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by

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Abstract. We extend Borel-Weil-Bott theory to the case of the quantum group $Sl_q(n)$. The quantum flag varieties at $q^l = 1$ are briefly studied.

1. The quantum group $Sl_q(n)$

1.1 Hopf algebras. We will think of quantum groups as Hopf algebras providing non-commutative generalizations of affine group schemes. In the commutative setup, Yoneda's lemma allows one to dualize the notion of an affine group scheme and his representation theory, by introducing a corresponding algebra: the Hopf algebra. Let us briefly recall the subject. Consider a representable functor from the category of commutative k-algebras with unity and the category of affine groups. By the Yoneda lemma, the representing k-algebra H "inherits" the group properties by means of three maps:
1) the *comultiplication* Δ, an algebra homomorphism $\Delta : H \to H \otimes H$, which is *coassociative* i.e. $(\Delta \otimes id_H) \circ \Delta = (id_H \otimes \Delta) \circ \Delta$,
2) the *counity* ϵ, which is an algebra homomorphism $\epsilon : H \to k$ such that $(\epsilon \bar{\otimes} id_H) \circ \Delta = (id_H \bar{\otimes} \epsilon) \circ \Delta = id_H$ where $(\phi \bar{\otimes} \psi)(a \otimes b) = \phi(a)\psi(b)$,
3) the *coinverse* or *antipode* S, which is an algebra anti-homomorphism $S : H \to H$ such that $(S \bar{\otimes} id_H) \circ \Delta = (id_H \bar{\otimes} S) \circ \Delta = \eta \circ \epsilon$, where η is the unity of H, namely the algebra homomorphism $\eta : k \to H$ defined by $\eta(c) = c \cdot 1 \in H$, with $c \in k$.
These maps are exactly the dual counterparts of the multiplication, the unity and the inverse of the group. A k-algebra H which possesses these maps is called a Hopf algebra, while a k-algebra together with a comultiplication and a counity is called a bialgebra.
There is also a notion corresponding to a close subgroup: the *Hopf ideal.* A subset I of a Hopf algebra H is called a Hopf ideal if it is an ideal and satisfies the following properties: $\Delta(I) \subseteq I \otimes H \oplus H \otimes I$, $\epsilon(I) = 0$ and $S(I) \subseteq I$. It follows that if I is a Hopf ideal then H/I inherits a Hopf algebra structure. Finally, the notion of representation of a group is dualized by the notion of a left (right) comodule (ρ, V) which is a k-vector space V with a map $\rho : V \to H \otimes V$ such that $(id_H \otimes \rho) \circ \rho = (\Delta \otimes id_V) \circ \rho$ ($\rho : V \to V \otimes H$, $(\rho \otimes id_H) \circ \rho = (id_V \otimes \Delta) \circ \rho$ respectively). It is also useful the notion of an H-comodule algebra (ρ, A), which is both an associative k-algebra A and an H-comodule such that the map ρ is a homomorphism of k-algebras.

1.2 Quantum matrix bialgebras. Here comes the example of non commutative Hopf

algebras we will deal with. We denote by $M_q(2)$ the algebra freely generated over the ring of Laurent polynomials $\mathbb{C}_q =: \mathbb{C}[q, q^{-1}]$ by indeterminates T_{ij}, with $1 \leq i, j \leq 2$ modulo the relations which, organizing the generators in a matrix as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

state that:
1) every line (column) generates a "quantum plane" (i.e. $T_{11}T_{12} = qT_{12}T_{11}$ etc.) with respect to the ordering from left to right (from top to bottom),
2) the off-diagonal elements commute,
3) the commutator of the diagonal elements reads $[T_{11}, T_{22}] = \lambda T_{12}T_{21}$, with $\lambda = q - q^{-1}$.
Definition. *The quantum matrix algebra $M_q(n)$ (of type A_{n-1}) is the ring over $\mathbb{C}_q$ generated by indeteminates T_{ij}, $(1 \leq i, j \leq n)$, modulo the relations given by imposing that the subalgebra $\mathbb{C}_q[T_{ij}, T_{ik}, T_{hj}, T_{hk}]$ $(1 \leq i < h \leq n,\ 1 \leq j < k \leq n)$ is isomorphic to $M_q(2)$.*

Let us introduce the standard counity $\epsilon(T_{ij}) = \delta_{ij}$ and the standard coproduct $\Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}$, then (see [M], [PW])
Proposition. *The standard comultiplication Δ and the standard counity ϵ above make $M_q(n)$ into a bialgebra.*

1.3 Quantum determinants. For $q = 1$ we have that $M_1(n)$ is the classical bialgebra associated to $M(n, \mathbb{C})$. The next step in the definition of the quantum group is the introduction of the quantum determinant. One mimics the classical case and generalizes to a generic q the notion of the fundamental comodules of $M(n, \mathbb{C})$ by introducing the q-Grassmann algebra which is a left $M_q(n)$-comodule algebra. The result is that if Σ_k is the symmetric group of k letters and $|\sigma|$ is the lenght of $\sigma \in \Sigma_k$, i.e. the number of strict inversions in σ, one defines the *q-minor*

$$|^{j_1 \ldots j_k}_{i_1 \ldots i_k}|_q =: \sum_{\sigma \in \Sigma_k} (-q)^{|\sigma|} T_{i_1 j_{\sigma(1)}} \ldots T_{i_k j_{\sigma(k)}}.$$

and calls it the q-determinat of the submatrix of T obtained by taking the rows $1 \leq i_i < \ldots < i_k \leq n$ and the columns $1 \leq j_1 < \ldots < j_k \leq n$. One extends to generic q the definition of determinant by setting $det_q T =: |^{1 \ldots n}_{1 \ldots n}|_q$. If we denote by $\hat{T}_{hk} =: |^{1 \ldots \hat{k} \ldots n}_{1 \ldots \hat{h} \ldots n}|_q$ the q-algebraic complement of T_{hk} and by $\tilde{T}^t$ the matrix with elements $(\tilde{T}^t)_{hk} = (-q)^{k-h}\hat{T}_{hk}$, then it is easy to show that $det_q T \cdot 1 = T\tilde{T} = \tilde{T}T$ so the q-determinant is a central element. Moreover, one shows also that it is group-like ($\Delta(det_q T) = det_q T \otimes det_q T$) and $\epsilon(det_q T) = 1$.

1.4 The quantum group $Sl_q(n)$ Since $det_q T$ is central and group-like, the bialgebra

$$Gl_q(n) =: M_q(n)[t, t^{-1}] / < det_q T - t >$$

is well defined. In analogy with the classical case this is called quantum general linear group. Now $(det_q T)^{-1} \in Gl_q(n)$ and it is central as well as $det_q T$. So, if we set

$$S(T_{hk}) =: (-q)^{h-k} \hat{T}_{kh} (det_q T)^{-1}$$

from Lemma 1.3 it follows that $\sum_m T_{im}S(T_{mj}) = \sum_m S(T_{im})T_{mj} = \delta_{ij}$ and it is therefore the antipode. We can also introduce the quantum special linear group

$$Sl_q(n) =: M_q(n)/ < det_q T - 1 >$$

and summarize the results above in the

Proposition. *$Gl_q(n)$ and $Sl_q(n)$ together with the coproduct Δ, the counity ϵ inherited from $M_q(n)$, and the coinverse S are (non-commutative) Hopf algebras.*

Classically, all algebraic groups are algebraic varieties in "His All-Embracing Majesty" $M(n, \mathbb{C})$, as Weyl called it. For generic q this is no more the case. While quantizing different series of simple algebraic groups one has to work with non-isomorphic quantum matrix bialgebras $M_q^R(n)$, R being the so-called Yang-Baxter matrix. These have been constructed for all simple groups. We refer to [RTF] or to [S] for an account on these developments, while we shall concentrate on $Sl_q(n)$ in the following.

2. The representation theory

2.1 Irreducible comodules. Let us now come to the representation theory. As for $q = 1$, $Sl_q(n)$ is a comodule on itself in two ways; we have the right regular coaction given by the comultiplication

$$\begin{array}{rcl} {}^*R : Sl_q(n) & \rightarrow & Sl_q(n) \otimes Sl_q(n) \\ a & \mapsto & \Delta(a) \end{array}$$

and the left regular coaction given by

$$\begin{array}{rcl} {}^*L_{-1} : Sl_q(n) & \rightarrow & Sl_q(n) \otimes Sl_q(n) \\ a & \mapsto & (S \otimes id) \circ \Delta(a) \end{array}$$

The coactions above are the starting point for our study of Borel-Weil-Bott theory. The basic facts which are true at $q = 1$ and depend only on the form of the coproduct will hold true also for generic q. In particular,

Fact 1. *The left regular coaction is not irreducible. The submodules $V_q^{c_1 \cdots c_k}$ given by the span of the k-minors $(1 \leq k \leq n)$ of the $k \times n$ submatrices of T with columns $c_1, ..., c_k$ are irreducible subcomodules.*

The comodules generated by the elements of each column of T are obviously subcomodules. The same is then true for their tensor products. It is well known that the k-antisymmetric tensor products are irreducible comodules. As in the classical case, the irreducible comodules $V^{c_1 \cdots c_k}$ are all isomorphic. To select just one of these, one uses the coaction from the right of the standard Borel-Hopf algebra $B_q = Sl_q(n)/I_q$ where $I_q = \mathbb{C}_q[T_{ij}], (i > j)$. Notice that I_q is a Hopf ideal exactly as in the classical case. We denote by ${}^*R_B$ the quotient coaction (from the right)

$$\begin{array}{ccc} Sl_q(n) & \stackrel{\Delta}{\rightarrow} & Sl_q(n) \otimes Sl_q(n) \\ & {}^*R_B \searrow & \downarrow (id, \pi_B) \\ & & Sl_q(n) \otimes B_q \end{array}$$

We look for $c_1,..c_k$ such that ${}^*R_B : V^{c_1...c_k} \to V^{c_1...c_k} \otimes {}^*\chi(X)$ for some cocharacter ${}^*\chi : \mathbb{C}[X,1/X] \to B_q$. Since this is by definition a Hopf-algebra homomorphism from $\mathbb{C}_q[X,1/X]$ every element in its image is a group like element. Indeed, $\Delta_B({}^*\chi(X)) = ({}^*\chi,{}^*\chi)\Delta(X) = {}^*\chi(X) \otimes {}^*\chi(X)$. Since $\Delta|^{c_1...c_k}_{r_1...r_k}|_q = \sum_{1\le m_1<...<m_k\le n} |^{m_1...m_k}_{r_1...r_k}|_q \otimes |^{c_1...c_k}_{m_1...m_k}|_q$, we need that $\pi_B|^{c_1...c_k}_{m_1...m_k}|_q = \sum_\sigma (-q)^{|\sigma|} b_{m_1 c_{\sigma(1)}}...b_{m_k c_{\sigma(k)}} = {}^*\chi(X)\delta^{c_1}_{m_1}...\delta^{c_k}_{m_k}$, where $b_{ij} = \pi_B T_{ij}$. Now the sum reduces to a single term if and only if $c_1,...,c_k = 1,...,k$. In this case we have

$$\Delta_B|^{1...k}_{r_1...r_k}|_q = |^{1...k}_{r_1...r_k}|_q \otimes \pi_B|^{1...k}_{1...k}|_q = |^{1...k}_{r_1...r_k}|_q \otimes b_{11}...b_{kk}.$$

Fact 2. *The comodule $V_q^{1...k}$ correspond to the cocharacter ${}^*\chi(X) = b_{11}.b_{22}...b_{kk}$ of B_q.*

2.2 The quantum Borel-Weil-Bott theorem. At $q = 1$ these comodules are precisely the fundamental representations of $Sl(n)$ and the ring generated by them is the ring of the Plücker coordinates of the grassmannian $\mathbb{G}(k,n)$. Their union is the ring of Plücker coordinates of the complete flag manifold $\mathbb{F}(1,...,n-1;n)$. It is then natural to call the quantum grasmannian $\mathbb{G}_q(k,n)$ the ring generated over $\mathbb{C}_q$ by the elements of $V_q^{1...k}$, and the quantum flag manifold $\mathbb{F}_q(k_1,...,k_s;n)$ the ring generated by the elements of $V_q^{1...k_1},...,V_q^{1...k_s}$. Although one calls them quantum spaces, these rings have a few prime ideals and therefore a few points; the q-analogue of BWB theorem is rather dull: it ends up with identifying the rings $\mathbb{F}_q(k_1,...,k_s;n)$ as subrings of the Hopf algebra $Sl_q(n)$. So, for $1 \le k_1 < .... < k_s < n$, we set

$$\mathbb{F}_q(k_1,...,k_s;n) =: \mathbb{C}_q[V_q^{1....k_1} \oplus .... \oplus V_q^{1...k_s}],$$

with the obvious identifications $\mathbb{G}_q(k,n) =: \mathbb{F}_q(k;n)$ and $\mathbb{P}_q(n) =: \mathbb{G}_q(1,n)$. Notice that, similarly to the classical case, these rings are the $\mathbb{C}_q$-span of elements of the form

$$|^{1...k_1}_{i_1...i_{k_1}}|_q^{\,d_1} \cdot ... \cdot |^{1...k_s}_{j_1...j_{k_s}}|_q^{\,d_s}$$

and we consider them as multigraded by $\mathbb{N}^s$. Notice that this is not the grading inherited from $Sl_q(n)$.

Proposition. *The quantum complete flag variety $\mathbb{F}_q(1,2,...;n)$ is the $\mathbb{N}^{n-1}$-multigraded ring generated over $\mathbb{C}_q$ by elements $\pi_{i_1....i_s}$ $(0 < s < n)$ with $deg(\pi_{i_1...i_s}) = (0,\ldots,1,\ldots,0)$ (1 in the s-th position) and*

1) alternating relations

$$\pi_{i_1....i_s} = 0 \qquad \textit{if two } i's \textit{ coincide},$$

$$\pi_{i_{\sigma(1)}...i_{\sigma(k)}} = (-q)^{|\sigma|}\pi_{i_1...i_s} \qquad \textit{if } 1 \le i_1 < ... < i_s \le n \textit{ and } \sigma \in \Sigma_s.$$

2) Young symmetry relations

$$\sum_{1\le\lambda_1<...<\lambda_r\le t+r} (-q)^{|\sigma|}\pi_{i_1...i_{\lambda_1}...i_{\lambda_r}...i_{t+r}}\pi_{i_{\lambda_1}...i_{\lambda_r}j_1...j_{s-r}} = 0$$

for $1 \le r \le s \le t \le n$ and for all choices of $1 \le i_1 < ... < i_{t+r} \le n$ and $1 \le j_1 < ... < j_{s-r} \le n$. Here $|\sigma|$ denotes the length of the permutation $\sigma \in \Sigma_t$ such that $\sigma(1,...,t) = (i_1,...,\hat{i}_{\lambda_1},...,\hat{i}_{\lambda_r},...,i_{\lambda_1},...,i_{\lambda_r})$.

3) commutation relations

$$\pi_{j_1\dots j_r}\pi_{i_1\dots i_s} = \sum_{1\le\lambda_1<\dots<\lambda_r\le s} (-q)^{-|\sigma|}\pi_{j_1\dots j_r i_1\dots \hat{i}_{\lambda_1}\dots \hat{i}_{\lambda_r}\dots i_s}\pi_{i_{\lambda_1}\dots i_{\lambda_r}}$$

for $1 \le r \le s \le n$ *and for all* $1 \le i_1 < \dots < i_s \le n$ *and* $1 \le j_1 < \dots < j_r \le n$. *Here* $\sigma \in \Sigma_s$ *is the permutation such that* $\sigma(1,\dots,s) = (\lambda_1,\dots,\lambda_r,1,\dots,\hat{\lambda}_1,\dots\hat{\lambda}_r,\dots,s)$.

Remarks 1) We limit ourselves to notice that this is essentially a consequence of the Laplace expansions and refer to [TT] for the proof that the substitution $\pi_{j_1\dots j_r} =: |^{1\dots r}_{j_1\dots j_r}|_q$ satisfies all the relations above. (There is an inessential replacement $q \mapsto q^{-1}$ between our notation and that of [TT]). They also remark that the relations 1) and 2) are the q-analogue of the antisymmetry of Plücker coordinates and of the "incidence relations" for the classical flag manifold. The commutation relations 3) is less simple to understand.

2) The relations defining the q-grassmannians $\mathbb{G}_q(k,n)$ can be read off from those listed above, by noticing that this is the subring of $\mathbb{F}_q(1,2,\dots;n)$ generated by the $\pi_{i_1\dots i_s}$ with $s = k$. In particular the commutation relations 3) become the trivial statement of the form $AB = AB$.

3) $R = \mathbb{F}_1(1,2,\dots;n)$ is the ring of the Plücker coordinates of the flag variety and we associate to it the projective scheme $Flag(R)$ which is the set of prime multihomogeneous ideals of R exept those which are left fixed under the $\mathbb{C}^\times \times \dots \times \mathbb{C}^\times$ action given by $(\pi_{I_1},\pi_{I_2},\dots,\pi_{I_{n-1}}) \mapsto (\lambda_1\pi_{I_1},\lambda_2\pi_{I_2},\dots,\lambda_{n-1}\pi_{I_{n-1}})$, where I_j is a j-multi-index. Similarly to the case of $Proj(S)$ (see [H] pag. 121 and 122), we define the multigraded R-module $\Gamma_*(\mathcal{F})$ associated to a sheaf $\mathcal{F}$ over $Flag(R)$ as the direct sum of the modules of sections of the twisted sheaves $\mathcal{F}(\delta)$, where $\delta \in \mathbb{Z}^{n-1}$. By using the classical theory of Borel-Weil-Bott one shows that, for $\mathcal{R}$ the sheaf over $Flag(R)$ associated to R, $\Gamma_*(\mathcal{R}) = R$.

Examples. 1) For $Sl(2)_q$ we have $\mathbb{P}^1_q = \mathbb{C}_q[T_{11},T_{21}]$, with $T_{11}T_{21} = qT_{21}T_{11}$.

2) For $Sl_q(3)$ the two fundamental comodules are

$$V^1_q = \mathbb{C}_q\{\pi_1,\pi_2,\pi_3\} = \mathbb{C}_q\{T_{11},T_{21},T_{31}\}$$

$$V^{12}_q = \mathbb{C}_q\{\pi_{12},\pi_{13},\pi_{23}\} = \mathbb{C}_q\{\hat{T}_{33},\hat{T}_{23},\hat{T}_{13}\}$$

and the rings $\mathbb{P}^2_q$, $\mathbb{G}_q(2,3)$ are quantum planes (with respect to the choosen ordering of the generators) i.e.

$$\pi_i\pi_j = q\pi_j\pi_i, \quad (i<j),$$

$$\pi_{ij}\pi_{hk} = q\pi_{hk}\pi_{ij} \quad (ij<hk),$$

where the last inequality is w.r.t. the lexicographic ordering $12 < 13 < 23$. The additional commutation relations for the flag manifold $\mathbb{F}_q(1,2;3) = \mathbb{C}_q[V^1_q \oplus V^{12}_q]$ are easily got by direct computation and read

$$\pi_i\pi_{hk} = \pi_{hk}\pi_i, \quad (i = h \text{ or } i = k)$$

$$\pi_1\pi_{23} = q\pi_{23}\pi_1, \qquad \pi_3\pi_{12} = q^{-1}\pi_{12}\pi_3$$

$$\pi_2\pi_{13} = q^{-1}\pi_{13}\pi_2 + \lambda\pi_{12}\pi_3.$$

Moreover there is the q-analogue of the incidence relation

$$\pi_1\pi_{23} - q\pi_2\pi_{13} + q^2\pi_3\pi_{12} = 0,$$

coming from the fact that the l.h.s. equals $|^{112}_{123}|_q$ and therefore vanishes identically.
3) With a little algebra on the presentation given by [S] or [TT], the ring $\mathbb{G}_q(2,4)$ can be explicitely described as generated over $\mathbb{C}_q$ by $\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{24}, \pi_{34}$ with the following relations

$$\begin{aligned}
\pi_{ij}\pi_{hk} &= q\pi_{hk}\pi_{ij} \\
\pi_{14}\pi_{23} &= \pi_{23}\pi_{14} \\
\pi_{12}\pi_{34} &= q^2\pi_{34}\pi_{12} \\
[\pi_{13}, \pi_{24}] &= \lambda\pi_{14}\pi_{23} \\
\pi_{12}\pi_{34} - q\pi_{13}\pi_{24} + q^2\pi_{14}\pi_{23} &= 0,
\end{aligned}$$

where $\{i,j,h,k\} \neq \{1,2,3,4\}$ and $ij < hk$ with respect to the lexicographic ordering $12 < 13 < 14 < 23 < 24 < 34$. The last relation is the q-analogue of the classical Plücker relation and it is identically satisfied (for $q^2 \neq -1$) because $0 = |^{1212}_{1234}|_q = (1+q^2)(\pi_{12}\pi_{34} - q\pi_{13}\pi_{24} + q^2\pi_{14}\pi_{23})$.
Remark. The last example rises a question as to the quantum analogue of the Plücker embedding. Indeed, forgetting the q-Plücker relation, the ring generated by the π_{ij} above is by no means isomorphic to the quantum projective space $\mathbb{P}^5_q$.

3. $Sl_q(n)$ at roots of unity

Consider for a moment $Sl_q(2)$. As well known, one can actually reconstruct the ordinary projective line with the quantum Zariski topology out of $\mathbb{C}_q[a,c]$ (see e.g. [S], p. 142.) Another way in which geometry plays a rôle is at roots of unity. Before entering this theme, we need to recall some computational facts. One defines p-integers, factorials and binomial coefficient as

$$(l)_p = 1 + p + p^2 + \ldots + p^{l-1}, \qquad (l)_p! = (l)_p(l-1)_p\ldots(2)_p,$$

$$\binom{l}{k}_p = \frac{(l)_p!}{(k)_p!(l-k)_p!} = \frac{(l)_p(l-1)_p\ldots(l-k+1)_p}{(k)_p!}$$

and $\binom{l}{0}_p = \binom{l}{l}_p = 1$. Notice the identities $\binom{l}{k}_p = \binom{l}{l-k}$ and $(mn)_p = (m)_p(n)_{p^m} = (m)_{p^n}(n)_p$. For every l one has that $(l)_\eta = 0$, when η is a primitive l-root of unity, because $1, \eta, \eta^2, \ldots, \eta^{l-1}$ is the whole of the cyclic group μ_l and the roots of unity are symmetrically distributed around the origin in $\mathbb{C}$.
Definition. *We say that $(l,\eta) \in \mathbb{N} \times \mathbb{C}^*$ is a special couple if either $\eta^l = 1$ (l odd) or $\eta^l = -1$ (l even) and η is primitive.*

The relevance of p-binomial coefficients comes from the following simple lemmas which are easily proved by using induction arguments
Lemma 3.1 *Let $AB = pBA$. Then $(A+B)^l = \sum_{k=0}^{l}\binom{l}{k}_p B^k A^{l-k}$. Let $p = q^2$. If (l,η) is special, then at $q = \eta$ for all $1 \leq k \leq l-1$, $\binom{l}{k}_p = 0$ and $(A+B)^l = A^l + B^l$.*

Lemma 3.2 *Let A, D, W satisfy the commutation rules $AD = DA + \lambda W$, $AW = pWA$, $DW = p^{-1}WD$ with $\lambda = q - q^{-1}$, $p = q^2$. Then:*
1) $AD^k = D^k A + \lambda(k)_p D^{k-1} W$
2) $A^k D = DA^k + \lambda(k)_p W A^{k-1}$
3) $(AD - qW)^l = \sum_{k=1}^{l} (-q)^k \binom{l}{k}_p A^{l-k} D^{l-k} W^k$
4) if (l, η) is special then $(AD - qW)^l = A^l d^l - W^l$ at $q = \eta$.

Moreover

Lemma 3.3 *Let $\hat{T}_{r1}$ denote the q-algebraic complement of the element T_{r1} in the matrix T. Then: $det_q T = \sum_k (-q)^{k-1} T_{k1} \hat{T}_{k1}$ and the following commutation rules hold*
1) $T_{r1}\hat{T}_{s1} = q\hat{T}_{s1}T_{r1}$ $(r \neq s)$
2) $T_{r1}\hat{T}_{r1} = \hat{T}_{r1}T_{r1} + \lambda W_r$
3) $T_{k1}W_k = q^2 W_k T_{k1}, \quad \hat{T}_{k1}W_k = q^{-2} W_k \hat{T}_{k1}$.
where $\lambda = q - q^{-1}$ and $W_r = \sum_{k=r+1}^{n} (-q)^{k-r-1} T_{k1}\hat{T}_{k1}$.

This is what we mean for a quantum group at a root of unity: let (l, η) be a special couple. It is known that $\mathbb{C}_q = \mathbb{C}[q, q^{-1}]$ is the Hopf algebra of the affine group $Gl(1, \mathbb{C}) = \mathbb{C}^\times$. Now, evaluating Laurent polynomials at $q = \eta$ gives an exact sequence

$$0 \to \mathrm{m}_\eta \to \mathbb{C}_q \stackrel{ev(\eta)}{\to} \mathbb{C} \to 0,$$

i.e. the kernel of the evaluation map at η is the maximal ideal $\mathrm{m}_\eta =< q - \eta >$. Accordingly, the quantum group at the root of unity η, $Sl_\eta(n)$, is the localization at m_η of $Sl_q(n)$, i.e. $Sl_\eta(n) = Sl_q(n)_{\mathrm{m}_\eta}$, and the same is valid for the quantum homogeneous spaces.

We can now show:

Proposition. *Let (l, η) be a special couple. Then, at $q = \eta$, T^l_{ij} generate an abelian sub-Hopf algebra $H \subset Sl_\eta(n)$ with the standard comultiplication, counity and the coinverse*

$$S(T^l_{ij}) = (-1)^{j-i} \hat{T}^{(l)}_{ji},$$

isomorphic to the coordinate ring of $Sl(n)$. Here $\hat{T}^{(l)}_{ij}$ is the algebraic complement of the ij element in the matrix $T^{(l)}$ with matrix elements T^l_{ij}.

Proof. The fact that T^l_{ij} commute follows from lemmas 3.1, and 3.2. As for the comultiplication, we have

$$\Delta(T^l_{ij}) = (\Delta(T_{ij}))^l = \left(\sum_{k=1}^{n} T_{ik} \otimes T_{kj} \right)^l$$

but, for $k_1 < k_2$, $(T_{ik_1} \otimes T_{k_1 j})(T_{ik_2} \otimes T_{k_2 j}) = \eta^2 (T_{ik_2} \otimes T_{k_2 j})(T_{ik_1} \otimes T_{k_1 j})$ so we can use lemma 3.1 to get the result. Then lemma 3.3 is used to show by induction that $(det_q T)^l = det(T^{(l)})$, thus $det(T^{(l)}) = 1$. Since the counity (coinverse) is an algebra (anti-) homomorphism, one has that $\epsilon(T_{ij})^l = (\epsilon(T_{ij}))^l = \delta_{ij}$ and $S(T^l_{ij}) = (S(T_{ij}))^l = (-\eta)^{l(j-i)} (\hat{T}_{ij})^l = (-1)^{j-i} (\hat{T}_{ij})^l$. Now every $k \times k$ submatrix $\tilde{T}$ of T generates a subalgebra isomorphic to $M_\eta(k)$ so the l-th power of its q-determinant is nothing but the determinant of the matrix $\tilde{T}^{(l)}$. This then implies that $S(T^l_{ij}) = (-1)^{j-i} \hat{T}^{(l)}_{ji}$ and completes the proof.

Remarks

1) The subalgebra H is central exactly when l is odd.
2) By use of lemmas like 3.1 and 3.2 one can show that most of the representations obtained from the quantum spaces are neither irreducible nor indecomposable.
3) During the proof of the proposition we have showed that

$$\left(|^{i_1\dots i_k}_{j_1\dots j_k}|_\eta\right)^l = |^{i_1\dots i_k}_{j_1\dots j_k}|^{(l)}_1$$

with obvious notations. This immediately implies that the l-th powers of the generators of the quantum spaces (see section 2), commuting with each other, generate a subalgebra isomorphic to the ring of homogeneous coordinates of the classical spaces, but generated by elements of degree l. This result is analogous to that for the quantum group itself.

The last remark, together with the proposition, assures that the relevant algebras of the theory of quantum groups are indeed polynomial identity rings when the deformation parameter is a primitive root of unity (see e.g. Corollary 13.1.13 of [McCR]). Let us go back to the quantum flag variety $R = \mathbb{F}_\eta(k_1,\dots,k_s;n)$, and write the generators of R as

$$\pi_{I_j} = |^{1\dots k_j}_{i_1\dots i_{k_j}}|_\eta$$

for I_j the multi-index $I_j = \{i_1,\dots,i_{k_j}\}$, $j \leq s$. Consider the abelian subring

$$A = \mathbb{C}[x_{I_j}] \subset R$$

where $x_{I_j} = \pi^l_{I_j}$. This is isomorphic to the homogeneous coordinate ring of the classical flag variety $\mathbb{F}(k_1,\dots,k_s;n)$, but we have to take care of the gradings. The grading inherited from $Sl_\eta(n)$ gives troubles even classically: since $deg(\pi_{I_j}) = k_j$, $Flag(\mathbb{F}_1(k_1,\dots,k_s;n))$ is not a projective scheme because the ring is not generated by the component of degree one. This is easily taken care of by redefining the grading, $deg'(\pi_{I_j}) = (0,\dots,1_j,\dots,0)$ (1 in the j-th position), and is what we did for the quantum flag variety.

Now A is generated by elements of multidegree $(0,\dots,l_j,\dots,0)$ and we can redefine the grading by setting $A' = A$ as a ring but $l \cdot deg'(x_{I_j}) = deg(x_{I_j})$. However R is non graded as an A'-module and hence it does not correspond to anything over $Flag(A')$. There are actually three possibilities:
1) leave everything as it stands and work over $Flag(A)$,
2) "enlarge" A to $\tilde{A}$ by "filling in" the missing degrees,
3) forget the gradings and work over $Spec(A)$.
We shall concentrate only on the first two of them.

1) As we know R is a graded algebra and also a finitely generated graded module over A. Therefore it corresponds to a sheaf of algebras $\mathcal{R}$ over $Flag(A)$ and we would like to see if $R = \mathbb{F}_\eta(k_1,\dots,k_s;n) = \Gamma_*(\mathcal{R}) := \oplus_{\delta\in\mathbb{N}^s} H^0(Flag(A),\mathcal{R}(\delta))$.

2) The fact that A is generated by elements of degree $(0,\dots,l_j,\dots,0)$ tells us that $Flag(A)$ comes together with a covering $\tilde{\mathbb{F}}(k_1,\dots,k_s;n) \to Flag(A)$. In terms of graded rings, this corresponds to the morphism $A \to \tilde{A}$ given by $x_{I_j} \mapsto z^l_{I_j}$ with $\tilde{A} = \mathbb{C}[z_{I_j}]$. In particular the

covering is branched at the locus where one of the Plücker coordinates of the flag variety is zero. Considering R as an A-module, we extend the scalars getting an $\tilde{A}$-module

$$\tilde{R} = \tilde{A} \otimes_A R$$

Now $\tilde{A}$ is generated by $\oplus_j \tilde{A}_{(0\ldots 1_j \ldots 0)}$ as an $\tilde{A}_0 = \mathbb{C}$ algebra, so the scheme associated to $\tilde{A}$ is actually projective. The injective homomorphism $\tilde{A} \to \tilde{R}$ given by $z_{I_j} \mapsto z_{I_j} \otimes 1$ is obviously graded and makes $\tilde{R}$ into a graded $\tilde{A}$-algebra. In other words, $\tilde{R}$ is the algebra generated over $\tilde{A}$ by $1, \pi_{I_j}$ with relations $\pi_{I_j}^l = z_{I_j}^l$, the commutation relations and the q-incidence relations. The grading is given by $|\pi_{I_j}| = |z_{I_j}| = 1$.

As an $\tilde{A}$-algebra $\tilde{R}$ corresponds to a sheaf of rings $\tilde{\mathcal{R}}$ over $\tilde{\mathbb{F}}(k_1, \ldots, k_s; n)$, moreover it easy to show that one regains the indecomposable representations of $Sl_q(n)$ as the spaces of sections of the twists of this sheaf which are invariant with respect to the deck transformations of the covering, that is $\Gamma_*^{inv.}(\tilde{\mathcal{R}}) = \mathbb{F}_\eta(k_1, \ldots, k_s; n)$.

Examples. In order to clearify the subject we look at the simplest example: the quantum group $Sl_\eta(2)$. Let us simplify notations by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} =: \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

First of all, by lemma 3.1 we have that $\Delta(a^l) = a^l \otimes a^l + b^l \otimes c^l$, $\Delta(c^l) = c^l \otimes a^l + d^l \otimes c^l$ and this implies that the representation of $Sl_\eta(2)$ of degree l is not any more irreducible, indeed it contains the subrepresentation spaned by a^l and c^l. It is also easy to show that it is not decomposable. More generally, one proves that all the representations of degree kl are reducible and indecomposable.

The quantum homogeneous space is the quantum projective line $\mathbb{P}^1_\eta$, that is the ring $R = \mathbb{C}[a, c]_\eta$ where the subscript η means that the commutation relation is $ac = \eta ca$.

As discussed above, a^l, b^l, c^l and d^l generate an abelian subring isomorphic to the coordinate ring of $Sl(n)$, while $w_0 = a^l$ and $w_1 = c^l$ generate a subring $A = \mathbb{C}[w_0, w_1] \subset R$ isomorphic to the homogeneous coordinates ring of the classical projective line . A inherits a gradation by $\mathbb{N}$ (i.e. $A = \oplus_{n \in \mathbb{N}} A_n$ with $A_n = 0$ if $n \neq 0$ $(modl)$). Let us take the first point of view and look at the stalks $\mathcal{R}_{(\mathfrak{p}_i)}$ of the sheaf $\mathcal{R}$ over $Proj(A)$ which corresponds to $\mathbb{P}^1_\eta$ at the homogeneous prime ideals $\mathfrak{p}_i =< w_i > \in Proj(A)$. The multiplicative sets $T_i = A \backslash \mathfrak{p}_i$ are isomorphic to $\mathbb{C}[w_j]$, $(i \neq j)$. The stalk $\mathcal{R}_{\mathfrak{p}_i}$ being the module of elements of degree 0 in $T_i^{-1} R$ are given by:

$$\mathcal{R}_{\mathfrak{p}_0} = A_{\mathfrak{p}_0}\{1, a^{l-1}c^{-(l-1)}, \ldots, ac^{-1}\}, \quad \mathcal{R}_{\mathfrak{p}_1} = A_{\mathfrak{p}_1}\{1, a^{-1}c, \ldots, a^{-(l-1)}c^{l-1}\}$$

with $A_{\mathfrak{p}_i} = \mathbb{C}[w_i/w_j]$ $(j \neq i)$. Since $a^{l-k}c^{-(l-k)}$ is proportional to $(w_0/w_1)a^{-k}c^k$, we see that the only global sections of $\mathcal{R}$ are the constants. Notice also that the stalks above are isomorphic to the group algebra of the cyclic group μ_l over the local rings $A_{\mathfrak{p}_i}$. More precisely, let $\hat{\mu}_l(\alpha)$ be the group algebra of the cyclic group generated by α, with $\alpha^l = 1$, then

$$\mathcal{R}_{\mathfrak{p}_0} \simeq A_{\mathfrak{p}_0} \otimes_{\mathbb{C}} \hat{\mu}(ac^{-1}), \quad \mathcal{R}_{\mathfrak{p}_1} \simeq A_{\mathfrak{p}_1} \otimes_{\mathbb{C}} \hat{\mu}(a^{-1}c).$$

Let us next twist $R = \mathbb{P}^1_\eta$ by s, so that $R(s)_d = R_{d+s} = \mathbb{C}\{a^{d+s-k}c^k,\ (0 \le k \le d+s)\}$. Since $T_i = \mathbb{C}[w_j]$ is generated in degree l only the modules $R(s)_d$ with d multiple of l will contribute to the localizations. Now

$$R(s)_{kl} = \mathbb{C}\{a^{kl+s-h}c^h,\ (0 \le h \le kl+s)\} = \mathbb{C}\{a^{h'}c^{kl+s-h'},\ (0 \le h' \le kl+s)\}$$

and therefore

$$\mathcal{R}(s)_{\mathfrak{p}_0} = \mathbb{C}[w_0/w_1]\{a^{h'}c^{s-h'},\ (0 \le h' \le l-1)\},$$
$$\mathcal{R}(s)_{\mathfrak{p}_1} = \mathbb{C}[w_1/w_0]\{a^{s-h}c^h,\ (0 \le h \le l-1)\}.$$

It follows that the space $H^0(Proj(A), \mathcal{R}(s))$ of global sections of $\mathcal{R}(s)$ is the span of the monomials $a^h c^{s-h}$, $0 \le h \le s$. The interest of the sheaf $\mathcal{R}$ comes from the following fact: let $\mathcal{R}$ be the sheaf over $Proj(A)$ corresponding to the graded A-module P^1_η, then

$$\Gamma_*(\mathcal{R}) =: \oplus_s H^0(Proj(A), \mathcal{R}(s)) = \mathbb{P}^1_\eta.$$

Notice that A is a homogeneous comodule for $\mathbb{C}[Sl(2)]$ and hence $Proj(A)$ is a homogeneous scheme. However the action of $Sl(2)$ over $Proj(A)$ does not lift to an action on $\mathcal{R}$.

The other point of view is that our $Proj(A)$ comes together with a covering $\tilde{\mathbb{P}}^1 \to Proj(A)$ of degree l. In terms of graded rings, this corresponds to the morphism $A \to \tilde{A}$ given by $w_i \mapsto z_i^l$ with $\tilde{A} = \mathbb{C}[z_0, z_1]$ and $\tilde{\mathbb{P}}^1 = Proj(\tilde{A})$. In particular the covering is branched at zero and infinity. The extended algebra $\tilde{R}$ defined by

$$\tilde{R} = \tilde{A} \otimes_A R$$

is generated over $\tilde{A}$ by $1, a, c$ with relations $a^l = z_0^l$, $c^l = z_1^l$, $ac = \eta ca$, and as an $\tilde{A}$-module it corresponds to a sheaf $\tilde{\mathcal{R}}$ over $\tilde{\mathbb{P}}^1 = Proj(\tilde{A})$. We work out the exercise of constructing the stalks of $\tilde{\mathcal{R}}$ at the maximal ideals $(z_i) \in Proj(\tilde{A})$, $(i = 0,1)$. The multiplicative set $T_i = \tilde{A} - (z_i)$ of homogeneous elements not in (z_i) is the polynomial ring $\mathbb{C}[z_j] \subset \tilde{A}$ with $j \ne i$. The stalk $\tilde{\mathcal{R}}_i$ at (z_i), being the group of elements of degree 0 in the localization $T_i^{-1}\tilde{R}$, inherits from $\tilde{R}$ an algebra structure. To simplify notations, let $x = z_0/z_1$ and concentrate on $\tilde{\mathcal{R}}_0$. Recall that $\mathbb{C}[x]$ is a local ring with maximal ideal (x) and residue field $\mathbb{C}$. As $\mathbb{C}[x]$-module $\tilde{\mathcal{R}}_0$ is finitely generated, i.e.

$$\tilde{\mathcal{R}}_0 = \mathbb{C}[x]\{(a/z_1)^m(c/z_1)^n,\ (m,n) \in \mathbb{Z}_l \times \mathbb{Z}_l\}.$$

The ring $\mathbb{P}^1_\eta$ is the subring of $\Gamma_*(\mathcal{R})$ given by the sections which are invariant under the deck trasformations, so this time we have a sheaf over $\mathbb{P}^1$ but the group of automorphisms must be restricted to those which commute with the covering group.

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