SISSA ISAS

Fermilab Library
0 1160 0048710 2
SISSA-149-95-FM
FERMILAB
LIBRARY

Lie-CR-structures on a real Lie-algebra

Daniele Gouthier

S.I.S.S.A. - I.S.A.S., via Beirut 2,4 - 34014 Trieste, Italia.

Abstract. Let g_0 be a real Lie-algebra and g its complexification. The aim of this paper is to study the Lie-CR-structures (in the following we shall call them just LCR-structures) on g_0. To a LCR-structure corresponds a CR-structure on the associated real Lie-group G_0 for which right and left translations are both CR-maps. Levi-Mal'cev decomposition permits to consider separately the semisimple and the solvable cases and to describe completely the LCR-structures of a generic real Lie-algebra g_0. Hence, we introduce and describe the semidirect sum by the adjoint derivation of the structures induced on the solvable radical and on a Levi-subalgebra.

Keywords: Lie-algebras, Lie-groups, CR-structures.
MS classification: 17B20, 17B30, 22E15, 32F40.

Ref. S.I.S.S.A. 149/95/FM

INTRODUCTION.

The aim of this paper is to classify the *Lie-CR-structures* of a real Lie-algebra $\mathbf{g_0}$. By a LCR-structure on a real Lie-algebra $\mathbf{g_0}$ we mean a triple $\Gamma_0 = (\mathbf{g_0}, \mathbf{p}, J)$ such that $\mathbf{p}$ is an ideal in $\mathbf{g_0}$ and J is an endomorphism of $\mathbf{p}$ whose square is minus the identity (so $\mathbf{p}$ has to be even-dimensional) and which commutes with the adjoint derivations $ad_X, \forall X \in \mathbf{g_0}$.

Of course LCR-structures are a particular kind of CR-structures [GT]. Remind that a *CR-structure* $(\mathbf{p}, J)$ on a real Lie-algebra $\mathbf{g_0}$ is given by a linear subspace $\mathbf{p}$ and by an endomorphism $J : \mathbf{p} \to \mathbf{p}$ such that

1. $J^2 = -id$
2. $[X, Y] - [JX, JY] \in \mathbf{p}, \forall X, Y \in \mathbf{p}$
3. $[JX, JY] = [X, Y] + J[X, JY] + J[JX, Y], \forall X, Y \in \mathbf{p}$.

If one denotes with $\mathbf{g}$ the r of $\mathbf{g_0}$ and with τ the conjugation of $\mathbf{g}$ with respect to $\mathbf{g_0}$, then the linear space $\mathbf{q} = \{X + iJX : X \in \mathbf{p}\}$ is really a complex subalgebra of $\mathbf{g}$ which does not intersect its conjugate $\overline{\mathbf{q}} = \tau\mathbf{q}$. Given such a $\mathbf{q}$, $\mathbf{g}$ can be written as a direct sum $\mathbf{g} = \mathbf{q} \oplus \overline{\mathbf{q}} \oplus V$, where $V = \oplus_{i=1}^{r} \mathbf{C}X_i$, $X_1 ... X_r \in \mathbf{g_0}$. The integer r is called the *real codimension* of the CR-structure.

It is quite easy to introduce some basic CR-concepts: CR-subalgebra, CR-ideal, CR-homomorphism, equivalence between CR-structures. In fact, consider a subalgebra $\mathbf{h_0}$ in $\mathbf{g_0}$, a natural question is "when does $\mathbf{h_0}$ assume a CR-structure induced by Γ_0?". Let $\mathbf{h}$ be its complexification. Let us denote by $\mathbf{k}$ and $\widetilde{\mathbf{k}}$ the subalgebras $\mathbf{h} \cap \mathbf{q}$ and $\mathbf{h} \cap \overline{\mathbf{q}}$, respectively. Since $\mathbf{h}$ is a linear subspace in $\mathbf{g}$, it is $\mathbf{h} = \mathbf{k} \oplus \widetilde{\mathbf{k}} \oplus \mathbf{h} \cap \mathbf{V}$. Obviously, $\mathbf{k} \cap \widetilde{\mathbf{k}} = \mathbf{k} \cap (\mathbf{h} \cap \mathbf{V}) = \widetilde{\mathbf{k}} \cap (\mathbf{h} \cap \mathbf{V}) = \{\mathbf{0}\}$. But, in the general case, $\mathbf{k}$ and $\widetilde{\mathbf{k}}$ are not

isomorphic. Take, for instance, $\mathbf{h} = \mathbf{q}$, then you have $\mathbf{k} = \mathbf{h}$ and $\widetilde{\mathbf{k}} = \{0\}$. So we are forced to say that *a real subalgebra* $\mathbf{h_0}$ *admits the CR-structure* $\mathbf{k}$ *induced by* Γ_0 if $\mathbf{k}$ and $\widetilde{\mathbf{k}}$ are isomorphic as Lie-algebras. In that case $\mathbf{h}$ is said a *CR-subalgebra* in $\mathbf{g}$. Let $\mathbf{h_0}$ be an ideal, then $\mathbf{h}$ is said a *CR-ideal.* One can easily notice that $\mathbf{q}$ is not a CR-subalgebra; every subalgebra of V is a trivial CR-subalgebra; let $\mathbf{h}$ be a CR-subalgebra which doesn't intersect V, then $\mathbf{k}$ is an almost complex structure on $\mathbf{h_0}$. In order to compare different CR-structures, we introduce the *CR-homorphisms*: given two CR-structure, $\Gamma_0 = (\mathbf{g_0}, \mathbf{p_g}, \mathbf{J_{g_0}})$ and $\Xi_0 = (\mathbf{h_0}, \mathbf{p_h}, \mathbf{J_{h_0}})$, a Lie-homomorphism (resp. Lie-derivation) $\sigma : \mathbf{g_0} \to \mathbf{h_0}$ is a *CR-homomorphism* (resp. *CR-derivation*) if $\sigma \mathbf{p_g} \subset \mathbf{p_h}$ and $\sigma J_{\mathbf{g_0}} = J_{\mathbf{h_0}} \sigma$.

Now, let $\Gamma_0 = (\mathbf{g_0}, \mathbf{p_g}, J)$ and $\Xi_0 = (\mathbf{h_0}, \mathbf{p_h}, J')$ be CR-structures and $\varphi : \mathbf{g_0} \to \mathbf{h_0}$ a CR-monomorphism. In that case, we say that Γ_0 is *φ-compatible* with $\Xi_0$and that Γ_0 is a *φ-CR-substructure* of H_0. Hence, if one considers the triple $\varphi^* \Gamma_0 = (\varphi \mathbf{g_0}, \varphi \mathbf{p_g}, \varphi J \varphi^{-1})$, one obtains a CR-structure. It is a fact that Γ_0 is φ-compatible with Ξ_0 if and only if $\varphi \mathbf{g}$ is a CR-subalgebra of $\mathbf{h}$. We can say that Γ_0 and Ξ_0 are *equivalent* (or *CR-isomorphic*) if there exist $\varphi : \mathbf{g_0} \to \mathbf{h_0}$ and $\psi : \mathbf{h_0} \to \mathbf{g_0}$ such that Γ_0 is φ-compatible with Ξ_0 and Ξ_0 is ψ-compatible with Γ_0. In that case, $\mathbf{g_0}$ and $\mathbf{h_0}$ are Lie-isomorphic (via φ and ψ); φ (resp. ψ) sends $\mathbf{p_g}$ in $\mathbf{p_h}$ (resp. $\mathbf{p_h}$ in $\mathbf{p_g}$); and $\varphi J_{\mathbf{g_0}} = J_{\mathbf{h_0}} \varphi$. Obviously we can replace the ψ-compatibility with the φ^{-1}-compatibility. Two equivalent CR-structures are given, for instance, by Γ_0 and $\varphi^* \Gamma_0$, for any CR-monomorphism from Γ_0 in another CR-structure Ξ_0.

If $\mathbf{G_0}$ is the associated Lie-group, giving a CR-structure Γ_0 on $\mathbf{g_0}$ is the same as consider a structure of CR-manifold on $\mathbf{G_0}$ for which the left translations are CR-maps.

If the CR-structure Γ_0 is such that $\mathbf{p}$ is a real Lie-subalgebra (i.e. $\mathbf{q} \oplus \overline{\mathbf{q}}$ is a complex one), the Lie-group $\mathbf{G}_0$ with the CR-structure Γ_0 is a Levi-flat manifold: such CR-manifolds have Levi-form vanishing at each point, [AHR], [BO], [WE]. Moreover, if $\mathbf{p}$ is an ideal (i.e. $\mathbf{q}$ is it), then $\mathbf{p}$ (with the complex structure J) is really a complex subalgebra, the Lie-group $\mathbf{G}_0$ is foliated with complex Lie-subgroups and both the translations are CR-maps. Such a case is studied in this paper. In these conditions we say that the CR-structure Γ_0 is a *LCR-structure.*

In [SN] the reader can find a study on left-invariant complex structure on reductive Lie-groups. Such results have been translated by [GT] in terms of CR-structures on reductive Lie-algebras of the first category (in these algebras the involution determined by a Cartan decomposition is an inner automorphism [HE]): the authors study, essentially, the case of real codimension 1.

In this paper we explore the case of LCR-structures without any hypothesis on the Lie-algebra. This exploration permits us to classify all the LCR-structures. We base our work on the classical Levi-Mal'cev theorem which assures that all the Lie-algebras admit a decomposition $\mathbf{g}_0 = \mathbf{r} \oplus_{ad} \mathbf{s}$, where $\mathbf{r}$ is the solvable radical and $\mathbf{s}$ is a semisimple subalgebra [VA]. Remark that we denote with $\oplus$ the direct sum of linear spaces; with $\oplus_\delta$ the semidirect sum by δ of Lie-algebras; with $\odot$ the direct sum of Lie-algebras.

Thanks to Levi-Mal'cev decomposition, we have to study LCR-structures in the semisimple and in the solvable cases (Sections 1 and 2): in the first one the LCR-structures are sums (in the sense of the last proposition of Section 1) of simple ideals endowed with a complex structure (described by Cartan in the classical classification [HE]); in the sec-

ond one they are given by even-dimensional ideals $\mathbf{p}$, decomposed as $\mathbf{p} = \mathbf{u} \oplus A\mathbf{u}$ and by $J = J_A \doteq \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}$, endomorphism of $\mathbf{p}$. Section 3 shall proof when one has a LCR-structure on a semidirect sum whose factors are endowed with two LCR-structures. Finally, Section 4 concludes with the Theorem 4: let $\mathbf{g_0}$ be decomposed following Levi-Mal'cev decomposition. Then $\Gamma_0 = (\mathbf{g_0}, \mathbf{p}, J)$ is a LCR-structure if and only if its projections on the factors are LCR-structures whose semidirect sum by ad is Γ_0 itself. Obviously this result let us describe all the LCR-structures. The only indetermination is due to the knowledge of the ideals of solvable Lie-algebras. The last Section is devoted to some examples.

1.THE SEMISIMPLE CASE.

In this Section we denote by $\mathbf{g_0}$ a real Lie-algebra and by B its Killing form (remind that if $\mathbf{g_0}$ is semisimple, B is nondegenerate). Just by computation, one can observe that J is an anti-isometry with respect to B. In fact, $ad_{JX}Y = -ad_Y JX = -Jad_Y X = Jad_X Y$. So we have $ad_{JX} = ad_X J = Jad_X$ and $B(JX, JY) = -B(X, Y)$.

In the sequel we shall divide our study in two subcases: the first one considers compact semisimple real Lie-algebras; the second one those noncompact. Remind that a Lie-algebra $\mathbf{g_0}$ is *compact* if there exists a compact Lie-group whose Lie-algebra is $\mathbf{g_0}$. That is equivalent to the decomposition $\mathbf{g_0} = \zeta(\mathbf{g_0}) \odot [\mathbf{g_0}, \mathbf{g_0}]$, where $\zeta(\mathbf{g_0})$ is the center of $\mathbf{g_0}$ and $[\mathbf{g_0}, \mathbf{g_0}]$ is semisimple [GR].

It is a classical fact that the existence of a complex structure on a compact (not semisimple) Lie-algebra implies the abelianity of the algebra itself [GR]. Moreover, if $\mathbf{p}$ is in the center of $\mathbf{g_0}$, Γ_0 is trivially a LCR-structure, so we can hopefully expect a CR analogous of the complex result. Such an analogous result is based on the

Lemma. *Given* $\Gamma_0 = (\mathrm{g}_0, \mathrm{p}, J)$ *LCR-structure,* p *admits a biinvariant metric if and only if* p *is abelian.*

Proof. A metric g is biinvariant iff $g([X,Y],Z) = g(X,[Y,Z])$. Suppose p is abelian, then any metric is, certainly, biinvariant. Let us prove the converse. We can impose that J is an isometry with respect to g (otherwise we substitute g with $g'(X,Y) \doteq g(X,Y) + g(JX,JY)$). With this hypothesis the following chain of equivalences is true, $\forall X,Y,Z \in \mathrm{p}$: $g([X,Y],Z) = g(J[X,Y],JZ) = g([X,JY],JZ) = g(X,[JY,JZ]) = -g(X,[Y,Z]) = -g([X,Y],Z) = 0$, therefore $g([X,Y],Z) = 0$, which concludes the Lemma. Q.E.D.

Since any compact Lie-algebra admits a biinvariant metric the above lemma implies the following

Proposition. *Given* g_0 *compact (not semisimple),* $\Gamma_0 = (\mathrm{g}_0, \mathrm{p}, J)$ *is a LCR-structure if and only if* p *is abelian.*

The previous proposition permits us to conclude about compact Lie-algebras with the

Theorem1. *There are no LCR-structures on a compact semisimple Lie-algebra. Furthermore, when* g_0 *is compact,* $\Gamma_0 = (\mathrm{g}_0, \mathrm{p}, J)$ *is a LCR-structure if and only if* p *is in the center* $\zeta(\mathrm{g}_0)$.

Proof. The nonexistence of abelian ideals in a semisimple Lie-algebra concludes the first part of the assertion. About the second one, remind that a compact Lie algebra g_0 takes the form $\mathrm{g}_0 = \zeta(\mathrm{g}_0) \odot [\mathrm{g}_0, \mathrm{g}_0]$, where $[\mathrm{g}_0, \mathrm{g}_0]$ is semisimple. Now suppose that g_0 supports a LCR-structure $\Gamma_0 = (\mathrm{g}_0, \mathrm{p}, J)$, then p takes the form $\mathrm{p}_1 \oplus \mathrm{p}_2$ where p_2 is an ideal in $[\mathrm{g}_0, \mathrm{g}_0]$. Suppose that J maps p_2 in itself, then $([\mathrm{g}_0, \mathrm{g}_0], \mathrm{p}_2, J)$ is a LCR-structure, that is impossible. Hence, $\mathrm{p} = \mathrm{p}_1 \subset \zeta(\mathrm{g}_0)$. Let us conclude proving that J maps p_2 in itself.

Consider the complex subalgebras $\mathbf{q}_j = \{X - iJX : X \in \mathbf{p}_j\}$. Obviously it is $\mathbf{q} = \mathbf{q}_1 \oplus \mathbf{q}_2$ and $\mathbf{q}_2$ is another LCR-structure of $\mathbf{g}_0$. Hence, it is given the endomorphism $J_2 : \mathbf{p}_2 \to \mathbf{p}_2$. Take $X \in \mathbf{p}_2$, then $X - iJX \in \mathbf{q}$ (resp. $X + iJX \in \overline{\mathbf{q}}$) and $X - iJ_2X \in \mathbf{q}_2 \subset \mathbf{q}$ (resp. $X + iJ_2X \in \overline{\mathbf{q}}_2 \subset \overline{\mathbf{q}}$), so $i(J_2X - JX) = (X - iJX) - (X - iJ_2X) = (X + iJ_2X) - (X + iJX) \in \mathbf{q} \cap \overline{\mathbf{q}} = \{0\}$, which means that J maps $\mathbf{p}_2$ in itself. Q.E.D.

In Section 2 we shall study LCR-structures on abelian Lie-algebras. Hence we will describe completely LCR-structures in the compact case. Now we move to the study of LCR-structures on semisimple noncompact Lie-algebras. The simple case is trivial. In fact, since there are no nontrivial ideals, a LCR-structure on a simple Lie-algebra is, really, an *ad*-invariant complex one. Moreover, it is well known that a semisimple Lie-algebra is direct sum of simple ideals. These facts bring us to the

Proposition. *A LCR-structure on a semisimple Lie-algebra is completely defined by its simple ideals endowed with a complex structure.*

Proof. Since $\mathbf{g}_0$ is semisimple, we can write $\mathbf{g}_0 = \mathbf{p}_1 \odot \ldots \odot \mathbf{p}_k$, where the $\mathbf{p}_j$ are simple ideals (such a decomposition is essentially unique). Let $\mathbf{p}$ be the ideal on which is given the CR-structure, then $\mathbf{p}$ is direct sum of some $\mathbf{p}_i$. As in Theorem 1, the restriction of J to $\mathbf{p}_i$ has image in $\mathbf{p}_i$. This fact concludes the proof. Q.E.D.

Hence, a LCR-structure on a semisimple Lie-algebra is given by the complex structures on some simple factors. Each of these factors is described in the Cartan's classification of the complex simple Lie-algebras

$\mathbf{g}$	$\mathbf{G}$	$\mathbf{U}$	$\zeta(\mathbf{U}')$	$dim\mathbf{U}$
$a_n(n \geq 1)$	$\mathrm{SL}(n+1,\mathbf{C})$	$\mathrm{SU}(n+1)$	$\mathbf{Z}_{n+1}$	$n(n+2)$
$b_n(n \geq 2)$	$\mathrm{SO}(2n+1,\mathbf{C})$	$\mathrm{SO}(2n+1)$	$\mathbf{Z}_2$	$n(2n+1)$
$c_n(n \geq 3)$	$\mathrm{Sp}(n,\mathbf{C})$	$\mathrm{Sp}(n)$	$\mathbf{Z}_2$	$n(2n+1)$
$d_n(n \geq 4)$	$\mathrm{SO}(2n,\mathbf{C})$	$\mathrm{So}(2n)$	$\mathbf{Z}_4, n = odd$	$n(2n-1)$
			$\mathbf{Z}_2 + \mathbf{Z}_2, n = even$	
e_6	$E_6^{\mathbf{C}}$	E_6	$\mathbf{Z}_3$	78
e_7	$E_7^{\mathbf{C}}$	E_7	$\mathbf{Z}_2$	133
e_8	$E_8^{\mathbf{C}}$	E_8	$\mathbf{Z}_1$	248
f_4	$F_4^{\mathbf{C}}$	F_4	$\mathbf{Z}_1$	52
g_2	$G_2^{\mathbf{C}}$	G_2	$\mathbf{Z}_1$	14

In the Table, $\mathbf{g}$ is a simple Lie-algebra over $\mathbf{C}$; n the dimension of a Cartan-subalgebra; $\mathbf{G}$ a connected Lie-group such that $Lie(\mathbf{G}) = \mathbf{g}^{\mathbf{R}}$; $\mathbf{U}$ an analytical subgroup such that $Lie(\mathbf{U})$ is a compact real form of $\mathbf{g}$ (i.e. $\mathbf{U}$ is a maximal compact subgroup); and $\mathbf{U}'$ is its universal covering [HE].

The present Section will be concluded by the complete description of LCR-structures on semisimple Lie-algebras. As already showed in Theorem 1, when the algebra is compact there are no LCR-structures. Otherwise, is given the following

Classification: *let $\mathbf{g}_0$ be semisimple and noncompact. Then we give the (essentially unique) decomposition $\mathbf{g}_0 = \mathbf{r}_1 \odot ... \odot \mathbf{r}_j \odot \mathbf{p}_1 \odot ... \odot \mathbf{p}_h$, where:*

1. *both $\mathbf{r}_i$ and $\mathbf{p}_i$ are simple real ideals;*
2. *on the $\mathbf{r}_i$ there are no complex structures;*
3. *any $\mathbf{p}_i$ takes one of the forms in the Table.*

With such a decomposition we may choose any sum $\mathbf{p} = \odot_{l=1}^{k} \mathbf{p}_{i_l}$ with the endomorphism $J = \begin{pmatrix} J_{i_1} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & J_{i_k} \end{pmatrix}$. The triple $(\mathbf{g}_0, \mathbf{p}, J)$ is the generic LCR-structure in the semisimple case.

2.THE SOLVABLE CASE.

A real Lie-algebra $\mathbf{g_0}$ is *solvable* if one of its *derived* subalgebras vanishes. Remind that the *derived series* is given by $\mathbf{D}^0\mathbf{g_0} = \mathbf{g_0}$, $\mathbf{D}^1\mathbf{g_0} = [\mathbf{g_0}, \mathbf{g_0}]$, $\mathbf{D}^n\mathbf{g_0} = \mathbf{D}^1(\mathbf{D}^{n-1}\mathbf{g_0})$. Since any ideal of $\mathbf{g_0}$ is solvable, a LCR-structure on $\mathbf{g_0}$ is an *ad*-invariant complex structure on a solvable ideal. Let us study such structures on solvable Lie-algebras. In the sequel, when we speak of a complex structure, we mean a *ad-invariant* one.

Lemma. *Suppose* $\mathbf{g_0}$ *is solvable and* $(\mathbf{g_0}, J)$ *is a complex structure. Then there exists a subspace* $\mathbf{u}$ *such that* $\mathbf{g_0} = \mathbf{u} \oplus J\mathbf{u}$ *and* $J = \begin{pmatrix} 0 & J'' \\ J' & 0 \end{pmatrix}$, *where* J' *(resp.* J''*) is the restriction of* J *to* $\mathbf{u}$ *(resp.* $J\mathbf{u}$*).*

Proof: since $\mathbf{g_0}$ is solvable there exists a codimension one ideal $\mathbf{p}_1$ [VA]. It is easy to show that $J\mathbf{p}_1 \neq \mathbf{p}_1$. Then, there exists $X_1 \in \mathbf{p}_1$ such that $\mathbf{g_0} = L(X_1, JX_1) \oplus \mathbf{p}_1 \cap J\mathbf{p}_1$. Moreover $(\mathbf{g_0}, \mathbf{p}_1 \cap J\mathbf{p}_1, J)$ is a LCR-structure. Now we repeat the same proof with $\mathbf{p}_1 \cap J\mathbf{p}_1$ and $\mathbf{p}_2$ (where $\mathbf{p}_2$ is a codimension one ideal in $\mathbf{p}_1 \cap J\mathbf{p}_1$) instead of $\mathbf{g_0}$ and $\mathbf{p}_1$. In that way, we find a family $X_1 \ldots X_k$, such that $\mathbf{g_0} = L(X_1 \ldots X_k, JX_1 \ldots JX_k)$ and the space $\mathbf{u} = L(X_1 \ldots X_k)$ is the desired one. Q.E.D.

Now we want to show the converse, in the sense that any solvable Lie-algebra admits a complex structure if and only if it is even dimensional; in that case we write $\mathbf{g_0}$ as the sum $\mathbf{g_0} = \mathbf{u} \oplus \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ have the same dimension. Chosen a linear monomorphism $A : \mathbf{u} \to \mathbf{g_0}$ such that $\mathbf{v} = A\mathbf{u}$, the complex structure $J = J_A \doteq \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}$ is generic: so the complex structures depend only on the splitting of $\mathbf{g_0}$ in equal-dimensional subspaces. Let us proof this fact by induction.

The simplest solvable algebras are the abelian ones, i.e. the ones whose first derived

vanishes. On these, a complex structure is just the "multiplication by i", in fact you have the

Lemma. *Let $\mathbf{g_0}$ be abelian. Then there exists a complex structure J if and only if $\mathbf{g_0}$ is even-dimensional. In that case there exist a linear subspace $\mathbf{u}$ and a linear monomorphism $A : \mathbf{u} \to \mathbf{g_0}$ such that*

1. $\mathbf{g_0} = \mathbf{u} \oplus A\mathbf{u}$
2. $J = J_A \doteq \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}$.

Moreover, all the pair $(\mathbf{g_0}, J_A)$ are isomorphic as complex Lie-algebras, independently on the subspace $\mathbf{u}$ and the morphism A. So we may say that the structure is unique.

Proof: suppose that $\mathbf{g_0}$ is endowed with a complex structure J, then previous lemma gives us the pair $(\mathbf{u}, J')$ desired. Vice versa, let $\mathbf{g_0}$ be even-dimensional. Then, choose $\mathbf{u}$ and A, such that $\mathbf{g_0} = \mathbf{u} \oplus A\mathbf{u}$. The endomorphism J_A is trivially a complex structure on $\mathbf{g_0}$. If one considers the automorphism $\phi_{AB} \doteq \begin{pmatrix} I & 0 \\ 0 & BA^{-1} \end{pmatrix}$, one has an isomorphism between $(\mathbf{g_0}, J_A)$ and $(\mathbf{g_0}, J_B)$. Hence, the complex structure does not depend on A. Finally, we show that does not depend neither on $\mathbf{u}$: let $(\mathbf{v}, C)$ be a pair such that $\mathbf{g_0} = \mathbf{v} \oplus C\mathbf{v}$. Then we have $\mathbf{v} = D\mathbf{u}$ and $\mathbf{g_0} = D\mathbf{u} \oplus AD\mathbf{u}$, where we have taken D Lie-isomorphism. It is easy to show that $(D\mathbf{u} \oplus AD\mathbf{u}, J_A)$ and $(\mathbf{u} \oplus D^{-1}AD\mathbf{u}, J_{D^{-1}AD})$ are isomorphic. Q.E.D.

In the previous Section we have shown that, given a compact Lie-algebra $\mathbf{g_0}$, $\Gamma_0 = (\mathbf{g_0}, \mathbf{p}, J)$ is a LCR-structure if and only if $\mathbf{p}$ is contained in the center $\zeta(\mathbf{g_0})$. The last Property permits us to describe these LCR-structures. In fact, suppose $\Gamma_0 = (\mathbf{g_0}, \mathbf{p}, J)$ is a LCR-structure, then $\mathbf{p}$ has to be even-dimensional and takes the form $\mathbf{p} = \mathbf{u} \oplus A\mathbf{u}$, with $J = J_A$. Finally, the datum of a LCR-structure on a compact Lie-algebra is equivalent to

the choice of an even-dimensional linear subspace of the center. Let us return to solvable algebras.

Theorem2. *A solvable Lie-algebra* $\mathbf{g_0}$ *admits a complex structure if and only if it is even-dimensional. Let* $(\mathbf{g_0}, J)$ *be a complex structure, then there exist two vector spaces* $\mathbf{u}$ *and* $\mathbf{v}$ *and an isomorphism* A *between* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $\mathbf{g_0} = \mathbf{u} \oplus A\mathbf{u}$ *and* $J = J_A$. *Moreover, all the pair* $(\mathbf{g_0}, J_A)$ *are isomorphic as complex Lie-algebras.*

Proof: let k be the minimum integer such that $\mathbf{D}^k\mathbf{g_0} = 0$, then make the proof by induction over k. The base of the induction is given by the abelian case. Now, let $\mathbf{g_0}$ be solvable but not abelian. In any case, $\mathbf{g_0}' \doteq \mathbf{g_0}/\mathbf{D}^1\mathbf{g_0}$ is abelian. Furthermore J maps $\mathbf{D}^1\mathbf{g_0}$ on itself, since $Jad_X = ad_X J$. So $(\mathbf{g_0}', J')$ is a complex structure, where J' is the quotient of J. If we apply the previous lemma, we have that $\mathbf{g_0}' = \mathbf{w}' \oplus J'|_{\mathbf{w}'}\mathbf{w}'$ and $J' = J_{J'|_{\mathbf{w}'}}$. If we choose a subspace $\mathbf{w}$ in the class $\mathbf{w}'$, we obtain $\mathbf{g_0} = \mathbf{w} \oplus J^+\mathbf{w} \oplus \mathbf{D}^1\mathbf{g_0}$ and $J = \begin{pmatrix} 0 & J^+ & 0 \\ J^+ & 0 & 0 \\ 0 & 0 & J_1 \end{pmatrix}$, where J^+ is the restriction at $\mathbf{w}$ and J_1 the one at $\mathbf{D}^1\mathbf{g_0}$. Finally, we apply the inductive hypothesis on the pair $(\mathbf{D}^1\mathbf{g_0}, J_1)$. Q.E.D.

Let $\mathbf{g_0}$ be solvable, then $\mathbf{g_0}$ admits one LCR-structure on each $2l$-dimensional ideal (in the hypothesis that it exists) given by an isomorphism J of the form J_A. Hence LCR-structures are essentially given by the choice of even-dimensional ideals. Remark that it is possible to have different LCR-structures of the same dimension. The study of LCR-structures on a solvable real Lie-algebra is now completely equivalent to the knowledge of the ideals of the algebra itself: all even-dimensional ideals support one, and only one, LCR-structure.

3.SEMIDIRECT SUMS.

Take two LCR-structures $\Gamma_0 = (\mathbf{g_0}, \mathbf{p}, J)$ and $\Gamma_0' = (\mathbf{g_0}', \mathbf{p}', J')$. Let $\delta : \mathbf{g_0}' \to Der(\mathbf{g_0})$ be a Lie-homomorphism. The *semidirect sum* of $\mathbf{g_0}$ and $\mathbf{g_0'}$ by δ is the Lie-algebra $\mathbf{g_0} \oplus_\delta \mathbf{g_0}'$ defined on the linear space $\mathbf{g_0} \oplus \mathbf{g_0}'$ by the product $[(X, X'), (Y, Y')]_\delta = ([X, Y] + \delta(X')Y - \delta(Y')X, [X', Y'])$. The actual problem is whether the triple $\Gamma_0 \oplus_\delta \Gamma_0' = (\mathbf{g_0} \oplus_\delta \mathbf{g_0}', \mathbf{p}^* = \mathbf{p} \oplus \mathbf{p}', J^* = \begin{pmatrix} J & 0 \\ 0 & J' \end{pmatrix})$ is a LCR-structure. When it is, one shall say $\Gamma_0 \oplus_\delta \Gamma_0'$ the *semidirect sum* by δ of Γ_0 and Γ_0'. We will answer by steps.

Lemma. Given the linear subspaces $\mathbf{p} \subset \mathbf{g_0}$ and $\mathbf{p}' \subset \mathbf{g_0}'$, their sum $\mathbf{p}^* = \mathbf{p} \oplus \mathbf{p}'$ is an ideal if and only if

a) $\mathbf{p}$ and $\mathbf{p}'$ are ideals;

b) $\delta(X')\mathbf{g_0} \subset \mathbf{p}$, $\forall X' \in \mathbf{p}'$;

c) $\delta(Y')\mathbf{p} \subset \mathbf{p}$, $\forall Y' \in \mathbf{g_0'}$.

Proof. Let us suppose that (X, X') stays in $\mathbf{p}^*$ and that (Y, Y') is the generic element of $\mathbf{g_0} \oplus_\delta \mathbf{g_0}'$. The relation $[(X, X'), (Y, Y')]_\delta \in \mathbf{p}^*$ is equivalent to

1. $[X, Y] + \delta(X')Y - \delta(Y')X \in \mathbf{p}$;

2. $[X', Y'] \in \mathbf{p}'$.

Particular choices of (X, X') and (Y, Y') imply that the first one is equivalent to the following three

1.1 $[X, Y] \in \mathbf{p}$;

1.2 $\delta(X')Y \in \mathbf{p}$;

1.3 $\delta(Y')X \in \mathbf{p}$.

The conditions 1.1 and 2. say that $\mathbf{p}$ and $\mathbf{p}'$ are ideals; while 1.2 (resp. 1.3) coincides with

letter b (resp. c). Q.E.D.

Moreover, impose that $ad_{(X,X')}$ is a CR-derivation, $\forall X \in \mathbf{g_0}$ and $\forall X' \in \mathbf{g_0'}$ (i.e., J^* is ad-invariant). Hence, one obtains the following necessary and sufficient conditions:

d) ad_X and ad'_X are CR-derivations;

e) $J\delta(X') = \delta(J'X')$, $\forall X' \in \mathbf{p}'$

f) $\delta(X')$ is a CR-derivation, $\forall X' \in \mathbf{g_0'}$.

The previous computations let us conclude with the

Theorem3. *Given two LCR-structures Γ_0 and Γ_0', their semidirect sum by δ is a LCR-structure if*

1.1 $\delta(X')\mathbf{g_0} \subset \mathbf{p}$, $\forall X' \in \mathbf{p}'$;

1.2 $J\delta(X') = \delta(J'X')$, $\forall X' \in \mathbf{p}'$;

2.1 $\delta(Y')\mathbf{p} \subset \mathbf{p}$, $\forall Y' \in \mathbf{g_0}'$;

2.2 $\delta(Y')J = J\delta(Y')$, $\forall Y' \in \mathbf{g_0}'$.

Obviously, direct sums of LCR-structures are LCR-structures: in fact, they correspond to $\delta = 0$. From the proposition it follows that if $\mathbf{h_0}$ is endowed with a complex structure and if $\delta(X)$ is holomorphic, $\mathbf{h_0} \oplus_\delta \mathbf{g_0}$ supports a LCR-structure, where $\mathbf{g_0}$ is a generic real Lie-algebra. That is the case of noncompact semisimple Lie-algebras where $\mathbf{g_0}$ is the sum of the real factors and $\mathbf{h_0}$ is the sum of the Cartan-classified ones. A basic example is given by a reductive Lie-algebra. In fact, in that case the algebra is the direct sum of its center and of a semisimple Lie-subalgebra. So, a LCR-structure is direct sum of an abelian LCR-structure with a semisimple one. Such a situation bring us to consider Levi-Mal'cev decomposition. In this decomposition one factor is semisimple while the other

one is solvable, a little more generic than abelian. Its study is the object of the following Section.

4.LEVI-MAL'CEV DECOMPOSITION.

The tool of this Section is the study of LCR-structures $\Gamma_0 = (\mathbf{g_0}, \mathbf{p}, J)$ on a generic Lie-algebra $\mathbf{g_0}$ decomposed, following Levi-Mal'cev, as $\mathbf{g_0} = \mathbf{r} \oplus_{\mathbf{ad}} \mathbf{s}$, where $\mathbf{r}$ is the solvable radical and $\mathbf{s}$ is a Levi-subalgebra. Remind that a Levi-subalgebra is semisimple.

Due to the Lemma in Section3, one decomposes $\mathbf{p}$ as $\mathbf{p} = \mathbf{p_r} \oplus \mathbf{p_s}$, where $\mathbf{p_r}$ and $\mathbf{p_s}$ satisfy the following relations

1.1) $[\mathbf{p_r}, \mathbf{r}] \subset \mathbf{p_r}$

1.2) $[\mathbf{p_s}, \mathbf{s}] \subset \mathbf{p_s}$

1.3) $[\mathbf{p_s}, \mathbf{r}] \subset \mathbf{p_r}$

1.4) $[\mathbf{p_r}, \mathbf{s}] \subset \mathbf{p_r}$;

1.1 (resp. 1.2) means that $\mathbf{p_r}$ (resp. $\mathbf{p_s}$) is an ideal in $\mathbf{r}$ (resp. $\mathbf{s}$); 1.3 and 1.4 coincide respectively with the letters b and c of the Lemma.

The second characterizing property of a LCR-structure is $J^2 = -id$. Just computing the square of $J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we obtain

2.1) $A^2 + BC = -I$

2.2) $AB + BD = 0$

2.3) $DC + CA = 0$

2.4) $D^2 + CB = -I$.

Finally, let $ad_{(X,X')}$ be a CR-derivation. That means $(A[U,V] + A[U,Y] + A[X,V] + B[X,Y], C[U,V]+C[U,Y]+C[X,V]+D[X,Y]) = ([U, AV+BY]+[U, CV+DY], +[X, AV+$

$BY], [X, CV + DY])$. In particular, one may consider the cases given by the conditions $X = Y = 0$, $X' = Y' = 0$, $X = Y' = 0$ and $X' = Y = 0$. The corresponding equations are

3.1) $A[U, V] = [U, AV] + [U, CV]$

3.2) $A[X, V] = [X, AV]$

3.3) $A[U, Y] = [U, BY] + [U, DY]$

3.4) $B[X, Y] = [X, BY]$

3.5) $C[U, V] = 0$

3.6) $C[U, Y] = 0$

3.7) $C[X, V] = [X, CV]$

3.8) $D[X, Y] = [X, DY]$.

Proposition. *The equations 1.1, ... 1.4, 2.1, ... 2.4, 3.1, ... 3.8 define a necessary and sufficient condition in order that $\Gamma_0 \oplus_\delta \Gamma'_0$ be a LCR-structure.*

The above Proposition is just a "translation" of the LCR-structure's definition on a semidirect sum. Now we have to study the role played by the solvability af $\mathbf{r}$ and the semisemplicity of $\mathbf{s}$. In this context, their most important consequence is the

Property. *The matrices B and C vanish.*

Proof. Since $C[X, V] = [X, CV]$, $C(\mathbf{p_r})$ is an ideal in $\mathbf{s}$. But $[CV, CV_1] = C[CV, V_1] = 0$, so $C(\mathbf{p_r})$ is abelian. Consequently $C(\mathbf{p_r})$ vanishes.

It is a classical fact that every ideal and every quotient of semisimple algebras are semisimple [SE], moreover $\mathbf{p_s}/KerB$ is semisimple. Otherwise, $\mathbf{r}$ solvable implies that every subspace $\mathbf{t} \subset \mathbf{r}$ verifies $\mathbf{D}^n\mathbf{t} = 0$. So $B(\mathbf{p_s})$ does. As linear spaces, we have that $\mathbf{p_s}/KerB \simeq B(\mathbf{p_s})$, via the isomorphism $jX^+ = BX$, where $X^+ = X + KerB \in \mathbf{p_s}/KerB$.

Let us compute $[jX^+, jY^+]$: first of all, $\forall X, Y \in \mathbf{p_s}$, $[BX, BY] = A[BX, Y] - [BX, DY] = AB[X, Y] - B[X, DY] = -BD[X, Y] - B[X, DY] = -2BD[X, Y]$. By (2.2), you see that D sends $KerB$ in $KerB$ and hence $[jX^+, jY^+] = -2j(D[X, Y])^+$. So, we can conclude that $\mathbf{D}^n(\mathbf{p_s}/KerB) = 0$, since $B(\mathbf{p_s})$ does. But the fact that $\mathbf{p_s}/KerB$ is semisimple implies that its solvable radical vanishes. So $\mathbf{p_s} = KerB$. Q.E.D.

Remark: the condition "$C = 0$" is true even in a more general case. The only required hypothesis is the semisemplicity of $\mathbf{s}$. So if you take a semidirect sum with second factor $\mathbf{g_0'}$ semisimple, a LCR-structure takes the form $(\mathbf{g_0} \oplus_\delta \mathbf{g_0'}, \mathbf{p} \oplus \mathbf{p'}, \begin{pmatrix} A & B \\ 0 & D \end{pmatrix})$.

The Property shows that J takes the particular form $J = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Hence, our list of equations simplify itself to

1.1) $[\mathbf{p_r}, \mathbf{r}] \subset \mathbf{p_r}$

1.2) $[\mathbf{p_s}, \mathbf{s}] \subset \mathbf{p_s}$

1.3) $[\mathbf{p_s}, \mathbf{r}] \subset \mathbf{p_r}$

1.4) $[\mathbf{p_r}, \mathbf{s}] \subset \mathbf{p_r}$,

2.1') $A^2 = -I$

2.4') $D^2 = -I$.

3.1') $A[U, V] = [U, AV]$

3.2) $A[X, V] = [X, AV]$

3.3') $A[U, Y] = [U, DY]$

3.8) $D[X, Y] = [X, DY]$.

1.1, 2.1' and 3.1' say that $R = (\mathbf{r}, \mathbf{p_r}, A)$ is a LCR; 1.2, 2.4' and 3.8' say that $S = (\mathbf{s}, \mathbf{p_s}, D)$ is a LCR-structure; finally 1.3, 3.3', 1.4 and 3.2 correspond to 1.1, 1.2, 2.1 and 2.2 of

Theorem3, respectively. So we can conclude with

Theorem4. *Let* $\mathbf{g_0} = \mathbf{r} \oplus_{ad} \mathbf{s}$ *be a real Lie-algebra. Suppose* $\Gamma_0 = (\mathbf{g_0}, \mathbf{p}, J)$ *is a LCR-structure; then* R *and* S *are, too; and* Γ_0 *is their semidirect sum by ad. Vice versa, if one considers two LCR-structures* $R = (\mathbf{r}, \mathbf{p_r}, A)$ *and* $S = (\mathbf{s}, \mathbf{p_s}, D)$ *which verify*

1.3) $[\mathbf{p_s}, \mathbf{r}] \subset \mathbf{p_r}$

1.4) $[\mathbf{p_r}, \mathbf{s}] \subset \mathbf{p_r}$,

3.2) $A[X, V] = [X, AV]$

3.3)' $A[U, Y] = [U, DY]$

their semidirect sum by ad is a LCR-structure on $\mathbf{g_0}$.

5.EXAMPLES: LCR ON LOW-DIMENSIONAL REAL LIE-ALGEBRAS.

In this Section we conclude the paper describing LCR-structures for low dimensional Lie-algebras $\mathbf{g_0}$. First of all, remind that there exist just two different bidimensional Lie-algebras: the abelian one and the Lie-algebra $\mathbf{h_0}$ of the matrices $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, a, b \in \mathbf{R}$, which is solvable. Both of them are endowed with the complex structure given by the "multiplication by i."

If one wants to consider LCR-structures on real Lie-algebras $\mathbf{g_0}$ which are not complex structures, it must be $\dim \mathbf{g_0} \geq 3$. Let us start with $\dim \mathbf{g_0} = 3$. Such Lie-algebras are completely classified in [MI]. The classification makes use of the map $\varphi : \mathbf{g_0} \to \mathbf{R} : X \mapsto tr(ad_X)$. Since $tr([ad_X, ad_Y]) = 0$, φ is a Lie-homomorphism. The kernel $\mathbf{u} \doteq ker\varphi$ is an ideal called *unimodular kernel*; $\mathbf{g_0}$ is said *unimodular* if $\mathbf{g_0} = \mathbf{u}$. An important result is given by the

Lemma. Let $\mathbf{g_0}$ be an unimodular 3-dimensional Lie-algebra endowed with a scalar

product. Then there exists an orthonormal base (E_1, E_2, E_3) such that

1. $[E_2, E_3] = \lambda_1 E_1$, $[E_3, E_1] = \lambda_2 E_2$ and $[E_1, E_2] = \lambda_3 E_3$;

2. $B(X, Y) = -2(\lambda_2\lambda_3 X^1 Y^1 + \lambda_1\lambda_3 X^2 Y^2 + \lambda_1\lambda_2 X^3 Y^3)$, where $X = X^i E_i, Y = Y^i E_i$

(for a proof, see [GR]).

The 3-dimensional Lie-algebras are classified by the following cases

1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$

2. $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = 0$

3. $\lambda_1\lambda_2 \neq 0$, $\lambda_3 = 0$

4. $\lambda_1\lambda_2\lambda_3 \neq 0$.

Case1: $\mathbf{g_0}$ is abelian and isomorphic to $\mathbf{R}^3$. Each plane supports a LCR: in fact, let $\mathbf{p} = L(X, Y)$ a fixed plane; a structure as desired is given by $J(X, Y) = (-Y, X)$.

Case2: the Lie-product is described by $[E_2, E_3] = \lambda_1 E_1$, $[E_3, E_1] = [E_1, E_2] = 0$. The planes $\mathbf{p}_2 = L(E_1, E_3)$ and $\mathbf{p}_3 = L(E_1, E_2)$ are abelian ideals endowed with the LCRs $J_2(E_1, E_3) = (-E_3, E_1)$ and $J_3(E_1, E_2) = (-E_2, E_1)$.

Case3: as in the Case before, the plane $\mathbf{p}_3 = L(E_1, E_2)$ is an abelian ideal endowed with the structure $J_3(E_1, E_2) = (-E_2, E_1)$.

Case4: B is nondegenerate, i.e. $\mathbf{g_0}$ is semisimple. But 3-dimensional semisimple Lie-algebras are simple. Hence $\mathbf{g_0}$ has no nontrivial ideals. So there are no LCR-structures on such a $\mathbf{g_0}$. A deeper analysis shows that if all the λ_i are positive $\mathbf{g_0}$ is isomorphic to $\mathbf{su}(2)$; while if one of them is negative it is isomorphic to $\mathbf{sl}(2, \mathbf{R})$. In both the cases $\mathbf{g_0}$ is a real form (compact or not) for $\mathbf{sl}(2, \mathbf{C})$.

The last case is when $\mathbf{g_0}$ is not unimodular. Which means that φ is a nonvanishing real

linear form. So its kernel $\mathbf{u}$ is an abelian 2-dimensional ideal.

Summarizing all the case, one obtains that a 3-dimensional real Lie-algebra $\mathbf{g_0}$ either is a (simple) real form of $\mathbf{sl}(2, \mathbf{C})$ either is endowed with (at least) one LCR-structure given on a 2-dimensional abelian ideal.

The study of LCR-structures on 2- and 3-dimensional Lie-algebras, make easy the classification on 5-dimensional ones (remark that, if one considers the 4-dimensional case, the only non-solvable Lie-algebra endowed with a LCR-structure is $\mathbf{R} \oplus \mathbf{s_0}$, where $\mathbf{s_0}$ is a real form of $\mathbf{sl}(2, \mathbf{C})$). Such a study is quite interesting since it makes use of Levi-Mal'cev decomposition as we have shown in Section4. In the sequel, let $\dim \mathbf{g_0} = 5$. Suppose that $\mathbf{g_0}$ is decomposed as $\mathbf{g_0} = \mathbf{r_0} \oplus_{ad} \mathbf{s_0}$. Let us consider the dimension $\dim \mathbf{r_0}$.

When $\dim \mathbf{r_0} = 0$, $\mathbf{g_0}$ is semisimple. Since there are no semisimple algebras of dimension 1 and 2, $\mathbf{g_0}$ may not have nonvanishing ideals. So $\mathbf{g_0}$ is simple and it has no LCR-structures (cfr the Table).

Let $\dim \mathbf{r_0} = 1$. Then $\mathbf{r_0} = \mathbf{R}$ is abelian and $\mathbf{s_0}$ is simple. Any LCR-structure, if it is, is in $\mathbf{s_0}$. But, $\mathbf{s_0}$ does not contain ideals. So $\mathbf{g_0}$ has no LCR-structures.

In the case $\dim \mathbf{r_0} = 2$, $\mathbf{r_0}$ either is abelian or it is the solvable algebra $\mathbf{h_0}$. The corresponding Levi-subalgebra $\mathbf{s_0}$ is simple and coincides either with $\mathbf{su}(2)$ or with $\mathbf{sl}(2, \mathbf{R})$. Even in this case, $\mathbf{s_0}$ does not admit LCR-structures. The only one is given by the solvable ideal $\mathbf{r_0}$ endowed with an endomorphism of the form J_A.

The cases $\dim \mathbf{r_0} = 3, 4$ can not occur, since $\mathbf{s_0}$ should be 2- or 1-dimensional.

The last case is $\dim \mathbf{r_0} = 5$. Then $\mathbf{g_0}$ is solvable and it admits LCR-structures on all its 2- and 4-dimensional ideals.

6.REFERENCES.

[AHR], H. Azad, A. Huckleberry, W. Richtofer, Homogeneous CR-manifolds, *J. Math.* V. 358 198 B. **358** (1985), 125-154.

[BO], A. Boggess, "CR Manifolds and the Tangential Cauchy-Riemann Complex", Studies in Advanced Mathematics, CRC Press.

[DG], S. Donnini, G. Gigante, Classification of left invariant CR-structures on $GL^+(3,\mathbf{R})$, *Riv. Mat. Univ. Parma* **16** (1990), 343-355.

[GR], A. Gray, "Notes on Lie Groups," to appear.

[GT], G. Gigante and G. Tomassini, CR-structures on a Real Lie Algebra, *Adv. Math.* **94** (1992), 67-81.

[HE], S. Helgason, "Differential Geometry and Symmetric Spaces", Academic Press, San Diego, 1962.

[JA], H. Jacobowitz, "An introduction to CR-structures", AMS, Providence Rhode Island, 1990.

[KN], S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry", vol. I and II, Interscience Publ, New York, 1969.

[MI], J. Milnor, Curvature of left invariant metrics on Lie groups, *Adv. Math.* **21** (1976), 293-329.

[SE], J.P. Serre, "Complex Semisimple Lie Algebras", Springer-Verlag, New York, 1987.

[SN], D. Snow, Invariant complex structures on reductive Lie groups, *J. Math. B.* **371** (1986), 191-215.

[VA], V.S. Varadarajan, "Lie groups, Lie algebras and their Representations", Prentice-

Hall, Englewood Cliffs, NJ, 1974.

[WE], S.W. Webster, Pseudo-hermitian structures on a real hypersurface, *J. Differential Geom.* **13** (1978), 25-41.