SISSA-136-95-FM

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KAM THEOREM FOR GENERIC ANALYTIC PERTURBATIONS OF THE EULER SYSTEM

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Abstract: We apply here KAM theory to the fast rotations of a rigid body with a fixed point, subject to a purely positional potential. The problem is equivalent to a small perturbation of the Euler system. The difficulty is that the unperturbed system is properly degenerate, namely the unperturbed Hamiltonian depends only on two actions. Following the scheme used by Arnol'd for the N–body problem, we use part of the perturbation to remove the degeneracy: precisely, we construct Birkhoff normal form up to a suitable finite order, thus eliminating the two fast angles; the resulting system is nearly integrable and (generically) no more degenerate, so KAM theorem applies. The resulting description of the motion is that, if the initial kinetic energy is sufficiently large, then for most initial data the angular momentum has nearly constant module, and moves slowly in the space, practically following the level curves of the initial potential averaged on the two fast angles; on the same time the body precedes around it, essentially as in the Euler–Poinsot motion. We also provide two simple physical examples, where the procedure does apply.

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Keywords: Rigid body, KAM theorem.

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1. INTRODUCTION

1.1. The aim of this paper is to apply KAM theory to generic analytic perturbation of the free rigid body with a fixed point (analytically perturbed Euler system). Using the action–angle variables $(I_1, I_2, J, \varphi_1, \varphi_2, j)$ of the free Euler system, the Hamiltonian is written:[1]

$$H = T(I_1, I_2) + \mu V(I_1, I_2, J, \varphi_1, \varphi_2, j) \tag{1.1}$$

(see section 2 for a description of such coordinates). The kinetic energy T is known to be a homogenous function of (I_1, I_2) of degree two. Concerning V we shall assume it is derived from a purely positional potential (a potential, say, that in the familiar Euler coordinates depends only on the angles); the immediate consequence is that V is a homogenous function of degree zero of the actions. Due to such homogeneity property, it is clear that the Hamiltonian problem (1.1) is equivalent, via a trivial rescaling, to the fast rotations of a rigid body in a fixed analytic potential (with $\mu \sim ||\omega||^{-2}$, if ω is the angular velocity of the motions one deals with).

The absence of the third action J from the unperturbed Hamiltonian reflects the so-called proper degeneracy of the system. In special although relevant cases, like the gravity, the perturbation is symmetric, namely is independent of j; one then immediately reduces to a system with only two degrees of freedom, so degeneracy disappears and KAM theory applies.[2] For a general non-symmetric perturbation, instead, degeneracy prevents a direct use of KAM theorem. The situation is in fact similar to the N-body problem studied by Arnol'd.[3] We follow here the same idea, namely we use the perturbation to remove the degeneracy. More precisely we proceed as follows:

i) We perform a finite number of Birkhoff perturbative steps, namely we introduce a canonical transformation

$$(I_1, I_2, J, \varphi_1, \varphi_2, j) = \Psi_\mu(I_1', I_2', J', \varphi_1', \varphi_2', j'), \qquad \|\Psi_\mu - \mathrm{Id}\| \sim \mu,$$

such that the new hamiltonian $H' = H \circ \Psi$ has the form:

$$H' = T'(I_1', I_2') + \mu \widehat{V}'(I_1', I_2', J', j') + \mu^{l+1} \widetilde{V}'(I_1', I_2', J', \varphi_1', \varphi_2', j'),$$

where $\widehat{V}' = \overline{V} + \mathcal{O}(\mu)$, $\overline{V}$ denoting the average of the potential on the fast angles (φ_1, φ_2).

ii) We profit of the fact that $T' + \mu \widehat{V}'$ depends only on one angle, and thus it is integrable, to give the Hamiltonian the new form:

$$H'' = T''(I_1'', I_2'') + \mu \widehat{V}''(I_1'', I_2'', I_3'') + \mu^{l+1} \widetilde{V}''(I_1'', I_2'', I_3'', \varphi_1'', \varphi_2'', \varphi_3'').$$

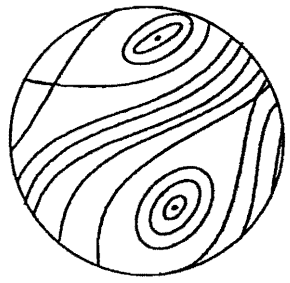

Fig. 1

More precisely, this can be done in the neighbourhood of every level curve of $\widehat{V}'$, at I_1', I_2' fixed, that does not contain singular points of $\widehat{V}'$ (see figure 1); let us notice that, for μ small, it is enough to assume that the corresponding level curve of $\overline{V}$ does not contain singular points.

iii) Then we make the "genericity assumption" that $\overline{V}$, and thus (for small μ) $\widehat{V}''$, does depend on I_3'', namely the degeneracy is removed and KAM theorem applies, $\varepsilon = \mu^{l+1}$ being the perturbative parameter. For sufficiently large l ($l = 4$ will suffice) the procedure actually works.

Let us remark that the possibility of such a procedure of removing the degeneracy of the Euler problem, is not unexpected, and certainly not surprising among people who are familiar with KAM theory. On the same time we believe it is worthwhile, and not trivial, to work out this procedure with all of the necessary details, making, whenever possible, precise estimates, determining the minimal number of the Birkhoff steps, and establishing, in particular, the relation between the small parameter μ appearing in (1.1) and the measure of the set of the preserved tori. This is indeed what we shall do in this paper. In addition, in order the treatment not to be abstract, we shall provide two simple examples where the procedure does apply. For simplicity we study here the case in which the potential has a "natural cutoff", namely it contains only a finite number of Fourier components in the fast angles φ_1, φ_2. The generalization is annoying, because of the presence of many logarithmic corrections, but is not difficult (see Section 3.4).

1.2. We provide here a rough description of our results; for a precise statement, see Section 3. The notation $\mathcal{A}$ indicates here the action space (see section 3.1 for a precise definition), slightly restricted by removing a small neighborhood of the gyroscopic rotations and also, for triaxial bodies, of the separatrices of the unstable rotation; such a restriction is necessary because, as is well known, the action–angle coordinates are there singular.

Our statement is essentially as follows:

Let us consider the Hamiltonian (1.1) defined in the domain $\mathcal{A} \times \mathbf{T}^3$. Consider any of the above described level curves of $\overline{V}$. If μ is small enough, and γ_1 is such that

$$\mu \leq \text{const.}\, \gamma_1^2,$$

then there exists a tube $\mathcal{P} \subset \mathcal{A} \times \mathbf{T}^3$ around this curve, which decomposes in two subsets

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$$

such that $\mathcal{P}_2$ has small Lebesgue measure:

$$\text{mes}\,(\mathcal{P}_2) \leq \text{const.}\, \gamma_1 \text{ mes}\,(\mathcal{P}),$$

while $\mathcal{P}_1$ is the union of three–dimensional invariant tori, and for every initial datum

$$(I_1^0, I_2^0, J^0, \varphi_1^0, \varphi_2^0, j^0) \in \mathcal{P}_1$$

the motion on the corresponding torus is described by

$$|I_i(t) - I_i^0| \leq \text{const.}\, \frac{\mu^{\frac{1}{2}}}{\gamma_1^5}, \quad |\varphi_i(t) - \varphi_i^0 - \omega_i t| \leq \text{const.}\, \frac{\mu}{\gamma_1^2}, \qquad i = 1, 2\,.$$

Correspondingly $(J(t), j(t))$ stay close to the level curves of the averaged potential at fixed $(I_1, I_2) = (I_1^0, I_2^0)$, namely one has

$$|\overline{V}(I_1, I_2, J(t), j(t)) - \overline{V}(I_1, I_2, J^0, j^0)| \leq \text{const.}\, \frac{\mu^{\frac{1}{2}}}{\gamma_1^6}\,.$$

In a word: if the potential removes the degeneracy, and μ is small enough (or equivalently, if the initial kinetic energy is sufficiently large), then for the majority of initial data the angular momentum has quasi-constant module, and moves slowly in the space, practically following the level curves of the initial potential averaged on the fast angles (φ_1, φ_2); on the same time the body precedes around it, essentially as in the Euler–Poinsot motion. Such a behavior is not so different from the one emerging from the Nekhoroshev study of the same problem;[4] as usual, in the Nekhoroshev approach, the description holds for all initial data, but only for finite, although long, time scales.

1.3. Now we introduce the version of KAM theorem that we use here as the reference statement. Such a statement essentially coincides with the one produced in [5], although

it was necessary to make it more precise in some details, in view of the application to the properly degenerate problem of the rigid body. We also included in the statement the time–dependent case, as it will be useful for one of our examples (namely a model of the satellite spin precession).

Let us fix some notations. We are concerned with quasi–integrable Hamiltonian systems with n degrees of freedom, of the form:

$$H(p,q,t) = h(p) + \varepsilon f(p,q,t), \tag{1.2}$$

with:

$$p = (p_1, \dots, p_n) \in \mathcal{B} \subset \mathbf{R}^n, \qquad q = (q_1, \dots, q_n) \in \mathbf{T}^n, \qquad t \in \mathbf{R},$$

where $\mathcal{B}$ is some ball; f is assumed to be periodic in t with given period T.

Denote by h_p the frequency map $h_p : p \mapsto \omega(p) = \frac{\partial h}{\partial p}(p)$, and define $\Omega = h_p(\mathcal{B})$. Given $\rho = (\rho_p, \rho_q, \rho_t)$, with $\rho_p, \rho_q, \rho_t > 0$, we let:

$$\mathcal{B}_\rho = \bigcup_{p \in \mathcal{B}} \Delta_\rho(p), \quad \text{with} \quad \Delta_\rho(p) = \{\overline{p} \in \mathbf{C}^n : |p_i - \overline{p}_i| \le \rho_p, \ i = 1, \dots, n\},$$
$$\mathbf{T}^n_\rho = \{q \in \mathbf{C}^n : |\mathrm{Im}\, q_i| \le \rho_q, \ i = 1, \dots, n\}, \quad \Sigma_\rho = \{t \in \mathbf{C} : |\mathrm{Im}\, t| \le \rho_t\},$$

and finally $\mathcal{P}_\rho = \mathcal{B}_\rho \times \mathbf{T}^n_\rho$; we call ρ_p, ρ_q, ρ_t the extension radii respectively in the variables p, q, t.

All functions are assumed to be real analytic (namely analytic and real for real variables), periodic of real period 2π in the angles, and periodic of real period T in time. For any $F : \mathcal{B}_{\tilde{\rho}} \times \mathbf{T}^n_{\tilde{\rho}} \times \Sigma_\rho \to \mathbf{C}$ and any $\tilde{\rho} = (\tilde{\rho}_p, \tilde{\rho}_q, \tilde{\rho}_t) \le (\rho_p, \rho_q, \rho_t)$ (the inequality is intended to work component by component), we let:

$$\|F\|_{\tilde{\rho}} = \sup_{(p,q,t) \in \mathcal{B}_{\tilde{\rho}} \times \mathbf{T}^n_{\tilde{\rho}} \times \Sigma_{\tilde{\rho}}} \|F(p,q)\|.$$

Finally, we denote by " mes" the Lebesgue measure.

KAM THEOREM. *Consider the Hamiltonian system (1.2). Suppose that H is real analytic in the domain $\mathcal{P}_\rho \times \Sigma_\rho$ with given ρ_p, ρ_q and $\rho_t = \rho_q \frac{T}{2\pi}$. Assume that h is bounded and non–degenerate, more precisely that the frequency map h_p is a global diffeomorphism $\mathcal{B} \to \Omega$, and there exist constants m and M, $0 < m \le M < \infty$, such that:*

$$m\|v\|_{\rho_p} \le \|h_{pp} v\|_{\rho_p} \le M\|v\|_{\rho_p}, \qquad v \in \mathcal{B}_{\rho_p}. \tag{1.3}$$

Finally, let γ be such that:

$$\varepsilon \le \delta\gamma^2, \tag{1.4}$$

where $\delta(n,\rho_p,\rho_q,M)$ is a suitable constant:

$$\delta(n,M,\rho_p,\rho_q)=\left(\frac{1}{3^2 2^{10n+14}5^{2n+1}[(2n-1)!]^2}\frac{\rho_p\rho_q^{2n+1}}{M{\rho_p}^2+\|f\|}\right)^2.$$

Then there exists a decomposition $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$, such that:

$$\text{mes}\,(\mathcal{P}_2)\le\frac{C(n)M^{n-1}\gamma}{d\min_{p\in\mathcal{B}}\{\det(h_{pp}(p))\}}\,\text{mes}\,(\mathcal{P}), \tag{1.5}$$

where d is the diameter of $\mathcal{B}$ and $C(n)=\frac{2^{n+1}\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-1}{2}+1)n\sqrt{\pi}}$, while $\mathcal{P}_1$ is the union of invariant n-dimensional tori $\mathcal{T}_\omega$ depending in a Lipschitz continuous way on $\omega\in\Omega^\gamma\subset\Omega$,

$$\Omega^\gamma=\left\{\omega:\langle\mathbf{k},\omega\rangle\ge\frac{\gamma}{|\mathbf{k}|^n}\ \forall\mathbf{k}\in\mathbb{Z}^n\setminus\{0\}\right\}.$$

More precisely, such tori are defined parametrically by equations of the form

$$p=p_0+L_\omega+f_\omega(Q,t),\qquad q=Q+g_\omega(Q,t),\quad \textit{with}\quad Q\in\mathbf{T}^n\quad\textit{and}\quad t\in\mathbf{R}, \tag{1.6}$$

where $p_0=h_p^{-1}(\omega)$, while L_ω is some constant, and f_ω, g_ω are real analytic functions, periodic in t with period T; L_ω, f_ω and g_ω depend in a Lipschitz continuous way on ω. The tori $\mathcal{T}_\omega$ are near the unperturbed ones $\{p_0\}\times\mathbf{T}^n$, more precisely one has

$$\|L_\omega\|\le c(n)\left(\frac{1}{2}M\rho_p^2+\|f\|\right)\frac{M\sqrt{\varepsilon}}{m\gamma} \tag{1.7a}$$

$$\|f_\omega\|\le\frac{c(n)\left(\frac{1}{2}M\rho_p^2+\|f\|\right)}{\rho_q^{2n+1}}\frac{\varepsilon}{\gamma} \tag{1.7b}$$

$$\|g_\omega\|\le\frac{c(n)\left(\frac{1}{2}M\rho_p^2+\|f\|\right)}{\rho_p\rho_q^{2n}}\frac{\sqrt{\varepsilon}}{\gamma}, \tag{1.7c}$$

$c(n)$ being a suitable constant depending only on the number of degrees of freedom.

As remarked above, the statement does not really differ in an essential way from the one appearing in [5]; the time–dependent case is there not explicitly treated, but the generalization is straightforward. The only point which is not treated in [5], but is crucial to our application to the degenerate problem, is the estimate (1.5) of the measure. A sketch of the proof of this point is deferred to Section 5.

1.4. The paper is organized as follows: in Section 2 we recall a few basic facts on the action-angle variables for the rigid body; in Section 3 we describe the procedure of removing the degeneracy, then we state our result and we prove it. Section 4 is devoted to two examples: the former is not physically relevant, but the application of our method is straightforward; the latter is a model for the satellite spin precession. In Section 5 we report the proofs of a few technical lemmas.

2. ON THE EULER–POINSOT SYSTEM

We recall here some basic facts on the dynamics the Euler–Poinsot system. For a detailed treatment, see for example [4,6].

2.1. *The Hamiltonian setting.* The Euler–Poinsot system has three degrees of freedom; the configuration space is SO_3 and the phase space is its cotangent bundle T^*SO_3 that can be identified with $\mathbb{R}^3 \times SO_3$ in two different ways according to consider the angular momentum in the body frame $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, as we do, or in the inertial frame $(O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. By body frame, of course, we mean the one whose axes are the principal axes of inertia.

The so-called Andoyer variables are defined as follows: if $\mathbf{m}$ is the angular momentum, then (see figure 2):

i) The Andoyer angles are three real functions $j(\mathrm{mod}2\pi), g(\mathrm{mod}2\pi), l(\mathrm{mod}2\pi)$ where j is the angle between the directions of $\mathbf{e}_x$ and of $\mathbf{e}_z \wedge \mathbf{m}$; g is the angle between $\mathbf{e}_z \wedge \mathbf{m}$ and $\mathbf{m} \wedge \mathbf{e}_3$; l is the angle between $\mathbf{m} \wedge \mathbf{e}_3$ and $\mathbf{e}_1$.

ii) The Andoyer momenta are:

$$J = \langle \mathbf{m}, \mathbf{e}_z \rangle, \qquad G = \langle \mathbf{m}, \mathbf{m} \rangle^{\frac{1}{2}}, \qquad L = \langle \mathbf{m}, \mathbf{e}_3 \rangle.$$

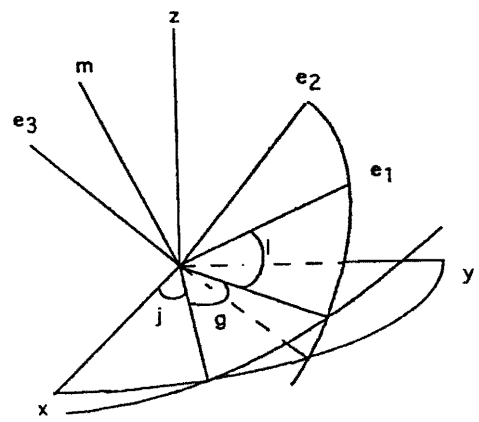

Fig. 2

Remark: the Andoyer variables are not defined for $\mathbf{m} \parallel \mathbf{e}_3$ and $\mathbf{m} \parallel \mathbf{e}_z$; since the choice of the $\mathbf{e}_z$ axis is arbitrary, the singularity $\mathbf{m} \parallel \mathbf{e}_z$ can be avoided by an appropriate choice of the z-axis. On the contrary the singularity $\mathbf{m} \parallel \mathbf{e}_3$ reflects a real singularity of the system (a family of two–dimensional tori collapse there into a periodic orbit) and must be removed from the phase space. The subset of the phase space $\mathbb{R}^3 \times SO_3$ obtained by removing such singularities will be denoted by $\mathcal{M}$.

More precisely the following holds:

i) *The map:*

$$\begin{array}{ccc} \mathcal{M} & \to & \Lambda \times \mathbf{T}^3 \\ (\mathbf{m}, \mathcal{R}) & \mapsto & (J, L, G, j, l, g) \,, \end{array}$$

where

$$\Lambda = \{(J, L, G) \in \mathbf{R}^3 : G > 0, |L| < G, |J| < G\} \,,$$

is a real analytic canonical diffeomorphism.

ii) *Denoting by θ, χ, ψ the familiar Euler angles, one has:*

$$\cos\vartheta = \Theta\left(\frac{J}{G}, \frac{L}{G}, g\right); \qquad \Theta(x, y, g) = xy - \sqrt{1-x^2}\sqrt{1-y^2}\cos g \quad (2.1a)$$

$$\cos(\chi - j) = \Phi\left(\frac{J}{G}, \frac{L}{G}, g\right); \qquad \Phi(x, y, g) = \frac{y - x\Theta(x, y, g)}{\sqrt{1-\Theta^2(x, y, g)}\sqrt{1-x^2}} \quad (2.1b)$$

$$\cos(\psi - l) = \Psi\left(\frac{J}{G}, \frac{L}{G}, g\right); \qquad \Psi(x, y, g) = \Phi(x, y, g) \quad (2.1c)$$

and:

$$p_\vartheta = \sqrt{G^2 - L^2}\left[1 - \Psi^2\left(\frac{J}{G}, \frac{L}{G}, g\right)\right], \qquad p_\chi = J, \qquad p_\psi = L. \quad (2.2)$$

For a detailed proof, see [4,6].

It is easy to prove (see for example [7]) that the kinetic energy, i.e. the unperturbed Hamiltonian, has the form:

$$\mathcal{H}(J, G, L, j, g, l) = (G^2 - L^2)\left(\frac{\sin^2 l}{2a_1} + \frac{\cos^2 l}{2a_2}\right) + \frac{L^2}{2a_3}, \quad (2.3)$$

where a_i are the principal moments of inertia; from the equations of motion associated to the Hamiltonian (2.3) it follows that (G, J, j) are constant (this expresses the fact that the angular momentum $\mathbf{m}$ is constant in the inertial frame). Moreover the energy is conserved, so there are four independent constants of motion for our three degrees of freedom system: the system is *superintegrable*, and correspondingly, the Hamiltonian is *properly degenerate.*

2.2. *The action-angle variables.* In the symmetric case the Andoyer variables are already action-angle variables, $I_1 = L$, $I_2 = G$ and $\varphi_1 = l$, $\varphi_2 = g$. In fact we can always assume $a_3 \neq a_1 = a_2$, and obtain

$$\mathcal{H}(J, G, L, j, g, l) = \frac{1}{2a_1}\left(G^2 + \eta L^2\right), \qquad \eta = \frac{a_1 - a_3}{a_3}. \quad (2.4)$$

For $\eta \neq 0$ (non spherical ellipsoid of inertia) the rank of the Hessian matrix in the actions (L, G) is two.

In the triaxial case the action–angle variables $(I_1, I_2, \varphi_1, \varphi_2)$ are also easily constructed (see for example [1]). The simplest procedure is to preliminarily introduce, for any fixed G, a pair of action–angle variables (I_1, φ_1) in place of (L, l); in particular one has:

$$I_1(G, T(L, G, l)) = \frac{1}{2\pi} \oint L dl = \frac{1}{2\pi} \oint \sqrt{\frac{\left|2T - G^2\left(\frac{\sin^2 l}{a_1} + \frac{\cos^2 l}{a_2}\right)\right|}{\frac{1}{a_3} - \left(\frac{\sin^2 l}{a_1} + \frac{\cos^2 l}{a_2}\right)}} dl\,, \tag{2.5}$$

where the integration path is given by the curves of constant energy, at constant G; energy levels corresponding to gyroscopic rotations must be avoided. Such a transformation is easily canonically extended to a transformation

$$(L, G, l, g) \mapsto (I_1, I_2, \varphi_1, \varphi_2),$$

where $I_2 = G$ (but $\varphi_2 \neq g$). The new Hamiltonian has then the form $T(I_1, I_2)$ and the following holds: [1]

The energy function $T(I_1, I_2)$ is continuous, is a homogeneous function of degree 2 and is analytic in the region

$$\Delta = \left\{(I_1, I_2) \in \mathbb{R}^2 : \ 0 < |I_1| < I_2, \ \frac{2T}{I_2^2} \neq \frac{1}{a_2}\right\}.$$

Moreover the canonical transformation:

$$(I_1, I_2, J, \varphi_1, \varphi_2, j) \mapsto (G, L, J, g, l, j)$$

is analytic in $\Delta \times \mathbb{T}^3$.

The domain Δ is the union of four connected subregions (see figure 3), bounded by the lines $|I_1| = I_2$, $I_2 = 0$ and $|I_1| = \hat{\beta} I_2$, where $\hat{\beta}$ is a constant defined as follows: one preliminarily observes that the level curves of $\frac{2T}{I_2^2}$ are straight lines through the origin, with angular coefficient β depending on $\frac{2T}{I_2^2}$:

$$\beta\left(\frac{2T}{I_2^2}\right) = \frac{1}{2\pi} \oint \sqrt{\frac{\left|\frac{2T}{I_2^2} - \left(\frac{\sin^2 l}{a_1} + \frac{\cos^2 l}{a_2}\right)\right|}{\frac{1}{a_3} - \left(\frac{\sin^2 l}{a_1} + \frac{\cos^2 l}{a_2}\right)}} dl\,.$$

The singularity $\frac{2T}{I_2^2} = \frac{1}{a_2}$ (that is the one associated to the separatrix) corresponds then to the line with angular coefficient $\hat{\beta} = \beta(\frac{1}{a_2})$.

If we consider $T(I_1, I_2)$ as the Hamiltonian of a system with two degrees of freedom, it turns out that it is non-degenerate; more precisely one proves that: [2,1]

The Hessian Γ of the function $T(I_1, I_2)$ is strictly positive in the subregions of Δ adjacent to the lines $|I_1| = I_2$, strictly negative in the remaining connected subregions.

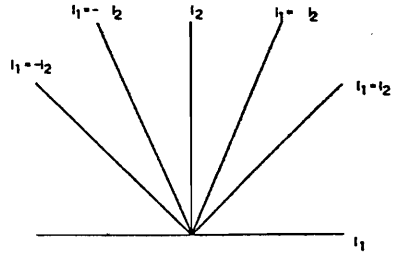

Fig. 3

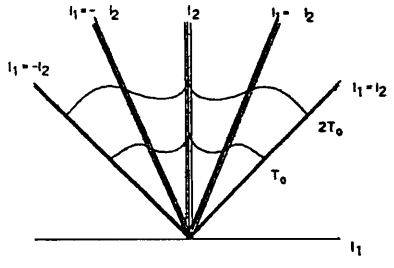

Fig. 4

3. KAM THEOREM FOR THE PERTURBED EULER SYSTEM

Let us consider the Hamiltonian of the triaxial rigid body with a fixed point:

$$H(I_1, I_2, J, \varphi_1, \varphi_2, j) = T(I_1, I_2) + \mu V(I_1, I_2, J, \varphi_1, \varphi_2, j)\,, \tag{3.1}$$

where

$$(I_1, I_2) \in \Delta, \quad |J| < I_2, \quad (\varphi_1, \varphi_2, j) \in \mathbb{T}^3\,.$$

3.1. *Regularity properties.* As already remarked, the kinetic energy T is real analytic in the whole infinite domain Δ. However, in order to dominate the Hessian of T, we shall restrict (I_1, I_2) to a subset of the form

$$\Delta(T_0) = \{(I_1, I_2) \in \Delta : T_0 \le T(I_1, I_2) \le 2T_0\}\ ;$$

moreover, to be later able to suitably extend the domain of H without introducing singularities, we further restrict the action domain by eliminating a narrow strip around the singular lines which border Δ; so we introduce the reduced domain (see figure 4)

$$\Delta(T_0, r_0) = \{(I_1, I_2) \in \Delta(T_0) : |I_1| > 2r_0;\ |I_1 - \hat{\beta} I_2| > 2r_0,\ I_2 - |I_1| > 2r_0\}\ , \tag{3.2}$$

where r_0 is a small free parameter.

So, the kinetic energy turns out to be analytic in every complex extension $\Delta_r(T_0, r_0)$ of $\Delta(T_0, r_0)$, with extension radius $r \leq r_0$, and the norm M of the Hessian matrix is there finite and its determinant Γ is bounded away from zero.

Let

$$\mathcal{A}(T_0, r_0) = \{(I_1, I_2, J) : (I_1, I_2) \in \Delta(T_0, r_0),\ |J| < I_2 - r_0\}\ , \tag{3.3}$$

and denote by $\mathcal{A}_r(T_0, r_0)$ the complex extension of such a set, with extension radius $r \leq r_0$. We shall assume that V is analytic and bounded in $\mathcal{A}_r(T_0, r_0) \times \mathbb{T}^3_{s_0}$ up to $r = r_0$. We stress that such assumption is certainly satisfied, for not too large r_0, s_0, if V is derived by a purely positional potential, which is a real analytic function of the Euler angles. The domain of the above Hamiltonian H coincides, of course, with the domain of V. For simplicity, in the following we shall use the short notation $\mathcal{A}_r$ in place of $\mathcal{A}_r(T_0, r_0)$.

As remarked in the Introduction, we shall consider the particular case of a potential with a finite number N of harmonics in the fast angles, but the extension to the general case is easy.

3.2. *Birkhoff steps.* In the following we shall denote by $\overline{V}(I_1, I_2, J, j)$ the average of $V(I_1, I_2, J, \varphi_1, \varphi_2, j)$ over the fast angles φ_1, φ_2 and by V_0 the sup–norm of V on $\mathcal{A}_{r_0} \times \mathbb{T}^3_{s_0}$. As remarked in the Introduction, we shall restrict ourselves to the case in which the Fourier series of V in the angles φ_1, φ_2 contains only a finite number of terms, say up to $|k| \leq N$ with $|k| = |k_1| + |k_2|$.

Lemma 1. *Consider the Hamiltonian (3.1), let it be real analytic in $\mathcal{A}_r \times \mathbb{T}^3_{s_0}$, and assume V has truncated Fourier series, with N as above. Assume γ_1 and r be such that:*

$$\mu \leq \frac{C}{V_0}\gamma_1 r s_0^3\ , \qquad r \leq \frac{\gamma_1}{M(4N-1)^3}\ , \tag{3.4}$$

where $C = \frac{1}{3^6 2^{22}}$, and consider the real sets

$$\Delta' = \left\{(I_1, I_2) \in \Delta(T_0, r_0) : |\omega_1 k_1 + \omega_2 k_2| \geq \frac{\gamma_1}{|k|^2},\ \forall k \in \mathbb{Z}^2 : |k| \leq 4N-1\right\}$$
$$\mathcal{A}' = \{(I_1, I_2, J) : (I_1, I_2) \in \Delta',\ |J| < I_2 - r_0\}\ ,$$

where $\omega_i = \frac{\partial T}{\partial I_i}$, $i = 1, 2$. Then there exists a real analytic canonical transformation

$$\Psi : \mathcal{A}'_{\frac{r}{2}} \times \mathbb{T}^3_{\frac{s_0}{2}} \to \mathcal{A}_r \times \mathbb{T}^3_{s_0}\ ,$$

$(I_1, I_2, J, \varphi_1, \varphi_2, j) = \Psi(I'_1, I'_2, J', \varphi'_1, \varphi'_2, j')$, that conjugates the Hamiltonian (3.1) to a new Hamiltonian $H' = H \circ \Psi$ of the form

$$H' = T'(I'_1, I'_2) + \mu\widehat{V}(I'_1, I'_2, J', j'; \mu) + \mu^5\widetilde{V}(I'_1, I'_2, J', \varphi'_1, \varphi'_2, j'; \mu)\ , \tag{3.5}$$

where $\widehat{V}$ is close to $\overline{V}$:

$$\|\widehat{V} - \overline{V}\|_{\frac{r}{2},\frac{s}{2}} \leq \mu \frac{8c_1 V_0^2}{\gamma_1 r s_0^3}, \tag{3.6}$$

and H' is real analytic on $\mathcal{A}'_{\frac{r}{2}} \times \mathbf{T}^3_{\frac{s_0}{2}}$. Moreover, one has:

$$\|\widehat{V}\|_{\frac{r}{2},\frac{s_0}{2}} \leq 2V_0, \qquad \|\widetilde{V}\|_{\frac{r}{2},\frac{s_0}{2}} \leq \mu^5 \frac{c_2 V_0^5}{(\gamma_1 r s_0^3)^4}, \tag{3.7}$$

and the canonical transformation Ψ is near the identity:

$$\begin{aligned} \frac{1}{r}\|\Psi^I(I_1, I_2, J) - \mathrm{Id}_{(I_1,I_2,J)}\|_{\frac{r}{2},\frac{s_0}{2}} &\leq \frac{c_1 \mu V_0}{\gamma_1 r s_0^3} \\ \frac{1}{s}\|\Psi^\varphi(I_1, I_2, J) - \mathrm{Id}_{(\varphi_1,\varphi_2,j)}\|_{\frac{r}{2},\frac{s_0}{2}} &\leq \frac{c_1 \mu V_0}{\gamma_1 r s_0^3}, \end{aligned} \tag{3.8}$$

with $c_1 = (2^{11}3^2 5!)^4$ and $c_2 = 2^9 3^2 5!$.

A sketch of the proof is reported in Section 5.

3.3. *Action–angle variables for the normal form Hamiltonian.* The next crucial point concerns the introduction of the action–angle coordinates for the normal form Hamiltonian $T'(I'_1, I'_2) + \mu \widehat{V}'(I'_1, I'_2, J', j')$. To this purpose, let us preliminary fix (I'_1, I'_2), and consider the one degree of freedom Hamiltonian

$$h'_{I'_1,I'_2}(J', j') = \widehat{V}'(I'_1, I'_2, J', j').$$

Such a system is obviously integrable and the domain of the variables (J', j') is the cylinder $\{|J'| < I'_2 - r_0,\ j' \in \mathbf{T}\}$, or equivalently the sphere of constant angular momentum from which one removed the poles and small bowls around them. Since $h'_{I'_1,I'_2}$ is an analytic function on this sphere, there are at most finitely many singular level curves, namely level curves containing points in which $\nabla h'_{I'_1,I'_2} = 0$.

Let us consider a regular level curve $h'_{I'_1,I'_2}(J', j') = h^*$; if Δh is small enough, then the tube

$$\mathcal{J}'_{I'_1,I'_2} = \left\{(J', j') : h^* - \Delta h \leq h'_{I'_1,I'_2}(J', j') \leq h^* + \Delta h\right\}$$

does not contain singular level curves. It is then possible to introduce, in $\mathcal{J}'_{I'_1,I'_2}$, the action-angle variables (I''_3, φ''_3) for the Hamiltonian $h'_{I'_1,I'_2}$, with $\varphi''_3 \in \mathbf{T}$ and I''_3 belonging to some interval $\mathcal{I}''_{I'_1,I'_2}$: namely there exists a canonical transformation $\Phi_{I'_1,I'_2}$, parametrized by (I'_1, I'_2), such that

$$(J', j') = \Phi_{I'_1,I'_2}(I''_3, \varphi''_3),$$

and $\Phi_{I'_1,I'_2}$ is analytic in a suitable complex neighbourhood of $\mathcal{I}''_{I'_1,I'_2} \times \mathbf{T}$.

Remark: We implicitly assumed that $\mathcal{J}'_{I'_1,I'_2}$ is completely contained in the domain of $h'_{I'_1,I'_2}$, namely in the cylinder $\{|J'| < I'_2 - r_0,\ j' \in \mathbf{T}\}$. But this can be always realized because the choice of the z axis is arbitrary, and for small Δh, one can always take it in such a way that it does not intersect $\mathcal{J}'_{I'_1,I'_2}$.

In order to apply KAM theorem to the whole Hamiltonian $T'(I'_1,I'_2)+\mu\widehat{V}'(I'_1,I'_2,J',j';\mu)$, one needs to extend the above canonical transformation to all coordinates. To this purpose, let us notice that by varying (I'_1,I'_2) in some disc $D'(R) \subset \Delta'$ of radius R, the singular level curves in general move: however, if R is small enough, then for every $(I'_1,I'_2) \in D'(R)$ the set $\left\{(J',j') : h^* - \Delta h \le h'_{I'_1,I'_2}(J',j') \le h^* + \Delta h\right\}$ is still a tube free from singularities, and it is possible to extend the above defined change of variables to a set $\mathcal{P}'$ defined by

$$\begin{aligned} &(I'_1,I'_2) \in D'(R), \qquad (\varphi'_1,\varphi'_2) \in \mathbf{T}^2, \\ &(J',j') : h^* - \Delta h \le h'_{I'_1,I'_2}(J',j') \le h^* + \Delta h; \end{aligned} \tag{3.9}$$

correspondingly, the interval of definition $\mathcal{I}''(R)$ of the new action I''_3 in general reduces, but for R small enough, such reduced interval has radius larger or equal to R. It is then clear that:

Lemma 2. *There exists R such that for every disc $D'(R) \subset \Delta'$ the above canonical transformation $\Phi_{I'_1,I'_2}$ extends to*

$$\Phi : B'' \times \mathbf{T}^3 \to \mathcal{P}',$$

$$(I'_1,I'_2,J',\varphi'_1,\varphi'_2,j') = \Phi(I''_1,I''_2,I''_3,\varphi''_1,\varphi''_2,\varphi''_3),$$

where B'' is a ball of radius R, and Φ is such that $I'_1 = I''_1$, $I'_2 = I''_2$, and $(J',j) = \Phi_{I'_1,I'_2}(I''_3,\varphi''_3)$; Φ conjugates the Hamiltonian (3.5) to $H'' = H' \circ \Phi$ of the form

$$H'' = T''(I''_1,I''_2) + \mu\widehat{V}''(I''_1,I''_2,I''_3;\mu) + \mu^5\widetilde{V}''(I''_1,I''_2,I''_3,\varphi''_1,\varphi''_2,\varphi''_3;\mu), \tag{3.10}$$

where $\widehat{V}''$ and $\widetilde{V}''$ satisfy (3.6) and (3.7). The new Hamiltonian H'' is real analytic on the domain $B''_\rho \times \mathbf{T}^3_\rho$, with some nonvanishing extension radii ρ_I and ρ_φ.

Remarks. (i) We cannot compute explicitly the radii ρ_I and ρ_φ, since they depend in an essential way on the form of the considered potential. Anyway since, by (3.4), r is small with μ, while ρ_I is finite, for μ small enough one has certainly $\rho_I \ge r$. Concerning ρ_φ and s_0, we shall denote by s the smaller between them. (ii) In order for R, ρ_I and ρ_φ to be appropriately defined, we made reference, in the above Lemma, to the level curves of the whole normalized Hamiltonian $\widehat{V}$. However, it is important to stress that, if μ is small, the regular level curves of $\widehat{V}$ do not differ significantly from the corresponding level curves

of the average $\overline{V}$. So, every assumption can be verified, for small μ, on $\overline{V}$. This is relevant because in any example $\overline{V}$, and not $\widehat{V}$, is immediately computed.

3.4. *Main result.* Every set $B'' \times \mathbf{T}^3$, as described above, is the union of invariant three-tori for the Hamiltonian $T'' + \mu\widehat{V}''$. We want to apply KAM theorem to prove the persistence of a set of large Lebesgue measure of such tori for the perturbed Hamiltonian H''.

In order to do this, one has to verify that $T'' + \mu\widehat{V}''$ is non-degenerate; to this purpose, as we shall see (Lemma 3), it is enough to assume that there exists $m > 0$ such that

$$\left|\frac{\partial^2 \widehat{V}''}{\partial I_3''^2}\right| \geq m \quad \text{in} \quad B_r'' \times \mathbf{T}_s^3 \,. \tag{3.11}$$

It is then natural to give the following definition:

Definition. *The potential energy $V(I_1, I_2, J, \varphi_1, \varphi_2, j)$ is said to remove the degeneracy, if the assumption (3.11) is satisfied.*

<u>Remark</u>: This assumption too could be formulated only for $\overline{V}$. Indeed, in order for an inequality of the form (3.11) to be satisfied for small μ, it is enough it be satisfied by $\widehat{V}''(I_1'', I_2'', I_3''; 0)$, and this is nothing but the function one would obtain by using $\overline{V}$ in place of $\widehat{V}$ in Lemma 2.

We can now state our main result.

Proposition. *Let the Hamiltonian*

$$H(I_1, I_2, J, \varphi_1, \varphi_2, j) = T(I_1, I_2) + \mu V(I_1, I_2, J, \varphi_1, \varphi_2, j)$$

be real analytic in the above defined domain

$$\mathcal{A}_r \times \mathbf{T}_s^3 ;$$

assume the potential removes the degeneracy. Denote $C_0 = c(V_0 M)^{-1}(4N-1)^{-3}$, $C_1 = \frac{cM^3 V_0^3 (4N-1)^6}{m}$, $C_2 = cMV_0(4N-1)^3$ and $C_3 = \frac{cM^4 V_0^4 (4N-1)^9}{m}$, c being some suitable constant depending only on s. If μ is small enough, and γ_1 is such that

$$C_0 \gamma_1^2 \geq \mu \,, \tag{3.12}$$

then for every disc $D(R) \subset \Delta'$ of sufficiently small radius R, and for every regular level curve $\mathcal{C}$ of the averaged potential at fixed $(I_1, I_2) \in D(R)$, there exists a tube $\mathcal{P}$ around $\mathcal{C}$, which decomposes as

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \,,$$

with $\mathcal{P}_2$ such that

$$\text{mes}\,(\mathcal{P}_2) = \frac{M^2}{m\Gamma R}\gamma_1 \text{ mes}\,(\mathcal{P}) \ , \tag{3.13}$$

while $\mathcal{P}_1$ is the union of three–dimensional invariant tori, and for every initial datum

$$(I_1^0, I_2^0, J^0, \varphi_1^0, \varphi_2^0, j^0) \in \mathcal{P}_1$$

the motion on the corresponding torus satisfies

$$|I_i(t) - I_i^0| \le C_1 \ \frac{\mu^{\frac{1}{2}}}{\gamma_1^5} \tag{3.14}$$

$$|\varphi_i(t) - \varphi_i^0 - \omega_i t| \le C_2 \ \frac{\mu}{\gamma_1^2} \ , \tag{3.15}$$

for $i = 1, 2$. Correspondingly $(J(t), j(t))$ stay close to the level curves of the averaged potential at fixed $(I_1, I_2) = (I_1^0, I_2^0)$, namely one has

$$|\overline{V}(I_1, I_2, J(t), j(t)) - \overline{V}(I_1, I_2, J^0, j^0)| \le C_3 \ \frac{\mu^{\frac{1}{2}}}{\gamma_1^6}. \tag{3.16}$$

<u>Remark</u>. In the case in which the Fourier series of V in the angles φ_1, φ_2 contains an infinite number of terms, the procedure still works, although it requires a few technical changes. Indeed in Lemma 1 one should cut-off the Fourier series at order N such that $(4N - 1)^3 e^{-(4N-1)s} = \mu$; this leads to logarithmic corrections in the estimates (3.7) and (3.8), in the sense that r acquires a logarithmic dependence on μ. Such corrections then appear in every successive estimate, in particular in our Proposition; in fact in the definition of $C_0, \ldots, C_3$, whenever N appears, a logarithmic dependence on μ is introduced. Everything runs then quite smoothly.

3.5. *Proof of the Proposition.* The proof is based on the following

Lemma 3. *Let us consider the Hamiltonian:*

$$T''(I_1'', I_2'') + \mu \widehat{V}''(I_1'', I_2'', I_3'', \mu) + \mu^5 \widetilde{V}''(I_1'', I_2'', I_3'', \varphi_1'', \varphi_2'', \varphi_3'', \mu),$$

real analytic on the domain $B'' \times \mathbf{T}^3$ with extension radii r and s, and such that $\widehat{V}''$, $\widetilde{V}''$ satisfy (3.7) and $\widehat{V}''$ is close to $\overline{V}''$, see (3.6). Suppose that:

$$\left| \frac{\partial^2 \widehat{V}''}{\partial I_3''^2} \right| \ge m, \quad \text{on} \quad B''_{\rho_I} \times \mathbf{T}^3_{\rho_\varphi} \ ,$$

where $m > 0$ is some constant, and let γ_2 be such that:

$$\mu^5 \le \frac{\bar{c}(s)}{V_0^6} r^6 \gamma_1^4 \gamma_2^2, \tag{3.17}$$

where $\bar{c}(s) = \frac{s^{26}}{3^9 4^{54} 5^{19}}$. *Then there exists a subset*

$$\mathcal{P}''_K \subset B'' \times \mathbf{T}^3$$

of measure

$$\operatorname{mes}(\mathcal{P}''_K) \geq \left(1 - \frac{\gamma_2 M^2}{\mu R m \Gamma}\right) \operatorname{mes}(B'' \times \mathbf{T}^3), \tag{3.18}$$

such that $\mathcal{P}''_K$ *is the union of invariant three-dimensional tori* $\mathcal{T}_{\omega''}$ *parametrized by* $\omega'' = (\omega''_1, \omega''_2, \omega''_3) \in \Omega^\gamma$, *where* $\omega''_i = \frac{\partial T'' + \mu \widehat{V}''}{\partial I''_i}$ *and* $\Omega^\gamma = \left\{\omega'' : \langle k, \omega'' \rangle \geq \frac{\gamma_2}{|k|^2}\ \forall k \in \mathbb{Z}^2 \setminus \{0\}\right\}$. *More precisely, such tori are defined parametrically by the equations:*

$$I'' = I''_0 + L_{\omega''} + f_{\omega''}(Q), \qquad \varphi'' = Q + g_{\omega''}(Q), \quad \text{with} \quad Q \in \mathbf{T}^3$$

where I''_0 *is such that* $\frac{\partial T'' + \mu \widehat{V}''}{\partial I''_i}(I''_0) = \omega''$, *while* L''_ω *is some constant, and* $f_{\omega''}$, $g_{\omega''}$ *are real analytic functions. The tori* $\mathcal{T}''_\omega$ *are near to the unperturbed ones* $\{I''_0\} \times \mathbf{T}^3$, *more precisely one has the estimates*

$$\|L''_\omega\| \leq c_3 V_0^3 \frac{M \mu^{\frac{3}{2}}}{m(\gamma_1 r)^2 \gamma_2}, \quad \|f''_\omega\| \leq c_4 V_0^5 \frac{\mu^5}{(\gamma_1 r s^3)^4 \gamma_2}, \quad \|g''_\omega\| \leq c_5 V_0^3 \frac{\mu^{\frac{5}{2}}}{r^3 \gamma_1^2 \gamma_2}, \tag{3.19}$$

where c_3, c_4 *and* c_5 *are some suitable constants depending on* s.

See Section 5 for the proof.

Proof of the Proposition. It is easy to check that, by the construction of the domain $\mathcal{A}$, our Hamiltonian satisfies the hypotheses of Lemma 1 (one should choose $r = \frac{\gamma_1}{M(4N-1)^3}$) and that it is possible to apply Lemma 2 to the new Hamiltonian (3.5). Then there exists R such that for every disc $D(R)$ and for every $\mathcal{P}'$, defined as in (3.9), we can apply the above canonical transformation which conjugates H' to the new Hamiltonian H'' real analytic on a set of the form $B''_r \times \mathbf{T}^3_s$, where B'' is a ball of radius R, $r = \frac{\gamma_1}{(4N-1)^3 M}$.

Let us then consider a regular level curve $\mathcal{C}$ of the averaged potential, free from singularities; then for small μ, thanks to the fact that $\widehat{V}$ is close to $\overline{V}$, see (3.6), and that Ψ is close to the identity, see (3.8), for μ small enough there exists a tube $\mathcal{P}$ around $\mathcal{C}$, composed of non-singular level curves of the averaged potential, such that the set $\mathcal{P}'$ defined by

$$\mathcal{P} = \Psi(\mathcal{P}') = \Psi \circ \Phi(B'' \times \mathbf{T}^3)$$

has the form (3.9). Now, with the natural[1] choice $\gamma_2 = \mu \gamma_1$, the new Hamiltonian H'' fulfills all the hypotheses of Lemma 3.

---

[1] By definition, γ_1 and γ_2 are some non-resonance parameters respectively for the two frequencies (ω_1, ω_2), and the three frequencies $(\omega''_1, \omega''_2, \omega''_3)$, where $\omega_{1,2} = \frac{\partial T}{\partial I_{1,2}}$, $\omega''_{1,2} = \frac{\partial T''}{\partial I''_{1,2}} = \omega_{1,2} + \mathcal{O}(\mu)$ and $\omega''_3 = \mu \frac{\partial \widehat{V}''}{\partial I''_3}$.

Applying Lemma 3 to the Hamiltonian H'', one obtains that there exists a subset $\mathcal{P}''_K \subset B'' \times \mathbf{T}^3$ which is the union of invariant three–dimensional tori for which the estimates (3.18), and (3.19) hold. Let us define

$$\mathcal{P}_1 = \Psi \circ \Phi(\mathcal{P}''_K) \subset \mathcal{P}. \tag{3.20}$$

Then $\mathcal{P}_1$ is also the union of invariant three–dimensional tori for the Hamiltonian H. The estimate (3.13) on the measure of $\mathcal{P}_2$ directly follows from the definition (3.20) and from the estimate (3.18).

We now prove the estimates (3.14), (3.15). To this purpose let us take

$$(I_1^0, I_2^0, J^0, \varphi_1^0, \varphi_2^0, j^0) \in \mathcal{P}_1;$$

thanks to the fact that Ψ is close to the identity, and that $I'_{1,2} = I''_{1,2}$, one has, for $i = 1, 2$:

$$\begin{aligned} |I_i(t) - I_i^0| &\leq |I_i(t) - I'_i(t)| + |I'_i(t) - I'_i(0)| + |I'_i(0) - I_i^0| \\ &\leq \frac{2c_1 V_0}{s^3}\frac{\mu}{\gamma_1} + |I''_i(t) - I''_i(0)|, \end{aligned}$$

and by the lemma 3

$$|I_i(t) - I_i^0| \leq 2c_3 \frac{M}{m} V_0^3 \frac{\mu^{\frac{3}{2}}}{(\gamma_1 r)^2 \gamma_2} + \frac{2c_1 V_0}{s^3}\frac{\mu}{\gamma_1}. \tag{3.21}$$

In the same way, thanks to the fact that Ψ is close to the identity, and that φ'' and φ' differ only for a change of the origin, one has:

$$|\varphi_i(t) - \varphi_i^0 - \omega_i t| \leq c_5 V_0^3 \frac{\mu^{\frac{5}{2}}}{\gamma_1^2 r^3 \gamma_2} + 2c_1 V_0 \frac{\mu}{r\gamma_1}, \qquad i = 1, 2. \tag{3.22}$$

Let us now study $(J(t), j(t))$, and to this purpose, let us consider the value of the averaged potential $\overline{V}$ at fixed (I_1, I_2). Thanks to (3.6) one has

$$\overline{V}(I_1^0, I_2^0, J(t), j(t)) = \widehat{V}'(I'_1(0), I'_2(0), J'(t), j'(t)) + 8c_1 \frac{\mu V_0^2}{\gamma_1 r s^3},$$

and since $\widehat{V}'(I'_1(0), I'_2(0), J'(t), j'(t)) = \widehat{V}''(I''_1(0), I''_2(0), I''_3(t))$, it follows

$$|\Delta\overline{V}| = \widehat{V}''(I''_1(0), I''_2(0), I''_3(t)) - \widehat{V}''(I''_1(0), I''_2(0), I''_3(0)) + 16c_1 \frac{\mu V_0^2}{\gamma_1 r s^3},$$

where $\Delta\overline{V} = \overline{V}(I_1^0, I_2^0, J(t), j(t)) - \overline{V}(I_1^0, I_2^0, J^0, j^0)$. Now by Lemma 3 one has

$$|\Delta\overline{V}| \leq 2c_3 \frac{M}{m} V_0^4 \frac{\mu^{\frac{3}{2}}}{\gamma_1^2 r^3 \gamma_2} + 16c_1 \frac{\mu V_0^2}{\gamma_1 r s^3}, \tag{3.23}$$

and for the above choice of the parameters γ_2 and r, using (3.21), (3.22) and (3.23), one easily gets the estimates (3.14), (3.15), and (3.16); in fact, for the (3.14) one has

$$c_3 \frac{M}{m} V_0^3 \frac{\mu^{\frac{3}{2}}}{(\gamma_1 r)^2 \gamma_2} + \frac{2c_1 V_0}{s^3} \frac{\mu}{\gamma_1} \leq \frac{cM^3 V_0^3 (4N-1)^6}{m} \frac{\mu^{\frac{1}{2}}}{\gamma_1^5} ,$$

while concerning (3.15) one finds,

$$c_5 V_0^3 \frac{\mu^{\frac{5}{2}}}{\gamma_1^2 r^3 \gamma_2} + 2c_2 V_0 \frac{\mu}{r\gamma_1} \leq cMV_0 (4N-1)^3 \frac{\mu^{\frac{1}{2}}}{\gamma_1^2} ,$$

and finally for(3.16) one has

$$2c_3 \frac{M}{m} V_0^4 \frac{\mu^{\frac{3}{2}}}{\gamma_1^2 r^3 \gamma_2} + 16c_1 \frac{\mu V_0^2}{\gamma_1 r s^3} \leq \frac{cM^4 V_0^4 (4N-1)^9}{m} \frac{\mu^{\frac{1}{2}}}{\gamma_1^6} .$$

4. EXAMPLES

4.1. *Elastic forces.* Let us consider a rigid body with a fixed point, subject to two elastic forces, namely one, of elastic constant k_1, acting between the point $\mathbf{S}_{(1)} = S_1\mathbf{e}_1 + S_2\mathbf{e}_2 + S_3\mathbf{e}_3$ and its projection on the (x,z)–plane, and the other, of elastic constant k_2, acting between the point $\mathbf{S}_{(2)} = S\mathbf{e}_3$ and its projection on (x,y)–plane (see figure 5).

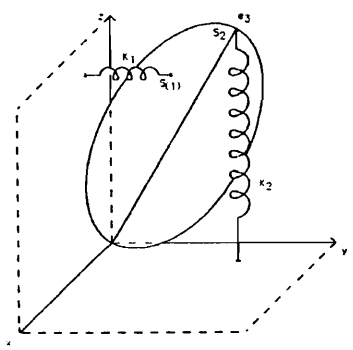

Fig. 5

The potential is positional, and considered as a function of the Euler coordinates has the form:

$$\begin{aligned} V =& \frac{1}{2}k_1\left[S_1(\sin\chi\cos\psi + \sin\psi\cos\chi\cos\vartheta) - S_2(\sin\chi\sin\psi \right. \\ & \left. - \cos\psi\cos\chi\cos\vartheta) - S_3\cos\chi\sin\vartheta\right]^2 + \frac{1}{2}k_2\left(S\cos\vartheta\right)^2. \end{aligned} \tag{4.1}$$

If we apply the transformation equations (2.1) to (4.1) we obtain the potential as a function of the Andoyer variables and to apply our result we have to check that the average of the potential on the fast angles (g, l) realizes our assumptions. Assuming, for simplicity, $S_1^2 + S_2^2 = S_3^2$, it turns out that:

$$\langle V\rangle_{l,g} = \frac{1}{2}k_1 S_3^2 + \frac{1}{2}k_2 S^2\left[\frac{J^2}{2G^2}\left(3\frac{L^2}{G^2} - 1\right) + \frac{1}{2}\left(1 - \frac{L^2}{G^2}\right)\right];$$

this means that the Andoyer variables coincide with the action–angle ones for the averaged motion and the change of coordinates of Section 3.3 is practically not necessary. One immediately obtains that:

$$\frac{\partial^2\langle V\rangle_{l,g}}{\partial J^2} = \frac{1}{2}k_2 S^2\frac{1}{G^2}\left(3\frac{L^2}{G^2} - 1\right),$$

so for $\frac{L^2}{G^2} \neq \frac{1}{3}$ the potential does remove the degeneracy, while for $\frac{L^2}{G^2} = \frac{1}{3}$ the averaged potential does not depend on J.

As a consequence, we can apply our Proposition on the domain $\mathcal{G}\times\mathcal{L}\times\mathcal{J}\times\mathbf{T}^3$ where:

$$\mathcal{L} = \left[\frac{1}{3}G + 2r_0, G - 2r_0\right] \cup \left[-\frac{1}{3}G + 2r_0, \frac{1}{3}G - 2r_0\right] \cup \left[-G + 2r, -\frac{1}{3}G + 2r_0\right],$$

$\mathcal{G} = \{G \in \mathbb{R} : G \geq 2r_0\}$ and $\mathcal{J} = \{J \in \mathbb{R} : |J| < G - 2r_0\}$. As a result we obtain the estimates (3.13), (3.14), (3.15), and (3.16), that is, for a large set of initial data, with large enough kinetic energy, the angular momentum has almost constant module and direction in the body frame, while it slowly precedes around the z-axis in the inertial-frame, essentially as in the ordinary top.

4.2 *Gravitational potential.* Let us now consider a symmetric homogenous rigid body $\mathcal{T}$ with figure axis $\mathbf{e}_3$, mass $M_\mathcal{T}$, and barycenter $B_\mathcal{T}$, that moves on an elliptic orbit of given eccentricity e and major axis $2a$. Let us suppose that $\mathcal{T}$ is subject to the gravitational field due to a point mass $M_\mathcal{S}$ that lies in one focus O. We choose the inertial frame (O, x, y, z) such that the (x, y)–plane contains the orbit. Let r denote the position of $B_\mathcal{T}$ in such inertial frame (see figure 6).

A Hamiltonian which suitably describes this physical situation is

$$H(L, G, J, g, j, t) = \frac{G^2 - L^2}{2a_1} + \frac{L^2}{2a_3} - \omega_\mathcal{T}^2 a^2 M_\mathcal{T}[1 - e\cos\lambda(t)] + \mathcal{V}(L, G, J, g, j, t), \tag{4.2}$$

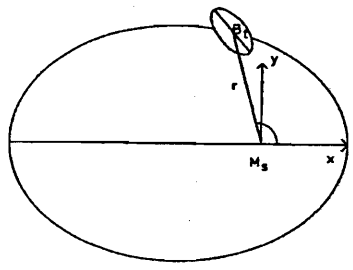

Fig. 6

where $\lambda(t) = \omega t + \lambda_0$ is the mean anomaly and:

$$\begin{aligned}\mathcal{V} =& \frac{3}{2}\omega_{\mathcal{T}}^2(a_3 - a_1)[1 - 3e\cos\lambda(t)]\Bigg\{ -\frac{J^2}{2G^2} + \left(1 - \frac{L^2}{G^2}\right)\cos^2(\lambda(t) - j)\sin^2 g \\ &+ \left[\frac{L^2}{G^2}\left(1 - \frac{J^2}{G^2}\right) + \frac{J^2}{G^2}\left(1 - \frac{L^2}{G^2}\right)\cos^2 g - 2\frac{LJ}{G^2}\sqrt{1 - \frac{L^2}{G^2}}\sqrt{1 - \frac{J^2}{G^2}}\cos g\right] \\ &\times \sin^2(\lambda(t) - j) - 2\sin g\sin(\lambda(t) - j)\cos(\lambda(t) - j)\sqrt{1 - \frac{L^2}{G^2}} \\ &\times \left[\frac{L}{G}\sqrt{1 - \frac{J^2}{G^2}} - \frac{J}{G}\sqrt{1 - \frac{L^2}{G^2}}\cos g\right]\Bigg\}.\end{aligned}$$

In fact, to obtain the above Hamiltonian, one approximates the gravitational potential

$$V = -k\frac{M_S M_{\mathcal{T}}}{|\mathcal{T}|}\int_{\mathcal{T}} \frac{dx}{|r + x|}$$

by:

$$V = -k\frac{M_S M_{\mathcal{T}}}{|r|} + k\frac{M_S}{2|r|^3}(a_3 - a_1)(3\cos^2\beta - 1), \tag{4.3}$$

where β is the angle between $\mathbf{e}_3$ and $\boldsymbol{r}$; such an approximation[8] is meaningful if the equatorial radius R is negligible with respect to $|r|$, and if the body has the ellipsoid of inertia close to a sphere. In the Euler coordinates one has:

$$\cos\beta = \sin\vartheta\sin(\chi - \lambda_{\mathcal{T}}),$$

where $\lambda_{\mathcal{T}}$ is the longitude of $B_{\mathcal{T}}$ on the ellipse, so that $|r| = \frac{a}{1 - e\cos\lambda_{\mathcal{T}}}$. In our model we neglect the terms of order e^2, and write

$$\frac{1}{|r|^3} = \frac{1 - e\cos\lambda(t)}{a^3}.$$

Using the third Kepler law with, for simplicity, $\frac{M_T+M_S}{M_S}=1$, we then obtains (4.2) from (4.3).

Let us consider the autonomous system associated to the (4.2):

$$H(L,G,J,g,j,\lambda)=T(G)+\omega_T\Lambda-\omega_T^2a^2M_T[1-e\cos\lambda]+\mathcal{V}(L,G,J,g,j,\lambda), \qquad (4.4)$$

where Λ is the action conjugate to the angle λ and L is considered as a parameter.

We can study the (4.4) in two ways: either we consider a_3-a_1 as a small perturbative parameter, $a_3-a_1=\mu a_3$ (this is meaningful because of our assumption on the ellipsoid of inertia) or, for a_3-a_1 small but finite, we consider the fast rotations with large angular velocity $\Omega=\frac{1}{\mu}$. In the former case we study the Hamiltonian:

$$H=T(G)+\omega_T\Lambda-\omega_T^2a^2M_T[1-e\cos\lambda]+\mu\mathcal{V}(G,J,g,j,\lambda),$$

which, after the canonical transformation

$$\varphi=\lambda, \qquad I=\Lambda-\omega_Ta^2M_T\cos\lambda,$$

takes the form:

$$H=T(G)+\omega_TI+\mu\mathcal{V}(G,J,g,j,\varphi).$$

This Hamiltonian is similar to the one considered in the first example, namely it has an integrable term $T(G)+\omega_TI$ depending only on two actions (G,Λ), and a perturbative term $\mu\mathcal{V}$ depending also on the third degree of freedom (J,j). Here of course, the degree of freedom (I,φ) plays the role of (L,l).

As before we have to check that the average of the potential on the fast angles (g,λ) satisfies our assumptions. The averaged Hamiltonian is:

$$K=\frac{G^2}{2a_1}+\omega_TI+\mu\frac{3a_3\omega_T^2}{2}\left[-\frac{J^2}{G^2}+\frac{1}{2}+\frac{L^2}{2G^2}-\frac{3L^2J^2}{2G^4}\right],$$

so in this case too, the Andoyer variables coincide with the action–angle ones for the averaged motion; moreover the potential removes the degeneracy:

$$\frac{\partial^2K}{\partial J^2}=-\mu\frac{3}{2}a_3\omega_T^2\frac{1}{G^2}\neq 0.$$

We can then apply our Proposition in the domain:

$$\left\{(G,L,J)\in\mathbb{R}^3: G\geq 2r_0,\ |L|<G-2r_0\ |J|<G-2r_0\right\},$$

and obtain the estimates (3.13), (3.14), (3.15), and (3.16). Their meaning is now that, for a large set of initial data, with finite kinetic energy, if the planet is sufficiently close to

a sphere, then the angular momentum has almost constant module, constant direction in the body frame, and, as in the previous example, it slowly precedes around the z-axis in the inertial-frame.

We can obtain analogous results also in the second case. Indeed, after a suitable rescaling, the Hamiltonian takes the form:

$$H = T(G) + \frac{\omega_T}{\Omega}\Lambda + \left(\frac{\omega_T}{\Omega}\right)^2 v(G, J, g, j, \lambda),$$

where $v(G, J, g, j, \lambda) = a^2 M_T(1 - e\cos\lambda) + \frac{\nu}{\omega_T^2}$, and so:

$$H = T(G) + \mu\Lambda + \mu^2 v(G, J, g, j, \lambda).$$

We now consider $T(G) + \mu\Lambda$ as the unperturbed part, properly degenerate, and $\mu^2 v$ as the perturbation. Since the unperturbed Hamiltonian contains also terms of order μ, we should slightly modify our procedure: in this case it turns out that four perturbative steps are not enough to remove the degeneracy, but it is easy to see that seven are enough; a part from this point the procedure is the same of the previous case.

As before, we have to study the properties of the Hamiltonian averaged on the angles (λ, g) which is

$$K = \frac{G^2}{2a_1} + \mu\Lambda + \mu^2\frac{3}{4}\omega_T^2(a_3 - a_1)\left[-\frac{J^2}{G^2} + \frac{1}{2} + \frac{L^2}{2G^2} - \frac{3L^2J^2}{2G^4}\right]$$

So once more the Andoyer variables coincide with the action–angle ones for the averaged motion, the potential removes the degeneracy and we can apply our Proposition (with minor modifications on the estimates) in the domain:

$$\left\{(G, L, J) \in \mathbb{R}^3 : G \geq 2r_0,\ |L| < G - 2r_0\ |J| < G - 2r_0\right\}.$$

The resulting description of the motion is the same of the former approach. In particular in both cases, if M and $\mathbf{e}_3$ are close each other (although not coinciding, for we need $|L| < G - 2r_0$) we obtain, in the above simplified model, a description of the planetary spin precession.

5. TECHNICAL PROOFS

5.1. *Proof of the estimate (1.5).* It is well known that $\mathcal{P}_2 = h_p^{-1}(\mathcal{R}^\gamma) \times \mathbf{T}^n$, where in the time-independent case it is

$$\mathcal{R}^\gamma = \left\{ \omega \in \Omega |\ \exists k \in \mathbb{Z}^n \setminus \{0\} : |\langle k, \omega \rangle| < \frac{\gamma}{|k|^n} \right\},$$

while in the time-dependent case one has

$$\mathcal{R}^\gamma = \left\{ \omega \in \Omega |\ \exists (k, k_0) \in \mathbb{Z}^{n+1} \setminus \{0\} : |\langle k, \omega \rangle + k_0| < \frac{\gamma}{(|k| + k_0)^n} \right\},$$

with $\Omega = h_p(\mathcal{B})$. Then we have:

$$\text{mes}\left(h_p^{-1}\left(\mathcal{R}^\gamma\right)\right) \leq \frac{1}{\min_p \left(\det h_{pp}(p)\right)} \text{mes}\left(\left(\mathcal{R}^\gamma\right)\right),$$

so we have to estimate the measure of $\mathcal{R}^\gamma$; in both cases it turns out that:

$$\text{mes}\left(\mathcal{R}^\gamma \cap \Omega\right) \leq \frac{2^{n+1}}{n} \gamma S,$$

with $S = \max_k S_k$ where S_k is the $(n-1)$–dimensional volume of the set $\{\omega \in \Omega : \langle k, \omega \rangle = 0\}$. Now it is easy to show that, thanks to (1.3):

$$S = \max_k S_k \leq (Md)^{n-1} c_{n-1},$$

where d is the radius of the ball $\mathcal{B}$ and $c_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$. Then we have:

$$\text{mes}\left(h_p^{-1}\left(\mathcal{R}^\gamma\right)\right) \leq \frac{(Md)^{n-1} c_{n-1} \frac{2^{n+1}}{n}}{d \min_{p \in \mathcal{B}} \{\det(h_{pp}(p))\}} \gamma \leq C(n) \frac{M^{n-1} \gamma}{d \min_{p \in \mathcal{B}} \{\det(h_{pp}(p))\}} \text{mes}\left(\mathcal{B}\right),$$

where $C(n) = \frac{2^{n+1} c_{n-1}}{n c_n} = \frac{2^{n+1} \Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-1}{2}+1) n \sqrt{\pi}}$. In conclusion:

$$\text{mes}\left(\mathcal{P}_2\right) \leq C(n) \frac{M^{n-1} \gamma}{d \min_{p \in \mathcal{B}} \{\det(h_{pp}(p))\}} \text{mes}\left(\mathcal{P}\right).$$

5.2. *Sketch of the proof of the lemma 1.* We perform four perturbative steps; at the l-th step we deal with an Hamiltonian of the form:

$$H_{l-1} = T_{l-1}(I_1, I_2) + \mu \widehat{V}_{l-1}(I_1, I_2, I_3, \varphi_3, \mu) + \mu^l \widetilde{V}_{l-1}(I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3),$$

defined on the domain $\mathcal{A}'_{r-2(l-1)\rho} \times \mathbf{T}^3_{s_0-3(l-1)\sigma}$, where $\rho = \frac{r}{16}$ and $\sigma = \frac{s_0}{24}$, such that:

$$\|\widehat{V}_{l_1}\| \leq \|V\| \sum_0^{l-2} \frac{1}{2^i} \quad \text{and} \quad \|\widetilde{V}_{l-1}\| \leq (4c)^{l-1} \frac{\|V\|^l}{(\gamma_1 r s_0^\nu)^{l-1}}.$$

When $l = 0$ we put $\mathcal{A}'_{r-2(l-1)\rho} \times \mathbf{T}^3_{s_0-3(l-1)\sigma} = \mathcal{A}'_r \times \mathbf{T}^3_{s_0}$, $\widehat{V}_{l-1} = 0$ and $\widetilde{V}_{l-1} = V$. To remove the dependence on the fast angles at the order μ^l, we construct a canonical transformation Ψ_l by the Lie method, that is Ψ_l is the time-one map of a Hamiltonian flow Φ^t_χ induced by a Hamiltonian χ. By the properties of the Poisson bracket one has

$$\begin{aligned} H_{l-1} \circ \Psi_l =& T_{l-1} + \{T_{l-1}, \chi\} + \mu \widehat{V}_{l-1} + \mu^l \widetilde{V}_{l-1} + \int_0^1 (1-t)\{\{T_{l-1}, \chi\}, \chi\} \circ \Phi^t_\chi dt \\ &+ \int_0^1 \{\mu \widehat{V}_{l-1} + \mu^l \widetilde{V}_{l-1}, \chi\} \circ \Phi^t_\chi dt; \end{aligned}$$

we then choose χ in such a way that

$$T_{l-1} + \{T_{l-1}, \chi\} + \mu^l \widetilde{V}_{l-1} = \mu^l \overline{\widetilde{V}}_{l-1} = \frac{\mu^l}{(2\pi)^3} \int \widetilde{V}_{l-1}(\varphi_1, \varphi_2) d\varphi_1 d\varphi_2.$$

By the hypotheses on r we obtain an estimate on the norm of χ, on the complex domain $\mathcal{A}'_{r-2l\rho} \mathbf{T}^3_{s_0-3l\sigma}$, namely

$$\frac{1}{\rho} \|\chi\|_{r-2l\rho, s_0-3l\sigma} \leq \mu^l \left(c \frac{V_0}{\gamma_1 r s_0^6} \right)^l \qquad \text{with} \qquad c = 2^9 3^2 5!,$$

and then:

$$\int_0^1 (1-t)\{\{T_{l-1}, \chi\}, \chi\} \circ \Phi^t_\chi dt + \int_0^1 \{\mu \widehat{V}_{l-1} + \mu^l \widetilde{V}_{l-1}, \chi\} \circ \Phi^t_\chi dt = \mathcal{O}(\mu^{l+1});$$

finally we obtain:

$$H \circ \Psi = T_l + \mu \widehat{V}_l + \mu^{l+1} \widetilde{V}_l,$$

where H is real analytic on the domain $\mathcal{A}'_{r-2l\rho} \times \mathbb{T}^3_{s_0-3l\sigma}$, $\|\widehat{V}_l\| \leq \left(\sum_0^{l-1} \frac{1}{2^i}\right) \|V\|$ and $\|\widetilde{V}_l\| \leq (4c)^l \frac{\|V\|^{l+1}}{(\gamma_1 r s_0^3)^l}$. Iterating four times this procedure, the lemma is easily proved.

5.3. *Proof of the lemma 3.* We want to show that it is possible to apply KAM theorem to the unperturbed Hamiltonian:

$$h'' = T''(I''_1, I''_2) + \mu \widehat{V}''(I''_1, I''_2, I''_3),$$

with the perturbation:

$$\varepsilon f = \mu^5 \widetilde{V}''(I_1'', I_2'', I_3'', \varphi_1'', \varphi_2'', \varphi_3''), \quad \|f\| \le V_0 \quad \text{and} \quad \varepsilon = c_2 \frac{V_0^4}{(\gamma_1 r s^3)^4} \mu^5 .$$

The full Hamiltonian $h'' + \varepsilon f$ is real analytic on $B_r'' \times \mathbf{T}_s^3$. Let us show that h'' is non degenerate. In fact, for μ small enough,

$$\det\left(\frac{\partial^2 T''}{\partial I_i'' \partial I_j''}\right) = \det\left(\frac{\partial^2 T}{\partial I_l \partial I_k}\frac{\partial I_l \partial I_k}{\partial I_i'' \partial I_j''}\right) = (1 + \mathcal{O}(\mu)) \det\left(\frac{\partial^2 T}{\partial I_l \partial I_k}\right);$$

and

$$\left|\det\left(\frac{\partial^2 h''}{\partial I_i'' \partial I_j''}\right)_{i,j=1..3}\right| = \mu \left|\frac{\partial^2 \widehat{V}''}{\partial I_3''^2}\right| \left|\det\left(\frac{\partial^2 T''}{\partial I_i'' \partial I_j''}\right)_{i,j=1,2}\right| + O(\mu^2),$$

for μ small enough one has then

$$\left|\det\left(\frac{\partial^2 h''}{\partial I_i'' \partial I_j''}\right)_{i,j=1..3}\right| = \mu \left|\frac{\partial^2 (\widehat{V}'')}{\partial I_3''^2}\right| \cdot \Gamma + O(\mu^2) \ge \mu m \Gamma + O(\mu^2) \ne 0 .$$

Now we find the constants $\widehat{m}, \widehat{M} : 0 < \widehat{m} \le \widehat{M} < \infty$ such that $\forall v \in B_{\rho_I}''$ one has

$$\widehat{m}\|v\|_{\rho_I} \le \|h_{I''I''} v\|_{\rho_I} \le \widehat{M}\|v\|_{\rho_I}.$$

We estimate $\widehat{M}$ and $\widehat{m}$ respectively by the maximal and the minimal module of the elements of the Hessian matrix of h. Concerning $\widehat{M}$ it is clear that the term due to T'' dominates, so $\widehat{M}$ is the same of the one computed for T, with an error of order μ. For $\widehat{m}$ instead the dominant term is due to $\mu \overline{V}''$ (in fact T'' is properly degenerate) and thanks to the hypothesis (3.11) one has $\widehat{m} = \mu m$. Now we show that γ_2 and ε satisfy:

$$\varepsilon \le \delta \gamma_2^2, \quad \text{with} \quad \delta(n, M, r, s) = \left(\widehat{c} \frac{r s^7}{\frac{1}{2} M r^2 + \|f\|}\right)^2,$$

where $\widehat{c} = \frac{1}{3^4 2^{50} 5^9}$. By the definition of ε and by the assumption (3.17), one has

$$\varepsilon = \frac{c_2 \mu^5 V_0^4}{(\gamma_1 r s^3)^4} \le \frac{c_2 \bar{c}(s) r^2}{V_0^2 s^3} \gamma_2^2 .$$

Now $\delta \ge \left(\frac{\hat{c} s^7 r}{2 V_0}\right)^2$ and, by definition, $\bar{c}(s)$ is equal to $\frac{\hat{c}^2 s^{26}}{4 c_2}$; as a consequence one has:

$$\varepsilon \le \frac{c_2 \bar{c}(s) r^2}{V_0^2 s^3} \gamma_2^2 \le \left(\frac{\hat{c} s^7 r}{2 V_0}\right)^2 \gamma_2^2 \le \delta \gamma_2^2 .$$

To sum up, the Hamiltonian $h'' + \varepsilon f$ satisfies all the hypotheses of KAM theorem; as a consequence there exists a subset $\mathcal{P}''_K \subset B'' \times \mathbf{T}^3$, of measure:

$$\mathrm{mes}(\mathcal{P}''_K) \geq \left(1 - \frac{M^2}{R\widehat{\Gamma}}\gamma_2\right) \mathrm{mes}(B'' \times \mathbf{T}^3),$$

where $\widehat{\Gamma} = \min_{I'' \in \mathcal{I}'}\{\det(h''_{I''I''}(I''))\} \geq \mu m \frac{\Gamma}{2}$, such that $\mathcal{P}''_K$ is the union of invariant three–dimensional tori $\mathcal{T}_{\omega''}$ parametrized by $\omega'' = \left(\frac{\partial h''}{\partial I''_1}, \frac{\partial h''}{\partial I''_2}, \frac{\partial h''}{\partial I''_3}\right) \in \Omega^\gamma$,

$$\Omega^\gamma = \left\{\omega'' : \langle k, \omega''\rangle \geq \frac{\gamma_2}{|k|^n} \ \forall k \in \mathbb{Z}^n \setminus \{0\}\right\}.$$

More precisely, such tori are defined parametrically by the equations:

$$I'' = I''_0 + L_{\omega''} + f_{\omega''}(Q), \qquad \varphi'' = Q + g_{\omega''}(Q), \quad \text{with} \quad Q \in \mathbf{T}^3,$$

where $I''_0 = h_I^{-1}(\omega'')$, while L''_ω is some constant, and $f_{\omega''}$, $g_{\omega''}$ are real analytic functions. The tori $\mathcal{T}''_\omega$ are near the unperturbed ones $\{I''_0\} \times \mathbf{T}^3$, more precisely one has the estimates (1.7 a, b, c) or, by the definition of ε, the (3.19).

ACKNOWLEDGEMENTS

The idea of this work came during a conversation with A. I. Neishtadt. I am indebted to Giancarlo Benettin, who introduced me to the field of Hamiltonian perturbation theory, for many helpful discussions and for having constantly addressed and encouraged my work. I like to thank Francesco Fassò who helped me many times and Heinz Hanßmann for some talks. This work has been developed with the support of the grant EC contract ERBCHRXCT940460 for the project Stability and universality in classical mechanics.

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