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EXISTENCE OF μ-STABLE VECTOR BUNDLES ON K3 SURFACES AND THE FOURIER-MUKAI TRANSFORM

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We consider a family of polarized K3 complex surfaces X which includes all generic Kummer surfaces. We prove a non-emptiness result for the moduli space of stable vector bundles on X for a small value of the second Chern class. For K3 surfaces in this family, the Fourier-Mukai transform preserves the μ-semistability of coherent sheaves. New results on quasi-homogeneous sheaves, and a novel algebraic proof of preservation of μ-stability are given as well.

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EXISTENCE OF μ-STABLE VECTOR BUNDLES ON K3 SURFACES AND THE FOURIER-MUKAI TRANSFORM

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ABSTRACT. We consider a family of polarized K3 complex surfaces X which includes all generic Kummer surfaces. We prove a non-emptiness result for the moduli space of stable vector bundles on X for small values of the second Chern class. For K3 surfaces in this family, the Fourier-Mukai transform preserves the μ-semistability of coherent sheaves. New results on quasi-homogeneous sheaves, and a novel algebraic proof of preservation of μ-stability are given as well.

1. INTRODUCTION

1.1. In [1] we introduced and studied some properties of the Fourier-Mukai transform of coherent sheaves on K3 surfaces. If X is a K3 surface, there are components $\widehat{X}$ of the moduli space of Gieseker-stable sheaves on X which are K3 surfaces. Under suitable assumptions, there is a universal sheaf on $X \times \widehat{X}$, and one can define a Fourier-Mukai transform mapping sheaves on X to sheaves on $\widehat{X}$ (a technical condition must be satisfied, namely, the sheaf on X must be WIT_i). We have proved by means of trascendental methods that the Fourier-Mukai transform of a μ-polystable bundle on X is polystable (μ-stability is Mumford-Takemoto's notion of stability).

For this theory to be meaningful we must prove that at least in some cases the moduli space $\widehat{X}$ is not empty. In Section 2 of this paper we prove that whenever X belongs to a wide class of K3 surfaces, that we call *reflexive*, there exists on X a Gieseker-stable bundle having the prescribed invariants. We emphasize that the bundles we find do not satisfy sufficient conditions for the existence of stable bundles established by Hirschowitz–Laszlo [4] and Sorger.[1]

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All points of the moduli space of Gieseker-stable sheaves $\widehat{X}$ thus obtained actually correspond to locally free μ-stable sheaves; moreover, $\widehat{X}$ turns out to be isomorphic, as an algebraic variety, to X, and X can be regarded as a moduli space of stable bundles on $\widehat{X}$. The two varieties are therefore in a duality relation, in analogy to abelian surfaces, and one can adapt to this setting Maciocia's algebraic proof of the preservation of μ-semistability and μ-stability (Section 3). We also construct algebraically a suitable polarization for $\widehat{X}$, by using Donaldson's results on the determinant line bundles on moduli spaces (our previous approach relied on the Kähler structure of the moduli space).

The algebraic analysis of the preservation of stability yields new information about the behaviour of sheaves under the Fourier-Mukai transform. We may for instance prove that the sheaves on X admitting a filtration whose quotients are dual of sheaves in $\widehat{X}$ (we call these sheaves quasi-homogeneous) are exactly the Fourier-Mukai transform of the skyscraper sheaves on $\widehat{X}$.

1.2. Given two schemes X, $\widehat{X}$ of finite type over an algebraically closed field k, and an element $\mathcal{Q}$ in the derived category $D(X\times Y)$ of $\mathcal{O}_{X\times Y}$-modules, Mukai [8] defined a functor $\mathcal{S}\colon D^-(X)\to D^-(\widehat{X})$ between the derived categories of $\mathcal{O}_X$-modules and $\mathcal{O}_{\widehat{X}}$-modules bounded from below,

$$\mathcal{S}(\mathcal{E}) = R\hat{\pi}_*(\mathcal{Q}\overset{L}{\otimes}\pi^*\mathcal{E})$$

(here $\pi : X\times\widehat{X}\to X$ and $\hat{\pi} : X\times\widehat{X}\to\widehat{X}$ are the natural projections). If $\mathcal{Q}$ is locally free, the Fourier-Mukai transform extends to the full derived categories, and maps the derived category $D_c(X)$ of complexes with coherent cohomology into $D_c(\widehat{X})$.

For a complex consisting in a single sheaf $\mathcal{E}$ on X, the i-th cohomology sheaf $\mathcal{S}^i(\mathcal{E})$ of $\mathcal{S}(\mathcal{E})$ is the higher direct image $\mathcal{S}^i(\mathcal{E}) = R^i\hat{\pi}_*(\mathcal{Q}\otimes\pi^*\mathcal{E})$. In this case, given an exact sequence of coherent sheaves

$$0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''\to 0\,,$$

there is a long exact sequence

$$0\to\mathcal{S}^0(\mathcal{E}')\to\mathcal{S}^0(\mathcal{E})\to\mathcal{S}^0(\mathcal{E}'')\to\mathcal{S}^1(\mathcal{E}')\to\mathcal{S}^1(\mathcal{E})\to\mathcal{S}^1(\mathcal{E}'')\to\dots \qquad (1)$$

A sheaf $\mathcal{E}$ is said to satisfy the ith Weak Index Theorem condition (i.e. it is WIT_i) if $\mathcal{S}^j(\mathcal{E}) = R^j\hat{\pi}_*(\mathcal{Q}\otimes\pi^*\mathcal{E}) = 0$ for $j\neq i$; similarly, the sheaf $\mathcal{E}$ is said to satisfy the ith Index Theorem condition (i.e. it is IT_i) if $H^j(X,\mathcal{Q}_\xi\otimes\mathcal{E}) = 0$ for $j\neq i$ and all $\xi\in\widehat{X}$, where $\mathcal{Q}_\xi = \mathcal{Q}_{|\hat{\pi}^{-1}(\xi)}$ is the sheaf on X corresponding to the point $\xi\in\widehat{X}$.

Whenever $\mathcal{E}$ is a WIT_i sheaf, we shall write

$$\widehat{\mathcal{E}} = \mathcal{S}^i(\mathcal{E}) = R^i\hat{\pi}_*(\mathcal{Q}\otimes\pi^*\mathcal{E})\,;$$

by abuse of language, we shall refer to $\widehat{\mathcal{E}}$ as the Fourier-Mukai transform of the WIT_i sheaf $\mathcal{E}$.

The base change theorem implies that for any i, the sheaf $\mathcal{E}$ is IT_i if and only if it is WIT_i and $\widehat{\mathcal{E}} = \mathcal{S}^i(\mathcal{E})$ is locally free. It turns out that the WIT_0 and IT_0 conditions are equivalent.

2. Nonemptiness of moduli space

2.1. In this paper the notions of Gieseker and Mumford-Takemoto (semi)stability refer only to torsion-free sheaves. We write '(semi)stable' for 'Gieseker-(semi)stable.'

Let X be a projective K3 complex surface polarized by an ample divisor H. For every sheaf $\mathcal{F}$ on X, we define the Mukai vector $v(\mathcal{F}) \in H^\bullet(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^{1,1}(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$ as the element $(\operatorname{rk}\mathcal{F}, c_1(\mathcal{F}), s(\mathcal{F}))$, where $s(\mathcal{F}) = \operatorname{rk}\mathcal{F} + \operatorname{ch}_2(\mathcal{F}) = \chi(\mathcal{F}) - \operatorname{rk}\mathcal{F}$.

For every vector $v \in H^*(X,\mathbb{Z})$, we denote by $M_H^G(X,v)$ the coarse moduli space of stable sheaves (with respect to H) $\mathcal{E}$ on X such that $v(\mathcal{E}) = v$. The space $M_H^G(X,v)$ is a quasi-projective subscheme of the moduli space of simple sheaves over X. Mukai [9,10] proved that whenever $M_H^G(X,v) = Y$ is nonempty and proper of dimension two, it is a K3 surface, isogeneous to the original K3 surface X.

For rank-two sheaves we have the following existence result (see [10], Theorems 5.1, 5.2).

Lemma 1. *If $v = (2,\ell,s)$ is isotropic and primitive, there exists a μ-semistable simple sheaf $\mathcal{E}$ on X with $v(\mathcal{E}) = v$. Moreover, for every divisor class D, $\mathcal{E}$ can be chosen so that $2\,D \cdot c_1(\mathcal{F}) \geq D \cdot \ell$ for every torsion free rank-one quotient sheaf $\mathcal{F}$ of $\mathcal{E}$ such that $\mu(\mathcal{E}) = \mu(\mathcal{F})$.* ■

(v is said to be primitive if it is not divisible by any integer.)

One can slightly generalize Corollary 3.10 of [10].

Lemma 2. *Let $\mathcal{E}$ be a simple μ-semistable sheaf with $v(\mathcal{E}) = (2,\ell,s)$, $\ell \cdot H = 0$, $\ell^2 = 4s$ and s odd. Then $\mathcal{E}$ is locally free.*

Proof. Notice that $v(\mathcal{E})$ is primitive and isotropic. If $\mathcal{E}$ is not locally free, Proposition 3.9 of [10] implies that $\mathcal{E}^{**}$ is rigid, that is, $\operatorname{Ext}^1(\mathcal{E}^{**},\mathcal{E}^{**}) = 0$, and that there is an exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}^{**} \to \kappa(p) \to 0 \tag{2}$$

for a point $p \in X$. It follows that $\operatorname{ch}_2(\mathcal{E}^{**}) = \operatorname{ch}_2(\mathcal{E})+1$, so that $v(\mathcal{E}^{**}) = (2,\ell,s+1)$ and $v(\mathcal{E}^{**})^2 = -4$. By Proposition 3.2 of [10] $\mathcal{E}^{**}$ is not simple, hence is not stable. We then have a destabilizing sequence

$$0 \to \mathcal{O}(D_1) \to \mathcal{E}^{**} \to \mathcal{I}_Z(D_2) \to 0$$

where $\mathcal{I}_Z$ is the ideal sheaf of a zero-dimensional subscheme, and D_1, D_2 are divisors of degree zero such that $\ell = D_1 + D_2$ and $D_1^2 \geq s-1$.

The divisors D_1, D_2 are not numerically equivalent, since we would have $\ell \equiv 2D_1$ and then $D_1^2 = s$, which is absurd; thus by the Hodge Index theorem,

$$0 > (D_1 - D_2)^2 = D_1^2 + D_2^2 - 2D_1 \cdot D_2 = \ell^2 - 4D_1 \cdot D_2 = 4s - 4D_1 \cdot D_2\,,$$

so that $D_1 \cdot D_2 > s$. Since $c_2(\mathcal{E}^{**}) = s+1$, the exact sequence (2) yields $s+1 = D_1 \cdot D_2 + \operatorname{length}(Z) > s + \operatorname{length}(Z)$ so that $\operatorname{length}(Z) = 0$ and $D_1 \cdot D_2 = s+1$. Then, $(D_1 - D_2)^2 = -4$ and we have

$$\dim \operatorname{Ext}^1(\mathcal{O}(D_2),\mathcal{O}(D_1)) = -\chi(X,\mathcal{O}(D_1 - D_2)) = 0\,,$$

which implies $\mathcal{E}^{**} \xrightarrow{\sim} \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$. Then, the exact sequence (2) implies that either $\mathcal{E} \xrightarrow{\sim} \mathcal{I}_p(D_1) \oplus \mathcal{O}(D_2)$ or $\mathcal{E} \xrightarrow{\sim} \mathcal{O}(D_1) \oplus \mathcal{I}_p(D_2)$, which is absurd since $\mathcal{E}$ is simple. ■

2.2. We shall now prove that the moduli space $M_H^G(X,v)$ is nonempty for a wide class of K3 surfaces. The nonemptiness of the moduli spaces $M_H^G(X,v)$ is in general known only for large values of the second Chern class. Sorger established a lower bound on the discriminant which ensures the existence of semistable bundles with the prescribed invariants. For K3 surfaces, Sorger's bound is $v^2 \geq 16$. However, we need the existence of stable bundles with $v^2 = 0$, so that we must resort to direct methods. To this end we restrict our consideration to K3 surfaces satisfying the following requirements.

Definition 1. A K3 surface is reflexive if it carries a polarization H and a divisor ℓ such that $H^2 = 2$, $H \cdot \ell = 0$ and $\ell^2 = -12$.

We also need a technical assumption:

the divisor $\ell + 2H$ is not effective.

Let us fix a Mukai vector $v = (2, \ell, -3)$.

Proposition 1. *There exists a μ-semistable simple locally free sheaf $\mathcal{E}$ on X with $v(\mathcal{E}) = v$ such that $\ell \cdot c_1(\mathcal{F}) \geq -6$ for every torsion free rank one quotient sheaf $\mathcal{F}$ of $\mathcal{E}$ of degree 0.*

Proof. By Lemma 1 with $D = \ell$ there is μ-semistable simple sheaf $\mathcal{E}$ on X fulfilling the remaining conditions. Moreover, $\mathcal{E}$ is locally free by Lemma 2. ■

Proposition 2. *There exists a stable locally free sheaf $\mathcal{E}$ on X with $v(\mathcal{E}) = v$, that is, the moduli space $M_H^G(X,v)$ is not empty. Moreover, every element in $M_H^G(X,v)$ is locally free.*

Proof. Let $\mathcal{E}$ be the sheaf provided by the previous Proposition. If it is not stable, there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_X(D_1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z(D_2) \longrightarrow 0\,,$$

where $\mathcal{I}_Z$ is the ideal of a zero-dimensional subscheme, D_1, D_2 are divisors of degree 0, and $D_1^2 \geq -5$, so that $D_1^2 \geq -4$ since D_1^2 is even. From $\ell = D_1 + D_2$, $-1 = c_2(\mathcal{E}) = D_1 \cdot D_2 + \mathrm{length}(Z)$ and $\ell \cdot D_2 \geq -6$, we obtain $-4 \leq D_1^2 \leq -5 + \mathrm{length}(Z)$, so that $\mathrm{length}(Z) \geq 1$. Moreover D_1, D_2 are not numerically equivalent, since we would have $\ell \equiv 2D_1$ and then $D_1^2 = -3$, which is absurd; thus by the Hodge index theorem,

$$0 > (D_1 - D_2)^2 = D_1^2 + D_2^2 - 2D_1 \cdot D_2 = -12 - 4D_1 \cdot D_2\,,$$

so that $D_1 \cdot D_2 > -3$ and $\mathrm{length}(Z) < 2$. Then $\mathrm{length}(Z) = 1$ and $D_1^2 = D_2^2 = -4$. Since $\ell + 2H = (D_1 + H) + (D_2 + H)$ and $D_1 + H$, and $D_2 + H$ are linearly equivalent to nodal curves of degree 2, this contradicts the fact that $\ell + 2H$ is not effective.

So the moduli space $M_H^G(X,v)$ is not empty. Moreover every element in $M_H^G(X,v)$ is locally free by Lemma 2. ■

Proposition 3. *Any stable bundle $\mathcal{E}$ on X with $v(\mathcal{E}) = v$ is μ-stable.*

Proof. If $\mathcal{E}$ is not μ-stable it can be destabilized by a sequence

$$0 \to \mathcal{O}_X(D_1) \to \mathcal{E} \to \mathcal{I}_Z(D_2) \to 0\,,$$

where $\mathcal{I}_Z$ is the ideal of a zero-dimensional subscheme, D_1, D_2 are divisors of degree 0, and $D_1 \cdot D_2 = -1 - \mathrm{length}(Z)$. Since $\mathcal{E}$ is stable,

$$D_1^2 < -5 \qquad D_2^2 > -5 + 2\,\mathrm{length}(Z)\,.$$

Moreover, since $D_2 \neq 0$, we have $\chi(X, \mathcal{O}_X(D_2)) \leq 0$, so that $D_2^2 \leq -4$. It follows that $\mathrm{length}(Z) = 0$, and we have an exact sequence

$$0 \to \mathcal{O}_X(D_1) \to \mathcal{E} \to \mathcal{O}_X(D_2) \to 0\,,$$

with D_1, D_2 of degree 0 and $D_1^2 = -6$, $D_2^2 = -4$. Then, $D_1 - D_2$ is not effective and we have

$$0 \to \mathrm{Hom}(\mathcal{O}(D_2), \mathcal{E}) \to \mathrm{Hom}(\mathcal{O}(D_2), \mathcal{O}(D_2)) = k$$

thus proving that $\mathcal{E} \xrightarrow{\sim} \mathcal{O}_X(D_1) \oplus \mathcal{O}_X(D_2)$, a contradiction. ■

Summing up, for a reflexive K3 surface (with $\ell + 2H$ non-effective), $M_H^G(X, v)$ is not empty and every element in $M_H^G(X, v)$ is μ-stable and locally free. As a consequence, using well-known results by Mukai [9,10] and Maruyama [6,7], we have:

Proposition 4. *$M_H^G(X, v)$ is a projective K3 surface and a fine moduli scheme parametrizing locally free μ-stable sheaves with vector v, so that there is a locally free rank 2 universal sheaf $\mathcal{Q}$ on $X \times M_H^G(X, v) \to M_H^G(X, v)$ which is relatively μ-stable with vector v.* ■

3. The Fourier-Mukai transform for reflexive K3 surfaces

3.1. In [1] we prove that the divisor $\ell + 2H$ is not effective if the following assumption is satisfied:

$D \cdot H > 2$ for every nodal curve D in X.

This condition holds generically in the moduli space of reflexive K3 surfaces. Indeed, the polarization H defines a double cover of $\mathbb{P}^2$ branched over a sextic; the image of a nodal curve of degree 1 is a line tritangent to the sextic, while the image of a nodal curve of degree 2 is a conic, tangent to the sextic at six points. Generically, neither situation can arise.

The subset of the moduli space of polarized K3 surfaces consisting of reflexive K3 surfaces satisfying the above condition on nodal curves is a quasi-projective scheme of dimension 18, which contains all generic Kummer surfaces (see [2]).

In the sequel of this paper we shall consider only reflexive K3 surfaces (X, H, ℓ) of that kind.

All results in the previous section do apply, and we denote by $\widehat{X}$ the moduli scheme $\widehat{X} = M_H^G(X, v)$ corresponding to the Mukai vector $v = (2, \ell, -3)$. We now consider the Fourier-Mukai transform

$$\mathcal{S}: D(X) \to D(\widehat{X})$$
$$\mathcal{S}(\mathcal{F}) = R\hat{\pi}_*(\mathcal{Q} \overset{L}{\otimes} \pi^*\mathcal{F}).$$

We normalize $\mathcal{Q}$ so that $R^1\hat{\pi}_*(\mathcal{Q}) \simeq \mathcal{O}_{\widehat{X}}$.

Proceeding as in [1], one can show that $\det(\mathcal{Q}) = \pi^*\mathcal{O}_X(\ell) \otimes \hat{\pi}^*\mathcal{O}_{\widehat{X}}(-\hat{\ell})$ for a suitable divisor $\hat{\ell}$ on $\widehat{X}$, and that $\mathcal{O}_X(H)$ is IT_0 and its Fourier-Mukai transform $\widehat{\mathcal{O}_X(H)}$ is a line bundle.

We now need to give $\widehat{X}$ a suitable polarization; this is given by the divisor

$$\widehat{H} = -c_1(\widehat{\mathcal{O}_X(H)}) - \hat{\ell}.$$

The proof of the ampleness of $\widehat{H}$ relies on the ampleness of the determinant line bundles on the moduli spaces proved by Donaldson. Let us consider the line bundle

$$\begin{aligned}\mathcal{L}_{mH} &= (\det[R\hat{\pi}_*(\mathcal{Q} \otimes \pi^*\mathcal{O}_X(mH))])^{-1} \otimes \det[R\hat{\pi}_*(\mathcal{Q} \otimes \pi^*\mathcal{O}_X(-mH))] \\ &= (\det[\mathcal{S}(\mathcal{O}_X(mH))])^{-1} \otimes \det[\mathcal{S}(\mathcal{O}_X(-mH))]\end{aligned}$$

for $m > 0$.

Proposition 5.

(1) *$\mathcal{O}_X(mH)$ is IT_0, $\mathcal{O}_X(-mH)$ is IT_2, and the determinants of the corresponding Fourier-Mukai transforms are the line bundles*

$$\det[\widehat{\mathcal{O}_X(mH)}] = \mathcal{O}_{\widehat{X}}(-m\widehat{H} - m^2\hat{\ell}), \quad \det[\widehat{\mathcal{O}_X(-mH)}] = \mathcal{O}_{\widehat{X}}(m\widehat{H} - m^2\hat{\ell}).$$

(2) *There are isomorphisms*

$$\mathcal{L}_{mH} \xrightarrow{\sim} (\det[\widehat{\mathcal{O}_X(mH)}])^{-1} \otimes \det[\widehat{\mathcal{O}_X(-mH)}] \xrightarrow{\sim} \mathcal{O}_{\widehat{X}}(2m\widehat{H}).$$

Proof. (1) We know that $\mathcal{O}_X(H)$ is IT_0 and $\widehat{\mathcal{O}_X(H)}$ is a line bundle. Moreover for every closed point $\xi = [\mathcal{E}] \in \widehat{X}$ we have

$$H^i(X, \mathcal{E}(-H)) \xrightarrow{\sim} H^{2-i}(X, \mathcal{E}^*(H))^*.$$

The sheaf $\mathcal{E}^*$ is μ-stable of degree zero with isotropic Mukai vector $(2, -\ell, -3)$, and $\chi(X, \mathcal{E}^*) = -1$. Lemma 3 of [1] applies here to give $h^j(X, \mathcal{E}^*(H)) = 0$ for $j > 0$ and $h^0(X, \mathcal{E}^*(H)) = 1$, which means that $\mathcal{O}_X(-H)$ is IT_2 and that $\widehat{\mathcal{O}_X(-H)}$ is a line bundle.

Since $\mathcal{O}_X$ is IT_1, we learn that $\mathcal{O}_H$ is IT_1 and $\mathcal{O}_H(H)$ is IT_0. Then, $\mathcal{O}_H(mH)$ is IT_0 and $\mathcal{O}_H(-mH)$ is IT_1 for every $m > 0$, by induction on m. This implies that $\mathcal{O}_X(mH)$ is IT_0 and $\mathcal{O}_X(-mH)$ is IT_2. The line bundles $\det[\widehat{\mathcal{O}_X(mH)}]$ and $\det[\widehat{\mathcal{O}_X(-mH)}]$ are identified by computing their first Chern classes.

(2) This follows from (1) and the definition of $\mathcal{L}_{mH}$. ∎

Corollary 1. *$\widehat{H}$ is an ample divisor on $\widehat{X}$, and $\widehat{H}^2 = 2$.*

Proof. The determinant bundles $\mathcal{L}_{mH}$ are ample by [3], §V. ■

The same result was proved in [1] by showing that $\widehat{H}$ equals, up to a positive factor, the class of the Weil-Petersson metric on the moduli space.

We summarize now some properties of $\widehat{X}$ which were proved in [1].

Proposition 6.

(1) *There is an isomorphism of schemes $\Psi\colon \widehat{X} \xrightarrow{\sim} X$.*

(2) *The Mukai vector $\hat{v} = (2, \hat{\ell}, -3)$ is isotropic (that is, $\hat{\ell}^2 = -12$) and $\hat{\ell}\cdot\widehat{H} = 0$, so that $\widehat{X}$ is a reflexive K3 surface.*

(3) *There is a natural isomorphism of schemes $X \xrightarrow{\sim} M^G_{\widehat{H}}(\widehat{X}, \hat{v})$, which identifies X with a moduli space of stable vector bundles on the moduli space $\widehat{X} = M^G_H(X, v)$, and the relevant universal sheaf on $\widehat{X} \times X \to X$ is the dual $\mathcal{Q}^*$ of the universal sheaf $\mathcal{Q}$.*

■

We now consider the Fourier-Mukai transform

$$\widehat{\mathcal{S}}\colon D(\widehat{X}) \to D(X), \qquad \widehat{\mathcal{S}}(\mathcal{G}) = R\pi_*(\hat{\pi}^*\mathcal{G} \overset{L}{\otimes} \mathcal{Q}^*)$$

defined by means of the universal sheaf $\mathcal{Q}^*$. The two functors $\mathcal{S}$ and $\widehat{\mathcal{S}}$ are actually the inverse of each other up to a shift of degrees ([1], Proposition 10).

Proposition 7. *For any objects $\mathcal{F}$ in $D(X)$ and $\mathcal{G}$ in $D(\widehat{X})$ we have functorial isomorphisms*

$$\widehat{\mathcal{S}}(\mathcal{S}(\mathcal{F})) \xrightarrow{\sim} \mathcal{F}[-2], \qquad \mathcal{S}(\widehat{\mathcal{S}}(\mathcal{G})) \xrightarrow{\sim} \mathcal{G}[-2]$$

in the derived categories $D(X)$ and $D(\widehat{X})$ respectively. ■

This means that for a single coherent sheaf $\mathcal{E}$ we obtain a convergent spectral sequence with $E_2^{p,q} = 0$ for $p, q \neq 0, 1, 2$, and

$$E_2^{p,q} = \widehat{\mathcal{S}}^p(\mathcal{S}^q(\mathcal{E})) \implies E_\infty^{p+q} = \begin{cases} \mathcal{E} & \text{if } p+q = 2 \\ 0 & \text{otherwise} \end{cases}$$

An analogous convergent spectral sequence is obtained for any coherent sheaf on $\widehat{X}$.

Proposition 8.

(1) *If a coherent sheaf $\mathcal{E}$ on X is WIT_i, its Fourier-Mukai transform $\widehat{\mathcal{E}} = S^i(\mathcal{E})$ is WIT_{2-i}, and there is an isomorphism $\mathcal{E} \xrightarrow{\sim} \widehat{\mathcal{S}}^{2-i}(\mathcal{S}^i(\mathcal{E}))$.*

(2) *For every coherent sheaf $\mathcal{E}$ on X, $\mathcal{S}^0(\mathcal{E})$ is WIT_2 and $\mathcal{S}^2(\mathcal{E})$ is IT_0.* ■

This inversion theorem has some important consequences; in particular, it provides an algebraic proof of the fact that μ-stability is preserved under the Fourier-Mukai transform, as we shall show in Section 3.3.

3.2. Recall that $\widehat{X} = M^G_H(X, v)$, with $v = (2, \ell, -3)$, and $X = M_{\widehat{H}}(\widehat{X}, \hat{v})$, with $\hat{v} = (2, \hat{\ell}, -3)$. For any sheaf $\mathcal{P}$ on $X \times \widehat{X}$ we write $\mathcal{P}_\xi = \mathcal{P}_{|\hat{\pi}^{-1}(\xi)}$ ($\xi \in \widehat{X}$) and $\mathcal{P}_p = \mathcal{P}_{|\pi^{-1}(p)}$ ($p \in X$).

Proposition 9.

(1) *If $\mathcal{E}$ is a μ-stable vector bundle of degree zero, and $v(\mathcal{E}^*) \neq (2, \ell, -3)$, then $\mathcal{E}$ is IT_1. In particular, every zero-degree line bundle on X is IT_1.*

(2) *If $\mathcal{E}$ is a μ-stable coherent non-locally free sheaf of degree zero, then $\mathcal{E}$ is IT_1 unless there is an exact sequence $0 \to \mathcal{E} \to \mathcal{Q}_\xi^* \to \mathcal{O}_Z \to 0$, for a point $\xi \in \widehat{X}$ and a zero-dimensional closed subscheme $Z \hookrightarrow X$.*

Proof. (1) is Lemma 6 of [1].

(2) If $\mathcal{E}$ is a μ-stable non-locally free sheaf of degree zero for every $\xi \in \widehat{X}$ we have $H^2(X, \mathcal{E} \otimes \mathcal{Q}_\xi)^* \xrightarrow{\sim} \mathrm{Hom}(\mathcal{E}, \mathcal{Q}_\xi^*)$. Since $\mathcal{E}$ and $\mathcal{Q}_\xi^*$ are μ-stable, for any non-zero homomorphism $f \in \mathrm{Hom}(\mathcal{E}, \mathcal{Q}_\xi^*)$ there is an exact sequence

$$0 \to \mathcal{E} \xrightarrow{f} \mathcal{Q}_\xi^* \to \mathcal{K} \to 0$$

with $\mathrm{rk}(\mathcal{K}) = 0$. Moreover $\mathcal{K}$ has degree zero, and then $\mathcal{K} \xrightarrow{\sim} \mathcal{O}_Z$ for a zero-dimensional closed subscheme Z, a situation we are excluding. Then $H^2(X, \mathcal{E} \otimes \mathcal{Q}_\xi)^* = 0$ for every $\xi \in \widehat{X}$. On the other hand, $H^0(X, \mathcal{E} \otimes \mathcal{Q}_\xi)^* \xrightarrow{\sim} \mathrm{Hom}(\mathcal{Q}_\xi^*, \mathcal{E})$. If $f \in \mathrm{Hom}(\mathcal{Q}_\xi^*, \mathcal{E})$ is non-zero, we have as above

$$0 \to \mathcal{Q}_\xi^* \xrightarrow{f} \mathcal{E} \to \mathcal{K}' \to 0$$

with $\mathrm{rk}(\mathcal{K}') = 0$. After dualizing we get again a contradiction. Then $H^0(X, \mathcal{E} \otimes \mathcal{Q}_\xi)^* = 0$ for every $\xi \in \widehat{X}$, thus concluding the proof. ∎

Definition 2. A coherent torsion-free sheaf $\mathcal{E}$ on X is quasi-homogeneous if there is a filtration by coherent sheaves $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E}$ such that every sheaf $(\mathcal{E}_{i+1}/\mathcal{E}_i)^*$ is μ-stable with Mukai vector v, namely, it is a sheaf defining points of $\widehat{X}$.

An analogous definition applies for sheaves on $\widehat{X}$. These quasi-homogeneous sheaves play, in a sense, the same rôle as homogeneous sheaves on abelian surfaces. In that case, sheaves that admit a filtration by line bundles of zero degree are just homogeneous sheaves, that is, sheaves invariant under translations by points of the abelian surface (cf. [8] or [5]). The relevance of this definition is shown by the following result.

Proposition 10.

(1) *For every closed point $p \in X$, the skyscraper sheaf $\kappa(p)$ is IT_0, and its Fourier-Mukai transform is the sheaf $\widehat{\kappa(p)} = \mathcal{Q}_p$; thus, the dual $\widehat{\kappa(p)}^*$ is the sheaf $\mathcal{Q}_p^*$ which defines the point of the moduli space $\widehat{\widehat{X}}$ corresponding to p by the isomorphism $X \xrightarrow{\sim} \widehat{\widehat{X}}$ of Proposition 6.*

(2) *If $\mathcal{T}$ is a coherent sheaf on X with zero-dimensional support, it is IT_0, and its Fourier-Mukai transform $\widehat{\mathcal{T}}$ is quasi-homogeneous.*

(3) *For every closed point $\xi = [\mathcal{E}^*] \in \widehat{X}$, the sheaf $\mathcal{E}$ is WIT_2 and its Fourier-Mukai transform $\widehat{\mathcal{E}}$ is the skyscraper sheaf $\kappa(\xi)$.*

(4) *If a sheaf $\mathcal{E}$ is quasi-homogeneous, then it is WIT_2 and its Fourier-Mukai transform $\widehat{\mathcal{E}}$ has zero-dimensional support.*

Proof. (1) One has an isomorphism

$$S^i(\kappa(p)) = R^i\hat{\pi}_*(\mathcal{Q} \otimes \pi^*\kappa(p)) \simeq R^i\hat{\pi}_*(j_*(\mathcal{Q}_p))$$

where $j\colon \widehat{X} \hookrightarrow X \times \widehat{X}$ is the inclusion of $\widehat{X}$ as the fibre $\pi^{-1}(p)$. Since j is a closed immersion and $\pi \circ j$ is the identity, we easily conclude.

(2) This is proved by induction on the length m of the support of $\mathcal{T}$. For $m = 1$ this reduces to the previous case. Otherwise, if p is a point of the support, there is an exact sequence

$$0 \to \mathcal{T}' \to \mathcal{T} \to \kappa(p) \to 0\,,$$

where $\mathcal{T}'$ has a zero-dimensional support of length $m-1$. By the inductive hypothesis, $\mathcal{T}'$ is IT_0 and $\widehat{\mathcal{T}'}$ is quasi-homogeneous. From the exact sequence (1) we see that $\mathcal{T}$ is IT_0, and that there is an exact sequence

$$0 \to \widehat{\mathcal{T}'} \to \widehat{\mathcal{T}} \to \widehat{\kappa(p)} \to 0\,,$$

which implies that $\widehat{\mathcal{T}}$ is quasi-homogeneous as well.

(3) This is the dual statement to (1).

(4) Let $\mathcal{E}$ be a quasi-homogeneous sheaf. We prove the statement for every sheaf $\mathcal{E}_i$ in the filtration of Definition 2. For $i = 1$ we are in the previous situation. By taking the exact sequence (1) associated with the exact sequence

$$0 \to \mathcal{E}_{i-1} \to \mathcal{E}_i \to \mathcal{E}_i/\mathcal{E}_{i-1} \to 0\,,$$

we conclude the proof by induction on i and by (3), since $(\mathcal{E}_i/\mathcal{E}_{i-1})^*$ defines a point of $\widehat{X}$. ■

Corollary 2. *Let $\mathcal{E}$ be a μ-stable sheaf of degree zero.*

(1) *If $\mathcal{E}$ is locally free it is either WIT_2 or IT_1 according to whether $v(\mathcal{E}^*) = (2, \ell, -3)$ or not.*

(2) *If $\mathcal{E}$ is not locally free and not IT_1, there is an exact sequence*

$$0 \to \mathcal{E} \to \mathcal{Q}^*_\xi \to \mathcal{O}_Z \to 0\,,$$

for a point $\xi \in \widehat{X}$ and a zero-dimensional closed subscheme $Z \hookrightarrow X$, so that $\mathcal{S}^0(\mathcal{E}) = 0$, $\mathcal{S}^1(\mathcal{E})$ is quasi-homogeneous and $\mathcal{S}^2(\mathcal{E}) \xrightarrow{\sim} \kappa(\xi)$. ■

3.3. In this section we study the behaviour of μ-(semi)stable sheaves on reflexive K3 surfaces under the Fourier-Mukai transform. Our treatment is inspired by techniques developed by Maciocia for abelian surfaces [5].

Degrees of coherent sheaves on X and $\widehat{X}$ are defined with respect to H and $\widehat{H}$. One has ([1], Proposition 11):

$$\deg(\widehat{\mathcal{S}}^0(\mathcal{E})) - \deg(\widehat{\mathcal{S}}^1(\mathcal{E})) + \deg(\widehat{\mathcal{S}}^2(\mathcal{E})) = -\deg(\mathcal{E})\,,$$

so that $\deg(\widehat{\mathcal{E}}) = (-1)^{i+1}\deg(\mathcal{E})$ for a WIT_i sheaf.

Lemma 3. *Let $\mathcal{E}$ be a coherent sheaf on X.*

(1) *If $\mathcal{E}$ is IT_0, then $\deg(\mathcal{E}) \geq 0$, and $\deg(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ has zero-dimensional support.*
(2) *If $\mathcal{E}$ is WIT_2, then $\deg(\mathcal{E}) \leq 0$, and $\deg(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a quasi-homogeneous sheaf.*

Proof. (1). Let us consider the exact sequence

$$0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{F} \to 0\,, \tag{3}$$

where $\mathcal{T}$ is the torsion subsheaf of $\mathcal{E}$. Then $\deg(\mathcal{T}) \geq 0$ so that $\deg(\mathcal{F}) \leq \deg(\mathcal{E})$ with equality if and only if $\mathcal{T}$ is supported in dimension zero. Suppose that $\deg(\mathcal{E}) \leq 0$; then $\deg(\mathcal{F}) \leq 0$. Since $\mathcal{T}$ has rank zero, $H^2(X, \mathcal{T} \otimes \mathcal{Q}_\xi) = 0$ for every $\xi \in \widehat{X}$, so that $\mathcal{S}^2(\mathcal{T}) = 0$ and the exact sequence (1) applied to (3) proves that $\mathcal{F}$ is IT_0. Then $\widehat{\mathcal{F}}$ is WIT_2 and locally free of degree $\deg(\widehat{\mathcal{F}}) = -\deg(\mathcal{F}) \geq 0$. This implies that

$$\mathrm{Hom}(\widehat{\mathcal{F}}, \mathcal{Q}_p) \xrightarrow{\sim} H^2(\widehat{X}, \widehat{\mathcal{F}} \otimes \mathcal{Q}_p^*)^* \neq 0$$

for every point $p \in X$, so that there exists a nonzero morphism $\widehat{\mathcal{F}} \to \mathcal{Q}_p$. We have now two cases, according to whether $\widehat{\mathcal{F}}$ is μ-stable or not.

a) If $\widehat{\mathcal{F}}$ is μ-stable, since $\mathcal{Q}_p$ is μ-stable as well, the only possibility is that $\deg(\widehat{\mathcal{F}}) = 0$, and there is an exact sequence

$$0 \to \widehat{\mathcal{F}} \to \mathcal{Q}_p \to \mathcal{P} \to 0$$

where $\mathrm{rk}(\mathcal{P}) = 0$ and $\deg(\mathcal{P}) = 0$. Then $\mathcal{P}$ has zero-dimensional support, hence is IT_0. But $\widehat{\mathcal{F}}$ and $\mathcal{Q}_p$ are WIT_2, so that $\mathcal{P}$ is WIT_2 as well; hence, $\mathcal{P} = 0$. Thus, $\widehat{\mathcal{F}} \xrightarrow{\sim} \mathcal{Q}_p$ and then $\mathcal{F}$ has zero-dimensional support by Proposition 10. But $\deg(\mathcal{F}) = 0$ implies $\deg(\mathcal{T}) = 0$; as a consequence, $\mathcal{T}$, and hence $\mathcal{E}$ have zero-dimensional support as well.

b) If $\widehat{\mathcal{F}}$ is not μ-stable, there is a destabilizing sequence

$$0 \to \mathcal{L} \to \widehat{\mathcal{F}} \to \mathcal{M} \to 0\,,$$

where $\mathcal{L}$ is a line bundle with $\mu(\mathcal{L}) \geq \mu(\widehat{\mathcal{F}}) \geq 0$. Then, $\mathcal{L}^* \otimes \mathcal{Q}_p$ is μ-stable of nonpositive degree for every $p \in X$, and Serre duality implies $H^2(\widehat{X}, \mathcal{L} \otimes \mathcal{Q}_p^*) = 0$, so that $\widehat{\mathcal{S}}^2(\mathcal{L}) = 0$. By the exact sequence (1) associated with the destabilizing sequence, we obtain $\widehat{\mathcal{S}}^0(\mathcal{L}) = 0$; hence $\mathcal{L}$ is WIT_1, and $\widehat{\mathcal{S}}^0(\mathcal{M}) \xrightarrow{\sim} \widehat{\mathcal{S}}^1(\mathcal{L})$. But by Proposition 9, $\widehat{\mathcal{S}}^0(\mathcal{M})$ is WIT_2 and $\widehat{\mathcal{S}}^1(\mathcal{L})$ is WIT_1, which is impossible. This proves (1).

(2). If $\mathcal{E}$ is WIT_2, then $\widehat{\mathcal{E}}$ is IT_0, so that $\deg(\widehat{\mathcal{E}}) \geq 0$ with equality if and only if $\widehat{\mathcal{E}}$ has zero-dimensional support, by (1) on $\widehat{X}$. Hence, by Proposition 10 we have $\deg(\mathcal{E}) \leq 0$ with equality if and only if $\mathcal{E}$ is quasi-homogeneous. ∎

Theorem 1. *Let $\mathcal{E}$ be a coherent sheaf on X. If $\mathcal{E}$ is μ-semistable and WIT$_1$ with* $\deg(\mathcal{E}) = 0$, *then its Fourier-Mukai transform $\widehat{\mathcal{E}}$ is μ-semistable.*

Proof. Since $\deg(\widehat{\mathcal{E}}) = 0$, if $\widehat{\mathcal{E}}$ is not μ-semistable there is an exact sequence

$$0 \to \mathcal{F} \to \widehat{\mathcal{E}} \to \mathcal{G} \to 0\,, \tag{4}$$

where $\mathcal{G}$ is torsion-free and $\deg(\mathcal{F}) \geq 0 \geq \deg(\mathcal{G})$. In general, we do not get strict inequalities because we cannot assume that $\widehat{\mathcal{E}}$ is torsion-free; the equalities $\deg(\mathcal{F}) = 0 = \deg(\mathcal{G})$ hold only if $\widehat{\mathcal{E}}$ has torsion and $\mathcal{F}$ has zero-dimensional support. The long exact sequence (1) applied to (4) gives $\widehat{\mathcal{S}}^0(\mathcal{F}) = \widehat{\mathcal{S}}^2(\mathcal{G}) = 0$ and exact sequences

$$0 \to \widehat{\mathcal{S}}^0(\mathcal{G}) \to \widehat{\mathcal{S}}^1(\mathcal{F}) \to \mathcal{K} \to 0\,, \quad 0 \to \mathcal{K} \to \mathcal{E} \to \widehat{\mathcal{S}}^1(\mathcal{G}) \to \widehat{\mathcal{S}}^2(\mathcal{F}) \to 0\,.$$

In particular, $\mathcal{F}$ is not IT$_0$ and, by Proposition 10, its support is not zero-dimensional, whence $\deg(\mathcal{F}) > 0$. Thus $\widehat{\mathcal{E}}$ is torsion-free.

Now, $\widehat{\mathcal{S}}^0(\mathcal{G})$ is WIT$_2$ and $\widehat{\mathcal{S}}^2(\mathcal{F})$ is IT$_0$ by Proposition 8, so that Lemma 3 implies $\deg(\widehat{\mathcal{S}}^0(\mathcal{G})) \leq 0$ and $\deg(\widehat{\mathcal{S}}^2(\mathcal{F})) \geq 0$. Then $\deg(\widehat{\mathcal{S}}^1(\mathcal{F})) = \deg(\mathcal{F}) + \deg(\widehat{\mathcal{S}}^2(\mathcal{F})) > 0$, so that $\deg(\mathcal{K}) = \deg(\widehat{\mathcal{S}}^1(\mathcal{F})) - \deg(\widehat{\mathcal{S}}^0(\mathcal{G})) > 0$, thus contradicting the semistability of $\mathcal{E}$. ■

Lemma 4. *If $\mathcal{F}$ is locally free and $\mathcal{T}$ has zero-dimensional support, every exact sequence $0 \to \mathcal{F} \to \mathcal{K} \to \mathcal{T} \to 0$ splits.*

Proof. By local duality the sheaves $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{T}, \mathcal{F})$ vanish for $i \neq 2$. Then $\mathrm{Ext}^1(\mathcal{T}, \mathcal{F}) = 0$ and the exact sequence splits. ■

Theorem 2. *Let $\mathcal{E}$ be a coherent μ-stable sheaf of degree zero. If either $\mathcal{E}$ is locally free and $v(\mathcal{E}^*) \neq (2, \ell, -3)$, or $\mathcal{E}$ is not of the form*

$$0 \to \mathcal{E} \to \mathcal{Q}^*_\xi \to \mathcal{O}_Z \to 0$$

for a point $\xi \in \widehat{X}$ and a zero-dimensional closed subscheme Z of X, then $\mathcal{E}$ is IT$_1$ and its Fourier-Mukai transform $\widehat{\mathcal{E}}$ is μ-stable as well.

Proof. $\mathcal{E}$ is IT$_1$ by Corollary 2 and $\widehat{\mathcal{E}}$ is μ-semistable by Theorem 1. Since $\widehat{\mathcal{E}}$ is locally free, if it is not μ-stable it can be destabilized by a sequence

$$0 \to \mathcal{F} \to \widehat{\mathcal{E}} \to \mathcal{G} \to 0\,,$$

where $\mathcal{F}$ is μ-stable and locally free, $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\widehat{\mathcal{E}})$ and $\mathcal{G}$ is torsion-free.

We have two cases:

a) $\mathcal{F}$ is IT$_1$. We have $\widehat{\mathcal{S}}^2(\mathcal{G}) = 0$ and two exact sequences

$$0 \to \widehat{\mathcal{S}}^0(\mathcal{G}) \to \widehat{\mathcal{F}} \to \mathcal{K} \to 0\,, \quad 0 \to \mathcal{K} \to \mathcal{E} \to \widehat{\mathcal{S}}^1(\mathcal{G}) \to 0\,. \tag{5a,b}$$

Again, $\widehat{\mathcal{S}}^0(\mathcal{G})$ is WIT$_2$ and $\deg(\widehat{\mathcal{S}}^0(\mathcal{G})) \leq 0$, so that $\deg(\mathcal{K}) = \deg(\widehat{\mathcal{F}}) - \deg(\widehat{\mathcal{S}}^0(\mathcal{G})) \geq 0$. Since $\mathcal{E}$ is μ-stable, the only possibility is $\deg(\mathcal{K}) = 0$ and $\mathrm{rk}(\mathcal{K}) = \mathrm{rk}(\mathcal{E})$, so that

$\mathrm{rk}(\widehat{\mathcal{S}}^1(\mathcal{G})) = 0$. Since $\deg(\widehat{\mathcal{S}}^1(\mathcal{G})) = 0$, $\widehat{\mathcal{S}}^1(\mathcal{G})$ has zero-dimensional support, hence it is IT_0 by Proposition 10. On the other hand, $\deg(\mathcal{K}) = 0$ implies $\deg(\widehat{\mathcal{S}}^0(\mathcal{G})) = 0$, and then $\widehat{\mathcal{S}}^0(\mathcal{G})$ is quasi-homogeneous, by Lemma 3 again. The sequence (5a) shows that $\mathcal{K}$ is WIT_1 and induces the exact sequence

$$0 \to \mathcal{F} \to \widehat{\mathcal{K}} \to \mathcal{S}^2(\widehat{\mathcal{S}}^0(\mathcal{G})) \to 0\,.$$

By Proposition 10, $\mathcal{S}^2(\widehat{\mathcal{S}}^0(\mathcal{G}))$ has zero-dimensional support, and since $\mathcal{F}$ is locally free and $\widehat{\mathcal{K}}$ is torsion-free, Lemma 4 forces $\widehat{\mathcal{S}}^0(\mathcal{G}) = 0$, and then $\mathcal{G}$ is WIT_1. But then $\widehat{\mathcal{S}}^1(\mathcal{G})$ is WIT_1 as well, and, since we have proved that it is IT_0, we have $\mathcal{G} = 0$, which is absurd as $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\widehat{\mathcal{E}})$.

b) $\mathcal{F}$ is not IT_1. Then, by Corollary 2 and Proposition 10, $\mathcal{F}$ is WIT_2 and $\widehat{\mathcal{F}} \xrightarrow{\sim} \kappa(p)$ for a closed point $p \in X$. The long exact sequence (1) shows that $\mathcal{G}$ is WIT_1 and yields the exact sequence

$$0 \to \mathcal{E} \to \widehat{\mathcal{S}}^1(\mathcal{G}) \to \kappa(p) \to 0\,.$$

By Lemma 4 this sequence splits, which is absurd since $\widehat{\mathcal{S}}^1(\mathcal{G})$ is WIT_1 and $\mathcal{E} \oplus \kappa(p)$ is not. ■

Corollary 3. *Let $\mathcal{E}$ be a WIT_1 torsion-free sheaf of degree zero on X. If $\mathcal{E}$ is strictly μ-semistable, then its Fourier-Mukai transform $\widehat{\mathcal{E}}$ is strictly μ-semistable.*

Proof. $\widehat{\mathcal{E}}$ is μ-semistable by Theorem 1. Since $\mathcal{E}$ is not μ-stable, Theorem 2 for $\widehat{X}$ implies that if $\widehat{\mathcal{E}}$ is μ-stable, then either it is locally free with $v(\widehat{\mathcal{E}}^*) = (2, \hat{\ell}, -3)$ or it is given by

$$0 \to \mathcal{E} \to \mathcal{Q}_p \to \mathcal{O}_W \to 0$$

for a point $p \in X$ and a zero-dimensional closed subscheme W of $\widehat{X}$. Both cases lead to a contradiction by Corollary 2 since $\widehat{\mathcal{E}}$ is WIT_1. ■

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