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FOURIER-MUKAI TRANSFORM OF HIGGS PAIRS

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FOURIER-MUKAI TRANSFORM OF HIGGS PAIRS †

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Abstract. We define a Fourier-Mukai transform for Higgs pairs and show that it preserves stability. This transform is compatible with the action of $\mathbb{C}^*$ on the space of Higgs pairs, and therefore defines a transformation of systems of Hodge bundles.

1. Introduction.

The transform which by now is usually called Fourier-Mukai transform was introduced by Mukai [13] as a functor between the derived categories of $\mathcal{O}$-modules on an abelian variety and of the dual variety (the transform is actually an equivalence of categories). The transformation of periodic instantons on $\mathbb{R}^4$ introduced by Nahm [14] was later recognized to coincide with the Fourier-Mukai transform [8]. Fahaloui and Laszlo [9] and Maciocia [11] showed that on 2-dimensional abelian varieties the Fourier-Mukai transform preserves the stability property of sheaves. A Fourier-Mukai transform for K3 surfaces was introduced and studied in [3] [4]; also this transform establishes an equivalence of derived categories and preserves stability. The proof given in [3] exploits partly transcendental techniques and allows for a generalization to higher-dimensional hyperkähler manifolds [5]; this includes abelian varieties of even dimension, and therefore provides a generalization of Fahaloui-Laszlo's and Maciocia's results.

Similar results can be proved for Higgs pairs. A Higgs pair on a complex manifold X is an $\mathcal{O}_X$-module $\mathcal{E}$ together with an element $\theta \in \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)$ (where Ω^1_X is the sheaf of holomorphic 1-forms) such that $\theta^2 = 0$ (cf. [10] [16]). For such pairs there is an appropriate notion of stability. We define a Fourier-Mukai transform for Higgs pairs in the general setting where one has two complex manifolds X, Y and a fixed holomorphic bundle Q on the product $X \times Y$. By standard algebraic techniques one can prove the preservation of stability for locally-free Higgs pairs (satisfying an additional technical condition) when X is a two-dimensional abelian variety, Y the dual variety, and Q the Poincaré bundle on the product; or when X is a reflexive K3 surface, Y is the "mirror" reflexive K3 surface, and Q the universal bundle on the product (cf. [3] [4] [6] for details on reflexive K3 surfaces). This is done in Section 4.

Simpson has proved a Kobayashi-Hitchin correspondence for Higgs pairs; namely, a locally-free Higgs pair (E, θ) on a Kähler manifold is polystable if and only if E can be equipped with a hermitian metric h such that the connection $D_h = \nabla_h + \theta + \bar{\theta}$ is

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Einstein-Hermite (here ∇_h is the hermitian connection of h). This generalizes to the higher-dimensional case Hitchin's result that stable Higgs pairs on Riemann surfaces correspond to solutions of the vortex equations. By exploiting this correspondence in Section 5 we study the Fourier-Mukai transform of Higgs pairs by taking X and Y as hyperkähler manifolds, and Q as a quaternionic instanton on $X \times Y$ [15]. In this case the preservation of stability is proved by transcendental techniques.

This definition of Fourier-Mukai transform for Higgs pairs appears to be quite natural in several respects. There is an obvious forgetful functor from the category of Higgs pairs to the category of coherent modules, and this commutes with the Fourier-Mukai transform. The latter also commutes with the action of $\mathbb{C}^*$ on the space of Higgs pairs (mapping (E, θ) to $(E, \lambda\theta)$, $\lambda \in \mathbb{C}^*$) and therefore, according to the results of [16], defines a transformation of systems of Hodge bundles.

2. Higgs pairs.

Let X be a complex manifold; we shall denote by $\mathcal{A}_X^k$, $\mathcal{A}_X^{p,q}$, and Ω_X^k the sheaves of smooth $\mathbb{C}$-valued k-forms, of differential forms of type (p, q), and of holomorphic k-forms, respectively.

DEFINITION 1. [10][15] *A Higgs pair $(\mathcal{E}, \theta)$ is a coherent $\mathcal{O}_X$-module $\mathcal{E}$ together with a morphism of $\mathcal{O}_X$-modules*

$$\theta: \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$$

such that $\theta^2 = 0$.

The morphism $\theta: \mathcal{E} \otimes \Omega_X^1 \to \mathcal{E} \otimes \Omega_X^2$ is of course defined as $\theta(s \otimes \omega) = \theta(s) \wedge \omega$.

The datum of a Higgs pair is equivalent to a pair $(\mathcal{E}^\infty, D)$, where $\mathcal{E}^\infty$ is a $\mathcal{C}_X^\infty$-module and $D: \mathcal{E}^\infty \to \mathcal{E}^\infty \otimes \mathcal{A}_X^1$ a morphism such that:

(i) D satisfies the Leibniz rule $D(fs) = fD(s) + s \otimes \bar{\partial} f$;

(ii) $D^2 = 0$;

(iii) the $\mathcal{O}_X$-module $\mathcal{E} = \ker D^{0,1}$ is coherent.

The morphism θ is recovered as $\theta = D^{1,0}$; the condition $D^2 = 0$ is equivalent to the integrability of $D^{0,1}$ and θ and to the holomorphicity of θ. Since $\mathcal{C}_X^\infty$ is a faithfully flat $\mathcal{O}_X$-module, if $\mathcal{E}^\infty$ is locally free, then $\mathcal{E}$ is a locally-free $\mathcal{O}_X$-module (in this case we shall call $(\mathcal{E}, \theta)$ a *Higgs bundle*).

For Higgs pairs one can introduce a notion of stability, cf. Section 4. If X is a Riemann surface, stable Higgs pairs correspond to solutions to vortex equations [10]. On a higher-dimensional manifold, the vortex equations are replaced by an Einstein-Hermite type condition, cf. Section 5.

3. Transforming Higgs pairs.

Assume now that X and Y are compact complex manifolds, and let $\mathcal{Q}$ be a holomorphic vector bundle on the product $X \times Y$. After denoting by π, $\hat{\pi}$ the projections from the product onto its factors, we consider the corresponding relative Dolbeault complexes

$$\pi^* \mathcal{A}_X^{p,0} \xrightarrow{\bar{\partial}_1} \pi^* \mathcal{A}_X^{p,1} \xrightarrow{\bar{\partial}_1} \dots, \qquad \hat{\pi}^* \mathcal{A}_Y^{p,0} \xrightarrow{\bar{\partial}_2} \hat{\pi}^* \mathcal{A}_Y^{p,1} \xrightarrow{\bar{\partial}_2} \dots$$

To each coherent $\mathcal{O}_X$-module $\mathcal{E}$ on X we associate a double complex of sheaves on Y according to the formula

$$\mathcal{F}^{p,q}(\mathcal{E}) = \hat{\pi}_*(\pi^*\mathcal{E} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{Q} \otimes_{\mathcal{O}_{X\times Y}} \pi^*\mathcal{A}_X^{0,p} \otimes_{\mathcal{C}^\infty_{X\times Y}} \hat{\pi}^*\mathcal{A}_Y^{0,q}),$$

whose differentials are the direct images via $\hat{\pi}$ of the twisted relative Dolbeault operators $\bar{\partial}_1$ and $\bar{\partial}_2$.[1]

In particular, the Fourier-Mukai functors

$$\mathcal{S}^k(\bullet) = R^k\hat{\pi}_*(\pi^*(\bullet) \otimes \mathcal{Q})$$

are the cohomology sheaves of the complex $\mathcal{F}^{0,\bullet}(\mathcal{E})$ [13].

Following Mukai's terminology we shall say that the sheaf $\mathcal{E}$ is WIT_k if $\mathcal{S}^i\mathcal{E} = 0$ for $i \neq k$, and that $\mathcal{E}$ is IT_k if $H^i(X, \mathcal{E} \otimes \mathcal{Q}_{|X\times\{y\}}) = 0$ for $i \neq k$ and all $y \in Y$. By the base change theorem, a sheaf is IT_k if and only if it is both WIT_k and the only non-vanishing direct image sheaf $\mathcal{S}^k\mathcal{E}$ is locally free.

Given a Higgs pair $(\mathcal{E}, \theta)$ on X, we aim to obtain Higgs pairs $(\mathcal{S}^k\mathcal{E}, \hat{\theta}^k)$ on Y. One way to define such transformations of the pair $(\mathcal{E}, \theta)$ is to define differential operators $\hat{D}^k$ on Y induced by the differential operator D corresponding to θ; to this end we shall exploit the formalism developed in [2] (cf. also [7]).

Let D be the operator on $\mathcal{E}^\infty$ corresponding to the Higgs pair $(\mathcal{E}, \theta)$, and define the differential operator on $X \times Y$

$$\nabla\colon \pi^*\mathcal{E} \otimes \mathcal{Q} \otimes \mathcal{C}^\infty_{X\times Y} \to \pi^*\mathcal{E} \otimes \mathcal{Q} \otimes \mathcal{A}^1_{X\times Y}$$
$$\nabla = \pi^*D + \bar{\partial}_Q$$

This induces a relative differential operator ∇_π with respect to the projection π. Let us now consider the naturally defined relative differential operators

$$\delta\colon \pi^*\mathcal{A}_X^{0,p} \to \pi^*\mathcal{A}_X^{0,p} \otimes \hat{\pi}^*\mathcal{A}_Y^1,$$

where $\delta_{(p)}\omega$ is $(-1)^p$ times the component of $d\omega$ on $\pi^*\mathcal{A}_X^{0,p} \otimes \hat{\pi}^*\mathcal{A}_Y^1$. One now introduces the relative differential operators

$$\tilde{\nabla} = \nabla_\pi \otimes 1 + 1 \otimes \delta$$

on the sheaves $\pi^*\mathcal{E} \otimes \mathcal{Q} \otimes \pi^*\mathcal{A}_X^p$ which satisfy the compatibility condition $\tilde{\nabla}^{0,1} = \bar{\partial}_2$ and the anticommutation property

$$\tilde{\nabla} \circ \bar{\partial}_1 = -\bar{\partial}_1 \circ \tilde{\nabla}.$$

This implies that the operators $\tilde{\nabla}$ induce differential operators on the cohomology groups of the complex $\mathcal{F}^{0,\bullet}(\mathcal{E})$

$$\hat{D}^k\colon \mathcal{S}^k\mathcal{E} \otimes \mathcal{C}^\infty_Y \to \mathcal{S}^k\mathcal{E} \otimes \mathcal{A}^1_Y;$$

$(\hat{D}^k)^{0,1}$ coincides with the Dolbeault operator of $\mathcal{S}^k\mathcal{E}$. Since $(\pi^*D + \bar{\partial}_Q)^2 = 0$, we have $(\hat{D}^k)^2 = 0$; then $\hat{D}^k$ determines a holomorphic morphism

$$\hat{\theta}^k\colon \mathcal{S}^k\mathcal{E} \to \mathcal{S}^k\mathcal{E} \otimes \Omega^1_Y$$

such that $(\hat{\theta}^k)^2 = 0$. So we have:

[1] To simplify notation we denote all inverse images by the same symbol, understanding that they are inverse images of $\mathcal{O}$-modules or $\mathcal{C}^\infty$-modules according to the relevant case.

PROPOSITION 1. *The pair* $(\mathcal{S}^k\mathcal{E}, \hat{\theta}^k)$ *is a Higgs pair on* Y.

When $\mathcal{E}$ is WIT_k we shall write $\widehat{\mathcal{E}} = \mathcal{S}^k\mathcal{E}$ and $\hat{\theta} = \hat{\theta}^k$.

REMARK 1. The holomorphic structure of the sheaf $\mathcal{E}$ is given by a Dolbeault operator $\bar{\partial}_{\mathcal{E}}$, and D can be represented as $D = \bar{\partial}_{\mathcal{E}} + \theta$. The holomorphic structures of the cohomology sheaves $\mathcal{S}^k\mathcal{E}$ are given by Dolbeault operators which can be obtained by applying to $\bar{\partial}_{\mathcal{E}}$ the construction previously developed for D. By comparing this with the transform of D we see that the morphisms $\hat{\theta}^k$ can also be constructed by defining the morphism

$$\Theta\colon \pi^*\mathcal{E} \otimes \mathcal{Q} \otimes_{\mathcal{C}^\infty_{X\times Y}} \to \pi^*\mathcal{E} \otimes \mathcal{Q} \otimes \mathcal{A}^1_{X\times Y}$$
$$\Theta = \pi^*\theta \otimes \mathrm{id}$$

by making it into a relative morphism Θ_π, and noticing that this satisfies the anticommutation property

$$\Theta_\pi \circ \bar{\partial}_1 = -\bar{\partial}_1 \circ \Theta_\pi\,;$$

it therefore induces morphisms on the cohomology sheaves $\mathcal{S}^k\mathcal{E}$ which coincide with $\hat{\theta}^k$. ■

4. Stable Higgs pairs on projective surfaces.

Let us fix a Kähler metric on X, and define the slope $\mu(\mathcal{E})$ of a sheaf $\mathcal{E}$ of positive rank as $\mu(\mathcal{E}) = \deg(\mathcal{E})/\operatorname{rk}(\mathcal{E})$. Given a Higgs pair $(\mathcal{E}, \theta)$ a subsheaf $\mathcal{F}$ of $\mathcal{E}$ is said to be θ-invariant if $\theta(\mathcal{F}) \subset \mathcal{F} \otimes \Omega^1_X$.

DEFINITION 2. *A Higgs pair* $(\mathcal{E}, \theta)$ *is stable (resp. semistable) if* $\mathcal{E}$ *is torsion-free and* $\mu(\mathcal{F}) < \mu(\mathcal{E})$ *(resp.* $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$*) for all* θ*-invariant proper subsheaves* $\mathcal{F}$ *of* $\mathcal{E}$.

LEMMA 1. *A Higgs pair* $(\mathcal{E}, \theta)$ *which is not stable can be destabilized by a* θ*-invariant subsheaf* $\mathcal{F}$ *such that the quotient* $\mathcal{E}/\mathcal{F}$ *is torsion-free.*

PROOF: If $\mathcal{F}'$ is a destabilizing subsheaf, $\mathcal{T}$ is the torsion of $\mathcal{G}' = \mathcal{E}/\mathcal{F}'$, and $\mathcal{G} = \mathcal{G}'/\mathcal{T}$, the kernel $\mathcal{F}$ of the epimorphism $\mathcal{E} \to \mathcal{G}$ is a subsheaf of $\mathcal{E}$ with torsion-free quotient such that $\mu(\mathcal{F}) \geq \mu(\mathcal{F}') > \mu(\mathcal{E})$. We must show that the morphism θ lifts to $\mathcal{F}$, that is, that the morphism $\theta\colon \mathcal{G}' \to \mathcal{G}' \otimes \Omega^1_X$ descends to $\mathcal{G}$. This is true because θ maps $\mathcal{T}$ into the torsion of $\mathcal{G}' \otimes \Omega^1_X$, which is the sheaf $\mathcal{T} \otimes \Omega^1_X$ since Ω^1_X is locally free. ■

A similar result is proved for non-semistable Higgs pairs.

To get an algebraic proof of the preservation of stability we need to assume that the Fourier-Mukai transform is invertible, and that its inverse is given by the functor $R^\bullet\pi_*(\hat{\pi}^*(\bullet) \otimes \mathcal{Q}^*)$. Examples of situations where this condition holds are:

1) X an algebraic torus, Y the dual torus, and $\mathcal{Q}$ the normalized Poincaré bundle [13].

2) X a reflexive K3 surface, Y the canonical "mirror" surface, and $\mathcal{Q}$ the normalized universal sheaf on the product [3] [6].

For a general discussion of this issue cf. [12]; in particular, one shows that the condition of invertibility, together with a property of preservation of the Ext groups (the so-called Parseval formula), force the canonical sheaves of X and Y to be trivial.

Therefore, we restrict ourselves to projective surfaces with trivial canonical sheaf. If X is abelian, with polarization H, we consider on the dual variety Y the polarization $\hat{H} = -c_1(\widehat{\mathcal{O}_X(H)})$. In the case when X is a reflexive K3 surface one shows that the "mirror" K3 surface Y has a natural polarization $\hat{H}$ (actually, Y is a component of the moduli space of stable sheaves on X, and $\hat{H}$ is half the polarization given by the determinant bundle on Y) [3] [4].

Let $(\mathcal{E}, \theta)$ be a Higgs pair on X, with $\mathcal{E}$ a WIT_k sheaf; we know that the sheaf $\widehat{\mathcal{E}}$ is WIT_{2-k}, and that $\widehat{\widehat{\mathcal{E}}} \simeq \mathcal{E}$. Under this isomorphism we have the identification $\widehat{\widehat{\theta}} = \theta$, which follows directly from the construction of $\hat{\theta}$.

Under the previous assumptions we have:

THEOREM 1. *Let $(\mathcal{E}, \theta)$ be a (semi)stable Higgs pair, with $\deg \mathcal{E} = 0$, and $\mathcal{E}$ a WIT_1 sheaf. Then the Higgs pair $(\widehat{\mathcal{E}}, \hat{\theta})$ is (semi)stable.*

PROOF: We shall consider only the stable case, the semistable one being entirely analogous. If $(\widehat{\mathcal{E}}, \hat{\theta})$ is not stable by Lemma 1 there is a sequence

$$0 \to \mathcal{F} \to \widehat{\mathcal{E}} \to \mathcal{G} \to 0$$

where $\deg \mathcal{F} \geq 0$, $\mathcal{F}$ is $\hat{\theta}$-invariant, and $\mathcal{G}$ is torsion-free. Since $\widehat{\mathcal{E}}$ is WIT_1 we have $\widehat{\mathcal{S}}^0 \mathcal{F} = \widehat{\mathcal{S}}^2 \mathcal{G} = 0$. Moreover we have commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{S}}^0 \mathcal{G} \arrow[r] \arrow[d] & \widehat{\mathcal{S}}^1 \mathcal{F} \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\mathcal{S}}^0 \mathcal{G} \otimes \mathcal{A}^1_X \arrow[r] & \widehat{\mathcal{S}}^1 \mathcal{F} \otimes \mathcal{A}^1_X \arrow[r] & \mathcal{K} \otimes \mathcal{A}^1_X \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
0 \arrow[r] & \mathcal{K} \otimes \mathcal{A}^1_X \arrow[r] & \mathcal{E} \otimes \mathcal{A}^1_X
\end{tikzcd}$$

where the vertical arrows are induced by $\hat{\theta}$. The arrow $\mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^1_X$ coincides with θ. As in the usual case (cf. [4]) one proves $\deg \mathcal{K} \geq 0$; since $\mathcal{K}$ is θ-invariant the Higgs pair $(\mathcal{E}, \theta)$ is not stable, which is absurd. ■

5. The index-theoretic construction.

The stability of the Fourier-Mukai transform of a stable Higgs bundle on a Kähler manifold can be proved under suitable hypotheses in the differential-geometric setting. Let E be the smooth bundle and D the differential operator $D: \mathcal{E}^\infty \to \mathcal{E}^\infty \otimes \mathcal{A}^1_X$ corresponding to a Higgs bundle $(\mathcal{E}, \theta)$ as explained in Section 1 (so $\mathcal{E}^\infty$ is the sheaf of smooth sections of E). We assume that $\mathcal{E}$ is IT_k for some k (one can weaken this assumption, cf. [5]). Let h be any hermitian metric on E, and define a connection D_h on E by letting

$$D_h = \nabla_h + \theta + \bar{\theta} \tag{1}$$

where ∇_h is the hermitian connection of (E,h) (the unique h-metric connection compatible with the complex structure of E) and $\bar\theta$ is defined as the pointwise h-adjoint of θ,

$$h(\bar\theta(s_1),s_2)=h(s_1,\theta(s_2))$$

with s_1, s_2 sections of E (note that the connection D_h is neither h-metric nor compatible with the complex structure of E). From [16] we have the following result:

PROPOSITION 2. *A Higgs bundle $(\mathcal{E},\theta)$ is polystable (i.e. is a direct sum of stable Higgs bundles having the same slope) if and only if there is a hermitian metric h on E such that D_h is an Einstein-Hermite connection, i.e. $\Lambda F_{D_h}=c\,\mathrm{Id}_E$.*

Here c is a constant, which turns out to be a multiple of the slope of E, F_{D_h} is the curvature of D_h, and Λ is the adjoint of the operation of wedging by the Kähler form of X. We also notice the relation

$$F_{D_h}=F_{\nabla_h}+[\theta,\bar\theta]$$

where F_{∇_h} is the curvature of the hermitian connection ∇_h, and $[\theta,\bar\theta]=\theta\wedge\bar\theta+\bar\theta\wedge\theta$ is the natural extension of the commutator of $\mathrm{End}\,E$ to $\mathrm{End}\,E$-valued 1-forms. From this we see that $F^{0,2}_{D_h}=0$, so that the connection D_h induces on E a second complex structure; we denote by $\mathcal{E}'$ the corresponding sheaf of holomorphic sections. Since D_h is Einstein-Hermite, the sheaf $\mathcal{E}'$ is polystable (so, if we assume $\deg E=0$, the pair (E,D_h) is an instanton, i.e. D_h is an $SU(n)$ anti-self-dual connection). We assume that the sheaf $\mathcal{E}'$ is IT_1; this happens in several situations.

REMARK 2. In the torus case it is enough to assume that $\mathcal{E}$ is not a flat line bundle, and then $\mathcal{E}'$ is IT_1 [9]. In the reflexive K3 case one takes Y as the moduli space of H-stable bundles on X with Mukai vector $v=(2,\ell,-3)$, where the polarization H and the divisor ℓ satisfy the relations $H^2=2$, $\ell^2=-12$, $H\cdot\ell=0$ (cf. [3]); one then needs to assume $v(\mathcal{E})\neq(2,-\ell,-3)$. ■

The preservation of the (poly)stability of the pair $(\mathcal{E},\theta)$, with $\deg E=0$, can be proved by transcendental techniques in a general setting where X and Y are hyperkähler manifolds, and $\mathcal{Q}$ is (the sheaf of sections of) a quaternionic instanton on the product [15] [5]. We do not need to assume that the Fourier-Mukai transform is invertible. Under these assumption we have:

THEOREM 2. *The Fourier-Mukai transform $(\hat E,\hat D_h)$ of (E,D_h) is an instanton, so that the sheaf $\widehat{\mathcal{E}'}$ of its sections which are holomorphic with respect to the holomorphic structure determined by $\hat D_h$ is polystable.*

Also the pair (E,∇_h) can be transformed; the resulting C^∞ bundle is isomorphic to $\hat E$ (the families of elliptic operators used in the two transforms differ by zero-order operators, so that the two index bundles are isomorphic). The transformed connection $\hat\nabla_h$ induces on $\hat E$ another holomorphic structure, whose corresponding sheaf of holomorphic sections we denote by $\hat{\mathcal{E}}$. Moreover on the bundle $\hat E$ a transformed hermitian metric $\hat h$ can be naturally defined, cf. [2]. By taking the $(1,0)$ and $(0,1)$ part of the difference $\hat D_h-\hat\nabla_h$ we obtain morphisms

$$\phi,\bar\phi\colon\hat{\mathcal{E}}^\infty\to\hat{\mathcal{E}}^\infty\otimes\mathcal{A}^1_Y\,.$$

PROPOSITION 3. *(1) $\bar{\phi}$ is the pointwise $\hat{h}$-adjoint of ϕ,*

$$\hat{h}(\bar{\phi}(s_1), s_2) = \hat{h}(s_1, \phi(s_2)).$$

(2) The morphism ϕ coincides with the Fourier-Mukai transform $\hat{\theta}$ of the morphism θ.
(3) The morphism ϕ is holomorphic (with respect to the complex structure $\hat{\mathcal{E}}$).
(4) $\phi^2 = 0$.

PROOF: In view of the observations in Remark 1 and the formula (1), and taking into account how the metric $\hat{h}$ is defined (cf. [8] or [2]), a direct calculation proves (1) and (2).
(3) One needs to prove that $[\phi, \bar{\partial}_{\hat{\mathcal{E}}}] = \phi \circ \bar{\partial}_{\hat{\mathcal{E}}} + \partial_{\hat{\mathcal{E}}} \circ \phi = 0$. This follows from $[\theta, \bar{\partial}_{\mathcal{E}}] = 0$.
(4) We first notice that

$$[\hat{D}_h, \hat{\nabla}_h]^{0,2} = [\bar{\partial}_{\hat{\mathcal{E}}'}, \bar{\partial}_{\hat{\mathcal{E}}}] = 0$$

since $[\bar{\partial}_{\mathcal{E}'}, \bar{\partial}_{\mathcal{E}}] = 0$. As a consequence, from

$$(\phi + \bar{\phi})^2 = F_{\hat{D}_h} + F_{\hat{\nabla}_h} - [\hat{D}_h, \hat{\nabla}_h]$$

we obtain $\bar{\phi}^2$=0, hence, in view of (1), $\phi^2 = 0$. ■

So $(\hat{E}, \phi)$ is a Higgs pair. Since the connection $\hat{D}_h$ is Einstein-Hermite [5], by Theorem 2 we have:

PROPOSITION 4. *The Higgs pair $(\hat{E}, \phi)$ is polystable.*

6. Concluding remarks.

There is an action of $\mathbb{C}^*$ on the space of Higgs pairs, given by $(\mathcal{E}, \theta) \mapsto (\mathcal{E}, \lambda\theta)$, with $\lambda \in \mathbb{C}^*$. Let us recall that a system of Hodge bundles is a Higgs bundle $(\mathcal{E}, \theta)$ for which there is a decomposition

$$\mathcal{E} = \bigoplus_{r+s=w} \mathcal{E}^{r,s} \qquad \text{with} \qquad \theta\colon \mathcal{E}^{r,s} \to \mathcal{E}^{r-1,s+1} \otimes \Omega^1_X.$$

Any system of Hodge bundles is a fixed point of the $\mathbb{C}^*$ action, $(\mathcal{E}, \theta) \simeq (\mathcal{E}, \lambda\theta)$. Conversely, if $\mathcal{E}$ is locally free, and $(\mathcal{E}, \theta) \simeq (\mathcal{E}, \lambda\theta)$ (where λ is not a root of unity) then the pair (E, θ) has a structure of system of Hodge bundles [16]. This structure is unique when the pair is stable.

In view of Remark 1 one easily shows that the $\mathbb{C}^*$ action commutes with the Fourier-Mukai transform, $(\widehat{\lambda\theta})^k = \lambda\hat{\theta}^k$. So, under the same hypotheses of Theorem 1, we have:

PROPOSITION 5. *Assume that a Higgs bundle $(\mathcal{E}, \theta)$ has a structure of system of Hodge bundles, and that $\mathcal{E}$ is IT_1. Then $(\hat{\mathcal{E}}, \hat{\theta})$ has a structure of system of Hodge bundles as well. This structure is stable if the first one is stable.*

An important role in the theory of Higgs pairs is played by the harmonic Higgs bundles; these are shown to be the stable Higgs bundles with vanishing first and second Chern class [16]. However, the Fourier-Mukai transform does not establish a transformation of harmonic Higgs bundles because these are never IT_1 (for instance, if E was a rank r IT_1 vector bundle on an abelian surface, the transformed bundle would have Chern character $(0, 0, -r)$, which is absurd).

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