \frac{b^1_{12}}{z-1}\right)y_2 = y_1' - \left(\frac{b^0_{11}}{z} + \frac{b^1_{11}}{z-1}\right)y_1$$

that gives the equation (2.57) for $r = -\frac{\beta(\gamma-\beta-1)}{(\beta-\alpha)b^0_{12}}$. QDE

To reduce the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ to the system (2.52) we need to diagonalize the matrices $A^0_1 + \Lambda = -A_\infty$ and Λ respectively, and to perform a suitable gauge transform. We need to introduce some notations. We denote $C^{\alpha,\beta,\gamma}_{0,1}$ are the connection matrices of the system (2.52). The matrices $J_{0,1}$ are the Jordan normal forms of $B_{0,1}$ and the matrices $G^{\alpha,\beta,\gamma}_{0,1}$ are such that $G^{\alpha,\beta,\gamma}_{0,1}J_{0,1}\left(G^{\alpha,\beta,\gamma}_{0,1}\right)^{-1} = B_{0,1}$. Then for the asymptotic behaviour of an appropriate fundamental matrix $Y(z,\alpha,\beta,\gamma)$ of the system (2.52), we have:

$$\begin{aligned} Y(z,\alpha,\beta,\gamma) &= \left(1+\mathcal{O}(\tfrac{1}{z})\right)z^{-\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}} && z\to\infty \\ &= G^{\alpha,\beta,\gamma}_0\left(1+\mathcal{O}(z)\right)z^{J_0}C^{\alpha,\beta,\gamma}_0 && z\to 0 \\ &= G^{\alpha,\beta,\gamma}_1\left(1+\mathcal{O}(z-1)\right)(z-1)^{J_1}C^{\alpha,\beta,\gamma}_1 && z\to 1. \end{aligned}$$

Some further remarks on the notations: from now on all the quantities with the hat are referred to the system $(\hat{\Sigma})$ and all the quantities with the tilde to the system $(\tilde{\Sigma})$. When we don't put any hat or tilde, the

formulae are true for both the systems, namely they hold true for the generic system (2.52), and, substituting all the quantities with the correspondent hat or tilde ones, the formulae hold true for the systems $(\hat{\Sigma})$ or $(\tilde{\Sigma})$ respectively.

We now choose the values of α, β, γ in relation with the eigenvalues of the matrices of the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$. Namely for $(\hat{\Sigma})$ we take:

$$\hat{\alpha} = \frac{\vartheta_\infty - \sigma_0}{2}, \qquad \hat{\beta} = -\frac{\vartheta_\infty + \sigma_0}{2}, \qquad \hat{\gamma} = 1 - \sigma_0, \tag{2.58}$$

and for $(\tilde{\Sigma})$ we take:

$$\tilde{\alpha} = \frac{\sigma_0}{2}, \qquad \tilde{\beta} = -\frac{\sigma_0}{2}, \qquad \tilde{\gamma} = 1. \tag{2.59}$$

With this choice of the values of α, β, γ, one has:

$$\hat{J}_0 = \begin{pmatrix} 1-\gamma & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{J}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{J}_1 = \tilde{J}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now we can reduce the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ to the system (2.52) via the following gauge transformations:

$$\hat{Y} = z^{\frac{\hat{\alpha}+\hat{\beta}}{2}} Y(z, \hat{\alpha}, \hat{\beta}, \hat{\gamma}), \qquad \tilde{Y} = G_0^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}} Y(z, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}), \tag{2.60}$$

where $G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}}$ is such that

$$\Lambda - \frac{\hat{\alpha}+\hat{\beta}}{2}\mathrm{Id} = G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} (G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}})^{-1}.$$

As a consequence the connection matrices of (2.52) are related to the ones of $(\hat{\Sigma})$ and $(\tilde{\Sigma})$ by the following formulae:

$$\hat{G}_1 = G_1^{\hat{\alpha},\hat{\beta},\hat{\gamma}}, \quad \hat{C}_1 = C_1^{\hat{\alpha},\hat{\beta},\hat{\gamma}}, \quad \hat{C}_0 = G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} C_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} \tag{2.61}$$

$$\tilde{G}_{0,x} = G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}} G_{0,1}^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}, \qquad \tilde{C}_{0,x} = C_{0,1}^{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}} (G_0^{\hat{\alpha},\hat{\beta},\hat{\gamma}})^{-1}. \tag{2.62}$$

Local behaviour of the solution of (2.52). The solutions of (2.56) around the singular points $0, 1, \infty$ are known and one can compute y_2 by (2.57). In this way one obtains the local behaviour of the fundamental solution Y for $z \to 0, 1, \infty$, and one can compute the connection matrices by the Kummer relations (which are the connection formulae for the hypergeometric equation). The difference w.r.t. the situation of [Jim] is that in our case the Gauss equation is degenerate, namely:

$$\hat{c} - \hat{a} - \hat{b} = 0, \qquad \tilde{c} - \tilde{a} - \tilde{b} = 0, \qquad \tilde{c} = 1.$$

Then we have to consider the logarithmic solutions of the Gauss equation around $z = 1$ for both the systems $(\hat{\Sigma})$ and $(\tilde{\Sigma})$, and around $z = 0$ for $(\tilde{\Sigma})$; moreover we shall use the extension of the Kummer relations to this logarithmic case (see [Nor]).

In what follows we denote $F(a, b, c, z)$ the hypergeometric function and with $g(a, b, z)$ its logarithmic counterpart for $c = 1$, namely:

$$F(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k,$$

$$g(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!} z^k [\ln z + \psi(a+k) + \psi(b+k) - 2\psi(k+1)],$$

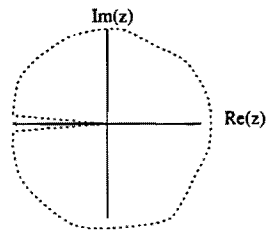

Fig.10. The branch cut $|\arg(z)| < \pi$.

with the branch cut $|\arg(z)| < \pi$ (see figure 10); here ψ is the logarithmic derivative of the gamma function, and the expressions of the parameters a, b, c via α, β, γ are given in the lemma 2.9.

Fundamental solution around ∞ Since $a - b \neq 0$, the solutions of (2.56) around ∞ are regular. We obtain

$$Y_\infty = \begin{pmatrix} z^{-\alpha} F(\alpha, -\beta, \alpha - \beta, \frac{1}{z}) & \frac{-\alpha\beta\, z^{-\beta-1} r}{(\beta-\alpha)(\beta-\alpha+1)} F(\beta+1, 1-\alpha, \beta-\alpha+2, \frac{1}{z}) \\ \frac{-\alpha\beta\, z^{-\alpha-1}}{r(\beta-\alpha)(\beta-\alpha-1)} F(\alpha+1, 1-\beta, \alpha-\beta+2, \frac{1}{z}) & z^{-\beta} F(\beta, \alpha, \beta-\alpha, \frac{1}{z}) \end{pmatrix}$$

namely:

$$Y_\infty \sim \left(1 + \mathcal{O}(\frac{1}{z})\right) z^{-\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}} \qquad z \to \infty$$

and the local monodromy around ∞ is $\begin{pmatrix} \exp(2\pi i\alpha) & 0 \\ 0 & \exp(2\pi i\beta) \end{pmatrix}$.

Fundamental solution around 1. Since $c - a - b = 0$, the solutions are logarithmic:

$$Y_1 = \begin{pmatrix} F(\alpha, \beta+1, 1, 1-z) & r\, g(\alpha, \beta+1, 1, 1-z) \\ \frac{1}{r} F(\alpha+1, \beta, 1, 1-z) & g(\alpha+1, \beta, 1, 1-z) \end{pmatrix},$$

namely:

$$Y_1 \sim G_1^{\alpha,\beta}\, (1-z)^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}},$$

with

$$G_1^{\alpha,\beta} = \begin{pmatrix} 1 & r[\psi(\alpha) + \psi(1+\beta) - 2\psi(1)] \\ \frac{1}{r} & \psi(1+\alpha) + \psi(\beta) - 2\psi(1) \end{pmatrix}$$

and the local monodromy around 1 is $\begin{pmatrix} 1 & 2i\pi r \\ 0 & 1 \end{pmatrix}$.

Fundamental solution around 0. We have to distinguish the case $(\hat{\Sigma})$, where the solutions of (2.56) around 0 are regular, and the case $(\tilde{\Sigma})$, where $c = 1$ and the solutions are logarithmic.

For $(\hat{\Sigma})$ one has:

$$\hat{Y}_0 = \begin{pmatrix} -\frac{\hat{\alpha}}{\hat{\beta}-\hat{\alpha}} z^{-\hat{\alpha}-\hat{\beta}} F(-\hat{\beta}, 1-\hat{\alpha}, 1-\hat{\alpha}-\hat{\beta}, z) & \hat{r}\frac{\hat{\beta}}{\hat{\beta}-\hat{\alpha}} F(\hat{\alpha}, \hat{\beta}+1, \hat{\alpha}+\hat{\beta}+1, z) \\ -\frac{\hat{\beta}}{\hat{r}(\hat{\beta}-\hat{\alpha})} z^{-\hat{\alpha}-\hat{\beta}} F(1-\hat{\beta}, -\hat{\alpha}, 1-\hat{\alpha}-\hat{\beta}, z) & \frac{\hat{\alpha}}{\hat{\beta}-\hat{\alpha}} F(\hat{\alpha}+1, \hat{\beta}, \hat{\alpha}+\hat{\beta}+1, z) \end{pmatrix}$$

namely

$$\hat{Y}_0 \sim G_0^{\hat{\alpha},\hat{\beta}}\, z^{\begin{pmatrix} -\hat{\alpha}-\hat{\beta} & 0 \\ 0 & 0 \end{pmatrix}}$$

where

$$G_0^{\hat{\alpha},\hat{\beta}} = \frac{1}{\hat{\beta}-\hat{\alpha}} \begin{pmatrix} -\hat{\alpha} & \hat{r}\hat{\beta} \\ -\frac{\hat{\beta}}{\hat{r}} & \hat{\alpha} \end{pmatrix}$$

and the local monodromy around 0 is $\begin{pmatrix} \exp(-2i\pi(\hat{\alpha}+\hat{\beta}) & 0 \\ 0 & 1 \end{pmatrix}$.

For $(\tilde{\Sigma})$ one has:

$$\tilde{Y}_0 = \begin{pmatrix} F(\tilde{\alpha}, 1-\tilde{\alpha}, 1, z) & \tilde{r}\, g(\tilde{\alpha}, 1-\tilde{\alpha}, 1, z) \\ -\frac{1}{\tilde{r}} F(\tilde{\alpha}+1, -\tilde{\alpha}, 1, z) & -g(\tilde{\alpha}+1, -\tilde{\alpha}, 1, z) \end{pmatrix},$$

namely:

$$\tilde{Y}_0 \sim G_0^{\tilde{\alpha}}\, z^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

with

$$G_0^{\tilde{\alpha}} = \begin{pmatrix} 1 & \tilde{r}[\psi(1-\tilde{\alpha}) + \psi(\tilde{\alpha}) - 2\psi(1)] \\ -\frac{1}{\tilde{r}} & -\psi(1+\tilde{\alpha}) - \psi(-\tilde{\alpha}) + 2\psi(1) \end{pmatrix}$$

and the local monodromy around 0 is $\begin{pmatrix} 1 & 2i\pi\tilde{r} \\ 0 & 1 \end{pmatrix}$.

Connection formulae. In order to compute the connection matrices we write Y_∞ in the form:

$$Y_\infty = \begin{pmatrix} \exp(-i\pi\alpha)\, U(\alpha, \beta+1, z) & \dfrac{-\alpha\beta \exp(-i\pi(\beta+1))r}{(\beta-\alpha)(\beta-\alpha+1)} U(\beta+1, \alpha, z) \\ \dfrac{-\alpha\beta \exp(-i\pi(\alpha+1))}{r(\beta-\alpha)(\beta-\alpha-1)} U(\alpha+1, \beta, z) & \exp(-i\pi\beta)\, U(\beta, \alpha+1, z) \end{pmatrix}$$

where $U(a,b,z) := \left(z^{-1}e^{i\pi}\right) F(a, 1-b, 1+a-b, \frac{1}{z})$. We have the following connection formulae:

$$\begin{aligned} U(a,b,z)\big|_{z\to 1} =& \frac{-\exp(i\pi a)\Gamma(1+a-b)}{\Gamma(a)\Gamma(1-b)} \big\{[i\pi + \psi(1-b) - \psi(b)] F(a,b,1,1-z) + \\ & g(a,b,1,1-z)\big\}, \end{aligned}$$

$$U(a, 1-a, z)\big|_{z\to 0} = \frac{-\Gamma(2a)}{\Gamma(a)^2} \left\{[-i\pi + \psi(a) - \psi(1-a)] F(a, 1-a, 1, z) + g(a, 1-a, 1, z)\right\},$$

$$\begin{aligned} U(a,b,z)\big|_{z\to 0} =& \frac{\Gamma(1+a-b)\Gamma(1-a-b)}{\Gamma(1-b)^2} F(a,b,a+b,z) + \frac{\Gamma(1+a-b)\Gamma(a+b-1)}{\Gamma(a)^2} \cdot \\ & \cdot z^{1-a-b} F(1-b, 1-a, 2-a-b, z). \end{aligned}$$

Using these relations we obtain the analytic continuation of Y_∞ around 0 and 1, and by the definition of the connection matrices:

$$Y_\infty\big|_{z\to 0,1} = Y_{0,1} C_{0,1}^{\alpha,\beta,\gamma},$$

we obtain, by straightforward computations:

$$C_0^{\hat{\alpha},\hat{\beta}} = \begin{pmatrix} \frac{\exp(i\pi\hat{\beta})\Gamma(\hat{\alpha}+\hat{\beta})\Gamma(\hat{\alpha}-\hat{\beta}+1)}{\Gamma(\hat{\alpha})\Gamma(1+\hat{\alpha})} & -\hat{r}\frac{\exp(-i\pi\hat{\alpha})\Gamma(\hat{\alpha}+\hat{\beta})\Gamma(1-\hat{\alpha}+\hat{\beta})}{\Gamma(\hat{\beta})\Gamma(1+\hat{\beta})} \\ \frac{1}{\hat{r}}\frac{\exp(-i\pi\hat{\alpha})\Gamma(-\hat{\alpha}-\hat{\beta})\Gamma(\hat{\alpha}-\hat{\beta}+1)}{\Gamma(1-\hat{\beta})\Gamma(-\hat{\beta})} & -\frac{\exp(-i\pi\hat{\beta})\Gamma(-\hat{\alpha}-\hat{\beta})\Gamma(1-\hat{\alpha}+\hat{\beta})}{\Gamma(-\hat{\alpha})\Gamma(1-\hat{\alpha})} \end{pmatrix},$$

$$C_1^{\hat{\alpha},\hat{\beta}} = \begin{pmatrix} -\frac{\Gamma(\hat{\alpha}-\hat{\beta})}{\Gamma(\hat{\alpha})\Gamma(-\hat{\beta})}[i\pi + \pi\cot(\pi\hat{\alpha})] & -\frac{\Gamma(-\hat{\alpha}+\hat{\beta})}{\Gamma(-\hat{\alpha})\Gamma(\hat{\beta})}[i\pi + \pi\cot(\pi\hat{\beta})] \\ -\frac{1}{\hat{r}}\frac{\Gamma(\hat{\alpha}-\hat{\beta})}{\Gamma(\hat{\alpha})\Gamma(-\hat{\beta})} & -\frac{\Gamma(-\hat{\alpha}+\hat{\beta})}{\Gamma(-\hat{\alpha})\Gamma(\hat{\beta})} \end{pmatrix}, \tag{2.63}$$

$$C_0^{\tilde{\alpha}} = \begin{pmatrix} \frac{\Gamma(2\tilde{\alpha})}{\Gamma^2(\tilde{\alpha})}\exp(-i\pi\tilde{\alpha})[i\pi + \pi\cot(\pi\tilde{\alpha})] & \tilde{r}\frac{\Gamma(-2\tilde{\alpha})}{\Gamma^2(-\tilde{\alpha})}\exp(i\pi\tilde{\alpha})[\pi\cot(\pi\tilde{\alpha}) - i\pi] \\ -\frac{1}{\tilde{r}}\frac{\Gamma(2\tilde{\alpha})}{\Gamma^2(\tilde{\alpha})}\exp(-i\pi\tilde{\alpha}) & \frac{\Gamma(-2\tilde{\alpha})}{\Gamma^2(-\tilde{\alpha})}\exp(i\pi\tilde{\alpha}) \end{pmatrix}, \tag{2.64}$$

$$C_1^{\tilde{\alpha}} = \begin{pmatrix} -\frac{\Gamma(2\tilde{\alpha})}{\Gamma^2(\tilde{\alpha})}[i\pi - \pi\cot(\pi\tilde{\alpha})] & -\tilde{r}\frac{\Gamma(-2\tilde{\alpha})}{\Gamma^2(-\tilde{\alpha})}[\pi\cot(\pi\tilde{\alpha}) + i\pi] \\ -\frac{1}{\tilde{r}}\frac{\Gamma(2\tilde{\alpha})}{\Gamma^2(\tilde{\alpha})} & -\frac{\Gamma(-2\tilde{\alpha})}{\Gamma^2(-\tilde{\alpha})} \end{pmatrix}.$$

Now we have to compute the monodromy matrices; using the formulae (2.51), (2.61) and (2.62) we have:

$$M_1 = (C_1^{\hat\alpha,\hat\beta})^{-1}\begin{pmatrix}1 & 2\pi i\hat r\\ 0 & 1\end{pmatrix}C_1^{\hat\alpha,\hat\beta}, \qquad M_{0,x} = (C_0^{\hat\alpha,\hat\beta})^{-1}(C_{0,1}^{\tilde\alpha})^{-1}\begin{pmatrix}1 & 2\pi i\tilde r\\ 0 & 1\end{pmatrix}C_{0,1}^{\tilde\alpha}C_0^{\hat\alpha,\hat\beta}.$$

Now we fix

$$\hat r = \frac{-\exp[-i\pi(\hat\alpha-\hat\beta)]\Gamma(\hat\alpha-\hat\beta)\Gamma(\hat\beta)\Gamma(-\hat\alpha)}{\Gamma(\hat\beta-\hat\alpha)\Gamma(-\hat\beta)\Gamma(\hat\alpha)}r$$

and

$$\tilde r = \frac{\exp(-2i\pi\tilde\alpha)\Gamma(2\tilde\alpha)\Gamma(-\tilde\alpha)^2}{\Gamma(-2\tilde\alpha)\Gamma(\tilde\alpha)^2}\tilde s.$$

In this way we immediately obtain the formula (2.40) for M_1 and it turns out that

$$C_0^{\hat\alpha,\hat\beta} = D^{\hat\alpha,\hat\beta}\cdot C$$

where C is given in the formula (2.44) and:

$$D^{\hat\alpha,\hat\beta} := \begin{pmatrix}\frac{\exp(i\pi\beta)\Gamma(\hat\beta+\hat\alpha)\Gamma(1-\hat\beta+\hat\alpha)}{\hat\alpha\Gamma(\hat\alpha)^2\sin\pi\hat\alpha} & 0\\ 0 & \frac{-\exp(-i\pi\beta)\Gamma(-\hat\beta-\hat\alpha)\Gamma(1+\hat\beta-\hat\alpha)}{\Gamma(1-\hat\alpha)\Gamma(-\hat\alpha)\sin\pi\hat\alpha}\end{pmatrix}.$$

As a consequence, one has:

$$CM_{0,x}C^{-1} = (D^{\hat\alpha,\hat\beta})^{-1}(C_{0,1}^{\tilde\alpha})^{-1}\begin{pmatrix}1 & 2\pi i\tilde r\\ 0 & 1\end{pmatrix}C_{0,1}^{\tilde\alpha}D^{\hat\alpha,\hat\beta}.$$

By straightforward computations one can easily check that, for

$$\tilde s = -\frac{\exp(2i\pi\beta)\Gamma(\hat\beta+\hat\alpha)\Gamma(1-\hat\beta+\hat\alpha)\Gamma(1-\hat\alpha)\Gamma(-\hat\alpha)}{\Gamma(-\hat\beta-\hat\alpha)\Gamma(1+\hat\beta-\hat\alpha)\hat\alpha\Gamma(\hat\alpha)^2}s$$

namely:

$$\frac{\tilde r}{\hat r} = -\frac{\Gamma(1-\sigma_0)^2\Gamma(\frac{\sigma_0}{2})^2}{\Gamma(1+\sigma_0)^2\Gamma(-\frac{\sigma_0}{2})^2}\frac{\Gamma(1+\frac{\vartheta_\infty+\sigma_0}{2})\Gamma(1+\frac{-\vartheta_\infty+\sigma_0}{2})}{\Gamma(1+\frac{\vartheta_\infty-\sigma_0}{2})\Gamma(1-\frac{\vartheta_\infty+\sigma_0}{2})}\frac{s}{r},$$

the formulae (2.41), (2.42) hold true.

To conclude the proof we have to prove the relation (2.43), namely we want to prove that $\frac{\tilde r}{\hat r} = -\frac{1}{4a_0}\frac{2\mu+\sigma_0}{2\mu-\sigma_0}$. To this aim we compute the matrices $A^0_{0,1,x}$ and Λ and then the asymptotic behaviour of y in terms of σ_0 and $\tilde r$. To compute the matrices $A^0_{0,1,x}$ and Λ we observe that, thanks to the gauges (2.60),

$$A_1^0 = \hat B_1, \qquad \Lambda = \hat B_0 + \frac{\hat\alpha+\hat\beta}{2}\mathbf{1}, \qquad A^0_{0,x} = G_0^{\hat\alpha,\hat\beta}\tilde B_{0,1}(G_0^{\hat\alpha,\hat\beta})^{-1}.$$

First of all one has to compute the $B_{0,1}$:

$$B_0 = G_0^{\alpha,\beta,\gamma}J_0\,(G_0^{\alpha,\beta,\gamma})^{-1}, \qquad B_1 = G_1^{\alpha,\beta,\gamma}J_1\,(G_1^{\alpha,\beta,\gamma})^{-1}$$

then

$$\hat B_0 = \frac{1}{\hat\beta-\hat\alpha}\begin{pmatrix}\alpha^2 & -\hat r\hat\alpha\beta\\ \frac{\hat\alpha\beta}{\hat r} & -\hat\beta^2\end{pmatrix}, \qquad \hat B_1 = \frac{\hat\alpha\hat\beta}{\hat\beta-\hat\alpha}\begin{pmatrix}-1 & \hat r\\ -\frac{1}{\hat r} & 1\end{pmatrix},$$

and

$$\tilde B_1 = \frac{\tilde\alpha}{2}\begin{pmatrix}-1 & \tilde r\\ -\frac{1}{\tilde r} & 1\end{pmatrix}, \qquad \tilde B_0 = -\frac{\tilde\alpha}{2}\begin{pmatrix}1 & \tilde r\\ -\frac{1}{\tilde r} & -1\end{pmatrix}.$$

It is then obvious that, referring to (2.49) and (2.50), $b = \tilde{r}$, $T = G_0^{\hat{\alpha},\hat{\beta}}$, namely, using the formula (2.18),

$$y(x) \sim -\frac{\hat{r}(\sigma_0 + 2\mu)}{4\tilde{r}(2\mu - \sigma_0)} x^{1-\sigma_0}.$$

This proves the formula (2.43) and concludes the proof of the theorem, in the case $\sigma_0 \neq 0$.

For completeness we write here the result for the matrices $A^0_{0,1,x}$ and Λ:

$$\Lambda = \frac{1}{4\vartheta_\infty}\begin{pmatrix} -\vartheta_\infty{}^2 - \sigma_0^2 & (-\vartheta_\infty{}^2 + \sigma_0^2)\hat{r} \\ \frac{-\sigma_0^2 + \vartheta_\infty{}^2}{\hat{r}} & \vartheta_\infty{}^2 + \sigma_0^2 \end{pmatrix}, \qquad A_1^0 = \frac{\vartheta_\infty{}^2 - \sigma_0^2}{4\vartheta_\infty}\begin{pmatrix} -1 & \hat{r} \\ -\frac{1}{\hat{r}} & 1 \end{pmatrix}, \tag{2.65}$$

$$A_0^0 = \frac{1}{8\vartheta_\infty}\begin{pmatrix} \frac{\vartheta_\infty{}^2 - \sigma_0^2}{2}\left(\frac{\tilde{r}}{\hat{r}} + \frac{\hat{r}}{\tilde{r}}\right) - \vartheta_\infty{}^2 - \sigma_0^2 & \hat{r}\left(\frac{(\vartheta_\infty - \sigma_0)^2\tilde{r}}{2\hat{r}} + \frac{(\vartheta_\infty + \sigma_0)^2\hat{r}}{2\tilde{r}} + \sigma_0^2 - \vartheta_\infty{}^2\right) \\ \frac{1}{\hat{r}}\left(-\frac{(\vartheta_\infty - \sigma_0)^2\hat{r}}{2\tilde{r}} - \frac{(\vartheta_\infty + \sigma_0)^2\tilde{r}}{2\hat{r}} + \vartheta_\infty{}^2 - \sigma_0^2\right) & -\frac{\vartheta_\infty{}^2 - \sigma_0^2}{2}\left(\frac{\tilde{r}}{\hat{r}} + \frac{\hat{r}}{\tilde{r}}\right) + \vartheta_\infty{}^2 + \sigma_0^2 \end{pmatrix}, \tag{2.66}$$

$$A_x^0 = \frac{1}{8\vartheta_\infty}\begin{pmatrix} \frac{\sigma_0^2 - \vartheta_\infty{}^2}{2}\left(\frac{\tilde{r}}{\hat{r}} + \frac{\hat{r}}{\tilde{r}}\right) - \vartheta_\infty{}^2 - \sigma_0^2 & \hat{r}\left(-\frac{(\vartheta_\infty - \sigma_0)^2\tilde{r}}{2\hat{r}} - \frac{(\vartheta_\infty + \sigma_0)^2\hat{r}}{2\tilde{r}} + \sigma_0^2 - \vartheta_\infty{}^2\right) \\ \frac{1}{\hat{r}}\left(\frac{(\vartheta_\infty - \sigma_0)^2\hat{r}}{2\tilde{r}} + \frac{(\vartheta_\infty + \sigma_0)^2\tilde{r}}{2\hat{r}} + \vartheta_\infty{}^2 - \sigma_0^2\right) & \frac{\vartheta_\infty{}^2 - \sigma_0^2}{2}\left(\frac{\tilde{r}}{\hat{r}} + \frac{\hat{r}}{\tilde{r}}\right) + \vartheta_\infty{}^2 + \sigma_0^2 \end{pmatrix}, \tag{2.67}$$

Case of $\sigma_0 = 0$. In this case the solution of the system $(\hat{\Sigma})$ has logarithmic behaviour around 0. Moreover, as seen before, it has a logarithmic behaviour around 1; then for this system we can use all the formulae derived for $(\tilde{\Sigma})$, substituting $\tilde{\alpha}$ by $\hat{\alpha}$. The treatment of the $(\tilde{\Sigma})$, it's even easier; in fact in this case Λ has zero eigenvalues and it is straightforward to solve the system (2.52) exactly. In fact in this case we have

$$B_0 + B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \det B_i = \mathrm{Tr} B_i = 0, \quad i = 0, 1.$$

Then the matrices B_0 and B_1 are uniquely determined up to an arbitrary parameter s:

$$B_0 = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1-s \\ 0 & 0 \end{pmatrix},$$

and we can solve the differential equation (2.52) explicitly:

$$\tilde{Y} = \begin{pmatrix} 1 & s\log z + (1-s)\log(z-1) \\ 0 & 1 \end{pmatrix}.$$

The solution $\tilde{Y}$ has the following asymptotic behaviour near the singular points:

$$\begin{aligned} \tilde{Y} &= \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) z^J, \qquad \text{as} \qquad z \to \infty, \\ &= \tilde{G}_0\,(1 + \mathcal{O}(z))\, z^J \tilde{C}_0, \qquad \text{as} \qquad z \to 0, \\ &= \tilde{G}_1\,(1 + \mathcal{O}(z-1))\,(z-1)^J \tilde{C}_1 \qquad \text{as} \qquad z \to 1, \end{aligned}$$

where $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It's easy to verify that

$$\tilde{C}_0 = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad \tilde{C}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-s \end{pmatrix}, \quad \tilde{G}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{s} \end{pmatrix}, \quad \tilde{G}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-s} \end{pmatrix}.$$

As a consequence the monodromy matrices of the system (2.52) are

$$\tilde{M}_0 = \begin{pmatrix} 1 & 2\pi i s \\ 0 & 1 \end{pmatrix}, \quad \tilde{M}_1 = \begin{pmatrix} 1 & 2\pi i(1-s) \\ 0 & 1 \end{pmatrix}. \tag{2.68}$$

The correspondent monodromy matrices of the full system (2.2) are given by:

$$M_1 = (C_1^{\hat{\alpha}})^{-1} \begin{pmatrix} 1 & 2i\pi\hat{r} \\ 0 & 1 \end{pmatrix} C_1^{\hat{\alpha}}, \quad \text{and} \quad M_{0,x} = (C_0^{\hat{\alpha}})^{-1} \tilde{M}_{0,1} C_0^{\hat{\alpha}},$$

where:

$$C_0^{\hat{\alpha}} = \begin{pmatrix} \frac{\pi\Gamma(2\hat{\alpha})}{\Gamma^2(\hat{\alpha})\sin\pi\hat{\alpha}} & \hat{r}\frac{\pi\Gamma(-2\hat{\alpha})}{\Gamma^2(-\hat{\alpha})\sin\pi\hat{\alpha}} \\ -\frac{1}{\hat{r}}\frac{\Gamma(2\hat{\alpha})\exp(-i\pi\hat{\alpha})}{\Gamma^2(\hat{\alpha})} & \frac{\Gamma(-2\hat{\alpha})\exp(i\pi\hat{\alpha})}{\Gamma^2(-\hat{\alpha})} \end{pmatrix},$$

and

$$C_1^{\hat{\alpha}} = \begin{pmatrix} \frac{\pi\Gamma(2\hat{\alpha})\exp(-i\pi\hat{\alpha})}{\Gamma^2(\hat{\alpha})\sin\pi\hat{\alpha}} & -\hat{r}\frac{\pi\Gamma(-2\hat{\alpha})\exp(i\pi\hat{\alpha})}{\Gamma^2(-\hat{\alpha})\sin\pi\hat{\alpha}} \\ -\frac{1}{\hat{r}}\frac{\Gamma(2\hat{\alpha})}{\Gamma^2(\hat{\alpha})} & -\frac{\Gamma(-2\hat{\alpha})}{\Gamma^2(-\hat{\alpha})} \end{pmatrix}.$$

We observe that

$$\hat{C}_{0,1}^{\hat{\alpha}} = C^{0,1} D^{\hat{\alpha}},$$

where:

$$D^{\hat{\alpha}} = \begin{pmatrix} \frac{\pi\Gamma(2\hat{\alpha})\exp(-i\pi\hat{\alpha})}{\Gamma(\hat{\alpha})^2\sin\pi\hat{\alpha}} & 0 \\ 0 & \frac{-\Gamma(-2\hat{\alpha})}{\Gamma(-\hat{\alpha})^2} \end{pmatrix},$$

and

$$C^0 = \begin{pmatrix} \exp(i\pi\hat{\alpha}) & \frac{-\hat{r}\pi}{\sin\pi\hat{\alpha}} \\ \frac{-\sin\pi\hat{\alpha}}{\pi\hat{r}} & \exp(i\pi\hat{\alpha}) \end{pmatrix}, \qquad C^1 = \begin{pmatrix} 1 & \frac{\hat{r}\pi\exp(i\pi\hat{\alpha})}{\sin\pi\hat{\alpha}} \\ \frac{-\sin\pi\hat{\alpha}\exp(i\pi\hat{\alpha})}{\pi\hat{r}} & 1 \end{pmatrix}.$$

We can factor out the diagonal matrix $D^{\hat{\alpha}}$ in (2.68), and take $\hat{r} = 1$. In this way, we obtain the formulae (2.45), (2.46), (2.47). The asymptotic behaviour of $y(x)$ can be computed as before. For $\sigma_0 = 0$ we obtain:

$$y \sim a_0 x, \quad \text{for} \quad a_0 = s.$$

This concludes the proof of the theorem.

The asymptotic behaviour around $1, \infty$ and the monodromy data. We can prove the analogous of the theorem 2.2 around 1 and ∞; namely for any pair of values (a_1, σ_1) there exists a unique branch of the solution of (PVI) with the asymptotic behaviour

$$y(x) \sim 1 - a_1 x^{1-\sigma_1} \quad \text{as} \quad x \to 1, \tag{2.69}$$

and it is possible to parameterise the monodromy matrices as in the theorem 2.2, substituting σ_0 with σ_1 and M_0 with M_1 and vice-versa. Analogously for any pair of values $(a_\infty, \sigma_\infty)$ there exists a unique branch of the solution of (PVI) with the asymptotic behaviour

$$y(x) \sim a_\infty x^{\sigma_\infty} \quad \text{as} \quad x \to \infty, \tag{2.70}$$

and it possible to parameterise the monodromy matrices as before, substituting σ_0 with σ_∞ and applying the braid β_2 to the monodromy matrices.

2.3. From the local asymptotic behaviour to the global one.

In this section we prove the theorem 2.1, which gives the asymptotic behaviour of the branches of the solutions in terms of the triplets (x_0, x_1, x_∞).

Lemma 2.10. *For the solution $y^{(0)}(x)$ of PVIμ behaving as*

$$y^{(0)}(x) = a_0 x^{1-\sigma_0}\left(1 + \mathcal{O}(x^\epsilon)\right) \quad \text{as} \quad x \to 0,$$

with $0 \le \sigma_0 < 1$ and $a_0 \neq 0$, $a_0 \neq 1$ for $\sigma_0 = 0$, the canonical form (1.19) of the monodromy matrices $M_0^{(0)}, M_x^{(0)}, M_1^{(0)}$ given by (2.42), (2.41), (2.40), or (2.46), (2.47), (2.45) for $\sigma_0 = 0$, is the following:

$$M_0 = \begin{pmatrix} 1 & -x_0^{(0)} \\ 0 & 1 \end{pmatrix}, \quad M_x = \begin{pmatrix} 1 & 0 \\ x_0^{(0)} & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 + \frac{x_1^{(0)} x_\infty^{(0)}}{x_0^{(0)}} & -\frac{(x_1^{(0)})^2}{x_0^{(0)}} \\ \frac{(x_\infty^{(0)})^2}{x_0^{(0)}} & 1 - \frac{x_1^{(0)} x_\infty^{(0)}}{x_0^{(0)}} \end{pmatrix},$$

where the triple $(x_0^{(0)}, x_1^{(0)}, x_\infty^{(0)})$ is defined, up to equivalence, by the following formulae, for $\sigma_0 \neq 0$:

$$\begin{aligned} x_0^{(0)} &= -2 \sin \frac{\pi \sigma_0}{2}, \\ x_1^{(0)} &= -\sqrt{2(\cos \pi\sigma_0 - \cos 2\pi\mu)} \frac{\sin \frac{\pi \phi}{2}}{\sin \frac{\pi \sigma + 0}{2}}, \\ x_\infty^{(0)} &= -\sqrt{2(\cos \pi\sigma_0 - \cos 2\pi\mu)} \frac{\cos \frac{\pi(\sigma_0 + \phi)}{2}}{\sin \frac{\pi \sigma + 0}{2}}, \end{aligned} \tag{2.71}$$

with ϕ given by

$$e^{i\pi\phi} = \frac{1}{4a_0} \frac{\sigma_0 + 2\mu}{\sigma_0 - 2\mu} \frac{\Gamma(1+\sigma_0)^2 \Gamma(1 - \frac{\sigma_0}{2})^2 \Gamma(1 + \mu - \frac{\sigma_0}{2}) \Gamma(1 - \mu - \frac{\sigma_0}{2})}{\Gamma(1-\sigma_0)^2 \Gamma(1 + \frac{\sigma_0}{2})^2 \Gamma(1 + \mu + \frac{\sigma_0}{2}) \Gamma(1 - \mu + \frac{\sigma_0}{2})}, \tag{2.72}$$

and for $\sigma_0 = 0$:

$$\begin{aligned} x_0^{(0)} &= 0, \\ x_1^{(0)} &= -|\sin \pi\mu| \sqrt{1 - a_0} \\ x_\infty^{(0)} &= -|\sin \pi\mu| \sqrt{a_0}. \end{aligned} \tag{2.73}$$

The proof of this lemma can be obtained by straightforward computations, using the algorithm of lemma 1.5. Similar formulae for the parameters $(x_0^{(1)}, x_1^{(1)}, x_\infty^{(1)})$ and $(x_0^{(\infty)}, x_1^{(\infty)}, x_\infty^{(\infty)})$ can be obtained respectively starting from a solution $y^{(1)}(x)$ of PVIμ behaving as

$$y^{(1)}(x) = 1 - a_1 (1-x)^{1-\sigma_1} (1 + \mathcal{O}(1-x)^\varepsilon) \quad \text{as} \quad x \to 1,$$

or from another solution $y^{(\infty)}(x)$ of PVIμ behaving as

$$y^{(\infty)}(x) = a_\infty x^{\sigma_\infty} \left(1 + \mathcal{O}(\frac{1}{x^\varepsilon})\right) \quad \text{as} \quad x \to \infty.$$

So, given an admissible triple (x_0, x_1, x_∞), with $x_i \in \mathbb{R}$, $|x_i| < 2$ for $i = 0, 1, \infty$, we choose the parameters μ, (a_0, σ_0), (a_1, σ_1) and $(a_\infty, \sigma_\infty)$ in such a way that (2.8) is satisfied and

$$x_i^{(0)} = x_i^{(1)} = x_i^{(\infty)} = x_i, \quad \text{for} \quad i = 0, 1, \infty.$$

Using the explicit formulae (2.71), (2.72), for $x_0 \neq 0$, we derive the expressions (2.10), (2.11), and using (2.73) for $x_0 = 0$, we derive the expression (2.12). In the same way, we derive the analogous expressions for (a_1, σ_1) and $(a_\infty, \sigma_\infty)$. The three correspondent branches $y^{(0)}(x), y^{(1)}(x), y^{(\infty)}(x)$o of solutions of PVI$\mu$, with μ given by (2.8), must coincide. In fact, the associated auxiliary Fuchsian systems have the same, modulo diagonal conjugation, monodromy matrices. This proves the existence of a solution of PVIμ with the asymptotic behaviour (2.1), with indices given by (2.9) and the coefficients specified as above, for any admissible triple (x_0, x_1, x_∞), with $x_i \in \mathbb{R}$, $|x_i| < 2$ for $i = 0, 1, \infty$. The uniqueness of such a branch follows from theorem 1.3.

Conversely, for any such a solution we obtain an admissible triple $(x_0, x_1, x_\infty) = (x_0^{(0)}, x_1^{(0)}, x_\infty^{(0)}) = (x_0^{(1)}, x_1^{(1)}, x_\infty^{(1)}) = (x_0^{(\infty)}, x_1^{(\infty)}, x_\infty^{(\infty)})$, using the formulae (2.71), (2.72) or (2.73) and their analogies. Let us

prove that the numbers (x_0, x_1, x_∞) are real and satisfy $|x_i| < 2$ for $i = 0, 1, \infty$. Indeed, from the definition of the parameters, it follows:

$$(x_0^{(0)})^2 = 4\sin^2 \pi\sigma_0, (x_1^{(1)})^2 = 4\sin^2 \pi\sigma_1, (x_\infty^{(\infty)})^2 = 4\sin^2 \pi\sigma_\infty.$$

This proves that our construction covers, for real μ, all the solutions of PVIμ with critical behaviour of algebraic type.

Finally, using corollary 1.5, we infer that the class of solutions of PVIμ, with real μ, having critical behaviour of algebraic type is invariant with respect to the analytic continuation. The law of transformation of the critical indices l_0, l_1, l_∞ of the expansions (2.1), is described by theorem 1.5.

2.4. The complete list

By theorem 1.6, there exist only five algebraic solutions, up to the symmetries (1.21) and (1.23), which are determined by the orbits, under the pure braid group, of the points (1.31), (1.32), (1.33), (1.34) and (1.35). As we remarked in the section 1.3, these algebraic solutions have 4, 3, 10, 10 and 18 branches respectively. By (2.8), the value of μ of every solution can be computed. Within the ambiguity $\mu \mapsto \pm\mu + n$, $n \in \mathbb{Z}$, we choose respectively $\mu = -\frac{1}{4}$, $\mu = -\frac{1}{3}$, $\mu = -\frac{1}{5}$, $\mu = -\frac{2}{5}$, and $\mu = -\frac{1}{3}$. We arrive at the following complete list:

Coxeter group $\mathbf{W(A_3)}$, of symmetries of tetrahedron. We have $(x_0, x_1, x_\infty) = (-1, 0, -1)$, then $\mu = -\frac{1}{4}$ and

$$y = \frac{(s-1)^2(1+3s)(9s^2-5)^2}{(1+s)(25-207s^2+1539s^4+243s^6)},$$

with

$$x = \frac{(s-1)^3(1+3s)}{(s+1)^3(1-3s)}.$$

The monodromy matrices, in the canonical form (1.19) are:

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

This solution was found in [Dub] in the implicit form (E.29). This was also obtained, independently, by N. Hitchin (see [Hit2]). To reduce it to the above form, we have to solve the cubic equation (E.29 b) with the substitution:

$$t = \frac{32(1-18s^2+81s^4)}{27(1+9s^2+27s^4+27s^6)}.$$

Then the three roots of (E.29 b) are:

$$\omega_1 = \frac{13-66s^2-27s^4}{3(1+3s^2)^2},$$
$$\omega_{2,3} = \frac{-5+42s^2 \pm 144s^3+27s^4}{3(1+3s^2)^2}.$$

Coxeter group $\mathbf{W(B_3)}$, of symmetries of cube. We have $(x_0, x_1, x_\infty) = (-1, 0, -\sqrt{2})$ and $\mu = -\frac{1}{3}$. The solution

$$y = \frac{(2-s)^2(1+s)}{(2+s)(5s^4-10s^2+9)},$$

with

$$x = \frac{(2-s)^2(1+s)}{(2+s)^2(1-s)},$$

was obtained in [Dub]. The canonical form for the monodromy matrices is:

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Coxeter group $\mathbf{W(H_3)}$, of symmetries of icosahedron. We have three possible choices of the point (x_0, x_1, x_∞) which lead to three different solutions:

i) $(x_0, x_1, x_\infty) = (-1, 0, -\frac{1+\sqrt{5}}{2})$, then $\mu = -\frac{2}{5}$ and

$$y = \frac{(s-1)^2(1+3s)^2(-1+4s+s^2)(7-108s^2+314s^4-588s^6+119s^8)^2}{(1+s)^3(-1+3s)P(s)}$$

with

$$x = \frac{(-1+s)^5\,(1+3\,s)^3\,(-1+4\,s+s^2)}{(1+s)^5\,(-1+3\,s)^3\,(-1-4\,s+s^2)},$$

and

$$\begin{aligned} P(s) =& 49 - 2133s^2 + 34308s^4 - 259044s^6 + 16422878s^8 - 7616646s^{10} + 13758708s^{12} \\ &+ 5963724s^{14} - 719271s^{16} + 42483s^{18}. \end{aligned}$$

The canonical form for the monodromy matrices is:

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ \frac{-3-\sqrt{5}}{2} & 1 \end{pmatrix}.$$

The above solution was already obtained in [Dub] in the implicit form (E.33). The above explicit formula can be obtained solving (E.33 b) in the form:

$$t = \frac{(1-4s-s^2)(-1-4s+s^2)(-1+5s^2)}{(1+3s^2)^3}$$

$$\omega_1 = \frac{25 - 585\,s^2 + 3530\,s^4 - 6690\,s^6 - 3955\,s^8 + 507\,s^{10}}{(1+3\,s^2)^5}$$

$$\omega_2 = \frac{-7 + 215\,s^2 - 1910\,s^4 - 4096\,s^5 + 5150\,s^6 + 20480\,s^7 + 6125\,s^8 - 357\,s^{10}}{(1+3\,s^2)^5}$$

$$\omega_3 = \frac{-7 + 215\,s^2 - 1910\,s^4 + 4096\,s^5 + 5150\,s^6 - 20480\,s^7 + 6125\,s^8 - 357\,s^{10}}{(1+3\,s^2)^5}.$$

The last two solutions with the icosahedral symmetry are new. To compute them we use the following algorithm. The leading terms of the Puiseaux expansions near the ramification points $0, 1, \infty$ of each branch can be computed by the formulae (2.9), (2.10), (2.11) and (2.12). Form this, the genus of the algebraic curve $F(y, x) = 0$ is easily computed. Namely the genus of (1.34) is 0 and the genus of (1.35) is 1. Since the symmetries of PVIμ preserve the indices l_0, l_1, l_∞ (up to permutations), they preserve the genus too.

Remark. The appearance of genus 1 in the last solution related to the great dodecahedron could seem less surprising, if we recall that the topology of this immersed two-dimensional surface is different from the topology of all the other polyhedra and star-polyhedra. In fact, this is a surface of genus 4, while all the others have genus 0 (see [Cox]).

ii) $(x_0, x_1, x_\infty) = (-1, 0, \frac{1-\sqrt{5}}{2})$, then $\mu = -\frac{1}{5}$ and

$$y = \frac{(-1+s)^4\,(1+3\,s)^2\,(-1+4\,s+s^2)\,(3-30\,s^2+11\,s^4)^2}{(1+s)\,(-1+3\,s)\,(1+3\,s^2)\,(9-342\,s^2+4855\,s^4-28852\,s^6+63015\,s^8-1942\,s^{10}+121\,s^{12})},$$

with

$$x = \frac{(-1+s)^5\,(1+3\,s)^3\,(-1+4\,s+s^2)}{(1+s)^5\,(-1+3\,s)^3\,(-1-4\,s+s^2)}.$$

The canonical form for the monodromy matrices is:

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ \frac{-3+\sqrt{5}}{2} & 1 \end{pmatrix}.$$

iii) The last solution $y(x)$ of PVIμ, with $\mu = -\frac{1}{3}$, as we have already said, is an algebraic function with 18 branches. It has two branch points of order 5, two of order 3 and two regular branches, over each ramification point $0, 1, \infty$. The branches $y_1(x), \cdots, y_{18}(x)$ near $x = 0$ have the form:

$$y_k(x) = e^{\frac{2\pi i k}{5}} \left(\frac{7}{13}\right)^2 6^{\frac{-2}{5}} x^{\frac{4}{5}} + \mathcal{O}(x), \qquad k = 1, \cdots, 5$$

$$y_{k+5}(x) = e^{\frac{2\pi i k}{5}} \frac{6^{\frac{4}{5}}}{19^2} x^{\frac{2}{5}} + \mathcal{O}(x^{\frac{4}{5}}), \qquad k = 1, \cdots, 5$$

$$y_{10+k}(x) = e^{\frac{2\pi i k}{3}} \frac{2^{\frac{2}{3}}}{18} \frac{1 + i\sqrt{15}}{4} x^{\frac{2}{3}} + \mathcal{O}(x), \qquad k = 1, \cdots, 3$$

$$y_{13+k}(x) = e^{\frac{2\pi i k}{3}} \frac{2^{\frac{2}{3}}}{18} \frac{1 - i\sqrt{15}}{4} x^{\frac{2}{3}} + \mathcal{O}(x), \qquad k = 1, \cdots, 3$$

$$y_{17,18}(x) = \frac{3 \pm \sqrt{5}}{6} x + \mathcal{O}(x^2).$$

The Puiseaux expansions of the branches near $x = 1$ and $x = \infty$ can be obtained from these formulae applying the symmetries

$$x \mapsto 1 - x, \qquad y \mapsto 1 - y,$$

and

$$x \mapsto \frac{1}{x}, \qquad y \mapsto \frac{1}{y},$$

respectively. Using these formulae, one can compute any term of the Puiseaux expansions of all the branches. Due to computer difficulties, at the moment, we do not manage to produce the algebraic equation for $y(x)$.
The canonical form for the monodromy matrices is:

$$M_0 = \begin{pmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ 0 & 1 \end{pmatrix} \quad M_1 = \begin{pmatrix} 1 & 0 \\ \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix} \quad M_0 = \begin{pmatrix} 1 & 0 \\ \frac{1+\sqrt{5}}{2} & 1 \end{pmatrix}.$$

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