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An Elementary Approach to the Rational Solutions of the KP Hierarchy

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Abstract

We construct in a fairly elementary and simple way the rational solutions of the KP hierarchy. Our starting point is a geometric approach to soliton equations, which relies on the concept of a bihamiltonian system. As a consequence, we establish a Wronskian formula for the polynomial τ-functions of the KP hierarchy. This formula, though known in the literature, is obtained in a very direct way.

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1 Introduction

Among the many remarkable properties of soliton equations one finds the existence of rational (with respect to the space variable x) solutions. It was discussed for the Korteweg-de Vries (KdV) equation as well as for the Kadomtsev-Petviashvili (KP) equation in many places, e.g., [1, 2, 3, 6, 12, 13, 15]. It was also observed that the poles of these solutions move according to finite-dimensional integrable Hamiltonian systems such as the Calogero-Moser hierarchy (see also the more recent papers [20] and [24]). Then, in the context of the Sato theory of soliton equations [19, 17], rational solutions of the KP *hierarchy* were seen as the solutions associated to polynomial τ-functions. These solutions are rational with respect to all times of the hierarchy, hence they represent very particular instances of rational solutions of the KP *equation*.

In this paper we recover the rational solutions of the KP hierarchy in a very elementary way using the framework introduced in [10], where a geometric approach to the linearization of the KP equations is presented. It starts from the geometry of bihamiltonian manifolds and arrives at a system of infinite-matrix Riccati equations, which are linearized by means of elementary methods. These equations already appeared in [22], where they are shown to describe linear flows on the Sato Grassmannian. For this reason, the above-mentioned system was called the Sato System in [10].

In this paper we explicitly determine the solutions of the Sato System when the initial condition has only a finite number of nonzero entries, and we show that these solutions give rise to the rational solutions of the KP hierarchy.

The plan for this article is the following: In Section 2 we recall how to pass from the KP hierarchy (in conservation-law form) to the Sato System, through another important object, which is the Central System. In Section 3 we review the linearization procedure for the Sato System and we add some information on the solution of the linearized system for arbitrary initial data. In Section 4 the initial condition is assumed to vanish outside the $m \times n$ upper (left) corner. Under those assumptions the Sato System can be explicitly solved, and the solutions can be expressed in terms of a polynomial τ-function. This is done in Section 5, and shows that they are the rational solutions of the KP hierarchy. Finally, an example is presented in Section 6.

2 The KP hierarchy, the Central System, and the Sato System

In this section (and at the beginning of the next one) we will recall some results from [10], where a description of the linearization of the KP hierarchy is given starting from the bihamiltonian approach to the KdV hierarchy. We refer to that paper for motivations,

details, and proofs.

First of all we introduce the KP equations in a form which, although equivalent, is different from the usual one — see, however, [5, 23]. Let $h(x,z) = z + \sum_{i\geq 1} h_i(x) z^{-i}$ be a Laurent series whose coefficients h_i belong to a given space $\mathcal{F}$ of C^∞-functions. Let us consider the Faà di Bruno iterates of h, defined as

$$h^{(j+1)} = \partial_x h^{(j)} + h h^{(j)}, \qquad h^{(0)} = 1. \tag{2.1}$$

They span a subspace $H_+ := \langle h^{(j)} \rangle_{j\geq 0}$ which is complementary to $H_- := \langle z^j \rangle_{j<0}$. Let π_+ and π_- be the corresponding projections, and let us call

$$H^{(k)} := \pi_+(z^k) \tag{2.2}$$

the *KP currents.* They are the unique Laurent series in H_+ having the asymptotic expansion $H^{(k)} = z^k + O(z^{-1})$.

Definition 2.1 *The KP equations are the equations*

$$\frac{\partial h}{\partial t_k} = \partial_x H^{(k)} \tag{2.3}$$

for the Laurent series h.

We refer to [10] for the relation with the usual description of the KP equations, in the language of pseudodifferential operators (see, e.g., [7, 9]).

Now we want to tackle the following problem: when h evolves according to the k-th KP flow, how do the currents $H^{(j)}$ evolve? The answer is given by

Proposition 2.2 *The KP currents $H^{(j)}$ evolve in such a way that*

$$\left(\frac{\partial}{\partial t_k} + H^{(k)} \right) (H_+) \subset H_+. \tag{2.4}$$

Explicitly, one has

$$\frac{\partial H^{(j)}}{\partial t_k} + H^{(k)} H^{(j)} = H^{(k+j)} + \sum_{l=1}^{k} H_l^j H^{(k-l)} + \sum_{l=1}^{j} H_l^k H^{(j-l)}. \tag{2.5}$$

It is important to observe that we can forget about the fact that the currents $H^{(i)}$ are constructed from h; then equations (2.5) define a hierarchy of vector fields in the space $\mathcal{H}$ of sequences $\{H^{(i)}\}_{i\geq 0}$ of Laurent series of the form $H^{(i)} = z^i + \sum_{l\geq 1} H_l^i z^{-l}$ (i.e., the space of $\mathbb{N} \times \mathbb{N}$ matrices).

Definition 2.3 *The hierarchy defined by equations (2.5) is called the Central System (CS).*

It can be shown that the CS flows commute. Another important point is that the CS is a family of *dynamical systems* in $\mathcal{H}$ (i.e., systems of ordinary differential equations), since the currents $H^{(i)}$ are no more supposed to depend on the space variable x.

Remark 2.4 The Central System admits some interesting reductions, described in [4]. For instance, one can obtain the fractional KdV hierarchies [8] and the stationary reductions of the Gelfand-Dickey equations [9] (see the paper by G. Falqui, F. Magri, and G. Tondo in these proceedings). In particular, the KP hierarchy can be seen as the projection of CS on the space of the solutions of the first CS flow.

We will show now that the solutions of CS can be obtained by the solutions of another system (the *Sato System*), which is easily linearizable. It is defined in the same phase-space of CS, but we will use different notations for the point, since the two systems are different. Hence we consider sequences $\{W^{(i)}\}_{i\geq 0}$ of Laurent series $W^{(i)} = z^i + \sum_{l\geq 1} W^i_l z^{-l}$, we introduce the subspace $W_+ := \langle W^{(i)}\rangle_{i\geq 0}$, and we define the Sato System as

$$\left(\frac{\partial}{\partial t_k} + z^k\right)(W_+) \subset W_+. \tag{2.6}$$

Explicitly, it reads

$$\frac{\partial W^{(j)}}{\partial t_k} + z^k W^{(j)} = W^{(k+j)} + \sum_{l=1}^{k} W^j_l W^{(k-l)}, \tag{2.7}$$

or

$$\frac{\partial W^j_m}{\partial t_k} + W^j_{k+m} - W^{k+j}_m = \sum_{l=1}^{k} W^j_l W^{k-l}_m. \tag{2.8}$$

Takasaki has shown ([22], p.29) that these equations represent linear flows on the Sato Grassmannian in suitable coordinates. He also showed [21] the connection with the pseudodifferential operators formulation of the KP hierarchy. The key point is that $W^{(0)}$ is the dressing operator, i.e., $W^{(0)} \exp \sum_{i\geq 1} t_i z^i$ is the Baker-Akhiezer function, see, e.g., [7, 16].

The relation with the Central System (2.5) is given by

Proposition 2.5 *Let $\mu : \mathcal{H} \to \mathcal{H}$ be the map defined as*

$$\{W^{(i)}\} \mapsto \left\{ H^{(i)} = \sum_{l=0}^{i} W^0_{i-l} W^{(l)}/W^{(0)} \right\}. \tag{2.9}$$

Then μ sends solutions of the Sato System into solutions of the Central System satisfying the constraint $H^{(0)} = 1$.

This result can be cast in the context of Darboux coverings, a concept introduced in [14] to study Darboux transformations (especially for the KP hierarchy).

In conclusion, we have reduced the study of the KP equations to the one of the Riccati equations (2.8). Once a solution of (2.8) is known, the corresponding solution of the KP equations (2.3) is simply given by $h(x;t) = H^{(1)}(x+t_1, t_2, \dots)$, where

$$H^{(1)} = (W^{(1)} + W_1^0 W^{(0)})/W^{(0)} = z + \frac{\partial}{\partial t_1} \log W^{(0)}, \tag{2.10}$$

the last equality being nothing but equation (2.7) for $j = 0$ and $k = 1$.

3 Linearization of the Sato System

Thanks to the results of the previous section, we can focus on the Sato System. Equations (2.8) can be written in matrix form as

$$\frac{\partial \mathcal{W}}{\partial t_k} + \mathcal{W} \cdot {}^T\Lambda^k - \Lambda^k \cdot \mathcal{W} = \mathcal{W}\Gamma_k\mathcal{W}, \tag{3.1}$$

where $\mathcal{W} = [W_m^j]_{j\geq 0, m\geq 1}$ and

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots \\ \vdots & & & & \ddots \end{bmatrix}, \qquad \Gamma_k = \begin{bmatrix} 0 & \cdots & & 1 & 0 & \cdots \\ \vdots & & 1 & 0 & \cdots & \cdots \\ & \cdot & \cdot & & & \\ 1 & 0 & & & & \\ \vdots & & & & & \end{bmatrix} \tag{3.2}$$

Equation (3.1) is a matrix Riccati equation, which is known [18] to be (formally) linearized by putting $\mathcal{W} = \mathcal{V} \cdot \mathcal{U}^{-1}$. Indeed, (3.1) is equivalent to the constant coefficients linear system

$$\begin{cases} \dfrac{\partial \mathcal{U}}{\partial t_k} = {}^T\Lambda^k \mathcal{U} - \Gamma_k \mathcal{V} \\ \dfrac{\partial \mathcal{V}}{\partial t_k} = \Lambda^k \mathcal{V} \end{cases} \tag{3.3}$$

The second equation is immediately solved, for all k, by

$$\mathcal{V}(t) = \exp(\sum_{i\geq 1} t_i \Lambda^i) C = \sum_{i\geq 0} p_i(t) \Lambda^i \cdot \mathcal{C}, \tag{3.4}$$

where $t = (t_1, t_2, \dots)$, the matrix $\mathcal{C}$ is constant, and the Schur polynomials $\{p_i(t)\}_{i\in\mathbb{Z}}$ are defined, as usual, by $\exp(\sum_{i\geq 1} t_i z^i) = \sum_{i\in\mathbb{Z}} p_i(t) z^i$ (clearly, $p_i(t) = 0$ for $i < 0$). Then we are left with the following equation for $\mathcal{U}$:

$$\frac{\partial \mathcal{U}}{\partial t_k} = {}^T\Lambda^k \mathcal{U} - \Gamma_k \cdot \sum_{i\geq 0} p_i(t)\Lambda^i \cdot \mathcal{C}. \tag{3.5}$$

After putting

$$\mathcal{U} = \exp(\sum_{i\geq 1} {t_i}^T \Lambda^i)(\widehat{\mathcal{U}}\mathcal{C} + \mathcal{D}), \tag{3.6}$$

where $\mathcal{D}$ is another constant matrix, equation (3.5) transforms into

$$\frac{\partial \widehat{\mathcal{U}}}{\partial t_k} = -\Omega_k, \tag{3.7}$$

where $\Omega_k = \exp(\sum_{i\geq 1} {t_i}^T \Lambda^i) \cdot \Gamma_k \cdot \exp(\sum_{i\geq 1} t_i \Lambda^i) = (\sum_{i\geq 0} {\tilde{p}_i}^T \Lambda^i) \cdot \Gamma_k \cdot \sum_{i\geq 0} p_i \Lambda^i$, and $\tilde{p}_i(t) = p_i(-t)$ (or, equivalently, $\{\tilde{p}_i(t)\}_{i\in\mathbb{Z}}$ are defined by $\exp(-\sum_{i\geq 1} t_i z^i) = \sum_{i\in\mathbb{Z}} \tilde{p}_i(t) z^i$). Taking into account the form of Γ_k, it is quite easy to show that

$$\Omega_k = \begin{bmatrix} \tilde{p}_{1-k} & \tilde{p}_{2-k} & \cdots & \tilde{p}_0 & 0 & \cdots \\ \tilde{p}_{2-k} & \tilde{p}_{3-k} & \cdots & \tilde{p}_1 & 0 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \end{bmatrix} \exp(\sum_{i\geq 1} t_i \Lambda^i). \tag{3.8}$$

Then, using the fact that $\dfrac{\partial \tilde{p}_k}{\partial t_j} = -\tilde{p}_{k-j}$, one immediately proves

Proposition 3.1 *The solution of equation (3.7) satisfying the initial condition $\widehat{\mathcal{U}}(0) = 0$ is*

$$\widehat{\mathcal{U}}(t) = \widetilde{\mathcal{H}} \cdot \exp(\sum_{i\geq 1} t_i \Lambda^i), \tag{3.9}$$

where $\widetilde{\mathcal{H}}$ is the Hankel matrix: $\widetilde{\mathcal{H}}_{ij} = \tilde{p}_{i+j-1}$, for $i, j \geq 1$.

We observe that, although (3.9) is a product of infinite matrices, every entry of $\widehat{\mathcal{U}}(t)$ can be explicitly computed, thanks to the special form of $\exp(\sum_{i\geq 1} t_i \Lambda^i) = \sum_{i\geq 0} p_i(t)\Lambda^i$. Of course, the general solution of equation (3.7) is found by adding a constant matrix to (3.9), but this simply changes the constant matrix $\mathcal{D}$ into the expression

$$\mathcal{U}(t) = \exp(\sum_{i\geq 1} {t_i}^T \Lambda^i)(\widetilde{\mathcal{H}} \cdot \exp(\sum_{i\geq 1} t_i \Lambda^i)\mathcal{C} + \mathcal{D}). \tag{3.10}$$

This matrix cannot be explicitly computed (and inverted) unless $\mathcal{C}$ and $\mathcal{D}$ (i.e., the initial conditions) are carefully chosen. In the next section we will show that this can be done if $\mathcal{W}(t_0)$ (where $t_0 = (t_{10}, t_{20}, \dots)$ is the vector of "initial times") has only a finite number of nonzero entries.

4 The finite-dimensional case

In this section we will write explicitly the general solution of the Sato System in the case where the initial condition $\mathcal{W}(t_0)$ satisfies

$$W_i^j(t_0) = 0 \qquad \text{for all } i > n,\ j \geq m. \tag{4.1}$$

In Section 5 we will show that in this way we obtain the rational solutions of the KP hierarchy.

First of all we observe that (4.1) implies

$$\frac{\partial W_i^j}{\partial t_k}(t_0) = 0 \qquad \text{for all } i > n,\ j \geq m; \tag{4.2}$$

this means that the flows of the Sato System are tangent to the subspace $\mathbb{W}_{m,n}$ of the matrices $\mathcal{W}$ such that $W_i^j = 0$ for $i > n$, $j \geq m$, and therefore restrict to $\mathbb{W}_{m,n}$. Hence equations (3.1) reduce to Riccati equations for *finite* matrices,

$$\frac{\partial \mathcal{W}_{m,n}}{\partial t_k} + \mathcal{W}_{m,n} \cdot \left({}^T\Lambda_{n,n}\right)^k - \left(\Lambda_{m,m}\right)^k \cdot \mathcal{W}_{m,n} = \mathcal{W}_{m,n} \left(\Gamma_k\right)_{n,m} \mathcal{W}_{m,n}, \tag{4.3}$$

where $\mathcal{M}_{m,n}$ denotes the $m \times n$ matrix obtained from the infinite matrix $\mathcal{M}$ by taking its $m \times n$ upper corner. *En passant*, we remark that from the matrix form (3.1) of the Sato System it also follows that

$$\mathcal{W} \in \mathbb{W}_{m,n} \Longrightarrow \frac{\partial \mathcal{W}}{\partial t_k} = 0 \qquad \text{for all } k \geq m+n, \tag{4.4}$$

so that the solution $\mathcal{W}(t)$ of the restriction to $\mathbb{W}_{m,n}$ of the Sato System depends only on times $(t_1, t_2, \cdots, t_{m+n-1})$.

For simplicity of notations, let us fix m and n, and let us redefine the matrices appearing in equation (4.3) as

$$\mathcal{W}_{m,n} = W, \quad \Lambda_{m,m} = A, \quad {}^T\Lambda_{n,n} = B, \quad (\Gamma_k)_{n,m} = G_k. \tag{4.5}$$

Then equation (4.3) takes the form

$$\frac{\partial W}{\partial t_k} + WB^k - A^k W = WG_k W \tag{4.6}$$

and can still be linearized (as we did in the previous section for the infinite case) by setting $W = VU^{-1}$, where V is an $m \times n$ matrix and U is an invertible $n \times n$ matrix, satisfying

$$\begin{cases} \dfrac{\partial U}{\partial t_k} = & B^k U - G_k V \\ \dfrac{\partial V}{\partial t_k} = & A^k V \end{cases} \tag{4.7}$$

It is not difficult to show that the general solution of this system is obtained by truncating the matrices $\mathcal{U}(t)$ and $\mathcal{V}(t)$ of Section 3, i.e., that

$$V(t) = (\mathcal{V}(t))_{m,n} = \exp(\sum_{i=1}^{m-1} t_i A^i)C = \sum_{i=0}^{m-1} p_i(t) A^i C \tag{4.8}$$

$$U(t) = (\mathcal{U}(t))_{n,n} = \exp(\sum_{j=1}^{n-1} t_j B^j)(D + \widetilde{H} \exp(\sum_{i=1}^{m-1} t_i A^i)C), \tag{4.9}$$

where C and D are constant (respectively $m \times n$ and $n \times n$) matrices and $\widetilde{H} = \widetilde{\mathcal{H}}_{n,m}$ (i.e., $\widetilde{H}$ is the $n \times m$ matrix whose entries are $\widetilde{H}_{ij} = \tilde{p}_{i+j-1}$). A simpler form of $U(t)$ is obtained by means of

Lemma 4.1 *Let H be the $n \times m$ matrix defined by $H_{ij} = p_{i+j-1}$; then*

$$\exp(\sum_{j=1}^{n-1} t_j B^j)\widetilde{H} \exp(\sum_{i=1}^{m-1} t_i A^i) = -H. \tag{4.10}$$

Proof. Equation (4.10) is equivalent to $\widetilde{H} \exp(\sum_{i=1}^{m-1} t_i A^i) = -\exp(-\sum_{j=1}^{n-1} t_j B^j)H$, or

$$\widetilde{H} \sum_{i=0}^{m-1} p_i A^i = -\sum_{j=0}^{n-1} \tilde{p}_j B^j H. \tag{4.11}$$

If we compute the (i,j) entry of both members of this equation, we see that it is equivalent to the relations $\sum_{l=0}^{m-1} \tilde{p}_{i+l} p_{j-1-l} = -\sum_{r=0}^{n-1} \tilde{p}_{i-r-1} p_{j+r}$, that is, $\sum_{l=i}^{m-1+i} \tilde{p}_l p_{j+i-1-l} = -\sum_{r=i-n}^{i-1} \tilde{p}_r p_{j+i-1-r}$. But this follows from $\sum_{l=0}^{j+i-1} \tilde{p}_l p_{j+i-1-l} = 0$, which is a consequence of $\left(\sum_{k\geq 0} \tilde{p}_k z^k\right)\left(\sum_{h\geq 0} p_h z^h\right) = 1$.

□

Combining equations (4.9) and (4.10), we obtain the following expression for $U(t)$:

$$U(t) = \exp(\sum_{j=1}^{n-1} t_j B^j)D - HC = \sum_{j=0}^{n-1} p_j B^j D - HC. \tag{4.12}$$

At this point, in order to determine the corresponding solution $W(t)$ of the Sato System (restricted to $\mathbb{W}_{m,n}$), we should invert $U(t)$ and compute

$$W(t) = V(t)U(t)^{-1} = (\sum_{i=0}^{m-1} p_i(t) A^i)C(\sum_{j=0}^{n-1} p_j B^j D - HC)^{-1}. \tag{4.13}$$

In the next section we will present a short cut to compute $W(t)$, based on the introduction of the τ-function.

5 The τ-function and rational solutions of KP

The aim of this section is to show a quicker way to find the solutions of the Sato System corresponding to the finite-dimensional case discussed in Section 4, and to prove that such solutions lead to the rational solutions of the KP hierarchy. The main object is the function $\tau = \det U$, whose explicit form is

$$\tau(t) = \begin{vmatrix} D_{11} - \sum_{j=1}^{m} p_j C_{j1} & \cdots & D_{1n} - \sum_{j=1}^{m} p_j C_{jn} \\ D_{21} + p_1 D_{11} - \sum_{j=1}^{m} p_{j+1} C_{j1} & \cdots & D_{2n} + p_1 D_{1n} - \sum_{j=1}^{m} p_{j+1} C_{jn} \\ \vdots & \vdots & \vdots \\ \sum_{j=0}^{n-1} p_j D_{n-j,1} - \sum_{j=1}^{m} p_{j+n-1} C_{j1} & \cdots & \sum_{j=0}^{n-1} p_j D_{n-j,n} - \sum_{j=1}^{m} p_{j+n-1} C_{jn} \end{vmatrix}. \tag{5.1}$$

This expression appears (in a slightly disguised form) in [11], p.338, where it is shown to be the most general polynomial solution of (the Hirota form) of the KP hierarchy. It is clear that one can obtain in particular the τ-functions associated with Young diagrams, that is, $\tau = \det \left[p_{f_i - i + j}\right]_{i,j=1,\ldots,n}$, where $f_1 \geq f_2 \geq \cdots \geq f_n > 0$.

Remark 5.1 The function τ is a Wronskian, where derivatives are taken with respect to t_1; more precisely, we have that

$$\tau(t) = \mathrm{Wr}\left[\sum_{j=0}^{n-1} p_j D_{n-j,n} - \sum_{j=1}^{m} p_{j+n-1} C_{jn}, \ldots, \sum_{j=0}^{n-1} p_j D_{n-j,1} - \sum_{j=1}^{m} p_{j+n-1} C_{j1}\right]. \tag{5.2}$$

This Wronskian property of $\tau = \det U$ could have been directly shown by multiplying the equation

$$\frac{\partial U}{\partial t_1} = BU - G_1 V \tag{5.3}$$

by the vectors $e_2 = [0, 1, 0, \ldots, 0]$, $e_3 = [0, 0, 1, \ldots, 0]$, $\ldots$, $e_n = [0, 0, \ldots, 0, 1]$.

Remark 5.2 In [22], p.29, the evolution equations for the τ-function were written. They can be easily obtained in our formalism noticing that

$$\frac{\partial}{\partial t_k} \log \tau = \frac{\partial}{\partial t_k} \log \det U = \mathrm{tr}\left(\frac{\partial U}{\partial t_k} U^{-1}\right) = \mathrm{tr}\left(B^k - G_k W\right) = -\mathrm{tr}\left(G_k W\right), \tag{5.4}$$

so that

$$\frac{\partial}{\partial t_k} \log \tau = -\sum_{l=1}^{k} W_l^{k-l}. \tag{5.5}$$

Now we want to show that τ allows us to determine the whole matrix $W(t)$, i.e., their rows $W^{(k)}$. We begin by proving that the link between τ and $W^{(0)}$ is expressed by the well-known formula (see, e.g., [7])

$$W^{(0)}(t) = \frac{\tau(t - [z^{-1}])}{\tau(t)}, \tag{5.6}$$

where, as usual, $[z^{-1}] = (z^{-1}, \frac{1}{2}z^{-2}, \frac{1}{3}z^{-3}, \dots)$. First we must recall from [25] the following two lemmas.

Lemma 5.3 *The Schur polynomials satisfy the relations*

$$p_k(t - [z^{-1}]) = p_k(t) - z^{-1}p_{k-1}(t). \tag{5.7}$$

Lemma 5.4 *If $v_1, v_2, \dots, v_{j+1}$ are row vectors in $\mathbb{C}^{j+1}$ and $\lambda \in \mathbb{C}$, then*

$$\det \begin{bmatrix} v_j - \lambda v_{j+1} \\ \vdots \\ v_1 - \lambda v_2 \end{bmatrix} = \det \begin{bmatrix} 1 & v_{j+1} \\ \lambda & v_j \\ \vdots & \vdots \\ \lambda^j & v_1 \end{bmatrix}. \tag{5.8}$$

Corollary 5.5 *The following formula holds,*

$$\tau(t - [z^{-1}]) =$$

$$\begin{vmatrix} 1 & -\sum_{j=1}^m p_{j-1}C_{j1} & \cdots & -\sum_{j=1}^m p_{j-1}C_{jn} \\ z^{-1} & D_{11} - \sum_{j=1}^m p_jC_{j1} & \cdots & D_{1n} - \sum_{j=1}^m p_jC_{jn} \\ z^{-2} & D_{21} + p_1D_{11} - \sum_{j=1}^m p_{j+1}C_{j1} & \cdots & D_{2n} + p_1D_{1n} - \sum_{j=1}^m p_{j+1}C_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ z^{-n} & \sum_{j=0}^{n-1} p_jD_{n-j,1} - \sum_{j=1}^m p_{j+n-1}C_{j1} & \cdots & \sum_{j=0}^{n-1} p_jD_{n-j,n} - \sum_{j=1}^m p_{j+n-1}C_{jn} \end{vmatrix}, \tag{5.9}$$

so that the coefficient of z^{-l} of $\tau(t-[z^{-1}])$, for $l=0,\dots,n$, is given by

$$(-1)^l \begin{vmatrix} -\sum_{j=1}^m p_{j-1}C_{j1} & \cdots & -\sum_{j=1}^m p_{j-1}C_{jn} \\ D_{11}-\sum_{j=1}^m p_jC_{j1} & \cdots & D_{1n}-\sum_{j=1}^m p_jC_{jn} \\ \vdots & \vdots & \vdots \\ (l+1)\text{-st} & \text{row} & \text{missing} \\ \vdots & \vdots & \vdots \\ \sum_{j=0}^{n-1} p_jD_{n-j,1}-\sum_{j=1}^m p_{j+n-1}C_{j1} & \cdots & \sum_{j=0}^{n-1} p_jD_{n-j,n}-\sum_{j=1}^m p_{j+n-1}C_{jn} \end{vmatrix}. \tag{5.10}$$

Proof. Just combine equation (5.1) with Lemmas 5.3 and 5.4.

□

In order to prove relation (5.6), we have to compute $W_l^0(t)=e^1W(t)e_l$, where $W(t)$ is given by (4.13) and e^i (resp. e_l) is the i-th row vector (resp. l-th column vector) of the standard basis. One has

$$W_l^0=e^1We_l=e^1\sum_{i=0}^{m-1}CU^{-1}e_l=\tau^{-1}[1,p_1,\dots,p_{m-1}]C\begin{bmatrix} U_{1l}^\dagger \\ \vdots \\ U_{nl}^\dagger \end{bmatrix}, \tag{5.11}$$

where $U^\dagger=\det(U)U^{-1}$ is the matrix of algebraic complements. Then

$$\tau W_l^0=\sum_{i=1}^m p_{i-1}\sum_{k=1}^n C_{ik}(-1)^{k+l}\times$$

$$\begin{vmatrix} D_{11}-\sum_{j=1}^m p_jC_{j1} & \cdots & D_{1n}-\sum_{j=1}^m p_jC_{jn} \\ D_{21}+p_1D_{11}-\sum_{j=1}^m p_{j+1}C_{j1} & \cdots & D_{2n}+p_1D_{1n}-\sum_{j=1}^m p_{j+1}C_{jn} \\ \vdots & \vdots & \vdots \\ l\text{-th row and} & k\text{-th column} & \text{missing} \\ \vdots & \vdots & \vdots \\ \sum_{j=0}^{n-1} p_jD_{n-j,1}-\sum_{j=1}^m p_{j+n-1}C_{j1} & \cdots & \sum_{j=0}^{n-1} p_jD_{n-j,n}-\sum_{j=1}^m p_{j+n-1}C_{jn} \end{vmatrix}, \tag{5.12}$$

which coincides with (5.10). Hence, we have shown that once $\tau(t)$ is computed via equation (5.1), $W^{(0)}$ can be found using the relation (5.6). Then, all the rows of the matrix $W(t)$ can

be recursively computed writing the first flow of the Sato System, which reads

$$\frac{\partial W^{(j)}}{\partial t_1} + zW^{(j)} = W^{(j+1)} + W_1^j W^{(0)}, \tag{5.13}$$

and gives $W^{(j+1)}$ in terms of the preceding $W^{(l)}$. Notice, however, that the corresponding solution of the KP equations (2.3) is simply given by

$$h(x;t) = z + \frac{\partial}{\partial t_1} \log W^{(0)}(x + t_1, t_2, \dots), \tag{5.14}$$

as explained at the end of Section 2.

6 An example

In this section we will explicitly find the solution of the Sato System (2.8) corresponding to the initial conditions

$$W_2^0(t_0) = W_2^2(t_0) = -2, \qquad W_i^j(t_0) = 0 \text{ otherwise}, \qquad \text{where } t_0 = (0, 1, 0, \dots). \tag{6.1}$$

We already know that we are allowed to consider the restriction of the Sato System to $\mathbb{W}_{3,2}$, and that only the times $(t_1, \dots, t_4)$ are involved. More precisely, we have to solve equations (4.6) in the case $m = 3$, $n = 2$, with initial condition

$$W(t_0) = \begin{bmatrix} 0 & -2 \\ 0 & 0 \\ 0 & -2 \end{bmatrix}. \tag{6.2}$$

We also know that the general solution of the system (4.6) is given by $W(t) = V(t)U(t)^{-1}$, with (see (4.12) and (5.6))

$$V(t) = \begin{bmatrix} 1 & p_1 & p_2 \\ 0 & 1 & p_1 \\ 0 & 0 & 1 \end{bmatrix} C, \qquad U(t) = \begin{bmatrix} 1 & 0 \\ p_1 & 1 \end{bmatrix} D - \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{bmatrix} C, \tag{6.3}$$

where C (resp. D) is a constant 3×2 (resp. 2×2) matrix. These matrices are determined from the initial value of U and V, that we can choose as $V(t_0) = W(t_0)$, $U(t_0) = I$. This

implies

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{6.4}$$

and therefore

$$V(t) = \begin{bmatrix} 0 & -2p_2 \\ 0 & -2p_1 \\ 0 & -2 \end{bmatrix}, \qquad U(t) = \begin{bmatrix} 1 & 2p_3 \\ p_1 & 2p_4 \end{bmatrix}, \tag{6.5}$$

so that

$$W(t) = \frac{2}{\tau} \begin{bmatrix} p_1p_2 & -p_2 \\ p_1^2 & -p_1 \\ p_1 & -1 \end{bmatrix}, \tag{6.6}$$

where $\tau = \det U = 2(p_4 - p_1p_3) = 2t_4 + t_2^2 - t_1^2t_2 - \frac{1}{4}t_1^4$. Hence we have

$$\begin{aligned} W^{(0)} &= 1 + 2\tau^{-1}(p_1p_2z^{-1} - p_2z^{-2}) \\ W^{(1)} &= z + 2\tau^{-1}(p_1^2z^{-1} - p_1z^{-2}) \\ W^{(2)} &= z^2 + 2\tau^{-1}(p_1z^{-1} - z^{-2}) \end{aligned} \tag{6.7}$$

and, clearly, $W^{(k)} = z^k$ for all $k \geq 3$. As an alternative, starting form τ we can directly compute

$$\begin{aligned} W^{(0)} &= \frac{\tau(t - [z^{-1}])}{\tau(t)} \\ W^{(1)} &= \frac{\partial W^{(0)}}{\partial t_1} + zW^{(0)} - W_1^0W^{(0)} \\ W^{(2)} &= \frac{\partial W^{(1)}}{\partial t_1} + zW^{(1)} - W_1^1W^{(0)} \end{aligned} \tag{6.8}$$

In any case, the corresponding solution of the KP hierarchy is given by (5.14).

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References

[1] M.J. Ablowitz, J. Satsuma, *Soliton and rational solutions of nonlinear evolution equations.* J. Math. Phys. **19** (1978), 2180–2186.

[2] H. Airault, H.P. McKean, J. Moser, *Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem.* Commun. Pure Appl. Math. **30** (1977), 95–148.

[3] F. Calogero, *Motion of poles and zeroes of special solutions of nonlinear and linear partial differential equations, and related "solvable" many–body problems.* Nuovo Cimento **43B** (1978), 177–241.

[4] P. Casati, G. Falqui, F. Magri, M. Pedroni, *A note on Fractional KdV Hierarchies.* J. Math. Phys. **38** (1997), 4606–4628.

[5] I. V. Cherednik, *Differential equations for the Baker-Akhiezer functions of Algebraic curves.* Funct. Anal. Appl. **12** (1978), 45–54 (Russian), 195–203 (English).

[6] D.V. Chudnovsky, G.V. Chudnovsky, *Pole expansions of nonlinear partial differential equations.* Nuovo Cimento **40B** (1977), 339–353.

[7] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Transformation Groups for Soliton Equations.* Proceedings of R.I.M.S. Symposium on Nonlinear Integrable Systems–Classical Theory and Quantum Theory (M. Jimbo, T. Miwa, eds.), World Scientific, Singapore, 1983, pp. 39–119.

[8] M.F. de Groot, T.J. Hollowood, J.L. Miramontes, *Generalized Drinfel'd-Sokolov Hierarchies.* Commun. Math. Phys. **145** (1992), 57–84.

[9] L. A. Dickey, *Soliton Equations and Hamiltonian Systems.* Adv. Series in Math. Phys. Vol. 12, World Scientific, Singapore, 1991.

[10] G. Falqui, F. Magri, M. Pedroni, *Bihamiltonian Geometry, Darboux Coverings, and Linearization of the KP Hierarchy.* To appear in Commun. Math. Phys. (1998).

[11] V.G. Kac, *Infinite dimensional Lie algebras*, third edition. Cambridge University Press, 1990.

[12] I.M. Krichever, *On rational solutions of the Kadomstev-Petviashvili equation and integrable systems of N particles on the line.* Funct. Anal. Appl. **12** (1978), 76–78 (Russian), 59–61 (English).

[13] M.D. Kruskal, *The Korteweg-de Vries equation and related evolution equations.* In: Nonlinear Wave Motion (A.C. Newell, ed.), Lect. Appl. Math. vol. 15, AMS, Providence, 1974, pp. 61–83.

[14] F. Magri, M. Pedroni, J.P. Zubelli, *On the Geometry of Darboux Transformations for the KP Hierarchy and its Connection with the Discrete KP Hierarchy.* Commun. Math. Phys. **188** (1997), 305–325.

[15] V.B. Matveev, *Some comments on the rational solutions of the Zakharov-Schabat equations.* Lett. Math. Phys. **3** (1979), 503–512.

[16] M. Mulase, *Algebraic Theory of the KP Equations.* Perspectives in Mathematical Physics (R. Penner and S–T. Yau, eds.), International Press, Boston, 1994, pp. 151–217.

[17] Y. Ohta, J. Satsuma, D. Takahashi, T. Tokihiro, *An Elementary Introduction to Sato Theory.* In: Recent Developments in Soliton Theory (J. Satsuma et al. eds.), Progr. Theoret. Phys. Suppl. **94** (1988), 210–241.

[18] W.T. Reid, *Riccati Differential Equations.* Academic Press, New York, 1972.

[19] M. Sato, Y. Sato, *Soliton equations as dynamical systems on infinite–dimensional Grassmann manifold.* In: Nonlinear PDEs in Applied Sciences (US-Japan Seminar, Tokyo), P. Lax and H. Fujita eds., North-Holland, Amsterdam, 1982, pp. 259–271.

[20] T. Shiota, *Calogero-Moser hierarchy and KP hierarchy.* J. Math. Phys. **35** (1994), 5844–5849.

[21] K. Takasaki, *Integrable System as deformations of $\mathcal{D}$-modules.* In: Theta Functions – Bowdoin 1987 (L. Ehrenpreis and R.C. Gunning, eds.), Proc. Symp. Pure Math. vol. 49 (1989), AMS, pp. 143–168.

[22] K. Takasaki, *Geometry of Universal Grassmann Manifold from Algebraic point of view.* Rev. Math. Phys. **1** (1989), 1–46.

[23] G. Wilson, *On two Constructions of Conservation Laws for Lax equations.* Quart. J. Math. Oxford **32** (1981), 491–512.

[24] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian.* Invent. math. **133** (1998), 1–41.

[25] J.P. Zubelli, *On the polynomial τ-functions for the KP hierarchy and the bispectral property.* Lett. Math. Phys. **24** (1992), 41–48.