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Existence of a global solution of the Whitham equations

Tamara Grava*
S.I.S.S.A. via Beirut 2-4
34014 Trieste, Italy

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Abstract

The Whitham equations are a collection of one-dimensional quasilinear hyperbolic systems. It is known that this collection of systems is enumerated by the genus g=0,1,2, ... of the corresponding Riemann surface. We study the Cauchy problem for the Whitham equations for monotone analytic initial data. We show that if the initial data $f(u)$ satisfies the conditions $f^{(2N+1)}(u) < 0$ for all $u \in \mathbb{R}$ except for a number of isolated points, then the solution of the Whitham equations has genus at most equal to N where $1 \leq N \in \mathbb{N}$.

1 Introduction

The Whitham equations are a collection of quasilinear hyperbolic systems of the form [1],[2],[3]

$$\frac{\partial u_i}{\partial t} + \lambda_i(u_1, u_2, \ldots, u_{2g+1})\frac{\partial u_i}{\partial x} = 0, \quad x, t, u_i \in \mathbb{R}, \quad i = 1, ..., 2g+1, \quad g = 0, 1, 2, \ldots, \tag{1.1}$$

with the ordering $u_1 > u_2 > \cdots > u_{2g+1}$. For a given g the system (1.1) is called g-phase Whitham equations. The zero-phase Whitham equation has the form

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} = 0\,. \tag{1.2}$$

This equation is also called Burgers equation.
For $g > 0$ the speeds $\lambda_i(u_1, u_2, \ldots, u_{2g+1})$, $i = 1, 2, \ldots, 2g+1$, depend through $u_1, \ldots, u_{2g+1}$ on complete hyperelliptic integrals of genus g.

*e-mail: grava@sissa.it

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The initial value problem of the Whitham equations for a monotone decreasing smooth initial data $x = f(u)|_{t=0}$ consists of the following:

1) for $t \geq 0$ the (x,t) plane is split into a number of domains D_g, where $g = 0, 1, \ldots$. In the inner part of each domain D_g we look for a solution $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ of the g-phase Whitham equations (1.1). For any $t \geq 0$ the functions $u_1(x,t) > u_2(x,t), \cdots > u_{2g+1}(x,t)$ can be plotted on the (x,u) plane as branches of a multivalued function. The g-phase solutions for different g must be glued together in order to produce a C^1-smooth curve in the (x,u) plane evolving smoothly with t (see Fig. 1.1).

2) At the time $t = 0$ we have only the D_0 domain for any x. The correspondent zero-phase solution $u(x,t)$ of the equation (1.2) must satisfy the initial data $x = f(u(x,0))$.

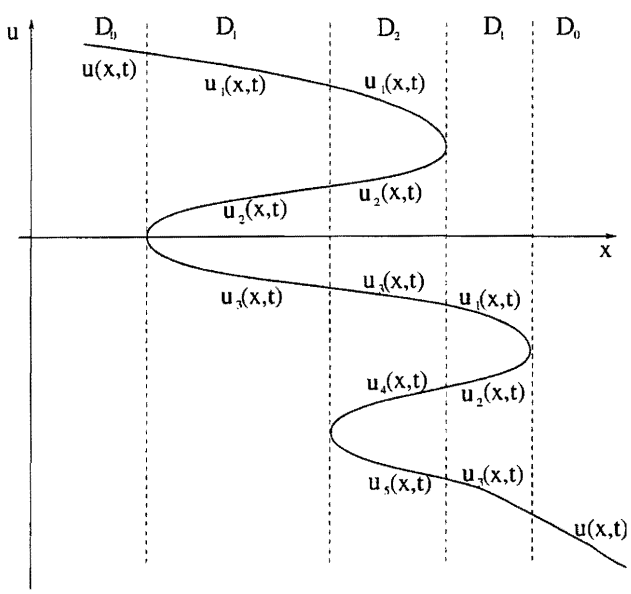

Figure 1.1: The solution of the Whitham equations for a certain $t > 0$.

We say that a solution of the initial value problem globally exists if it has a bounded genus, namely if it is defined for any x and $t > 0$ and the domains D_g are empty for $g > g_0$ where $g_0 < +\infty$.

The equations (1.1) were found by Whitham [1] in the single phase case $g = 1$ and more generally by Flaschka, Forest and McLaughlin [2] in the multiphase case. The Whitham equations were also found in [3] when studying the zero dispersion limit of the Korteweg de Vries equation. The hyperbolic nature of the equations was found by Levermore [4].

Dubrovin and Novikov [5] found the geometric Hamiltonian structure of the equations (1.1). Based on this structure, Tsarev [6] showed that equations (1.1) can be integrated by a generalization of the method of characteristics. This result was put into an algebro-geometric setting by Krichever [7].

The investigation of the initial value problem of the Whitham equations was initiated by Gurevich and Pitaevskii [8]. In the case $g \leq 1$ they solved the initial value problem of system (1.1) for step-like initial data and studied numerically the case of cubic initial data. The Cauchy problem for the Whitham equations has been extensively studied by Tian in the case $g \leq 1$ [9]. He showed that if the monotone smooth initial data $x = f(u)$ satisfies the condition $f'''(u) < 0$ except at one point, then the global solution of the Whitham equations has genus at most one.
In this paper we extend partially Tian's result. Namely we prove that if the monotone analytic initial data $x = f(u)$ satisfies the condition

$$\frac{d^{2N+1}}{du^{2N+1}} f(u) := f^{(2N+1)}(u) < 0, \quad 1 \leq N \in \mathbb{N}$$

for all real u except for a number of isolated points, then the global solution of the Whitham equations has genus at most N.

2 Riemann surfaces and Abelian differentials: notations and definitions

Let

$$\mathcal{S}_g := \left\{ P = (r, \mu),\ \mu^2 = \prod_{j=1}^{2g+1} (r - u_j) \right\}, \quad u_1 > u_2 > \cdots > u_{2g+1}, \quad u_i \in \mathbb{R}, \tag{2.3}$$

be the hyperelliptic Riemann surface of genus $g \geq 0$. We shall use the standard representation of $\mathcal{S}_g$ as a two-sheeted covering of $C\mathbb{P}^1$ with cuts along the intervals

$$[u_{2k}, u_{2k-1}], \quad k = 1, \ldots, g+1, \quad u_{2g+2} = -\infty. \tag{2.4}$$

We will call the above intervals bands while the intervals

$$[u_{2k+1}, u_{2k}], \quad k = 0, \ldots, g, \quad u_0 = +\infty \tag{2.5}$$

will be called gaps. We choose the basis $\{\alpha_j, \beta_j\}_{j=1}^g$ of the homology group $H_1(\Gamma_g)$ so that β_j lies fully on the upper sheet and encircles clockwise the interval $[u_{2j}, u_{2j-1}]$, $j = 1, \ldots, g$, while α_j emerges on the upper sheet at the point u_{2j+1}, passes to the point u_{2j} and return to the initial point u_{2j} through the lower sheet.
The one-forms that are analytic on the closed Riemann surface $\mathcal{S}_g$ except for a finite number of points are called Abelian differentials.
We define on $\mathcal{S}_g$ the following differentials [10]:
1) The canonical basis of holomorphic one-forms or Abelian differentials of the first kind $\phi_1, \phi_2 \ldots \phi_g$:

$$\phi_k(r) = \frac{r^{g-1}\gamma_1^k + r^{g-2}\gamma_2^k + \cdots + \gamma_g^k}{\mu(r)} dr, \quad k = 1, \ldots, g. \tag{2.6}$$

The constants γ_i^k are uniquely determined by the normalization conditions

$$\int_{\alpha_j} \phi_k = \delta_{jk}\,, \quad j,k = 1,\ldots,g. \tag{2.7}$$

2) The set σ_k, $k \geq 0$, of abelian differentials of the second kind with a pole of order $2k+2$ at infinity, with asymptotic behavior

$$\sigma_k(r) = \left[r^{k-\frac{1}{2}} + O(r^{-\frac{3}{2}})\right] dr \quad \text{for large } r \tag{2.8}$$

and normalized by the condition

$$\int_{\alpha_j} \sigma_k = 0, \quad j = 1,\ldots,g\,. \tag{2.9}$$

We use the notation

$$\sigma_0(r) = dp(r)\,, \qquad 12\sigma_1(r) = dq(r)\,. \tag{2.10}$$

In literature the differentials $dp(r)$ and $dq(r)$ are called quasi-momentum and quasi-energy respectively [5]. The explicit formula for the differentials σ_k, $k \geq 0$, is given by the expression

$$\sigma_k(r) = \frac{Q_k(r,\vec{u})}{\mu(r)} dr\,, \quad Q_k(r,\vec{u}) = r^{g+k} + a_1^k r^{g+k-1} + a_2^k r^{g+k-2} \cdots + a_{g+k}^k\,, \tag{2.11}$$

where $\vec{u} = (u_1, u_2, \ldots, u_{2g+1})$ and the coefficients $a_i^k = a_i^k(\vec{u})$, $i = 1,\ldots,g+k$, are uniquely determined by (2.8) and (2.9).

3) The abelian differential of the third kind $\omega_{qq_0}(r)$ with first order poles at the points $P = (q, \mu(q))$ and $P_0 = (q_0, \mu(q_0))$ with residues ± 1 respectively. Its periods are normalized by the relation

$$\int_{\alpha_j} \omega_{qq_0}(r) = 0, \quad j = 1,\ldots,g\,. \tag{2.12}$$

This differential satisfies the identity [11]

$$d_q[\omega_{qq_0}(p)] = d_p[\omega_{pp_0}(q)]\,, \tag{2.13}$$

where d_q and d_p denote differentiation with respect to q and p.

3 Preliminaries on the theory of the Whitham equations

The speeds $\lambda_i(u_1, u_2, \ldots, u_{2g+1})$ of the g-phase Whitham equations (1.1) are given by the ratio [1],[2]:

$$\lambda_i(\vec{u}) = \left.\frac{dq(r)}{dp(r)}\right|_{r=u_i}\,, \quad i = 1,2,\ldots,2g+1\,, \tag{3.14}$$

where $dp(r)$ and $dq(r)$ have been defined in (2.10). In the case $g = 0$

$$dp(r) = \frac{dr}{\sqrt{r-u}}, \quad dq(r) = \frac{12r - 6u}{\sqrt{r-u}} dr,$$

so that one obtains the zero-phase Whitham equation (1.2).
We consider monotone decreasing analytic initial data of the form

$$x = f(u) = c_0 + c_1 u + \cdots + c_k u^k + \ldots . \tag{3.15}$$

For such initial data the solution of the zero-phase equation (1.2) is obtained by the method of characteristic [1] and is given by the expression

$$x = 6tu + f(u). \tag{3.16}$$

The zero-phase solution is globally well defined only for $0 \leq t < t_c$ where $t_c = \frac{1}{6} \min_{u \in \mathbb{R}}[-f'(u)]$ is the time of gradient catastrophe of the solution (3.16). The breaking is cause by an inflection point in the initial data. Thus for $t < t_c$ we have only the domain D_0. For $t \geq t_c$ we expect to have single, double and higher phase domains. The solution of the Whitham equations is obtained gluing together C^1-smoothly solutions of different genera. The g-phase solution is obtained by a generalization of the method of characteristic and reads [6],[12]

$$x = \lambda_i(\vec{u})\, t + w_i(\vec{u}) \quad i = 1, \ldots, 2g+1, \tag{3.17}$$

where $\lambda_i(\vec{u})$ has been defined in (3.14) and $w_i(\vec{u}) = \left.\frac{ds}{dp}\right|_{r=u_i}$, $i = 1, \ldots 2g+1$. The differential ds is given by the expression

$$ds = \sum_{k=0}^{\infty} \frac{2^k k!}{(2k-1)!!} c_k \sigma_k, \tag{3.18}$$

where the differentials σ_k, $k \geq 0$ have been defined in (2.11).
The solution (3.17) of the g-phase Whitham equations can also be written in an equivalent algebro-geometric form [7, 12]. Let be

$$\Omega(r, \vec{u}) = -x dp(r) + t dq(r) + ds(r) \tag{3.19}$$

where dp, dq and ds have been defined in (2.10) and (3.18) respectively. Then the g-phase solution is determined by the equations

$$\Omega(r, \vec{u}) \,|_{r=u_i} = 0, \quad i = 1, 2 \ldots, 2g+1. \tag{3.20}$$

The solution $u_1 > u_2 > \cdots > u_{2g+1}$ of the g-phase Whitham equations (1.1) is implicitly defined as a function of x and t by the equations (3.17) or (3.20). The solution is uniquely defined only for those x and t such that the functions $u_i(x,t)$ are real and $\partial_x u_i(x,t)$, $i = 1, \ldots, 2g+1$, are not vanishing.
One of the problems in the theory of the Whitham equations is to determined when (3.17) or (3.20) are solvable for real $u_1, \ldots, u_{2g+1}$ as functions of x and t. We will study this problem in the next section. First we need to determine the phase transition boundaries of the domains D_g, $g = 0, 1, 2, \ldots$.

3.1 Boundaries of the domains D_g

The solution of the Whitham equations (1.1) with initial data (3.15) is obtained attaching C^1-smoothly solutions of different genera. From this condition we can determined the boundary between the domain D_g and D_{g+1}. To the common boundary of these domains it correspond the degenerate Riemann surface

$$\mu_v^2 = (r-v)^2 \prod_{j=1}^{2g+1} (r-u_j), \quad v \in \mathbb{R} \quad u_1 > u_2 > \cdots > u_{2g+1} \quad v \neq u_j, \ j = 1, \ldots 2g+1.$$

The point $v \in \mathbb{R}$ must be determined requiring that the g-phase solution and the $g+1$-phase solution are attached C^1-smoothly when two branch points of the Riemann surface $\mathcal{S}_{g+1}$ coalesce to the point v. We first consider the case in which v belongs to one of the bands defined in (2.4). We call the corresponding boundary *trailing edge*. In this case the common boundary of the domains D_g and D_{g+1} is determined by (3.20) and by the equations

$$\Omega(v, \vec{u}) = 0, \qquad \frac{\partial}{\partial r}(\mu(r)\Omega(r, \vec{u}))|_{r=v} = 0 \tag{3.21}$$

where the differential $\Omega(r, \vec{u})$ has been defined in (3.19). Equations (3.20) and (3.21) determined a system of $2g+3$ equations in $2g+3$ real unknowns $v, t \geq 0$ and $u_1 > u_2 > \cdots > u_{2g+1}$. When this system is solvable we can determine the function $t = t(x)$ which describes on the $x-t$ plane the boundary between the domains D_g and D_{g+1}.

When v belongs to one of the gaps defined in (2.5), namely $u \in (u_{2k+1}, u_{2k})$, $0 \leq k \leq g$, we call the corresponding boundary *leading edge*. In this case the common boundary of the domains D_g and D_{g+1} is determined by (3.20) and by the equations

$$\Omega(v, \vec{u}) = 0, \qquad \int_{u_{2k+1}}^{v} \Omega(r, \vec{u}) = 0\,. \tag{3.22}$$

We also consider the limiting case in which $v \to u_j$, $1 \leq j \leq 2g+1$. In this case the g-phase solution has a point of gradient catastrophe on the u_l-branch. The differential $\Omega(r, \vec{u})$ has a triple zero at $r = u_l$.

Example 3.1

We study the boundary between the domains D_0 and D_1 (see Fig 3.2). On the Riemann surface $\mathcal{S}_0 := \{\mu^2 = r - u\}$ the differential $\Omega(r, u)$ reads [9]

$$\Omega(r, u) = \frac{-x + 6tu + f(u)}{\sqrt{r-u}} dr + \left[\int_u^r \frac{6t + f'(\xi)}{\sqrt{r-\xi}} d\xi \right] dr,$$

so that the zero-phase solution (3.20) is given by

$$x = 6tu + f(u) \tag{3.23}$$

On the trailing edge where $u_2 = u_3 = v$ and $u_1 = u$ the equations (3.21) become

$$\int_u^v \frac{6t + f'(\xi)}{\sqrt{v-\xi}} d\xi = 0, \qquad \int_u^v \frac{f''(\xi)}{\sqrt{v-\xi}} d\xi = 0 \tag{3.24}$$

On the leading edge where $u_1 = u_2 = v$ and $u_3 = u$ the equations (3.22) read

$$\int_u^v \frac{6t + f'(\xi)}{\sqrt{v-\xi}} d\xi = 0, \qquad \int_u^v (6t + f'(\xi))\sqrt{v-\xi} d\xi = 0 \tag{3.25}$$

The equations in (3.24) and (3.25) are equivalent to the ones obtained in [9].

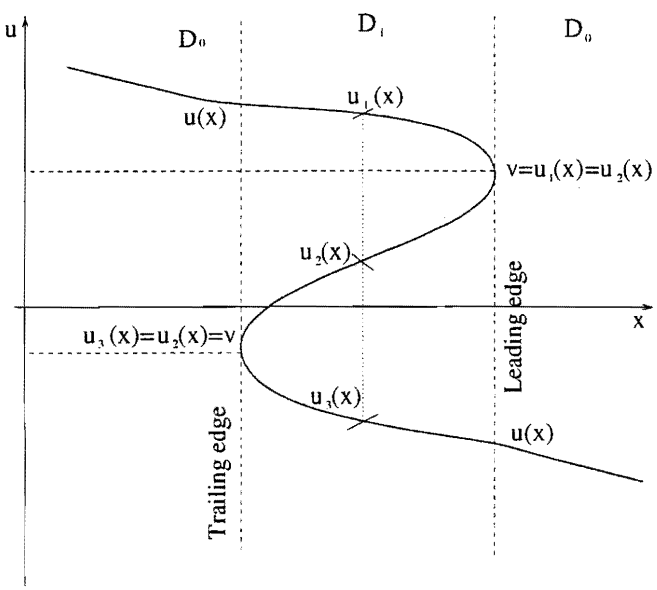

Figure 3.2: Boundaries between the domains D_0 and D_1.

When $f(u) = -u^3$ the solution of the Whitham equations has genus at most equal to one [13]. On the $x - t$, $t \geq 0$, plane we have only the domains D_0 and D_1. In this case we can solve explicitly the systems obtained from (3.23), (3.24) and (3.25). On the trailing edge we obtain $t = 1/(2^{1/3}6)(-x)^{2/3}$, $x \leq 0$, while on the leading edge we have $t = 3/(4\sqrt{5})^{\frac{2}{3}} x^{\frac{2}{3}}$, $x \geq 0$. This is a well known result obtained in [14].

4 Main result

In this section we show that if the monotone analytic initial data $f(u)$ defined in (3.15) satisfies the condition $f^{(2N+1)}(u) < 0$ for all $u \in \mathbb{R}$ except for a number of isolates points, then the global solution of the Whitham equations has genus at most N.

For the purpose, we first need to study some properties of the Abelian differentials.

4.1 Properties of the Abelian differentials

Let ω_1 and ω_2 be two Abelian differentials on $\mathcal{S}_g$. Suppose that all the residues of ω_1 are equal to zero. Then the integral $d^{-1}\omega_1$ does not have logarithmic singularity on $\mathcal{S}_g$. We have the following relation:

$$\sum_{j=1}^{g}\left[\int_{\alpha_j}\omega_1\int_{\beta_j}\omega_2-\int_{\alpha_j}\omega_2\int_{\beta_j}\omega_1\right]=2\pi i\sum_{\mathcal{S}_g}\mathrm{Res}(d^{-1}\omega_1)\omega_2\,. \tag{4.26}$$

This formula is known as the Riemann bilinear period relation [10].

Let $\omega_z(r)$ be the normalized Abelian differential of the third kind with simple poles at the points $P^{\pm}(z)=(z,\pm\mu(z))$ with residue ±1 respectively. The differential $\omega_z(r)$ is given by the expression

$$\omega_z(r)=\frac{dr}{\mu(r)}\frac{\mu(z)}{r-z}-\sum_{k=1}^{g}\phi_k(r)A_k(z,\vec{u})\,,\quad A_k(z,\vec{u})=\int_{\alpha_k}\frac{dt}{\mu(t)}\frac{\mu(z)}{t-z}\,, \tag{4.27}$$

where $\phi_k(r)$, $k=1,\dots,g$, is the normalized basis (2.6) of holomorphic differentials.
In order to obtain the explicit expression of the coefficients $A_k(z,\vec{u})$ we apply the Riemann bilinear relation (4.26) to $\omega_1(r)=\sigma_m(r)$ and $\omega_2(r)=\omega_z(r)$ getting

$$\begin{aligned}\int_{P^-(z)}^{P^+(z)}\sigma_m(\xi)&=-\operatorname*{Res}_{r=\infty}[\omega_z(r)d^{-1}\sigma_m(r)],\quad m=0,\dots,g-1,\\&=-\frac{4}{2m+1}\sum_{k=1}^{g}\sum_{j=1}^{m+1}A_k(z,\vec{u})\gamma_j^k\Gamma_{m+1-j}\,.\end{aligned} \tag{4.28}$$

In the above formula the coefficients γ_j^k have been defined in (2.6) and the Γ_l's are the coefficients of the expansion for $\xi\to\infty$ of

$$\frac{d\xi}{\mu(\xi)}=\xi^{-g-\frac{1}{2}}(\Gamma_0+\frac{\Gamma_1}{\xi}+\frac{\Gamma_2}{\xi^2}+\dots+\frac{\Gamma_l}{\xi^l}+\dots)d\xi.$$

Inverting (4.28) and introducing the quantities $B_j(z,\vec{u})=\sum_{k=1}^{g}A_k(z,\vec{u})\gamma_j^k$ we obtain

$$\begin{pmatrix}B_1(z,\vec{u})\\B_2(z,\vec{u})\\\dots\\B_g(z,\vec{u})\end{pmatrix}=\begin{pmatrix}\tilde{\Gamma}_0&0&0&\dots&0\\\tilde{\Gamma}_1&\tilde{\Gamma}_0&0&\dots&0\\\dots&\dots&\dots&\dots&\dots\\\tilde{\Gamma}_{g-1}&\tilde{\Gamma}_{g-2}&\dots&\tilde{\Gamma}_1&\tilde{\Gamma}_0\end{pmatrix}\begin{pmatrix}-\frac{1}{4}\int_{P^-(z)}^{P^+(z)}\sigma_0(\xi)\\-\frac{3}{4}\int_{P^-(z)}^{P^+(z)}\sigma_1(\xi)\\\dots\\-\frac{2g-1}{4}\int_{P^-(z)}^{P^+(z)}\sigma_{g-1}(\xi)\end{pmatrix} \tag{4.29}$$

where the $\tilde{\Gamma}_k$'s are the coefficients of the expansion for $\xi\to\infty$ of

$$\mu(\xi)d\xi=\xi^{g+\frac{1}{2}}(\tilde{\Gamma}_0+\frac{\tilde{\Gamma}_1}{\xi}+\frac{\tilde{\Gamma}_2}{\xi^2}+\dots+\frac{\tilde{\Gamma}_l}{\xi^l}+\dots)d\xi.$$

The following proposition is important for our subsequent considerations.

Proposition 4.1 *The Abelian differentials of the second kind $\sigma_k(r)$, $k \geq 0$, satisfy the relations*

$$\sigma_k(r) = \operatorname*{Res}_{z=\infty}[\omega_z(r)\, z^{k-\frac{1}{2}} dz] = -\frac{2}{2k+1} d_r \operatorname*{Res}_{z=\infty}[\omega_r(z)\, z^{k+\frac{1}{2}}], \tag{4.30}$$

where $\omega_r(z)$ is the normalized Abelian differential of the third kind with simple poles at the points $P^{\pm}(r) = (r, \pm\mu(r))$ with residue ± 1 respectively and d_r denotes differentiation with respect to r.

Proof: We prove the first equality in (4.30). Since the differential $\omega_z(r)$ is normalized, the differential $\mathrm{Res}_{z=\infty}[\omega_z(r)\, z^{k-\frac{1}{2}} dz]$ is also normalized. Furthermore the differential $\mathrm{Res}_{z=\infty}[\omega_z(r)\, z^{k-\frac{1}{2}} dz]$ has only a pole at $r = \infty$ and asymptotic behavior

$$\operatorname*{Res}_{z=\infty}[\omega_z(r)\, z^{k-\frac{1}{2}} dz] = \left[r^{k-\frac{1}{2}} + O(r^{-\frac{3}{2}})\right] dr\,, \quad \text{for large } r.$$

Consequently the first equality in (4.30) holds true. For the second equality in (4.30) we use the following argument. The quantity $\omega_z(r)\, z^{k+\frac{1}{2}}$ is a single-valued function in the z variable in the neighborhood of $z = \infty$ where it has a pole. Consequently

$$\begin{aligned} \frac{1}{2\pi i}\int_{C_\infty} d_z(\omega_z(r)\, z^{k+\frac{1}{2}}) = 0 &= \operatorname*{Res}_{z=\infty}[z^{k+\frac{1}{2}} d_z\omega_z(r) + (k+\frac{1}{2})\omega_z(r) z^{k-\frac{1}{2}} dz] \\ &= \operatorname*{Res}_{z=\infty}[z^{k+\frac{1}{2}} d_r\omega_r(z) + (k+\frac{1}{2})\omega_z(r) z^{k-\frac{1}{2}} dz] \end{aligned} \tag{4.31}$$

where C_∞ is a contour surrounding the point at infinity and the last equality in (4.31) is obtained from the identity $d_z\omega_z(r) = d_r\omega_r(z)$ which follows from (2.13). □

Let be

$$F(u) = 6tu + f(u) - x\,, \tag{4.32}$$

where $f(u)$ is the initial data (3.15). We define the Abel transform of F to be equal to

$$\mathcal{F}(z) = \int_0^z \frac{F(\xi)}{\sqrt{z-\xi}} d\xi, \quad \partial_z\mathcal{F}(z) = \frac{F(0)}{\sqrt{z}} + \int_0^z \frac{F'(\xi)}{\sqrt{z-\xi}} d\xi\,. \tag{4.33}$$

Using (4.30) the next proposition follows.

Proposition 4.2 *The differential $\Omega(r, \vec{u})$ defined in (3.19) satisfies the following identity*

$$\Omega(r, \vec{u}) = \operatorname*{Res}_{z=\infty}[\omega_z(r) d_z\mathcal{F}(z)] = -d_r \operatorname*{Res}_{z=\infty}[\omega_r(z)\mathcal{F}(z)]\,, \tag{4.34}$$

where $\mathcal{F}(z)$ is the Abel transform (4.33) of $F(u)$ and $\omega_z(r)$ has been defined in (4.27).

The proof is obtained by straightforward calculations.

4.2 Bound to the genus of the solution of the Whitham equations

Under certain hypothesis on the initial data, we give an upper bound to the genus of the solution of the Whitham equations. For the purpose we first write the solution (3.17) or (3.20) of the g-phase Whitham equations (1.1) in a new equivalent form.

From (4.27), (4.29) and from the second equality in (4.34) we can write the differential $\Omega(r,\vec{u})$ in the form

$$\begin{aligned}\Omega(r,\vec{u}) =& d_r[\mu(r)\Psi(r,\vec{u})] - \sum_{k=1}^{g} d_r B_k(r,\vec{u}) q_k(\vec{u}) \\ =& \left[\partial_r \Psi(r,\vec{u}) + \sum_{i=1}^{2g+1} \partial_{u_i}\Psi(r,\vec{u})\right]\mu(r)dr + \sum_{i=1}^{2g+1} \frac{\mu(r)}{r-u_i}\partial_{u_i} q_g(\vec{u}) - \sum_{k=1}^{g} d_r B_k(r,\vec{u}) q_k(\vec{u})\end{aligned} \tag{4.35}$$

where

$$\Psi(r,\vec{u}) = -\operatorname*{Res}_{z=\infty}\left[\frac{\mathcal{F}(z)dz}{\mu(z)(z-r)}\right], \quad q_k(\vec{u}) = -\operatorname*{Res}_{z=\infty}\left[\frac{z^{g-k}}{\mu(z)}\mathcal{F}(z)dz\right], \quad k=1,\dots,g \tag{4.36}$$

and for $g=0$ we define

$$q_0(\vec{u}) = -\operatorname*{Res}_{z=\infty}\left[\frac{\mathcal{F}(z)}{\mu(z)}dz\right]. \tag{4.37}$$

From the relation (4.29) it follows that

$$d_r B_k(r,\vec{u}) = -\frac{2k-1}{2}\sum_{l=1}^{k}\tilde{\Gamma}_{k-l}\sigma_{l-1}(r).$$

Hence $\sum_{k=1}^{g} d_r B_k(r,\vec{u}) q_k(\vec{u})$ in (4.35) is a differential of the second kind with a pole at infinity of order $2g$. Consequently we can write (4.35) in the form

$$\Omega(r,\vec{u}) = \frac{Z(r,\vec{u})}{\mu(r)}dr = \frac{\Phi(r,\vec{u})\prod_{j=1}^{2g+1}(r-u_j) + R_{2g}(r,\vec{u})}{\mu(r)}dr \tag{4.38}$$

where

$$\Phi(r,\vec{u}) = \partial_r\Psi(r,\vec{u}) + \sum_{i=1}^{2g+1}\partial_{u_i}\Psi(r,\vec{u}) \tag{4.39}$$

and $R_{2g}(r,\vec{u})$ is a polynomial in the variable r of degree $2g$ and is equal to

$$R_{2g}(r,\vec{u}) = \sum_{i=1}^{2g+1}\partial_{u_i} q_g(\vec{u})\prod_{j=1,j\neq i}^{2g+1}(r-u_j) + \sum_{k=1}^{g}\frac{2k-1}{2}q_k(\vec{u})\sum_{l=1}^{k}\tilde{\Gamma}_{k-l}Q_{l-1}(r,\vec{u}) \tag{4.40}$$

and $Q_{l-1}(r,\vec{u})$ has been defined in (2.11).
The solution (3.20) of the g-phase Whitham equations (1.1) can be written in the equivalent form

$$Z(r,\vec{u})|_{r=u_i} = 0, \quad i = 1,\dots,2g+1, \tag{4.41}$$

where $Z(r,\vec{u})$ has been defined in (4.38). From the expression of $Z(r,\vec{u})$ in (4.38), we deduce that the solution (4.41) of the g-phase Whitham equations (1.1) is equivalent to the condition

$$R_{2g}(r,\vec{u}) = \sum_{i=1}^{2g+1} \partial_{u_i} q_g(\vec{u}) \prod_{j=1,j\neq i}^{2g+1} (r-u_j) + \sum_{k=1}^{g} \frac{2k-1}{2} q_k(\vec{u}) \sum_{l=1}^{k} \tilde{\Gamma}_{k-l} Q_{l-1}(r,\vec{u}) \equiv 0\,. \tag{4.42}$$

Indeed $R_{2g}(r,\vec{u})$ is a polynomial of degree $2g$ and because of (4.41) it must have $2g+1$ zeros, consequently $R_{2g}(r,\vec{u})$ must be identically zero. The condition (4.42) determines a system of $2g+1$ equations in the variable $u_1,\dots,u_{2g+1}$ with parameters x and t. A solution of the g-phase Whitham equations exists when (4.41) or equivalently (4.42) are solvable for real $u_1 > u_2 > \cdots > u_{2g+1}$ as a functions of x and t and the derivatives $\partial_x u_i(x,t)$ are not vanishing for $i = 1,\dots,2g+1$.

Proposition 4.3 *If the Whitham equations (1.1) with the initial data (3.15) admit a g-phase solution, then the function $\Phi(r,\vec{u})$ defined in (4.39) has at least g real zeros.*

Proof: If the Whitham equations admit a g-phase solution then the system defined by (4.42) is solvable for real $u_1 > u_2 > \cdots > u_{2g+1}$ as functions of x and t. On the solution $u_1(x,t) > u_2(x,t) > \cdots > u_{2g+1}(x,t)$ of the g-phase Whitham equations (1.1) the differential $\Omega(r,\vec{u})$ in (4.38) reads

$$\Omega(r,\vec{u}) = \mu(r)\Phi(r,\vec{u})dr\,. \tag{4.43}$$

The differential $\Omega(r,\vec{u})$ defined on $\mathcal{S}_g$ satisfies the normalization conditions (2.9), consequently $\Omega(r,\vec{u})$ must have at least one real zero in each of the gaps (u_{2j+1},u_{2j}), $j = 1,\dots,g$.
Hence the function $\Phi(r,\vec{u})$ must have at least g real zeros in the r variable in order for $\Omega(r,\vec{u})$ to be normalized. □

Proposition 4.4 *On the common boundary of the domains D_g and D_{g+1} the function $\Phi(r,\vec{u})$ in (4.39) defined on the Riemann surface S_g must have $g+2$ real zeros.*

Proof: The domains D_g and D_{g+1} have a common boundary when a couple of branch points of the Riemann surface $\mathcal{S}_{g+1}$ coalesce to a single point $v \in \mathbb{R}$.
The trailing edge (see Sec. 3.1) of the common boundary of the domains D_g and D_{g+1} is determined from equations (3.21). Using (4.43) we can write (3.21) in the equivalent form

$$\Phi(v,\vec{u}) = 0, \quad \left.\frac{\partial\Phi(r,\vec{u})}{\partial r}\right|_{r=v} = 0\,. \tag{4.44}$$

Now let us consider the leading edge of the common boundary of the domains D_g and D_{g+1}. When v belongs to one of the gaps defined in (2.5), let be $u \in (u_{2l+1}, u_{2l})$, $0 \leq l \leq g$, the common boundary of the domains D_g and D_{g+1} is determined from equation (3.22). Using (4.43) we can write (3.22) in the form

$$\Phi(v, \vec{u}) = 0, \quad \int_{u_{2l+1}}^{v} \mu(r)\Phi(r, \vec{u})dr = 0\,. \tag{4.45}$$

Both equations (4.44) and (4.45) have a real solution if the function $\Phi(r, \vec{u})$ has two real zeros in the r variable. Combining this result with the one in Proposition 4.3, we deduce that on the common boundary of the domains D_g and D_{g+1} the function $\Phi(r, \vec{u})$ must have $g+2$ real zeros. These zeros in the generic case are distinct. □

Theorem 4.5 (Main Theorem) *If the initial data $f(u)$ defined in (3.15) satisfies the condition*

$$\frac{d^{2N+1}}{du^{2N+1}} f(u) < 0, \quad 1 \leq N \in \mathbb{N}, \tag{4.46}$$

for all $u \in \mathbb{R}$ except for a number of isolated points, then the solution of the Whitham equations (1.1) has genus at most N.

Proof:
The solution of the Whitham equations (1.1) with initial data (3.15) determines a decomposition of the $x-t$ plane, $t \geq 0$, into a number of domains D_g with g=0,1,2 .... To the inner part of each domain D_g it corresponds the g-phase solution (3.20) of the Whitham equations (1.1). The common boundaries of the domains D_g and D_{g+1} are determined from equations (3.20), (3.22) and (3.21).

We will show that each domain D_g, $g \leq N$, does not have a common boundary with any of the domains D_m, $m > N$. Since the set of domains $\{D_g\}_{g \leq N}$ is not empty because $D_0 \neq \emptyset$ the set $\{D_g\}_{g>N}$ must be empty. Indeed on the contrary the $x-t$ plane, $t \geq 0$, which is a connected set, would be split into a number of domains whose union forms a disconnected set.

From Proposition 4.4 it follows that when the domains D_g and D_{g+1} have a common boundary the function $\Phi(r, \vec{u})$ defined on $\mathcal{S}_g$ has $g+2$ real zeros for some value of x and t. We can easily generalize Proposition 4.4 considering the common boundary of the domains D_g and D_m, $m > g \geq 0$. When these domains have a common boundary, the function $\Phi(r, \vec{u})$ defined on $\mathcal{S}_g$ must have at least $2m-g$ real zeros for some values of x and t.

We estimate the number of real zeros of the function $\Phi(r, \vec{u})$ defined on $\mathcal{S}_g$ when $g \leq N$. For the purpose we need the following two lemmas.

Lemma 4.6 *The function $\Psi(r, \vec{u})$ defined in (4.36) is the unique solution symmetric in the variables*

$u_1, \ldots, u_{2g+1}$ *of the Cauchy problem for the linear over-determined system*

$$\begin{cases} \frac{\partial}{\partial u_i}\Psi(r,\vec{u}) - \frac{\partial}{\partial u_j}\Psi(r,\vec{u}) = 2(u_i - u_j)\frac{\partial^2}{\partial u_i \partial u_j}\Psi(r,\vec{u}), \quad i \neq j, \ i,j = 1,\ldots 2g+1 \\ \frac{\partial}{\partial r}\Psi(r,\vec{u}) - 2\frac{\partial}{\partial u_j}\Psi(r,\vec{u}) = 2(r - u_j)\frac{\partial^2}{\partial r \partial u_j}\Psi(r,\vec{u}), \quad j = 1,\ldots 2g+1 \\ \Psi(r,\underbrace{r,\ldots,r}_{2g+1}) = \frac{2^{g+1}}{(2g+1)!!}F^{(g)}(r) \end{cases} \tag{4.47}$$

where $F^{(g)}(r)$ is the gth derivative of the function $F(r)$ defined in (4.32). The solution $\Psi(r,\vec{u})$ of the Cauchy problem (4.47) can be written in the form

$$\Psi(r,\vec{u}) = \frac{1}{K}\int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} d\xi_1 d\xi_2 \ldots d\xi_{2g+1}(1+\xi_1)^g(1+\xi_2)^{g-\frac{1}{2}}\ldots(1+\xi_{2g})^{\frac{1}{2}} \times$$

$$\frac{F^{(g)}(\frac{1+\xi_1}{2}(\ldots(\frac{1+\xi_{2g}}{2}(\frac{1+\xi_{2g+1}}{2}r + \frac{1-\xi_{2g+1}}{2}u_1) + \frac{1-\xi_{2g}}{2}u_2) + \ldots) + \frac{1-\xi_1}{2}u_{2g+1})}{\sqrt{(1-\xi_1)(1-\xi_2)\ldots(1-\xi_{2g+1})}} \tag{4.48}$$

where $K = \frac{(2g+1)!!}{2^{g+1}}K_1K_2\ldots K_{2g+1}$ *and* $K_i = \int_{-1}^{1}\frac{(1+\xi)^{\frac{2g-i+1}{2}}}{\sqrt{1-\xi}}d\xi$.

The proof is obtained from Lemma 3.4 in [9]. We state also the next elementary lemma.

Lemma 4.7 *If the real analytic function $g(u)$ satisfies the condition*

$$\frac{d^m}{du^m}g(u) < 0 \quad \forall u \in \mathbb{R},\ 0 \leq m \in \mathbb{N},$$

then $g(u)$ has at most m real zeros.

We continue the proof of the main theorem.
If the initial data $f(u)$ satisfies (4.46), then $\frac{\partial^{2N-g}}{\partial r^{2N-g}}\Phi(r,\vec{u}) < 0$ for all $r \in \mathbb{R}$ and for all real $u_1 > u_2 > \cdots > u_{2g+1}$. Indeed using (4.32), (4.36),(4.39), (4.46) and (4.48) we obtain

$$\frac{\partial^{2N-g}}{\partial r^{2N-g}}\Phi(r,\vec{u}) = \int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} d\xi_1 d\xi_2 \ldots d\xi_{2g+1}\frac{(1+\xi_1)^{2N}(1+\xi_2)^{2N-\frac{1}{2}}\ldots(1+\xi_{2g+1})^{2N-g}}{2^{(2g+1)(2N-g)}K} \times$$

$$\frac{F^{(2N+1)}(\frac{1+\xi_1}{2}(\ldots(\frac{1+\xi_{2g}}{2}(\frac{1+\xi_{2g+1}}{2}r + \frac{1-\xi_{2g+1}}{2}u_1) + \frac{1-\xi_{2g}}{2}u_2) + \ldots) + \frac{1-\xi_1}{2}u_{2g+1})}{\sqrt{(1-\xi_1)(1-\xi_2)\ldots(1-\xi_{2g+1})}} < 0 \tag{4.49}$$

for all real $u_1 > u_2 > \cdots > u_{2g+1}$. Combining (4.49) and Lemma 4.7 it follows that the function $\Phi(r,\vec{u})$ has at most $2N-g$ real zeros in the r variable for any real $u_1 > u_2 > \cdots > u_{2g+1}$ and any

x and $t \geq 0$. If the domains D_g and D_m, $0 \leq g \leq N$, $m > g$, have a common boundary the function $\Phi(\tau, \vec{u})$ has $2m - g$ real zeros for some values of x and t. Hence

$$2N - g \geq 2m - g \quad \text{or} \quad m \leq N. \tag{4.50}$$

This shows that set of domains $\{D_g\}_{0 \leq g \leq N}$ does not have common boundaries with any of the domains in the set $\{D_m\}_{m>N}$. The theorem is proved. □

Example 4.6 We consider the initial data

$$x = f(u) = -\frac{1}{5!}(u^5 + cu^4 + u^3) - 2u - \sin u \quad c^2 \leq \frac{15}{4}.$$

The function $f(u)$ is monotone decreasing and $f^{(5)}(u) = -1 - \cos u < 0$ for all real u except for $u = (2k+1)\pi$, k integer. Hence applying Theorem 4.5 the global solution of the Whitham equations for such initial data has genus at most equal to 2.

5 Conclusion

In this work we have proved that under certain conditions on the initial data the Whitham equations have a global solution. Indeed if the monotone analytic initial data $f(u)$ satisfies the condition $f^{(2N+1)}(u) < 0$ for all $u \in \mathbb{R}$ except for a number of isolated points, then the global solution of the Whitham equations has genus at most N. We believe that this results holds true also for smooth initial data. We are still working on this problem.

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