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VARIATIONAL PROPERTIES OF AN IMAGE SEGMENTATION FUNCTIONAL DEPENDING ON CONTOURS CURVATURE

G. BELLETTINI(†) AND M. PAOLINI(*)

Abstract. We study a problem of image segmentation involving the lenght and the curvature of the contours. The (possibly linear) curvature term in the energy functional is introduced to reconstruct the hidden parts of the image occluded by regions which are closer to the observer. To deal with corners and cusps, which are viewed as points where curvature is concentrated, we introduce the class of the continuous curves having derivative of class BV, and we study their arguments. The proof is based on the techniques of relaxation theory in the Calculus of Variations. We estimate the lower semicontinuous envelope of the energy functional, and we compute it in some special cases.

AMS(MOS) subject classifications (1985 revision). 49J45, 49Q20

1. Introduction. A great attention has been recently devoted to the reconstruction of the contours of a picture given by a camera. One of the approaches considered to solve this problem if of variational type (see, among others, [16,6,24,23,21,13,19,25,26,27,12,1, 4,10]), where the solution is viewed as a minimum point of suitable energy functionals. More recently, in a paper by Mumford and Nitzberg [22] an integral term depending on the curvature of the contours has been added in the energy, in order to reconstruct the hidden boundaries, that is those parts of the image which are occluded by other regions closer to the observer. In this paper, starting from the suggestions of [22], we study the behaviour of a functional involving lenght and curvature of boundaries of (possibly non smooth) sets, from the viewpoint of relaxation theory. More precisely, let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function having the following properties:

(i) $\psi(\eta, \cdot)$ is convex on $\mathbf{R}$ for any $\eta \in \mathbf{R}$;

(ii) ψ is lower semicontinuous on $\mathbf{R} \times \mathbf{R}$;

(iii) $\psi(\cdot, 0)$ is locally bounded;

(iv) there exist two constants $a > 0$ and $b \geq 0$ such that

$$\psi(\eta, \xi) \geq a|\xi| - b \geq 0 \qquad \forall (\eta, \xi) \in \mathbf{R} \times \mathbf{R}; \tag{1.1}$$

(v) $\psi(\eta, \xi) = \psi(\eta + \pi, \xi)$ for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}$;

(vi) $\psi(\eta, \xi) = \psi(\eta, -\xi)$ for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}$.

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Let $n \in \mathbf{N}$, $n \geq 1$, and let $g \in L^\infty(\mathbf{R}^2)$ be a function with compact support. The aim of this paper is to study the functional $\mathcal{P}_\psi$ defined by

$$\mathcal{P}_\psi(E_1, \dots, E_n) = \sum_{i=1}^{n} \int_{\partial E_i} [1 + \psi(\alpha_i, \kappa_i)] \; d\mathcal{H}^1 + \sum_{i=1}^{n+1} \int_{E_i'} (u_i - g)^2 \; dz,$$

where $E_1, \dots, E_n \subseteq \mathbf{R}^2$ are bounded open sets of class $\mathcal{C}^2$, $\alpha_i(z)$ is the angle that the tangent unit vector of ∂E_i at $z \in \partial E_i$ forms with the positive direction of the x-axis, $\kappa_i(z)$ is the curvature of ∂E_i at z, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure in $\mathbf{R}^2$, and

$$\begin{aligned} &E_1' = E_1, \; E_i' = E_i \setminus \bigcup_{j=1}^{i-1} E_j, \; i = 2, \dots, n, \quad E_{n+1}' = \mathbf{R}^2 \setminus \bigcup_{j=1}^{n} E_j, \\ &u_i = \fint_{E_i'} g \; dz, \qquad i = 1, \dots, n, \quad u_{n+1} = 0 \end{aligned} \tag{1.2}$$

(see Section 7). Note that conditions (v) and (vi) ensure that the functional $\mathcal{P}_\psi$ is independent of the orientation of the boundaries ∂E_i.

In [22] the function ψ has the specific form

$$\psi(\eta, \xi) = \psi(|\xi|) = \begin{cases} c_1 \xi^2 & \text{if } |\xi| < T, \\ c_2 |\xi| - c_3 & \text{if } |\xi| \geq T, \end{cases} \tag{1.3}$$

with $T > 0$, $c_1 > 0$, $c_1 T^2 = c_2 T - c_3$, $c_2 = 2c_1 T$. Each partition $E_1, \dots, E_n$ is endowed with the following ordering: $E_i > E_j$ (i.e., E_i is closer than E_j to the observer) if $i < j$. This ordering represents relative depth (or occlusion); in this context, if $i = 1, \dots, n$, the sets E_i' defined in (1.2) are the "visible" part of E_i, and E_{n+1}' stands for the background. The partition minimizing the energy is called an optimal overlapping segmentation of the image g.

The special case $\psi(\eta, \xi) = |\xi|^2$ and $n = 1$ has been studied in [3]. Here the quadratic growth of ψ at infinity is the main difference with (1.3), and reveals very different effects with respect to the linear case (see also [2]).

To study the functional $\mathcal{P}_\psi$, which, in our approach is considered as a function of the sets $E_1, \dots, E_n$ rather than of their boundaries $\partial E_1, \dots, \partial E_n$, we need a detailed study of the variational properties of the functional

$$F_\psi(E) = \int_{\partial E} [1 + \psi(\alpha, \kappa)] \; d\mathcal{H}^1.$$

As usual in the relaxation method (see [7]), if $\mathcal{M}$ denotes the class of all Lebesgue measurable subsets of $\mathbf{R}^2$, we define $\overline{F_\psi} : \mathcal{M} \to [0, +\infty]$ as the lower semicontinuous envelope of F_ψ with respect to the $L^1(\mathbf{R}^2)$-topology, i.e.,

$$\overline{F_\psi}(E) = \inf\{\liminf_{h \to +\infty} F_\psi(E_h) : E_h \to E \text{ in } L^1(\mathbf{R}^2)\}. \tag{1.4}$$

The main purpose of this paper is to study the functional $\overline{F_\psi}$ and to compute $\overline{F_\psi}(E)$ for some special sets E.

Simple examples show that there exist sets $E \in \mathcal{M}$ with $\overline{F_\psi}(E) < +\infty$, whose boundary is not smooth, as polygons and sets with cusp points. In particular, let us consider the set of Figure 1.1, and let ψ be as in (1.3). Then the approximating sequence $\{E_h\}_h$ of Figure 1.2, constructed by smoothing the corner p with circular arcs, shows that

$$\overline{F_\psi}(E) \leq \mathcal{H}^1(\partial E) + \int_{\partial E \setminus \{p\}} \psi(\alpha, \kappa)\, d\mathcal{H}^1 + c_2(\pi - \alpha).$$

Note that the corner p is penalized, in terms of the energy, of a factor which is proportional to $\pi - \alpha$.

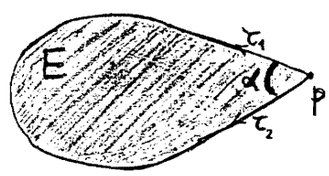

Fig. 1.1: A set E with $\overline{F_\psi}(E) < +\infty$ and whose boundary is not smooth.

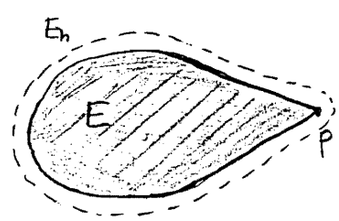

Fig. 1.2: This approximating sequence $\{E_h\}_h$ shows that $\overline{F_\psi}(E) < +\infty$.

This penalization corresponds to the variation made by the unit tangent vector of the curves parametrizing ∂E_h in a neighbourhood of p, in the limit as $h \to +\infty$. This example shows that, if one deals with curves parametrized by arc lenght, it can be useful to identify the tangent vectors as points of the unit circle $\mathbf{S}^1$ and to consider the jumps at the corners in terms of the corresponding angles. We are then led to look at the set ∂E of Figure 1.1 as the trace of a continuous closed curve γ with $\dot{\gamma} \in BV$ (the space of the functions of bounded variation) and with the constraint $|\dot{\gamma}| = 1$ (see also [17]). The convergence in this class of curves must be considered $\pmod{2\pi}$, as explained in Section 4, thus showing that some care is needed in the definition of the energies.

We describe now in detail the content of the paper.

In Section 2 we fix some notation and, in particular, we define the concept of system of curves of class $\mathcal{B}$, that is a finite family of closed curves of class $W^{1,1}$ whose derivatives are functions of class BV (Definition 2.2).

In Section 3 we introduce the argument of $\dot{\gamma}$ (see Lemma 3.1), where γ is a curve of class $\mathcal{B}$ parametrized with constant velocity.

In Section 4 we study some properties of the space $BV([a,b];\mathbf{R}/2\pi)$, which, in turn, are useful for the definition of the energy functional given in Section 5.

Section 6 is devoted to the study of the lower semicontinuity of F_ψ. Precisely, we prove (Theorem 6.1) that *given a sequence $\{E_h\}_h$ of bounded open sets of class C^2 converging in $L^1(\mathbf{R}^2)$ to a bounded open set E of class C^2, then*

$$F_\psi(E) \leq \liminf_{h\to+\infty} F_\psi(E_h).$$

A similar result holds if we localize F_ψ on an open subset Ω of $\mathbf{R}^2$ (Corollary 6.1).

In Section 7 we show how to reduce the study of the functional $\mathcal{P}_\psi$ to the study of the simpler functional F_ψ.

In Section 8 we show that, if E is a bounded open set that can be written, locally, as the subgraph of a Lipschitz function having derivative of class BV, then $\overline{F_\psi}(E) < +\infty$.

Finally, in Section 9 we focus our attention to the case (1.3), and we prove the following results (see Theorem 9.1 and Corollary 9.2): *let E be the disjoint union of a finite number of curvilinear polygons $P_1,\dots,P_q$. For any $i=1,\dots,q$ denote by $\alpha_i^1,\dots,\alpha_i^{h(i)}$ the angles of the polygon P_i. Then*

$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q}\sum_{j=i}^{h(i)} \min(\alpha_i^j, 2\pi-\alpha_i^j, |\pi-\alpha_i^j|) \leq \overline{F_\psi}(E) \leq$$

$$\leq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q}\sum_{j=i}^{h(i)} |\pi-\alpha_i^j|,$$

where $\partial_r E$ denotes ∂E without the vertices, and κ denotes the curvature of $\partial_r E$. If in particular $\frac{\pi}{2} \leq \alpha_i^j \leq \frac{3\pi}{2}$ for any $i=1,\dots,q$ and any $j=1,\dots,h(i)$, then

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q}\sum_{j=i}^{h(i)} |\pi-\alpha_i^j|. \tag{1.5}$$

Moreover, we show that (1.5) does not necessarily hold if $\alpha_i^j < \frac{\pi}{2}$ or $\alpha_i^j > \frac{3}{2}\pi$ for some i,j, the result depending on the relative distance between the corners. Indeed (Theorem 9.2) *let $E = E_1 \cup E_2$ be the set of Figure 9.1, and assume that*

$$0 < \alpha < \frac{\pi}{2}. \tag{1.6}$$

Then if the distance D betweeen p_1 and p_2 is sufficiently small one has

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2D + 2c_2\alpha,$$

where $\partial_r E = \partial E \setminus \{p_1, p_2\}$.

In particular, if D is sufficiently small,

$$\overline{F_\psi}(E) < \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2c_2(\pi - \alpha).$$

As the proof of this result has local character, we deduce that there are non convex sets E having some angles α_i verifying condition (1.6) such that the their energy (i.e., the value of $\overline{F_\psi}(E)$) is strictly less to the energy of the smooth part of the boundary $\partial_r E$ plus the extra term $c_2 \sum_i(\pi - \alpha_i)$. This lower value of the energy can be reached, ideally, by joining properly the angles with a "hidden" line.

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2. Notations and preliminary definitions.

$\mathcal{H}^h$ denotes the h-dimensional Hausdorff measure in $\mathbf{R}^2$ for $h = 0, 1$ [15]; $\mathcal{L}^2$ denotes the Lebesgue measure in $\mathbf{R}^2$. For any $z_0 \in \mathbf{R}^2$ and $\varrho > 0$, $B_\varrho(z_0) = \{z \in \mathbf{R}^2 : |z - z_0| < \varrho\}$ is the ball centered at z_0 with radius ϱ.

$\mathcal{M}$ denotes the class of all Lebesgue measurable subets of $\mathbf{R}^2$. We shall identify $\mathcal{M}$ with a closed subset of $L^1(\mathbf{R}^2)$ by means of the map $E \mapsto \chi_E$, where χ_E denotes the characteristic function of E, i.e., $\chi_E(z) = 1$ if $z \in E$, $\chi_E(z) = 0$ if $z \notin E$. The $L^1(\mathbf{R}^2)$-topology on $\mathcal{M}$ is, therefore, the topology on $\mathcal{M}$ induced by the distance $d(E_1, E_2) = \mathcal{L}^2(E_1 \triangle E_2)$, where $E_1, E_2 \in \mathcal{M}$ and $\triangle$ is the symmetric difference of sets.

For any subset C of $\mathbf{R}^2$, we denote by $\mathrm{int}(C)$ the interior of C, by $\overline{C}$ the closure of C, and by ∂C the topological boundary of C. Let $E \subseteq \mathbf{R}^2$; we say that E is of class $\mathcal{C}^2$ if E is open, and, near each point $z \in \partial E$, the set E is the subgraph of a function of class $\mathcal{C}^2$ with respect to a suitable orthogonal coordinate system. Note that, if ∂E can be parametrized, locally, by arcs of regular curves of class $\mathcal{C}^2$, and E lies locally on one side of its boundary, then E is of class $\mathcal{C}^2$. We denote by $\mathcal{C}^2_b(\mathbf{R}^2)$ the class of all bounded sets of class $\mathcal{C}^2$.

2.1. Systems of curves. A plane curve $\gamma : [0,1] \to \mathbf{R}^2$ of class $\mathcal{C}^1$ is said to be regular if $\frac{d\gamma(t)}{dt} \neq 0$ for every $t \in [0,1]$. Each closed regular curve $\gamma : [0,1] \to \mathbf{R}^2$ will be identified, in the usual way, with a map $\gamma : \mathbf{S}^1 \to \mathbf{R}^2$, where $\mathbf{S}^1$ denotes the oriented unit circle. By $(\gamma) = \gamma([0,1]) = \{\gamma(t) : t \in [0,1]\}$ we denote the trace of γ and by $l(\gamma)$ its length; s denotes the arc length parameter.

If $z \in \mathbf{R}^2 \setminus (\gamma)$, $I(\gamma, z)$ is the index of z with respect to γ (see, for instance, [5], and [8, III.8]).

DEFINITION 2.1. *A system of curves is a finite family* $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ *of closed curves of class* $W^{1,1}$ *such that* $|\frac{d\gamma^i}{dt}|$ *is constant almost everywhere on* $[0,1]$ *for any* $i = 1, \dots, m$. *The trace* (Γ) *of* Γ *is defined as* $\bigcup_{i=1}^m (\gamma^i)$, *and its lenght* $l(\Gamma)$ *as* $\sum_{i=1}^m l(\gamma^i)$. *If* $z \in \mathbf{R}^2 \setminus (\Gamma)$, *we define the index* $I(\Gamma, z)$ *of* z *with respect to* Γ *as* $\sum_{i=1}^m I(\gamma^i, z)$.

We say that a system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ is disjoint if $(\gamma^i) \cap (\gamma^j) = \emptyset$ for any $i, j = 1, \dots, m$, $i \neq j$.

Let $E \in \mathcal{C}_b^2(\mathbf{R}^2)$; we say that a disjoint system Γ of curves is an oriented parametrization of ∂E if any curve of the system is simple, $(\Gamma) = \partial E$, and

$$E = \{z \in \mathbf{R}^2 : I(\Gamma, z) = 1\}, \quad \mathbf{R}^2 \setminus \overline{E} = \{z \in \mathbf{R}^2 : I(\Gamma, z) = 0\}.$$

It is not difficult to prove that each $E \in \mathcal{C}_b^2(\mathbf{R}^2)$ admits an oriented parametrization of class $\mathcal{C}^2$.

2.2. The space BV. If λ is a scalar or vector-valued Radon measure, its total variation will be denoted by $|\lambda|$. Let I be a real interval, $B \subseteq I$ be a Borel set and $f : B \to \mathbf{R}$ be a Borel function; the integral of f on B with respect to λ will be indicated by $\int_B f\lambda$. By $\lambda \llcorner B$ we mean the measure defined by $\lambda \llcorner B(F) = \lambda(B \cap F)$ for any Borel set F.

If μ is a scalar Radon measure on I, we have a unique decomposition $\lambda = \lambda^a + \lambda^s$, where λ^a is absolutely continuous and λ^s is singular with respect to μ. The density of λ^a with respect to μ, which is a function belonging to L^1_μ, will be indicated by $\frac{d\lambda}{d\mu}$ or by $\frac{\lambda}{\mu}$, and will be called the Radon-Nikodym derivative of λ with respect to μ. Then

$$\frac{d\lambda}{d\mu}(t) = \lim_{\varrho \to 0} \frac{\lambda(B_\varrho(t))}{\mu(B_\varrho(t))} \qquad \text{for } \mu - \text{ a.e. } t \in I, \tag{2.1}$$

and $\lambda(B) = \int_B \frac{d\lambda}{d\mu} d\mu + \lambda^s(B)$ for every Borel set $B \subseteq I$.

The space $BV(I)$ is defined as the space of the functions $f \in L^1_{\text{loc}}(I)$ whose distributional gradient $\dot{f}$ is a Radon measure with bounded total variation in I. We say that $f = (f_1, f_2) \in BV(I; \mathbf{R}^2)$ if $f_i \in BV(I)$ for any $i = 1, 2$. We recall that any function in $BV(I)$ coincides with a function of bounded variation in the classical sense.

If $f \in BV(I)$ we indicate by S_f the jump set of f, and by $f(t-)$, $f(t+)$ the approximate lower and upper limits of f at the point $t \in I$. It is well known that S_f is at most countable. If $f = (f_1, f_2) \in BV(I; \mathbf{R}^2)$, by S_f we mean $S_{f_1} \cup S_{f_2}$. We shall write

$$\dot{f} = \dot{f}^a \, dt + \dot{f}^s.$$

Here $\dot{f}^a$ denotes the density of the absolutely continuous part of $\dot{f}$ with respect to the Lebesgue measure dt and coincides almost everywhere with the pointwise derivative of the function $f(t+)$; the singular part $\dot{f}^s$ can be written on a Borel set $B \subseteq I$ as

$$\dot{f}^s(B) = \int_{B \cap S_f} (f(t+) - f(t-)) d\mathcal{H}^0(t) + (Cf)(B),$$

where Cf is the so-called Cantor part, and is a measure such that $(Cf)(\{t\}) = 0$ for any $t \in B$.

We say that a sequence $\{f_h\}_h \subseteq BV(I;\mathbf{R}^2)$ is weakly convergent in BV to a function $f \in BV(I;\mathbf{R}^2)$ if $f_h \to f$ in L^1 and $\dot{f}_h \rightharpoonup \dot{f}$ weakly as measures as $h \to +\infty$.

We recall the following result (see [28, Sec. 13.2]).

THEOREM 2.1. *Let $f \in BV(I)$, and let $\phi \in C^1(\mathbf{R})$ be a Lipschitz continuous function. Then the composite function $\phi(f)$ is of class $BV(I)$, and its distributional derivative $\dot{\phi(f)}$ can be written as*

$$\dot{\phi(f)} = \dot{\phi}(f)\dot{f}\llcorner(I \setminus S_f) + \sum_{t \in S_f} [\phi(f(t+)) - \phi(f(t-))]\delta_t, \tag{2.2}$$

where δ_t is the Dirac mass at the point $t \in S_f \subseteq I$.

For the definitions and the main properties of the functions of bounded variation we refer to [28,9,18,15,20].

2.3. Systems of curves of class $\mathcal{B}$.

DEFINITION 2.2. *We say that a system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ is of class $\mathcal{B}$ if $\gamma^i \in W^{1,1}([0,1])$ and $\frac{d\gamma^i}{dt} \in BV([0,1];\mathbf{R}^2)$ for any $i = 1, \dots, m$.*

DEFINITION 2.3. *We say that a sequence $\{\Gamma_h\}_h$ of systems of curves of class C^2 is weakly convergent to a system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ if the number of curves of each system Γ_h is the same as the number of curves of Γ for h large enough, i.e., $\Gamma_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ for any h, and, in addition, $\gamma_h^i \to \gamma^i$ uniformly in C^0, $\frac{d\gamma_h^i}{dt} \rightharpoonup \frac{d\gamma^i}{dt}$ weakly in BV as $h \to +\infty$, for any $i = 1, \dots, m$.*

DEFINITION 2.4. *We say that Γ is a limit system of curves if Γ is the weak limit of a sequence of oriented parametrizations of bounded open sets of class C^2.*

It follows from the definitions that, if Γ is a limit system of curves, then $\Gamma \in \mathcal{B}$, and $I(\Gamma, z) \in \{0,1\}$ for any $z \in \mathbf{R}^2 \setminus (\Gamma)$.

Note that if the sequence of systems $\{\Gamma_h\}_h$ of class C^2 is weakly convergent to $\Gamma = \{\gamma^1, \dots, \gamma^m\} \in \mathcal{B}$, then, for any $i = 1, \dots, m$ we have

$$l(\gamma_h^i) = \int_0^1 |\frac{d\gamma_h^i}{dt}|\, dt \to \int_0^1 |\frac{d\gamma^i}{dt}|\, dt = l(\gamma^i) \qquad \text{as } h \to +\infty.$$

Possibly passing to a subsequence, we have that $|\frac{d\gamma_h^i}{dt}| \to |\frac{d\gamma^i}{dt}|$ almost everywhere on $[0,1]$ as $h \to +\infty$. Hence $|\frac{d\gamma^i}{dt}|$ is constant almost everywhere on $[0,1]$, so that each curve γ^i can be parametrized by arc lenght.

2.4. Relaxation. Given a functional $\mathcal{L} : \mathcal{M} \to [0, +\infty]$, we denote by $\overline{\mathcal{L}}$ the lower semicontinuous envelope (or relaxed functional) of $\mathcal{L}$ with respect to the topology of $L^1(\mathbf{R}^2)$ (see (1.4)). For the main properties of the relaxed functionals we refer to [7].

3. The argument functions.

Let us prove the following Lemma.

LEMMA 3.1. *Let $[a,b] \subseteq \mathbf{R}$ be a bounded interval, and let $f \in BV([a,b];\mathbf{R}^2)$ be such that $|f(t)| = R$ for almost every $t \in [a,b]$, for a suitable $R > 0$. Then there exists a function $\Theta : [a,b] \to \mathbf{R}$, that we shall call an argument of f, having the following properties:*

(a) $\Theta \in BV([a,b])$ and $f(t) = (R\cos\Theta(t), R\sin\Theta(t))$ for almost every $t \in [a,b]$;

(b) $S_\Theta = S_f$;

(c) $|\Theta(t+) - \Theta(t-)| \leq \pi$ for every $t \in]a,b[$.

Moreover condition (c) can be replaced by one of the following stronger conditions:

$$-\pi < \Theta(t+) - \Theta(t-) \leq \pi \qquad \forall t \in]a,b[, \tag{3.1}$$

or

$$-\pi \leq \Theta(t+) - \Theta(t-) < \pi \qquad \forall t \in]a,b[. \tag{3.2}$$

When (3.1) or (3.2) hold then Θ is unique up to a global translation of an integer multiple of 2π.

PROOF. Set $O_1 = \partial B_R(0) \cap \{x > 0\}$, $O_2 = \partial B_R(0) \cap \{y > 0\}$, $O_3 = \partial B_R(0) \cap \{x < 0\}$, $O_4 = \partial B_R(0) \cap \{y < 0\}$. If the image of $f = (f_1, f_2)$ is contained in one of the sets O_i, for instance in O_1, then the function Θ is simply defined as $\Theta(t) = \arcsin(\frac{1}{R} f_2(t))$.

Let us consider now the general case. Let us fix $0 < \sigma < \frac{R}{4}$. We claim that there exists a finite set $\{s_1, \ldots, s_m\}$ of points of $]a,b[$ such that

$$|\dot{f}|(J) < 2\sigma < \frac{R}{2}, \tag{3.3}$$

where J is any open interval contained in $[a,b] \setminus \{s_1, \ldots, s_m\}$.

The set $S_f = \{t \in]a,b[: |\dot{f}|(\{t\}) > 0\}$ is at most countable, and we denote it by $\{t_i\}_i$, for $i \in \mathbf{N}, i \geq 1$, and $t_i < t_{i+1}$. Hence there exist a sequence $\{v_i\}_i \subseteq \mathbf{R}^2$ and an $\mathbf{R}^2$-valued Radon measure μ on $[a,b]$ such that

$$\dot{f} = \mu + \sum_{i=1}^{+\infty} v_i \delta_i, \qquad \sum_{i=1}^{+\infty} |v_i| < +\infty, \qquad \text{and} \qquad \mu(\{t\}) = 0 \quad \forall t \in [a,b],$$

where δ_i is the Dirac mass at the point t_i.

Let $k \in \mathbf{N}$, $k \geq 1$, be such that

$$\sum_{i=k}^{+\infty} |v_i| < \sigma. \tag{3.4}$$

Let $g : [a,b] \to \mathbf{R}$ be defined by $g(t) = |\mu|([a,t]) = |\mu|(]a,t[)$. Since $\mu(\{t\}) = 0$ for any $t \in [a,b]$, g is continuous, and hence uniformly continuous on $[a,b]$. As a consequence, there exists a finite number of points $\{\tau_1, \dots, \tau_r\} \subseteq]a,b[$ such that, if I denotes any interval contained in $[a,b] \setminus \{\tau_1, \dots, \tau_r\}$, then

$$|\mu|(I) < \sigma. \tag{3.5}$$

Denote by $\{s_1, \dots, s_m\}$ the ordered set of points of $]a,b[$ given by the union of $\{t_1, \dots, t_k\}$ and $\{\tau_1, \dots, \tau_r\}$, $m \le k + r$. Set $s_0 = a$, $s_{m+1} = b$. Then, for any $j = 0, \dots, m$ and any $t \in]s_j, s_{j+1}]$, by (3.5) and (3.4) we have

$$|f(t-) - f(s_j+)| \le |\dot{f}|(]s_j,t[) \le |\mu|(]s_j,t[) + \sum_{i \ge k : t_i \in]s_j,t[} |v_i| \le 2\sigma < \frac{R}{2}, \tag{3.6}$$

and this proves claim (3.3).

From (3.6) it follows that

$$\forall j = 0, \dots, m \quad \exists n \in \{1, \dots, 4\} \text{ such that } f(]s_j, s_{j+1}[) \subset\subset O_n.$$

It is then sufficient to define the function Θ separately on each interval $]s_j, s_{j+1}[$ by means of functions Θ_j, $0 \le j \le m$, and then to glue together all these functions, possibly adding to the values of Θ_j suitable integer multiples of 2π, in order to fulfill condition (c). Assertion (3.1) (respectively (3.2)) can be easily obtained by properly gluing the functions Θ_j. The uniqueness of Θ (up to global translations of an integer multiple of 2π) follows from the previous contruction. □

Let γ be a curve of class $\mathcal{B}$ such that $|\dot{\gamma}(s)| = 1$ for almost every $s \in [0, l(\gamma)] = I$. Let Θ be an argument of $\dot{\gamma}$. Using formula (2.2), (a) of Lemma 3.1, and the uniqueness of the Lebesgue decomposition of a measure, it is not difficult to show that

$$\ddot{\gamma}^a = (-\sin\Theta, \cos\Theta)\dot{\Theta}^a \qquad \text{a.e. in } I \tag{3.7}$$

and, if $B \subseteq I$ is a Borel set, then

$$\begin{aligned} (\ddot{\gamma}^s)(B) = &\int_{B \cap (I \setminus S_\Theta)} (-\sin\Theta, \cos\Theta)\dot{\Theta}^s + \\ &\sum_{s \in B \cap S_\Theta} (\cos\Theta(s+) - \cos\Theta(s-), \sin\Theta(s+) - \sin\Theta(s-))\delta_s. \end{aligned} \tag{3.8}$$

Therefore (3.7) yields

$$|\ddot{\gamma}^a| = |\dot{\Theta}^a| \qquad \text{a.e in } I.$$

Note also that

$$|\ddot{\gamma}^s|(\{s\}) < |\dot{\Theta}^s|(\{s\}) \qquad \forall s \in S_\Theta.$$

Indeed, for any $s \in S_\Theta$, using (3.8) and the equality $2(1 - \cos\phi) = 4\sin^2(\frac{\phi}{2})$, we have

$$\begin{aligned} (|\ddot{\gamma}^s|(\{s\}))^2 &= |(\cos\Theta(s+) - \cos\Theta(s-), \sin\Theta(s+) - \sin\Theta(s-))|^2 = \\ 4\sin^2\Big(\frac{\Theta(s+) - \Theta(s-)}{2}\Big) &\le (\Theta(s+) - \Theta(s-))^2 = (|\dot{\Theta}^s|(\{s\}))^2. \end{aligned}$$

4. The space $BV([a,b]; \mathbf{R}/2\pi)$.

DEFINITION 4.1. *Let $\theta_1, \theta_2 \in BV([a,b])$. We say that θ_1 and θ_2 are equivalent, and we write $\theta_1 \sim \theta_2$ if for almost every $t \in [a,b]$ there exists $k(t) \in \mathbf{Z}$ such that $\theta_1(t) = \theta_2(t) + 2k(t)\pi$.*

Given $\theta \in BV([a,b])$, we denote by $[\theta]$ the equivalence class of θ, i.e., $[\theta] = \{\underline{\theta} \in BV([a,b]) : \underline{\theta} \sim \theta\}$. We set

$$BV([a,b];\mathbf{R}/2\pi) = \{[\theta] : \theta \in BV([a,b])\}.$$

Let $f \in BV([a,b];\mathbf{R}^2)$ be such that $|f(t)| = R > 0$ for almost every $t \in [a,b]$. Note that, if $\theta_1, \theta_2 : [a,b] \to \mathbf{R}$ are two arguments of f, then $\theta_1 \sim \theta_2$. Even more, the function $\theta_1 - \theta_2$ is constant on each open interval contained in $[a,b] \setminus \{t \in [a,b] : f(t+) = -f(t-)\}$, and its values are integer multiples of 2π.

DEFINITION 4.2. *Let $[\theta] \in BV([a,b];\mathbf{R}/2\pi)$, and let $\{[\theta_h]\}_h \subseteq BV([a,b];\mathbf{R}/2\pi)$ be a sequence. We say that $[\theta_h] \to [\theta]$ in $L^1([a,b];\mathbf{R}/2\pi)$ as $h \to +\infty$ if, for any $\underline{\theta} \in [\theta]$ and any $h \in \mathbf{N}$, there exists $\underline{\theta}_h \in [\theta_h]$ such that $\underline{\theta}_h \to \underline{\theta}$ in $L^1([a,b])$ as $h \to +\infty$.*

Note that if $\theta \in BV([a,b])$, $\{\theta_h\}_h \subseteq BV([a,b])$, and $\theta_h \to \theta$ in $L^1([a,b])$ as $h \to +\infty$, then $[\theta_h] \to [\theta]$ in $L^1([a,b];\mathbf{R}/2\pi)$ as $h \to +\infty$.

THEOREM 4.1. *Let $[\theta] \in BV([a,b];\mathbf{R}/2\pi)$. Then there exists a function $\Theta \in [\theta]$, that we shall call a minimal representative of $[\theta]$, such that*

$$|\dot{\Theta}|([a,b]) = \inf\{|\dot{\underline{\theta}}|([a,b]) : \underline{\theta} \in [\theta]\}. \tag{4.1}$$

PROOF. Let $\underline{\theta} \in [\theta]$; as $\underline{\theta} \in BV([a,b])$, the set $\{t \in]a,b[: |\underline{\theta}(t+) - \underline{\theta}(t-)| > \pi\}$ has a finite number of elements $\{t_1, \dots, t_N\}$, where $a < t_1 < \cdots < t_N < b$. Starting from $\underline{\theta}$, we construct $\Theta \sim \underline{\theta}$, with $S_\Theta \subseteq S_{\underline{\theta}}$, by adding to $\underline{\theta}$ separately on each connected component of $[a,b] \setminus \{t_1, \dots, t_N\}$, a suitable integer multiple of 2π, in such a way that

$$|\Theta(t+) - \Theta(t-)| \le \pi \qquad \forall t \in]a,b[\tag{4.2}$$

(for instance, set $\Theta = \underline{\theta}$ on $[a, t_1[$, and $\Theta = \underline{\theta} + 2n_1\pi$ on $]t_1, b]$, where $n_1 \in \mathbf{Z}$ is such that $|\Theta(t_1+) - \Theta(t_1-)| \le \pi$, and continue).

Let us prove (4.1). Assume by contradiction that there exists $\theta_1 \in [\theta]$ such that

$$|\dot{\theta}_1|([a,b]) < |\dot{\Theta}|([a,b]). \tag{4.3}$$

As $\Theta \in BV([a,b])$, there exists a finite set of points $a < \tau_0 < \cdots < \tau_n < b$ such that

$$|\dot{\Theta}|(]\tau_{i-1}, \tau_i[) < \frac{\pi}{2} \qquad \forall i = 1, \dots, n. \tag{4.4}$$

As $\theta_1 \sim \Theta$, for any $i = 1, \dots, n$ there exists $k_i \in \mathbf{Z}$ such that $\theta_1(\tau_i+) - \theta_1(\tau_i-) = \Theta(\tau_i+) - \Theta(\tau_i-) + 2k_i\pi$; since $|\Theta(\tau_i+) - \Theta(\tau_i-)| \leq \pi$ by (4.2), we get $|\theta_1(\tau_i+) - \theta_1(\tau_i-)| \geq |\Theta(\tau_i+) - \Theta(\tau_i-)|$. Therefore, by (4.3) we deduce that there exists $j \in \{1, \dots, n\}$ such that, letting $I =]\tau_{j-1}, \tau_j[$,

$$|\dot{\theta}_1|(I) < |\dot{\Theta}|(I). \tag{4.5}$$

Using (4.4), we have that $\Theta(I)$ is contained, up to a set of zero Lebesgue measure, into a real interval of lenght less than $\frac{\pi}{2}$, and by (4.5) the same property holds for $\theta_1(I)$. Let us fix $\tau \in I$ a continuity point of Θ and θ_1, let $k \in \mathbf{Z}$ be such that $\theta_1(\tau) = \Theta(\tau) + 2k\pi$, and let $\theta_2 = \theta_1 - 2k\pi$. Then $\theta_2 \sim \theta_1 \sim \Theta$, $\dot{\theta}_2 = \dot{\theta}_1$, and $\theta_2(I) \cap \Theta(I)$ is non empty, since it contains $\Theta(\tau)$. Hence $\theta_2(I) \cup \Theta(I)$ is contained, up to a set of zero Lebesgue measure, into a real interval of lenght less than π. As $\theta_2 \sim \Theta$, we necessarily have that $\theta_2(I)$ coincides with $\Theta(I)$ up to a set of zero Lebesgue measure. It follows that $|\dot{\theta}_1|(I) = |\dot{\theta}_2|(I) = |\dot{\Theta}|(I)$, and this contradicts (4.5). □

We shall denote by $\mathcal{M}[\theta]$ the set of the minimal representatives Θ of $[\theta] \in BV([a,b]; \mathbf{R}/2\pi)$ such that $\Theta(a+) \in [0, 2\pi[$.

Let $\theta \in BV([a,b]; \mathbf{R}/2\pi)$, and let $\underline{\theta} \in [\theta]$. Observe that, if $\underline{\theta}(a+) \in [0, 2\pi[$ and $|\underline{\theta}(t+) - \underline{\theta}(t-)| \leq \pi$ for any $t \in [a,b]$, then $\underline{\theta} \in \mathcal{M}[\theta]$. In addition, if $\Theta_1, \Theta_2 \in \mathcal{M}[\theta]$, then

$$|\dot{\Theta}_1^s|(B) = |\dot{\Theta}_2^s|(B) \qquad \text{for any Borel set } B \subseteq [a,b]. \tag{4.6}$$

Note that, given $[\theta] \in BV([a,b]; \mathbf{R}/2\pi)$ and $\{[\theta_h]\}_h \subseteq BV([a,b]; \mathbf{R}/2\pi)$, then $[\theta_h] \to [\theta]$ in $L^1([a,b]; \mathbf{R}/2\pi)$ as $h \to +\infty$ if and only if for any $\Theta \in \mathcal{M}[\theta]$ and any $h \in \mathbf{N}$ there exists $\underline{\theta}_h \in [\theta_h]$ such that $\underline{\theta}_h \to \Theta$ in $L^1([a,b])$ as $h \to +\infty$. Note also that, in general, if $\Theta_h \in \mathcal{M}[\theta_h]$, the sequence $\{\Theta_h\}_h$ does not converge to a minimal representative Θ of $[\theta]$ in $L^1([a,b])$, unless all jumps of Θ are less than or equal to π and $\Theta(a+) \in [0, 2\pi[$.

LEMMA 4.1. *Let $[\theta] \in BV([a,b]; \mathbf{R}/2\pi)$ and let $\theta_1, \theta_2 \in [\theta]$. Then*

$$|\theta_1^a| = |\theta_2^a| \qquad \text{a.e. in } [a,b]. \tag{4.7}$$

PROOF. As θ_1, $\theta_2 \in BV([a,b])$ there exists a finite set of points $a = \tau_0 < \tau_1 < \dots < \tau_n < \tau_{n+1} = b$ such that $|\dot{\theta}_i|(]\tau_j, \tau_{j+1}[) < \frac{\pi}{4}$ for any $i = 1, 2$ and any $j = 0, \dots, n$. The assertion then follows reasoning as at the end of the proof of Theorem 4.1 (see (4.5) and below). □

We conclude this section with a result that will be useful in the proof of Theorem 6.1.

PROPOSITION 4.1. *Let $\gamma \in \mathcal{B}$, let $\{\gamma_h\}_h \subseteq \mathcal{B}$ be a sequence of curves such that $\dot{\gamma}_h \to \dot{\gamma}$ in $L^1([0,1]; \mathbf{R}^2)$ as $h \to +\infty$. Assume that γ and γ_h are parametrized with constant*

velocity on $[0,1]$, *for any* $h \in \mathbf{N}$. *Let* $\theta, \theta_h \in BV([0,1])$ *be arguments of* $\dot\gamma, \dot\gamma_h$ *respectively, for any* $h \in \mathbf{N}$. *Then* $\theta_h \to \theta$ *in* $L^1([0,1];\mathbf{R}/2\pi)$ *as* $h \to +\infty$.

PROOF. Write $\dot\gamma(t) = (R\cos\theta(t), R\sin\theta(t))$, and $\dot\gamma_h(t) = (R_h\cos\theta_h(t), R_h\sin\theta_h(t))$ for almost every $t \in [0,1]$ and any $h \in \mathbf{N}$ (see Lemma 3.1). We have $\lim_{h\to+\infty} R_h = R$. Using the notation of Lemma 3.1, there exists a finite set $\{s_1,\dots,s_m\}$ of points of $]0,1[$ such that, if J is any open interval contained in $[a,b]\setminus\{s_1,\dots,s_m\}$, then

$$\dot\gamma(J) \subset\subset A \subset\subset O_n, \tag{4.8}$$

for a suitable arc $A \subseteq \partial B_R(0)$ and a suitable $n \in \{1,\dots,4\}$. Let $\underline{\theta} \in [\theta]$; we have to prove that for any $h \in \mathbf{N}$ there exists $\underline{\theta}_h \in [\theta_h]$ such that $\underline{\theta}_h \to \underline{\theta}$ in $L^1([0,1])$ as $h \to +\infty$. It will be sufficient to construct the sequence $\{\underline{\theta}_h\}_h$ on each interval J, and to prove that $\underline{\theta}_h \to \underline{\theta}$ in $L^1(J)$. Let us fix such an interval J; without loss of generality, we can assume that $\dot\gamma(J) \subset\subset A \subset\subset O_1$. Let $B \subseteq \partial B_R(0)$ be an arc such that $A \subset\subset B \subset\subset O_1$ and, for any $h \in \mathbf{N}$, let $A_h \subset\subset B_h \subset\subset O_1^h = \partial B_{R_h}(0) \cap \{x > 0\}$ be the corresponding arcs on $\partial B_{R_h}(0)$ omothetic to A, B, and O_1, respectively. Let $\tau \in I$ be a continuity point of $\underline{\theta}$. For any $h \in \mathbf{N}$, define $\underline{\theta}_h : J \to \mathbf{R}$, $\underline{\theta}_h \in [\theta_h]$, as follows:

$$\underline{\theta}_h(t) = \begin{cases} \theta_h(t) & \text{for a.e. } t \in J \text{ such that } \dot\gamma_h(t) \in B_h, \\ \alpha_h(t) & \text{for a.e. } t \in J \text{ such that } \dot\gamma_h(t) \notin B_h, \end{cases}$$

where $\alpha_h \in BV([0,1])$ is an arbitrary function belonging to $[\theta_h]$ such that

$$|\alpha_h(t) - \theta(\tau)| \le \pi \qquad \text{for a.e. } t \in J. \tag{4.9}$$

To prove that $\underline{\theta}_h \to \underline{\theta}$ in $L^1(J)$ as $h \to +\infty$, we show that there exists a constant $M > 0$ such that

$$|\underline{\theta}_h(t) - \underline{\theta}(t)| \le M|\dot\gamma_h(t) - \dot\gamma(t)| \qquad \text{for a.e. } t \in J,\ \forall h \in \mathbf{N}. \tag{4.10}$$

Let $h \in \mathbf{N}$ and $t \in J$; if $\dot\gamma_h(t) \in B_h$, then (4.10) follows from the lipschitzianity of arcsin (see the definition of the argument in the proof of Lemma 3.1), since B_h is relatively compact in O_1^h, and $R_h \to R$ as $h \to +\infty$. If $\dot\gamma_h(t) \notin B_h$, then, as $\dot\gamma(J) \subset\subset A$ and $R_h \to R$ as $h \to +\infty$, there exists a constant $c > 0$ depending only on $B \setminus A$ such that $|\dot\gamma_h(t) - \dot\gamma(t)| \ge c$. From (4.8) we have that $|\underline{\theta}(\tau) - \underline{\theta}(t)| < \pi$; hence, using (4.9), we obtain

$$|\underline{\theta}_h(t) - \underline{\theta}(t)| \le |\alpha_h(t) - \underline{\theta}(\tau)| + |\underline{\theta}(\tau) - \underline{\theta}(t)| \le \pi + \pi \le \frac{2\pi}{c}|\dot\gamma_h(t) - \dot\gamma(t)|,$$

and this concludes the proof of (4.10). □

5. The energy functionals.

Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying properties (i)-(vi) listed in Section 1. Denote by ψ_∞ the recession function of ψ with respect to ξ, i.e.,

$$\psi_\infty(\eta, \xi) = \lim_{t \to 0^+} t\psi(\eta, \frac{\xi}{t}) \qquad \forall (\eta, \xi) \in \mathbf{R} \times \mathbf{R}.$$

Let $\theta \in BV([a,b]; \mathbf{R}/2\pi)$, $\underline{\theta} \in [\theta]$ and $\Theta \in \mathcal{M}[\theta]$; for any open interval $I \subseteq [a,b]$ define

$$K_\psi(\theta, I) = K_\psi^a(\theta, I) + K_\psi^s(\theta, I),$$

where

$$K_\psi^a(\theta, I) = \int_I \psi(\underline{\theta}, \dot{\underline{\theta}}^a)\, dt,$$

and

$$K_\psi^s(\theta, I) = \int_I \fint_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau, \frac{d\dot{\Theta}^s}{d|\dot{\Theta}^s|})\, d\tau\, d|\dot{\Theta}^s|.$$

Observe that by condition (v) and by (4.7) the term $K_\psi^a(\theta, I)$ does not depend on the choice of $\underline{\theta} \in [\theta]$, and by (4.6) the term $K_\psi^s(\theta, I)$ does not depend on the choice of $\Theta \in \mathcal{M}[\theta]$. In addition

$$K_\psi^s(\theta, I) \le \int_I \fint_{\underline{\theta}(s-)}^{\underline{\theta}(s+)} \psi_\infty(\tau, \frac{d\dot{\underline{\theta}}^s}{d|\dot{\underline{\theta}}^s|})\, d\tau\, d|\dot{\underline{\theta}}^s| \qquad \forall\, \underline{\theta} \in [\theta]. \tag{5.1}$$

Note that, using condition (vi),

$$K_\psi^s(\theta, I) = \int_{I \setminus S_\Theta} \psi_\infty(\Theta(s+), 1)\, d|\dot{\Theta}^s| + \sum_{s \in I \cap S_\Theta} \int_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau, 1)\, d\tau, \tag{5.2}$$

and, if $\theta \in \mathcal{C}^2([a,b])$, then

$$K_\psi(\theta, I) = K_\psi^a(\theta, I) = \int_I \psi(\theta, \dot{\theta})\, dt.$$

5.1. Semicontinuity and relaxation of $K_\psi(\theta, [a,b])$.

THEOREM 5.1. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying properties (i)-(vi), and let $I \subseteq [a,b]$ be an open interval. Let $[\theta] \in BV(I; \mathbf{R}/2\pi)$, and let $\{[\theta_h]\}_h \subseteq BV(I; \mathbf{R}/2\pi)$ be such that $[\theta_h] \to [\theta]$ in $L^1(I; \mathbf{R}/2\pi)$ as $h \to +\infty$. Then*

$$K_\psi(\theta, I) \le \liminf_{h \to +\infty} K_\psi(\theta_h, I). \tag{5.3}$$

PROOF. We can suppose that the right hand side of (5.3) is finite, otherwise the result is trivial. Let $\{[\theta_k]\}_k$ be a subsequence of $\{[\theta_h]\}_h$ such that

$$\lim_{k\to+\infty} K_\psi(\theta_k, I) = \liminf_{h\to+\infty} K_\psi(\theta_h, I) < +\infty.$$

Let $\Theta \in \mathcal{M}[\theta]$ and $\underline{\theta}_k \in [\theta_k]$ be such that $\underline{\theta}_k \to \Theta$ in $L^1(I)$ as $k \to +\infty$. By (1.1) we have

$$\sup_k |\dot{\Theta}_k|(I) \le \sup_k |\dot{\theta}_k|(I) < +\infty.$$

Hence [18] there exist a subsequence (still denoted by $\{\Theta_k\}_k$) and a function $\alpha \in BV(I)$ such that $\Theta_k \to \alpha$ in $L^1(I)$ as $k \to +\infty$. It follows that $[\Theta_k] \to [\alpha]$ in $L^1(I; \mathbf{R}/2\pi)$ as $k \to +\infty$, so that $\alpha \in [\theta]$. Note that α does not necessarily belong to $\mathcal{M}[\theta]$. Using (5.1) and [11] we then have

$$\begin{aligned}
&K_\psi(\theta, I) = \int_I \psi(\alpha, \dot{\alpha}^a)\, dt + \int_I \fint_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau, \frac{d\dot{\Theta}^s}{d|\dot{\Theta}^s|})\, d\tau\, d|\dot{\Theta}^s| \le \\
&\int_I \psi(\alpha, \dot{\alpha}^a)\, dt + \int_I \fint_{\alpha(s-)}^{\alpha(s+)} \psi_\infty(\tau, \frac{d\dot{\alpha}^s}{d|\dot{\alpha}^s|})\, d\tau\, d|\dot{\alpha}^s| \le \\
&\liminf_{k\to+\infty} \left(\int_I \psi(\Theta_k, \dot{\Theta}_k^a)\, dt + \int_I \fint_{\Theta_k(s-)}^{\Theta_k(s+)} \psi_\infty(\tau, \frac{d\dot{\Theta}_k^s}{d|\dot{\Theta}_k^s|})\, d\tau\, d|\dot{\Theta}_k^s| \right) = \\
&\lim_{k\to+\infty} K_\psi(\theta_k, I) = \liminf_{h\to+\infty} K_\psi(\theta_h, I).
\end{aligned}$$

□

THEOREM 5.2. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying properties (i)-(vi). Let $I \subseteq [a, b]$ be an open interval. For any $[\theta] \in BV(I; \mathbf{R}/2\pi)$ we have*

$$K_\psi(\theta, I) = \inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{[\theta_h]\}_h \subseteq BV(I; \mathbf{R}/2\pi), [\theta_h] \to [\theta] \text{ in } L^1(I; \mathbf{R}/2\pi)\}.$$

PROOF. Let $[\theta] \in BV(I; \mathbf{R}/2\pi)$; by Theorem 5.1 we deduce

$$K_\psi(\theta, I) \le \inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{[\theta_h]\}_h \subseteq BV(I; \mathbf{R}/2\pi), [\theta_h] \to [\theta] \text{ in } L^1(I; \mathbf{R}/2\pi)\}.$$

Let us prove the opposite inequality. Let $G : BV(I) \to [0, +\infty]$ be the map defined by

$$G(\theta, I) = \begin{cases} \int_I \psi(\theta, \dot{\theta})\, dt & \text{if } \theta \in W^{1,1}(I), \\ +\infty & \text{elsewhere on } BV(I). \end{cases}$$

In view of properties (i)-(iv), by [11] we have, for any $\theta \in BV(I)$,

$$\begin{aligned}
&K_\psi(\theta, I) = \overline{G}(\Theta, I) = \\
&\inf\{\liminf_{h\to+\infty} G(\theta_h, I) : \{\theta_h\}_h \subseteq W^{1,1}(I), \theta_h \to \Theta \text{ in } L^1(I)\} = \\
&\inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{\theta_h\}_h \subseteq W^{1,1}(I), \theta_h \to \Theta \text{ in } L^1(I)\} \geq \\
&\inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{\theta_h\}_h \subseteq BV(I), \theta_h \to \Theta \text{ in } L^1(I)\} \geq \\
&\inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{\theta_h\}_h \subseteq BV(I), [\theta_h] \to [\theta] \text{ in } L^1(I; \mathbf{R}/2\pi)\},
\end{aligned}$$

and this concludes the proof. □

Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be a system of curves of class $\mathcal{B}$ parametrized by arc lenght, and, for any $i = 1, \dots, m$ let Θ_i be an argument of $\dot\gamma^i$ on $[0, l(\gamma^i)]$. We define

$$K_\psi(\Gamma) = K^a_\psi(\Gamma) + K^s_\psi(\Gamma), \qquad F_\psi(\Gamma) = l(\Gamma) + K_\psi(\Gamma),$$

where

$$K^a_\psi(\Gamma) = \sum_{i=1}^m K^a_\psi(\Theta_i, [0, l(\gamma^i)]), \qquad K^s_\psi(\Gamma) = \sum_{i=1}^m K^s_\psi(\Theta_i, [0, l(\gamma^i)]).$$

Fix $i \in \{1, \dots, m\}$, and let for simplicity $\gamma^i = \gamma$, $\Theta_i = \Theta$, $\lambda = l(\gamma^i) > 0$. Let $T^\lambda : [0,1] \to [0, \lambda]$ be the map defined by $T^\lambda(t) = \lambda t$ for any $t \in [0,1]$, and denote by g the reparametrization of γ on $[0,1]$, i.e., $g(t) = \gamma(T^\lambda(t))$. Let ω be an argument of $\dot g$ on $[0,1]$. It is not difficult to prove that $\dot\omega(B) = \dot\Theta(T^\lambda(B))$ for any Borel set $B \subseteq [0,1]$. From the uniqueness of the Lebesgue decomposition of a measure, it follows that $\dot\omega^s(B) = \dot\Theta^s(T^\lambda(B))$. Hence, by (2.1), for almost every $t \in [0,1]$, we have

$$\dot\omega^a(t) = \lim_{\varrho\to0+} \frac{\dot\omega(]t-\varrho, t+\varrho[)}{2\varrho} = \lambda \lim_{\varrho\to0+} \frac{\dot\Theta(]\lambda t - \lambda\varrho, \lambda t + \lambda\varrho[)}{2\lambda\varrho} = \lambda \dot\Theta^a(T^\lambda(t)),$$

and, for $|\dot\omega^s|$-almost every $t \in [0,1]$, we have

$$\frac{d\dot\omega^s}{d|\dot\omega^s|}(t) = \lim_{\varrho\to0+} \frac{\dot\omega^s(]t-\varrho, t+\varrho[)}{|\dot\omega^s|(]t-\varrho, t+\varrho[)} = \lim_{\varrho\to0+} \frac{\dot\Theta^s(]\lambda t - \lambda\varrho, \lambda t + \lambda\varrho[)}{|\dot\Theta^s|(]\lambda t - \lambda\varrho, \lambda t + \lambda\varrho[)} = \frac{d\dot\Theta^s}{d|\dot\Theta^s|}(T^\lambda(t)).$$

We then have

$$K^a_\psi(\gamma) = \int_0^{l(\gamma)} \psi(\Theta, \dot\Theta^a)\, ds = l(\gamma) \int_0^1 \psi(\omega, l(\gamma)^{-1}\dot\omega^a)\, dt, \tag{5.4}$$

and

$$K^s_\psi(\gamma) = \int_0^{l(\gamma)} \fint_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau, \frac{d\dot\Theta^s}{d|\dot\Theta^s|})\, d\tau d|\dot\Theta^s| = \int_0^1 \fint_{\omega(t-)}^{\omega(t+)} \psi_\infty(\tau, \frac{d\dot\omega^s}{d|\dot\omega^s|})\, d\tau d|\dot\omega^s|. \tag{5.5}$$

6. Definition and semicontinuity of $F_\psi(E)$.

Let $E \in \mathcal{C}_b^2(\mathbf{R}^2)$, and let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be an oriented parametrization of ∂E. The normal vector $n(z)$ at $z \in \partial E$ will point towards the inside of E. For any $z \in \partial E$ let $\alpha(z)$ be the angle that the tangent unit vector at $z \in \partial E$ forms with the positive direction of the x-axis, and let $\kappa(z)$ be the curvature of ∂E. If ψ is a function satisfying properties (i)-(vi), we define the map $F_\psi : \mathcal{M} \to [0, +\infty]$ as

$$F_\psi(E) = \begin{cases} \mathcal{H}^1(\partial E) + \displaystyle\int_{\partial E} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) & \text{if } E \in \mathcal{C}_b^2(\mathbf{R}^2), \\ +\infty & \text{elsewhere on } \mathcal{M}. \end{cases} \tag{6.1}$$

We set $F_\psi(\emptyset) = 0$. Note that if $E \in \mathcal{C}_b^2(\mathbf{R}^2)$ and $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ is an oriented parametrization of ∂E of class $\mathcal{C}^2$ such that each γ^i is parametrized by arc lenght, i.e., $\gamma^i(s) = (\cos \Theta_i(s), \sin \Theta_i(s))$ for any $s \in [0, l(\gamma^i)]$, then

$$\int_{\partial E} \psi(\alpha, \kappa) d\mathcal{H}^1(z) = \sum_{i=1}^m \int_0^{l(\gamma^i)} \psi(\Theta_i, \dot{\Theta}_i)\, ds = K_\psi(\Gamma),$$

$$F_\psi(E) = l(\Gamma) + K_\psi(\Gamma) = F_\psi(\Gamma).$$

Note that if $p > 1$ and $\psi(\eta, \xi) = |\xi|^p$ for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}$, then

$$F_\psi(E) = \int_{\partial E} [1 + |\kappa|^p]\, d\mathcal{H}^1(z),$$

which reduces to the case studied in [3].

LEMMA 6.1. *Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be a system of curves of class $\mathcal{C}^2$. Then*

$$m \le \frac{\max(1, b)}{2\pi a} F_\psi(\Gamma), \tag{6.2}$$

where a and b are defined in (1.1).

PROOF. For any $i \in \{1, \dots, m\}$, if Θ_i is an argument of $\dot{\gamma}^i$, we have ([14, Th. 5.7.3])

$$2\pi \le \int_0^{l(\gamma^i)} |\dot{\Theta}_i(s)|\, ds.$$

Using (1.1) it follows

$$2\pi a \le \int_0^{l(\gamma^i)} \psi(\Theta_i, \dot{\Theta}_i)\, d\mathcal{L}^1(s) + l(\gamma^i) b \le \max(1, b) F_\psi(\gamma^i).$$

Summing over $i = 1, \dots, m$ we get

$$2\pi a m \le \max(1, b) F_\psi(\Gamma),$$

which is the assertion. □

PROPOSITION 6.1. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying conditions (i)-(vi). Let $\{\Gamma_h\}_h$ be a sequence of systems of curves of class $\mathcal{C}^2$ such that*

$$\sup_h l(\Gamma_h) < +\infty, \qquad \sup_h K_\psi(\Gamma_h) < +\infty, \tag{6.3}$$

and such that

(*) the traces (Γ_h) are contained in a bounded subset of $\mathbf{R}^2$ independent of h;

(**) there exists a positive constant c such that each curve γ_h^i of the system Γ_h satisfies $l(\gamma_h^i) \geq c$, for any h.

Then $\{\Gamma_h\}_h$ has a subsequence weakly convergent to a system of curves Γ of class $\mathcal{B}$.

PROOF. From (6.2), using (6.3) it follows that the number m_h of the curves of the system Γ_h is uniformly bounded with respect to h. Hence, for a subsequence $\{\Gamma_h\}_h$, there exists an integer m such that $\Gamma_h = \{\gamma_h^1, \ldots, \gamma_h^m\}$ for any h. Let us fix $i \in \{1, \ldots, m\}$; using condition (**) and (6.3) we have $c \leq l(\gamma_h^i) \leq C$ for any h, where C is an absolute positive constant. Hence the curve γ_h^i can be parametrized with constant velocity on $[0,1]$, and from (1.1) and (6.3), we get $\sup_h \int_0^1 |\frac{d^2\gamma_h^i}{d^2t}|\, dt < +\infty$. Therefore, using (*), the family $\{\gamma_h^i\}_h$ is uniformly bounded in $H^{2,1}$ with respect to h. It follows that ([18, Th. 1.19]), for a subsequence, there exist m curves $\gamma^1, \ldots, \gamma^m$ of class $\mathcal{B}$ such that $\gamma_h^i \to \gamma^i$ uniformly in $\mathcal{C}^0$, $\frac{d\gamma_h^i}{dt} \rightharpoonup \frac{d\gamma^i}{dt}$ weakly in BV for $h \to +\infty$, for any $i = 1, \ldots, m$. The system $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ verifies the assertion. □

The following result gives some properties of those sets E such that $\overline{F_\psi}(E) < +\infty$.

LEMMA 6.2. *Let $E \subseteq \mathbf{R}^2$ be a measurable set, let $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ be a sequence satisfying*

$$\sup_h \mathcal{H}^1(\partial E_h) < +\infty, \qquad \sup_h \int_{\partial E_h} \psi(\alpha_h, \kappa_h)\, d\mathcal{H}^1(z), \tag{6.4}$$

and suppose that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Let us define

$$E^\star = \{z \in \mathbf{R}^2 : \exists r > 0\ \mathcal{L}^2(B_r(z) \setminus E) = 0\}, \quad F^\star = \{z \in \mathbf{R}^2 : \exists r > 0\ \mathcal{L}^2(B_r(z) \cap E) = 0\}.$$

Then $E^\star$ and $F^\star$ are open, $E^\star$ is bounded, $\mathcal{L}^2(E \triangle E^\star) = 0$, $E^\star = \mathrm{int}(\mathbf{R}^2 \setminus F^\star)$, $F^\star = \mathrm{int}(\mathbf{R}^2 \setminus E^\star)$, and

$$\partial E^\star = \partial F^\star = \{z \in \mathbf{R}^2 : 0 < \mathcal{L}^2(B_r(z) \cap E) < \mathcal{L}^2(B_r(z))\ \forall r > 0\}.$$

Moreover there exists a limit system of curves Γ of class $\mathcal{B}$ with the following properties:

$E^\star = \mathrm{int}\big(A_\Gamma \cup (\Gamma)\big) \supseteq A_\Gamma$, *where* $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$;

$F^\star = \mathrm{int}\big(B_\Gamma \cup (\Gamma)\big) \supseteq B_\Gamma$, *where* $B_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 0\}$;

$$\partial E^\star = \partial F^\star = \partial A_\Gamma \cap \partial B_\Gamma \subseteq (\Gamma).$$

PROOF. We shall prove the existence of the limit system of curves $\Gamma \in \mathcal{B}$ such that $\mathcal{L}^2(E \triangle A_\Gamma) = 0$, since all the remaining assertions can be proven as in [3, Lemma 3.3].

For any h, from (6.3) and from Lemma 6.1 we have that ∂E_h has an oriented parametrization Δ_h of class $\mathcal{C}^2$, and the number m_h of curves of the system Δ_h is uniformly bounded with respect to h. Hence, for a subsequence (still denoted by $\{\Delta_h\}_h$), there exists an integer m such that $\Delta_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ for any h.

For any h, we want to replace the system Δ_h with a suitable disjoint system of curves Γ_h satisfying all hypotheses of Proposition 6.1.

Following [3, Lemma 3.3], we can replace Δ_h with a disjoint system of curves $\Lambda_h = \{\gamma_h^{i_1}, \dots, \gamma_h^{i_k}\}$ of class $\mathcal{C}^2$, with $1 \le i_1 < \dots < i_k \le m$, having the following properties:

$$\text{there exists } R > 0 \text{ such that } (\Lambda_h) \subseteq B_R(0) \text{ for any } h; \tag{6.5}$$

for any $M > 0$ there exists $h_M \in \mathbf{N}$ such that

$$I(\Lambda_h, z) = I(\Delta_h, z) \tag{6.6}$$

for any $h \ge h_M$ and any $z \in B_M(0) \setminus (\Delta_h)$. Note that from (6.5) and (6.6) it follows that, if h is large enough, $I(\Lambda_h, z) \in \{0, 1\}$ for any $z \in \mathbf{R}^2 \setminus (\Lambda_h)$.

For simplicity, write $\Lambda_h = \{\gamma_h^1, \dots, \gamma_h^k\}$. We need a further substitution, in order to fulfill hypothesis (**) of Proposition 6.1.

Let us consider the sequence $\{\gamma_h^1\}_h$. If $\lim_{h \to +\infty} l(\gamma_h^1) = 0$, replace Λ_h by the syst $\Lambda_h^1 = \{\gamma_h^2, \dots, \gamma_h^k\}$. Observe that (6.5) yields $(\gamma_h^1) \subseteq \overline{B_R(0)}$ for any h, so that, po passing to a subsequence $\{\gamma_h^1\}_h$, we can find $r_1 > 0$ e $z_1 \in B_R(0)$ such that, for any $0 < r \le r_1$, there exists $h_r \in \mathbf{N}$ with $(\gamma_h^1) \subseteq B_r(z_1)$ for any $h \ge h_r$. Thus $I(\gamma_h^1, z) = 0$ for any $h \ge h_r$ and any $z \in \mathbf{R}^2 \setminus B_r(z_1)$.

If $\{l(\gamma_h^1)\}_h$ does not converge to zero as $h \to +\infty$, there exist a subsequence $\{\gamma_h^1\}_h$ and a constant $c_1 > 0$ such that $l(\gamma_h^1) \ge c_1$ for any h. In this case, let us define $\Lambda_h^1 = \Lambda_h$. Starting from $\{\Lambda_h^1\}_h$, we repeat the same procedure for $\{\gamma_h^2\}_h$, and we obtain a new sequence of systems of curves $\{\Lambda_h^2\}_h$. After k steps, we end up with a sequence of systems of curves $\{\Lambda_h^k\}_h$, which we shall denote by $\{\Gamma_h\}_h$.

By construction, for any h, Γ_h is a disjoint system of curves of class $\mathcal{C}^2$, with $\Gamma_h = \{\gamma_h^{i_1}, \dots, \gamma_h^{i_n}\}$, $1 \le i_1 < \dots < i_n \le k$, and $(\Gamma_h) \subseteq B_R(0)$. Let us suppose that $l = k - n$ eliminations have been made to replace the system Λ_h with the system Γ_h, and let $r_1, \dots, r_l$ be the corresponding numbers (respectively $z_1, \dots, z_l$ be the corresponding points) defined above. Then, for any $0 < r \le \min_{j=1,\dots,l} r_j$, there exists $h_r \in \mathbf{N}$ sufficiently large such that

$$I(\Gamma_h, z) = I(\Lambda_h, z) \in \{0, 1\} \tag{6.7}$$

for any $h \geq h_r$ and any $z \in \left(\mathbf{R}^2 \setminus \bigcup_{j=1}^l B_r(z_j)\right) \setminus (\Lambda_h)$. Using (6.6), for any $M > 0$ and any $0 < r \leq \min_{j=1,\dots,l} r_j$, there exists $h_{r,M} \in \mathbf{N}$ such that

$$I(\Gamma_h, z) = I(\Delta_h, z) \tag{6.8}$$

for any $h \geq h_{r,M}$ and any $z \in \left(B_M(0) \setminus \bigcup_{j=1}^l B_r(z_j)\right) \setminus (\Delta_h)$. Define $A_h = \{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) = 1\}$. Then $A_h \in \mathcal{C}_b^2(\mathbf{R}^2)$, and $\partial A_h = (\Gamma_h)$. Since $\sup_h F_\psi(\Gamma_h) \leq \sup_h F_\psi(E_h) < +\infty$ by (6.4), by Proposition 6.1 there exists a subsequence $\{\Gamma_h\}_h$ weakly convergent to a limit system of curves $\Gamma \in \mathcal{B}$; moreover $(\Gamma) \subseteq \overline{B_R(0)}$ and, using (6.7), we have that, if $z \in \mathbf{R}^2 \setminus (\Gamma)$, then $I(\Gamma, z) \in \{0, 1\}$. Let $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$. It is clear that $A_\Gamma \subseteq B_R(0)$. Since $\chi_{A_h}(z) = I(\Gamma_h, z)$ for any $z \in \mathbf{R}^2 \setminus (\Gamma_h)$, and $\chi_{A_\Gamma}(z) = I(\Gamma, z)$ for any $z \in \mathbf{R}^2 \setminus (\Gamma)$, by the continuity properties of the index and by the Dominated Convergence Theorem, we have $A_h \to A_\Gamma$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$.

Let us prove that $\mathcal{L}^2(E \triangle A_\Gamma) = 0$. By (6.8), for any $M > 0$ and any $0 < r \leq \min_{j=1,\dots,l} r_j$, we have $(A_h \cap B_M(0)) \setminus \bigcup_{j=1}^l B_r(z_j) = \left(E_h \cap B_M(0)\right) \setminus \bigcup_{j=1}^l B_r(z_j)$ for $h \geq h_{r,M}$ large enough. Passing to the limit as $r \to 0$, we get $\mathcal{L}^2\left((E \triangle A_\Gamma) \cap B_M(0))\right) = 0$. As M is arbitrary, we deduce $\mathcal{L}^2(E \triangle A_\Gamma) = 0$. □

The next comparison result is crucial for the proof of the semicontinuity Theorem 6.1.

LEMMA 6.3. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying conditions (i)-(vi). Let $E \in \mathcal{C}_b^2(\mathbf{R}^2)$, let $\Gamma \in \mathcal{B}$ be a system of curves such that $(\Gamma) \supseteq \partial E$. Then*

$$\int_{\partial E} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) \leq K_\psi^a(\Gamma). \tag{6.9}$$

PROOF. Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$, and, for any $i = 1, \dots, m$ let Θ_i be an argument of $\dot{\gamma}^i$. For any measurable set $T \subseteq \partial E$ and any $i = 1, \dots, m$ let

$$V_i = \{s \in [0, l(\gamma^i)] : \gamma^i(s) \in T\}, \qquad T_i = \gamma^i(V_i).$$

Since $\partial E \subseteq (\Gamma)$, we have $T = \bigcup_{i=1}^m T_i$. To prove the thesis it will be enough to show that

$$\int_{T_i} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) \leq K_\psi^a(\Theta_i, V_i) \quad \text{for any } i = 1, \dots, m, \tag{6.10}$$

where T varies over a suitable finite measurable partition $\mathcal{P}$ of ∂E. In fact, from (6.10) it follows that

$$\int_T \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) \leq \sum_{i=1}^m \int_{T_i} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) \leq \sum_{i=1}^m K_\psi^a(\Theta_i, V_i),$$

hence

$$\int_{\partial E} \psi(\alpha,\kappa)\, d\mathcal{H}^1(z) = \sum_{T\in\mathcal{P}} \int_{T\cap\partial E} \psi(\alpha,\kappa)\, d\mathcal{H}^1(z) \le \sum_{i=1}^{m} K^a_\psi(\Theta_i,[0,l(\gamma^i)]) = K^a_\psi(\Gamma),$$

that proves the assertion. We choose the finite partition $\mathcal{P}$ of ∂E as follows. Any element of $\mathcal{P}$ must be contained in a rectangle in which ∂E is a cartesian graph. This means that for any $T \in \mathcal{P}$ we can choose a suitable orthogonal coordinate system, two bounded open intervals I, J, and a function $f : I \to J$ of class C^2 such that

$$(I\times J)\cap E = \{(x,y)\in I\times J : y < f(x)\},$$

$$(I\times J)\cap \partial E = \{(x,f(x)) : x\in I\}, \qquad T\subseteq (I\times J)\cap\partial E.$$

Let us fix $i \in \{1,\dots,m\}$ and let $\gamma^i(s) = (\gamma^i_1(s),\gamma^i_2(s))$. For any $s \in V_i$ we have $\gamma^i_2(s) = f(\gamma^i_1(s))$. Hence $\dot\gamma^i_2(s) = \frac{d}{ds} f(\gamma^i_1(s))$ for almost every $s \in V_i$, and, since $|\dot\gamma(s)| = 1$, we have

$$\cos\Theta_i(s) = \dot\gamma^i_1(s) = \frac{\pm 1}{\sqrt{1+(f'(\gamma^i_1(s)))^2}}, \qquad \sin\Theta_i(s) = \dot\gamma^i_2(s) = \frac{\pm f'(\gamma^i_1(s))}{\sqrt{1+(f'(\gamma^i_1(s)))^2}},$$

for almost every $s \in V_i$.

Set

$$V_i^\pm = \{s\in V_i : \dot\gamma^i_1(s) = \frac{\pm 1}{\sqrt{1+(f'(\gamma^i_1(s)))^2}}\}, \qquad T_i^\pm = \gamma^i(V_i^\pm).$$

Note that, from $\dot\gamma^i_2(s) = \frac{d}{ds} f(\gamma^i_1(s))$, we have

$$\dot\gamma^i_2(s) = \frac{\pm f'(\gamma^i_1(s))}{\sqrt{1+(f'(\gamma^i_1(s)))^2}} \qquad \text{for a.e. } s\in V_i^\pm. \tag{6.11}$$

As $\dot\gamma^i_1 \in BV$, and $\frac{\pm 1}{\sqrt{1+(f'(\gamma^i_1))^2}} \in W^{1,1}$ and these two functions coincide almost everywhere on $V_i^\pm$, it follows (see [15, Th. 4.5.9]) that their approximate derivatives (i.e., the absolutely continuous part of these derivatives with respect to the Lebesgue measure) must coincide almost everywhere on $V_i^\pm$. We then have

$$\ddot\gamma^{i\,a}_1(s) = \frac{\mp f''(\gamma^i_1(s))f'(\gamma^i_1(s))\dot\gamma^i_1(s)}{(1+(f'(\gamma^i_1(s)))^2)^{\frac{3}{2}}} = \frac{\mp f''(\gamma^i_1(s))\dot\gamma^i_2(s)}{(1+(f'(\gamma^i_1(s)))^2)^{\frac{3}{2}}} \qquad \text{for a.e. } s\in V_i^\pm.$$

Similarly, by (6.11),

$$\ddot\gamma^{i\,a}_2(s) = \frac{\pm f''(\gamma^i_1(s))\dot\gamma^i_1(s)}{(1+(f'(\gamma^i_1(s)))^2)^{\frac{3}{2}}} \qquad \text{for a.e. } s\in V_i^\pm.$$

It follows that

$$\ddot{\gamma}^{ia}(s) = (-\sin\Theta_i(s), \cos\Theta_i(s))\frac{\pm f''(\gamma_1^i(s))}{(1+(f'(\gamma_1^i(s)))^2)^{\frac{3}{2}}} \qquad \text{for a.e. } s \in V_i^{\pm}.$$

On the other hand, from (3.7) we have

$$\ddot{\gamma}^{ia}(s) = (-\sin\Theta_i(s), \cos\Theta_i(s))\dot{\Theta}_i(s) \qquad \text{for a.e. } s \in V_i^{\pm},$$

so we conclude

$$\dot{\Theta}_i^a(s) = \frac{\pm f''(\gamma_1^i(s))}{(1+(f'(\gamma_1^i(s)))^2)^{\frac{3}{2}}} = \pm\kappa(\gamma^i(s)) \quad \text{for a.e. } s \in V_i^{\pm}. \tag{6.12}$$

Since

$$\Theta_i(s) = \alpha(\gamma^i(s)) \quad (\text{mod } 2\pi) \text{ for a.e. } s \in V_i^{\pm}, \tag{6.13}$$

by (6.13), (6.12) and properties (v) and (vi) (see Section 1) we find

$$K_\psi^a(\Theta_i, V_i) = \int_{V_i} \psi(\Theta_i, \dot{\Theta}_i^a)\, ds = \int_{V_i} \psi(\alpha(\gamma^i), \kappa(\gamma^i))\, ds. \tag{6.14}$$

But [15, Th. 3.2.6] ensures

$$\int_{V_i} \psi(\alpha(\gamma^i), \kappa(\gamma^i))\, ds \geq \int_{T_i} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z).$$

This, together with (6.14), gives (6.10) and concludes the proof of the Lemma. □

THEOREM 6.1. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying properties (i)-(vi). Let $E \in \mathcal{C}_b^2(\mathbf{R}^2)$, and let $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ be a sequence converging to E in $L^1(\mathbf{R}^2)$. Then*

$$F_\psi(E) \leq \liminf_{h\to+\infty} F_\psi(E_h). \tag{6.15}$$

PROOF. We can suppose that the right hand side of (6.15) is finite, otherwise the result is trivial. Let $\{E_{hk}\}_k$ be a subsequence of $\{E_h\}_h$ with the property that

$$\lim_{k\to+\infty} F_\psi(E_{hk}) = \liminf_{h\to+\infty} F_\psi(E_h) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. Using Lemma 6.1, we have that, for any k, ∂E_k has a finite number of connected components. Let Δ_k be an oriented parametrization of ∂E_k. Since $\{E_k\}_k$ satisfies (6.4), the

sequence $\{\Delta_k\}_k$ satisfies (6.3). Reasoning as in the proof of Proposition 6.1, we can suppose that $\Delta_k = \{\gamma_k^1, \dots, \gamma_k^m\}$ with m independent of k. Let $\{\Gamma_k\}_k = \{\gamma_k^{i_1}, \dots, \gamma_k^{i_n}\}$ be the sequence and let $\Gamma \in \mathcal{B}$ be the limit system of curves constructed in the proof of Lemma 6.2. Then $n \le m$, $\Gamma = \{\gamma^{i_1}, \dots, \gamma^{i_n}\}$, and, for any $j = 1, \dots, n$ we have ${\gamma_k}^{i_j} \to \gamma^{i_j}$ uniformly in C^0 and $\frac{d{\gamma_k}^{i_j}}{dt} \rightharpoonup \frac{d\gamma^{i_j}}{dt}$ weakly in BV as $k \to +\infty$. Denote by $\Theta_k^{i_j}$ (respectively Θ_{i_j}) an argument of $\dot\gamma_k^{i_j}$ (respectively of $\dot\gamma^{i_j}$). From Proposition 4.1 we have that $\Theta_k^{i_j} \to \Theta_{i_j}$ in L^1 as $k \to +\infty$. We have

$$\liminf_{h\to+\infty} F_\psi(E_h) = \lim_{k\to+\infty} F_\psi(E_k) \ge \lim_{k\to+\infty} \sum_{j=1}^n l(\gamma_k^{i_j}) +$$

$$\liminf_{k\to+\infty} \sum_{j=1}^n \int_0^{l(\gamma_k^{i_j})} \psi(\Theta_k^{i_j}, \dot\Theta_k^{i_j})\, ds = \sum_{j=1}^n l(\gamma^{i_j}) +$$

$$\liminf_{k\to+\infty} \sum_{j=1}^n l(\gamma_k^{i_j}) \int_0^1 \psi(\Theta_k^{i_j}, l(\gamma_k^{i_j})^{-1}\dot\Theta_k^{i_j})\, dt \ge$$

$$l(\Gamma) + \sum_{j=1}^n l(\gamma^{i_j}) \liminf_{k\to+\infty} \int_0^1 \psi(\Theta_k^{i_j}, l(\gamma_k^{i_j})^{-1}\dot\Theta_k^{i_j})\, dt.$$

Using (5.3), (5.4), and (5.5), we then get

$$\liminf_{h\to+\infty} F_\psi(E_h) \ge l(\Gamma) + \sum_{j=1}^n K_\psi(\Theta_{i_j}, [0, l(\gamma^{i_j})]) = l(\Gamma) + K_\psi(\Gamma) = F_\psi(\Gamma).$$

Recalling that Lemma 6.2 ensures that $(\Gamma) \subseteq \partial E$, from Lemma 6.3 we obtain $F_\psi(\Gamma) \ge F_\psi(E)$, and this concludes the proof. □

THEOREM 6.2. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F_\psi}(E) < +\infty$. Then $E^\star$ satisfies the following properties:*

$E^\star$ is bounded, open, and $\mathcal{L}^2(E \triangle E^\star) = 0$;

$\mathcal{H}^1(\partial E^\star) < +\infty$;

there exists a limit system of curves Γ of class $\mathcal{B}$ such that $(\Gamma) \supseteq \partial E^\star$ and $E^\star = \mathrm{int}(A_\Gamma \cup (\Gamma))$, where $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$;

$\overline{F_\psi}(E) \ge \inf\{F_\psi(\Gamma) : \Gamma \in \mathcal{A}(E)\}$, where $\mathcal{A}(E)$ is the collection of all limit systems of curves Γ of class $\mathcal{B}$ satisfying (iii).

PROOF. The first three assertions follow from Lemma 6.2. Let us prove the last assertion. Since $\overline{F_\psi}(E) < +\infty$, there exists a sequence $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ satisfying (6.4) and converging to E in $L^1(\mathbf{R}^2)$. Using the same notation as in the proof of Theorem 6.1, for a subsequence $\{E_k\}_k$ we have

$$\liminf_{h\to+\infty} F_\psi(E_h) = \lim_{k\to+\infty} F_\psi(E_k) \ge l(\Gamma) + \liminf_{k\to+\infty} K_\psi(\Gamma_k) \ge l(\Gamma) + K_\psi(\Gamma) = F_\psi(\Gamma).$$

Taking the infimum first with respect to the system $\Gamma \in \mathcal{A}(E)$ and then with respect to the approximating sequence $\{E_h\}_h$, the last assertion follows. □

6.1. Localization. Let $\Omega \subseteq \mathbf{R}^2$ be an open set. We say that a measurable set $E \subseteq \mathbf{R}^2$ is of class $\mathcal{C}^2(\Omega)$ if E is bounded, open, and $\Omega \cap \partial E$ is of class $\mathcal{C}^2$. We define

$$F_\psi(E,\Omega) = \int_{\Omega\cap\partial E} (1+\psi(\alpha,\kappa)\ d\mathcal{H}^1(z).$$

COROLLARY 6.1. *Let E be a set of class $\mathcal{C}^2(\Omega)$, and let $\{E_h\}_h$ be a sequence of sets of class $\mathcal{C}^2(\Omega)$ converging to E in $L^1(\Omega)$. Then*

$$F_\psi(E,\Omega) \le \liminf_{h\to+\infty} F_\psi(E_h,\Omega). \tag{6.16}$$

PROOF. We can assume that the right hand side of (6.16) is finite, otherwise the result is trivial. It will be enough to show that

$$F_\psi(E,A) \le \liminf_{h\to+\infty} F_\psi(E_h,A) \tag{6.17}$$

for any open set A which is relatively compact in Ω. Let $\{E_{hk}\}_k$ be subsequence of $\{E_h\}_h$ such that

$$\lim_{k\to+\infty} F_\psi(E_{hk},A) = \liminf_{h\to+\infty} F_\psi(E_h,A) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. Since

$$\sup_h \mathcal{H}^1(\Omega\cap\partial E_h) < +\infty, \quad \sup_h \int_{\Omega\cap\partial E_h} \psi(\alpha_h,\kappa_h)\ d\mathcal{H}^1(z) < +\infty,$$

it follows that, for any k,

$$A\cap\partial E_k \subseteq \bigcup_{i=1}^{m_k}(\gamma_k^i) \cup \bigcup_{j=1}^{r_k}(\beta_k^j),$$

where

$\{\gamma_k^1,\dots,\gamma_k^{m_k}\}$ is a disjoint system of curves of class $\mathcal{C}^2$ such that $\overline{A}\cap(\gamma_k^i) \ne \emptyset$ for any $i=1,\dots,m_k$;

$\{\beta_k^1,\dots,\beta_k^{r_k}\}$ is a finite family of simple regular curves of class $\mathcal{C}^2$ such that $|\frac{d\beta_k^j}{dt}|$ is constant, $\beta_k^j(0),\beta_k^j(1) \in \partial\Omega$ and $(\beta_k^j)\cap\partial A \ne \emptyset$ for any $j=1,\dots,r_k$;

the sets $(\gamma_k^1),\dots,(\gamma_k^{m_k})$, $(\beta_k^1),\dots,(\beta_k^{r_k})$ are mutually disjoint;

$$F_\psi(\gamma_k^i) + \sum_{j=1}^{r_k} l(\beta_k^j) + \sum_{j=1}^{r_k}\int_0^{l(\beta_k^j)} \psi(\dot\beta_k^j,\ddot\beta_k^j)\ ds < +\infty. \tag{6.18}$$

Using Lemma 6.1 it follows that $\{m_k\}_k$ is uniformly bounded with respect to k. As $l(\beta_k^j) \geq 2\mathrm{dist}(\partial A, \partial\Omega) > 0$ for any $j = 1, \ldots, r_k$, using (6.18) we have that $\{r_k\}_k$ is also uniformly bounded with respect to k. Possibly passing to a suitable subsequence, we can suppose that m_k and r_k are independent of k. Denote these numbers by m and r, respectively. As all curves of the family $\{\gamma_k^1, \ldots, \gamma_k^m, \beta_k^1, \ldots, \beta_k^r\}$ meet $\overline{A}$, from (6.18) it follows that their traces are contained in a bounded subset of $\mathbf{R}^2$ independent of k. Let us consider only the subfamily $\{\gamma_k^{i_1}, \ldots, \gamma_k^{i_n}\}$ of $\{\gamma_k^1, \ldots, \gamma_k^m\}$ of those curves which do not have infinitesimal lenght; repeating the arguments of Proposition 6.1, by compactness there exists a family $\Gamma = \{\gamma^{i_1}, \ldots, \gamma^{i_n}, \beta^1, \ldots, \beta^r\}$ of curves of class $\mathcal{B}$ such that $\gamma_k^{i_j} \to \gamma^{i_j}$ and $\beta_k^l \to \beta^l$ uniformly in $\mathcal{C}^0$, $\frac{d\gamma_k{}^{i_j}}{dt} \rightharpoonup \frac{d\gamma^{i_j}}{dt}$ and $\frac{d\beta_k^l}{dt} \rightharpoonup \frac{d\beta^l}{dt}$ weakly in BV as $k \to +\infty$, for any $j = 1, \ldots, n$ and any $l = 1, \ldots, r$. Therefore, as $E \in \mathcal{C}^2(\Omega)$, it follows that $A \cap (\Gamma) \supseteq A \cap \partial E$ (see [3, Th. 7.1]). Repeating the last part of the proof of Theorem 6.1 relatively to A, we get (6.17). □

7. Energy of an ordered partition.

In this section, following the approach of [22], we study the lower semicontinuous envelope of the functional $\mathcal{P}_\psi$ defined in Section 1.

Let $n \in \mathbf{N}$, $n \geq 1$, let $g \in L^\infty(\mathbf{R}^2)$ be a function with compact support, and let us define the map $\mathcal{P}_\psi : (L^1(\mathbf{R}^2))^n \to [0, +\infty]$ by

$$\mathcal{P}_\psi(E_1, \ldots, E_n) = \begin{cases} \displaystyle\sum_{i=1}^{n} F_\psi(E_i) + \sum_{i=1}^{n+1} \int_{E_i'} (u_i - g)^2 \, dz & \text{if } E_1, \ldots, E_n \in (\mathcal{C}_b^2(\mathbf{R}^2))^n, \\ +\infty & \text{elsewhere on } \mathcal{M}^n, \end{cases}$$

where, for any $i = 1, \ldots, n+1$, the sets E_i' and the numbers u_i are defined in (1.2).

THEOREM 7.1. *If $\overline{\mathcal{P}_\psi}$ denotes the lower semicontinuous envelope of $\mathcal{P}_\psi$ with respect to the product topology induced on $\mathcal{M}^n$ by $(L^1(\mathbf{R}^2))^n$, we have*

$$\overline{\mathcal{P}_\psi}(E_1, \ldots, E_n) = \sum_{i=1}^{n} \overline{F_\psi}(E_i) + \sum_{i=1}^{n} \int_{E_i'} (u_i - g)^2 \, dz.$$

PROOF. By the definition of the product topology and the continuity of the term $\int_{E_i'} (u_i - g)^2 \, dz$ with respect to the $L^1(\mathbf{R}^2)$-topology, one easily obtains

$$\overline{\mathcal{F}_\psi}(E_1, \ldots, E_n) \geq \sum_{i=1}^{n} \overline{F_\psi}(E_i) + \sum_{i=1}^{n+1} \int_{E_i'} (u_i - g)^2 \, dz.$$

Let us prove the opposite inequality. Let $(E_1, \ldots, E_n) \in \mathcal{M}^n$; we can assume that $\overline{F_\psi}(E_i) < +\infty$ for any $i = 1, \ldots, n$, otherwise the result is trivial. For any $i = 1, \ldots, n$ let $\{E_i^h\}_h \subseteq$

$C_b^2(\mathbf{R}^2)$ be a sequence converging to E_i in $L^1(\mathbf{R}^2)$ and such that $\lim_{h\to+\infty} F_\psi(E_i^h) = \overline{F_\psi}(E_i) < \infty$. Then, as $(E_1^h,\dots,E_n^h) \to (E_1,\dots,E_n)$ in $(L^1(\mathbf{R}^2))^n$ as $h \to +\infty$, we have

$$\sum_{i=1}^n \overline{F_\psi}(E_i) + \int_{E_i'} (u_i - g)^2\, dz = \lim_{h\to+\infty} \left[\sum_{i=1}^n F_\psi(E_i^h) + \sum_{i=1}^{n+1} \int_{E_i^{h\prime}} (u_i - g)^2\, dz\right]$$
$$\geq \overline{\mathcal{P}_\psi}(E_1,\dots,E_n).$$

This concludes the proof. □

This result shows that, to compute $\overline{\mathcal{P}_\psi}$, one needs to compute $\overline{F_\psi}$. This will be the goal of the next sections.

8. An upper bound for $\overline{F_\psi}$.

LEMMA 8.1. *Let γ be a closed regular curve of class C^2 parametrized by arc lenght, and set $\dot\gamma(s) = (\cos\Theta(s), \sin\Theta(s))$ for any $s \in [0, l(\gamma)]$. Let $s_1, s_2 \in [0, l(\gamma)]$, $s_1 < s_2$, be such that $\gamma(s_1) = \gamma(s_2)$. Then there exist $\sigma_1, \sigma_2 \in [s_1, s_2]$ such that $|\Theta(\sigma_1) - \Theta(\sigma_2)| \geq \pi$. In particular, $\int_{s_1}^{s_2} |\dot\Theta| ds \geq \pi$.*

PROOF. Since $\gamma(s) = \int_0^s (\cos\Theta(\tau), \sin\Theta(\tau)) d\tau + \gamma(0)$ for any $s \in [0, l(\gamma)]$, and $\gamma(s_1) = \gamma(s_2)$, we have

$$\int_{s_1}^{s_2} \cos\Theta(\tau) d\tau = \int_{s_1}^{s_2} \sin\Theta(\tau) d\tau = 0.$$

Therefore $\int_{s_1}^{s_2} \sin(\Theta(\tau) - \alpha) d\tau = 0$ for any $\alpha \in [0, 2\pi]$. It follows that, for any $\alpha \in [0, 2\pi]$, there exists $\sigma_\alpha \in [s_1, s_2]$ such that $\sin(\Theta(\sigma_\alpha) - \alpha) = 0$, hence $\Theta(\sigma_\alpha) = \alpha + k\pi$, for some $k \in \mathbf{Z}$. Since this equality holds for any $\alpha \in [0, 2\pi]$, we have that, given $p \in \mathbf{S}^1$, then either $p \in \Theta([s_1, s_2])$, or $p + \pi \in \Theta([s_1, s_2])$. This easily implies that there exist $\sigma_1, \sigma_2 \in [s_1, s_2]$ such that $|\Theta(\sigma_1) - \Theta(\sigma_2)| \geq \pi$. □

LEMMA 8.2. *Let $\gamma \in \mathcal{B}$ and let Θ be an argument of $\dot\gamma$ on $[0, l(\gamma)]$. Let $s_1, s_2 \in [0, l(\gamma)]$, $s_1 < s_2$, be such that $\gamma(s_1) = \gamma(s_2)$. Then $\int_{s_1}^{s_2} |\dot\Theta| \geq \pi$.*

PROOF. We can suppose that there exists $k \in \mathbf{Z}$ such that $|\Theta(0+) - \Theta(l(\gamma)-) - 2k\pi| < \pi$. Let us extend Θ on $\mathbf{R}$ as follows: $\Theta(s + l(\gamma)) = \Theta(s) + 2k\pi$. Let $\{\varrho_h\}_h$ be a sequence of symmetric mollifiers, and define $\alpha^h = \Theta \star \varrho_h$. Then $\alpha^h \rightharpoonup \Theta$ weakly in $BV_{\mathrm{loc}}(\mathbf{R})$ as $h \to +\infty$, $\lim_{h\to+\infty} \alpha_h(s) = \dfrac{\Theta(s+) - \Theta(s-)}{2}$ for any $s \in \mathbf{R}$, and $\alpha^h(s + l(\gamma)) = \alpha^h(s) + 2k\pi$. Set

$$f_h(s) = \int_0^s (\cos\alpha^h, \sin\alpha^h)\, d\tau + \gamma(0) \qquad \forall s \in \mathbf{R}.$$

We have that f_h is parametrized by arc lenght, $\lim_{h\to+\infty} f_h = \gamma$ uniformly on $[0, l(\gamma)]$, $\lim_{h\to+\infty} \dot f_h = \dot\gamma$ in $L^1([0, l(\gamma))]$, $\lim_{h\to+\infty} \dot f_h(s) = \frac{\dot\gamma(s+) - \dot\gamma(s-)}{2}$ for any $s \in [0, l(\gamma)]$, and f_h is, in general, neither closed nor simple. For any h define

$$d_h^i = \int_{s_1}^{s_2} (\cos\alpha^h, \sin\alpha^h)\, d\tau, \qquad g_h(t) = f_h(t) - \frac{t}{s_2 - s_1} d_h \qquad \forall t \in \mathbf{R}.$$

By definition it follows that $\lim_{h\to+\infty}|d_h| = 0$, $\lim_{h\to+\infty} g_h = \gamma$ uniformly on $[0, l(\gamma)]$, $\lim_{h\to+\infty}\dot{g}_h = \dot{\gamma}$ in $L^1([0, l(\gamma))]$, $\lim_{h\to+\infty}\dot{g}_h(s) = \frac{\dot{\gamma}(s+)-\dot{\gamma}(s-)}{2}$ for any $s \in [0, l(\gamma)]$, and $g_h(s_1) = g_h(s_2)$, so that $g_{h|[s_1,s_2]}$ is a regular closed curve of class C^2. Let us reparametrize $g_{h|[s_1,s_2]}$ by arc lenght, that is, write

$$\gamma_h(s) = g_h(t_h(s)),$$

where $t_h(s)$ is the inverse of the map $s_h = s_h(t) = \int_{s_1}^t |\dot{g}_h|\, d\tau$ for any $t \in [s_1, s_2]$. Set $\dot{\gamma}_h(s) = (\cos\phi_h(s), \sin\phi_h(s))$ for any $s \in [0, l(\gamma_h)]$. Let $\{s_1^h\}_h$, $\{s_2^h\}_h$ be two sequences of points of $[0, l(\gamma_h)]$ converging to s_1 and s_2, respectively. By Lemma 8.1 we have

$$\pi \le \lim_{h\to+\infty} |\phi_h(s_1^h) - \phi_h(s_2^h)| = \frac{1}{2}|\Theta(s_1+) - \Theta(s_1-) - \Theta(s_2+) + \Theta(s_2-)| \le \int_{s_1}^{s_2} |\dot{\Theta}|.$$

□

In the sequel of this section $E \subseteq \mathbf{R}^2$ will be a bounded open set that can be written, locally, as the subgraph of a Lipschitz function having derivative of class BV, with respect to a suitable orthogonal coordinate system. As ∂E is locally Lipschitz continuous, ∂E can have corners but can not have cusp points. One can prove that there exists a disjoint system of Lipschitz curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ such that $\dot{\gamma}^i \in BV$ for any $i = 1, \dots, m$, $(\Gamma) = \partial E$, and

$$E = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}.$$

For any $i = 1, \dots, m$ let Θ_i be an argument of $\dot{\gamma}_i$. Note that, as ∂E is locally Lipschitz, we have that $|\Theta_i(s+) - \Theta_i(s-)| < \pi$ for any $s \in]0, l(\gamma^i)[$ and any $i = 1, \dots, m$. Moreover we can suppose that for any $i = 1, \dots, m$ there exists $k_i \in \mathbf{Z}$ such that $|\Theta_i(0+) - \Theta_i(l(\gamma^i)-) - 2k_i\pi| < \pi$. The aim of this section is to prove

$$\overline{F_\psi}(E) \le \sum_{i=1}^m F_\psi(\Theta_i, [0, l(\gamma^i)]), \tag{8.1}$$

which in particular gives $\overline{F_\psi}(E) < +\infty$.

As $\Theta_i \in BV([0, l(\gamma^i)])$, we can find $\delta > 0$ such that

$$|\Theta_i(s+) - \Theta_i(s-)| \le \pi - \delta \qquad \forall s \in [0, l(\gamma^i)],\ \forall i = 1, \dots, m. \tag{8.2}$$

We shall use the same notation and some arguments of Lemma 8.2, with γ replaced by γ^i, and Θ replaced by Θ_i. Let us extend Θ_i by periodicity $\pmod{2\pi}$ on $\mathbf{R}$. Inequality (8.2) holds then for any $s \in \mathbf{R}$.

Fix $i \in \{1, \dots, m\}$; define $\alpha_i^h = \Theta_i \star \varrho_h$. Then $\alpha_i^h \rightharpoonup \Theta_i$ weakly in $BV_{\mathrm{loc}}(\mathbf{R})$ as $h \to +\infty$,

$$\lim_{h\to+\infty} |\dot{\alpha}_i^h|(A) = |\dot{\Theta}_i|(A) \tag{8.3}$$

for any bounded open set $A \subseteq \mathbf{R}$ such that $|\dot{\Theta}_i|(\partial A) = 0$. Following [11, Section 3] one can check that $\{\alpha_i^h\}_h$ is a minimizing sequence for $K_\psi(\Theta_i, [0, l(\gamma^i)])$, i.e.,

$$\lim_{h\to+\infty} K_\psi(\alpha_i^h, [0, l(\gamma^i)]) = K_\psi(\Theta_i, [0, l(\gamma^i)]) \qquad \forall i = 1, \dots, m.$$

Set

$$f_h^i(s) = \int_0^s (\cos\alpha_i^h, \sin\alpha_i^h)\, d\tau + \gamma^i(0) \qquad \forall s \in \mathbf{R},$$

$$d_h^i = \int_0^{l(\gamma^i)} (\cos\alpha_i^h, \sin\alpha_i^h)\, d\tau, \qquad g_h^i(t) = f_h^i(t) - \frac{t}{l(\gamma^i)} d_h^i \qquad \forall t \in \mathbf{R}.$$

By definition it follows that $\lim_{h\to+\infty} |d_h^i| = 0$, that g_h^i is closed, i.e., $g_h^i(0) = g_h^i(l(\gamma^i))$, and that $\lim_{h\to+\infty} g_h^i = \gamma^i$ and $\lim_{h\to+\infty} |\dot{\gamma}_h^i| = 1$ uniformly on $[0, l(\gamma^i)]$. Let us reparametrize g_h^i by arc lengh, that is, write

$$\gamma_h^i(s) = g_h^i(t_h(s)),$$

where $t_h(s)$ is the inverse of the map $s_h = s_h(t) = \int_0^t |\dot{g}_h^i|\, d\tau$. Then, using the fact that $\dot{f}_h^i \cdot \ddot{f}_h^i = 0$, we have

$$\dot{t}_h(s) = \frac{1}{|\dot{g}_h^i(t_h(s))|}, \qquad \ddot{t}_h(s) = -\frac{\dot{g}_h^i(t_h(s)) \cdot \ddot{g}_h^i(t_h(s))}{|\dot{g}_h^i(t_h(s)|^4} = \frac{\ddot{f}_h^i \cdot d_h^i}{l(\gamma^i)|\dot{g}_h^i(t_h(s))|^4}.$$

Hence

$$\ddot{\gamma}_h^i(s) = (\dot{t}_h(s))^2 \ddot{f}_h^i(t_h(s)) + \ddot{t}_h(s)\dot{g}_h^i(t_h(s)) =$$
$$((\dot{t}_h(s))^2 + \frac{\dot{g}_h^i \otimes d_h^i}{l(\gamma^i)|\dot{g}_h^i(t_h(s))|^4})\ddot{f}_h^i(t_h(s)).$$

Since $\lim_{h\to+\infty} \dot{t}_h = 1$ uniformly and $\lim_{h\to+\infty} |d_h^i| = 0$, setting $\dot{\gamma}_h^i(s) = (\cos\phi_h^i(s), \sin\phi_h^i(s))$ for any $s \in [0, l(\gamma_h^i)]$, then $\{\phi_h^i\}_h$ is also a minimizing sequence for $K_\psi(\Theta_i, [0, l(\gamma^i)])$, i.e.,

$$\lim_{h\to+\infty} K_\psi(\phi_h^i, [0, l(\gamma_h^i)]) = K_\psi(\Theta_i, [0, l(\gamma^i)]) \qquad \forall i = 1, \dots, m.$$

In addition, by (8.3), we have also

$$\lim_{h\to+\infty} |\dot{\phi}_h^i|(A) = |\dot{\Theta}_i|(A) \tag{8.4}$$

for any bounded open set $A \subseteq \mathbf{R}$ such that $|\dot{\Theta}_i|(\partial A) = 0$.

Let us prove that the curves γ_h^i are simple for h large enough. Assume by contradiction that there exist two sequences $\{s_h\}_h, \{\sigma_h\}_h$ of points of $[0, l(\gamma_h^i)]$ such that $\gamma_h^i(s_h) = \gamma_h^i(\sigma_h)$ for any h, and such that γ_h^i is simple between s_h and σ_h, for any h. Since γ^i is simple by

assumption, it follows that $\lim_{h\to+\infty}|s_h-\sigma_h|=0$. Let $p\in(\gamma^i)$, $p=\lim_{h\to+\infty}\gamma_h^i(\sigma_h)$, and $s_0=\gamma^{i^{-1}}(p)$. Possibly passing to a subsequence, we can assume that there exist $\phi_1=\lim_{h\to+\infty}\phi_h^i(s_h), \phi_2=\lim_{h\to+\infty}\phi_h^i(\sigma_h)$. By Lemma 8.1 we have $|\phi_h^i(s_h)-\phi_h^i(\sigma_h)|\geq\pi$ for any h, so that $|\phi_1-\phi_2|\geq\pi$. Hence $|\Theta_i(s_0-)-\Theta_i(s_0+)|\geq\pi$, a contradiction with (8.2).

We conclude that $\{\gamma_h^i\}_h$ is a sequence of closed simple curves of class C^∞; moreover, since $\gamma_h^i\to\gamma^i$ uniformly for any $i=1,\dots,m$ we have that $(\gamma_h^i)\cap(\gamma_h^j)=\emptyset$ for any $i\neq j$, provided h is sufficiently large. Consider the disjoint system of curves $\Gamma_h=\{\gamma_h^1,\dots,\gamma_h^m\}$, and define

$$E_h=\{z\in\mathbf{R}^2\setminus(\Gamma_h):I(\Gamma_h,z)\equiv 1\pmod 2\}.$$

By the continuity property of the index and by the Dominated Convergence Theorem (see also [8, III.8.8 (v)]), it follows that $E_h\to\{z\in\mathbf{R}^2\setminus(\Gamma):I(\Gamma,z)\equiv 1\pmod 2\}$ in $L^1(\mathbf{R}^2)$ as $h\to+\infty$.

In addition $F_\psi(E_h)=F_\psi(\Gamma_h)$, so that

$$\overline{F_\psi}(E)\leq\lim_{h\to+\infty}F_\psi(E_h)=\lim_{h\to+\infty}F_\psi(\Gamma_h)=l(\Gamma)+K_\psi(\Gamma),$$

which proves (8.1).

Note that

$$\int_0^{l(\gamma^i)}|\dot\Theta_i|=2\pi$$

provided that $\gamma^i(0)$ is not a corner. Indeed, it sufficient to observe that, in this case, $\Theta_i(0)$ in not a jump point for Θ_i (extended on $\mathbf{R}$), and to apply (8.4) with $A=]0,l(\gamma^i)[$.

9. Computation of $\overline{F_\psi}$ in some model examples.

Let $T>0$; in what follows we shall consider the convex function $\psi:\mathbf{R}\times\mathbf{R}\to[0,+\infty[$ defined by

$$\psi(\eta,\xi)=\begin{cases}c_1\xi^2 & \text{if } |\xi|<T,\\ c_2|\xi|-c_3 & \text{if } |\xi|\geq T,\end{cases}\tag{9.1}$$

where $c_1>0$, $c_1T^2=c_2T-c_3$, $c_2\geq 2c_1T$.

This function has been recently suggested in [22] to study some problems in image segmentation (see Section 1). Obviously $\psi_\infty(\eta,\xi)=c_2|\xi|$, and ψ is of class C^1 if and only if $c_2=2c_1T$. As ψ depend only on $|\xi|$, we shall write $\psi=\psi(|\xi|)$.

Let $\Gamma=\{\gamma^1,\dots,\gamma^m\}\in\mathcal{B}$, and for any $i=1,\dots,m$ let Θ_i be an argument of $\dot\gamma^i$; for any $p\in(\Gamma)$ we set

$$F_\psi(\Gamma,\{p\})=c_2\sum_{i=1}^m\sum_{s\in\gamma^{i^{-1}}(p)}|\Theta_i(s+)-\Theta_i(s-)|,$$

and

$$T_{(\Gamma)}(p) = \bigcup_{i=1}^{m} \bigcup_{s\in\gamma^{i-1}(p)} \dot\gamma^i(s+) \cup (-\dot\gamma^i(s-)).$$

THEOREM 9.1. *Let ψ be defined as in (9.1), and let E be the set of Figure 1.1, with $0 \le \alpha < 2\pi$. Then*

$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \min(\alpha, 2\pi - \alpha, |\pi - \alpha|) \le \overline{F_\psi}(E) \le$$
$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2|\pi - \alpha|,$$

where $\partial_r E = \partial E \setminus \{p\}$.

PROOF. It is not difficult (see Figure 1.2) to find a sequence $\{E_h\}_h \subseteq \mathcal{C}^2_b(\mathbf{R}^2)$ converging to E in $L^1(\mathbf{R}^2)$ and such that

$$\lim_{h\to+\infty} F_\psi(E_h) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2|\pi - \alpha|.$$

Consequently

$$\overline{F_\psi}(E) \le \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2|\pi - \alpha|.$$

It remains to show

$$\overline{F_\psi}(E) \ge \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \min(\alpha, 2\pi - \alpha, |\pi - \alpha|). \tag{9.2}$$

Let $\{E_h\}_h \subseteq \mathcal{C}^2_b(\mathbf{R}^2)$ be a sequence converging to E in $L^1(\mathbf{R}^2)$ such that $\overline{F_\psi}(E) = \lim_{h\to+\infty} F_\psi(E_h) < +\infty$. For any h let Γ_h be an oriented parametrization of ∂E_h of class $\mathcal{C}^2$, let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be the limit system of curves of class $\mathcal{B}$ given by Lemma 6.2. For any $i = 1, \dots, m$ let Θ_i be an argument of $\dot\gamma^i$. We recall (see Lemma 6.2) that

$$E = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}. \tag{9.3}$$

Theorem 5.1 gives

$$\overline{F_\psi}(E) = \lim_{h\to+\infty} F_\psi(E_h) = \lim_{h\to+\infty} F_\psi(\Gamma_h) \ge F_\psi(\Gamma) = l(\Gamma) + K^a_\psi(\Gamma) + K^s_\psi(\Gamma). \tag{9.4}$$

Using the inclusion $\partial E \subseteq (\Gamma)$ proven in Lemma 6.2 and reasoning as in Lemma 6.3 we deduce

$$l(\Gamma) + K^a_\psi(\Gamma) \ge \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z). \tag{9.5}$$

Moreover, the definition of K^s_ψ yields

$$K^s_\psi(\Gamma) \geq F_\psi(\Gamma, \{p\}). \tag{9.6}$$

From (9.4), (9.5) and (9.6) it follows

$$\overline{F_\psi}(E) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + F_\psi(\Gamma, \{p\}). \tag{9.7}$$

In view of (9.7), to conclude the proof of (9.2) it remains to show

$$F_\psi(\Gamma, \{p\}) \geq c_2 \min(\alpha, 2\pi - \alpha, |\pi - \alpha|). \tag{9.8}$$

We shall restrict ourselves to the case $m = 1$, i.e., when the system Γ reduces to one closed curve γ of class $\mathcal{B}$ (the case $m > 1$ is similar). We denote by Θ an argument of $\dot\gamma$.

Observe that there exists a positive constant ω such that

$$1 + \psi(|\xi|) \geq \omega|\xi| \qquad \forall \xi \in \mathbf{R}, \tag{9.9}$$

hence, if $s_0, s_1 \in [0, l(\gamma)]$, $s_0 < s_1$, we have

$$F_\psi(\gamma_{|]s_0,s_1[}) \geq \omega \int_{s_0}^{s_1} |\dot\Theta| \geq \omega|\Theta(s_1-) - \Theta(s_0+)|.$$

Using the previous inequality and Lemma 8.2, one can show that the set $\gamma^{-1}(p)$ is finite.

If $j \in \{\pm 1\}$, as a notation we set

$$\dot\gamma(s \text{ sign}(j)) = \begin{cases} \dot\gamma(s+) & \text{if } j = 1, \\ \dot\gamma(s-) & \text{if } j = -1. \end{cases}$$

For any $v \in \mathbf{S}^1$, let

$$A_v = \{(s, j) \ : \ s \in \gamma^{-1}(p),\ j \in \{\pm 1\},\ v = j\dot\gamma(s \text{ sign}(j))\}.$$

In the sequel we shall consider only those unit vectors v such that $A_v \neq \emptyset$ (which exist by the inclusion $\partial E \subseteq (\gamma)$).

Then A_v is a finite set, it is composed by distinct elements, and $v_1 \neq v_2 \Rightarrow A_{v_1} \cap A_{v_2} = \emptyset$. If $s_0 \in \gamma^{-1}(p)$ and γ is of class C^1 in a neighbourhood of s_0, then, if $v = \dot\gamma(s_0)$, and if $\dot\gamma(s+), \dot\gamma(s-) \neq v$ for any $s \in \gamma^{-1}(p)$, $s \neq s_0$, we have $A_v = (s_0, +)$.

If γ stops at p and goes back along the same direction $\mathbf{R}v$, i.e., $v = \dot\gamma(s+) = -\dot\gamma(s-)$ for some $s \in \gamma^{-1}(p)$, the corresponding elements of A_v are $(s, +1)$ and $(s, -1)$, which are obviously distinct. Geometrically, given $v \in \mathbf{S}^1$, each element of A_v is in correspondence

with a small arc of (γ) containing p lying on the same part of v and having v as the tangent unit vector at p, considered with its orientation. Define $N(v)$ as the number of elements of A_v.

Let τ_1, τ_2 be the two tangent unit vectors of ∂E at p chosen as in Figure 1.1. We claim that for any $v \in \mathbf{S}^1$ such that $v \neq \tau_1, \tau_2$, then $N(v)$ is even, while $N(\tau_1)$ and $N(\tau_2)$ are odd, unless $\tau_1 = \tau_2$ (i.e., p is a cusp point of ∂E, and $\alpha = 0$). Let $v \in \mathbf{S}^1$, $v \neq \tau_1, \tau_2$, and assume that v points towards the complement of E (the case in which v points towards E is similar). Suppose by contradiction that $N(v)$ is odd. Since $\gamma^{-1}(p)$ is a finite set and since there exist $\dot\gamma(s+)$, $\dot\gamma(s-)$ at each $s \in \gamma^{-1}(p)$, we can find a small half-cone C with vertex at p, in the direction of v, with $C \subseteq \mathbf{R}^2 \setminus E$, in such a way that $C \cap (\gamma)$ contains only the small arcs of (γ) having $\mathbf{R}v$ as the tangent line at p (which are in an odd number by assumption). Choose two points z_1, z_2 in $C \setminus (\gamma)$, one on each side of v, and close enough to v. Then (see [5, Lemma 9.2.5])

$$|I(\gamma, z_1) - I(\gamma, z_2)| \equiv 1 \pmod 2,$$

and this contradicts (9.3). Similarly, one can prove that $N(\tau_1)$ and $N(\tau_2)$ are odd. Hence the claim is proven.

Set $\gamma^{-1}(p) = \{s_1, \dots, s_k\}$, $k \geq 1$, and define the unit vectors $v_1, \dots, v_{2k}$ by

$$v_i = \dot\gamma(s_i+), \qquad v_{k+i} = -\dot\gamma(s_i-), \qquad \forall i = 1, \dots, k. \tag{9.10}$$

Observe that, for any $v \in \mathbf{S}^1$ there is a one to one correspondence between A_v and the set $\{i \in \{1, \dots, 2k\} : v_i = v\}$.

We want to construct a finite subfamily $\{v_{\alpha_1}, \dots, v_{\alpha_h}, v_{\beta_1}, \dots, v_{\beta_h}\}$ of $\{v_1, \dots, v_{2k}\}$, with $1 \leq h \leq k$, such that

$$\begin{aligned}
&\alpha_1, \dots, \alpha_h, \beta_1, \dots, \beta_h \text{ are distinct indices between 1 and } 2k;\\
&v_{\alpha_1} = \tau_1, \quad v_{\beta_h} = \tau_2,\\
&\forall n = 1, \dots, h, \text{ if } \alpha_n \leq k \text{ then } \beta_n = \alpha_n + k,\\
&\forall n = 1, \dots, h, \text{ if } \alpha_n > k \text{ then } \beta_n = \alpha_n - k;\\
&v_{\alpha_n} = v_{\beta_{n-1}} \qquad \forall n = 2, \dots, h.
\end{aligned}$$

We shall suppose $\tau_1 \neq \tau_2$, since the case $\tau_1 = \tau_2$ can be treated similarly. We shall reason inductively. As $N(\tau_1)$ is odd, there exists $l \in \{1, \dots, 2k\}$ such that $v_l = \tau_1$; set $\alpha_1 = l$. If $\alpha_1 \leq k$, set $\beta_1 = \alpha_1 + k$. If $\alpha_1 > k$, set $\beta_1 = \alpha_1 - k$. If $v_{\beta_1} = \tau_1$, then, as the set $\{l \in \{1, \dots, 2k\} : v_l = \tau_1\}$ has an odd number of elements, there exists $\alpha_2 \in \{l \in \{1, \dots, 2k\} : v_l = \tau_1\}$, with $\alpha_2 \neq \alpha_1, \beta_1$, such that $v_{\alpha_2} = \tau_1$. In this case we delete the indices α_1, β_1, and we restart the algorithm by defining $\alpha_2 = \alpha_1$. Therefore we shall assume that $v_{\beta_1} \neq \tau_1$. If $v_{\beta_1} = \tau_2$ then the algorithm stops. Otherwise, if $v_{\beta_1} \neq \tau_2$,

as the set $\{l \in \{1,\dots,2k\} : v_l = v_{\beta_1}\}$ has an even number of elements, there exists $\alpha_2 \in \{l \in \{1,\dots,2k\} : v_l = v_{\beta_1}\}$, with $\alpha_2 \neq \alpha_1, \beta_1$, such that $v_{\alpha_2} = v_{\beta_1}$. Assume that the sequence $\alpha_1,\dots,\alpha_q,\beta_1,\dots,\beta_{q-1}$ of indices has been constructed, with the required properties. Set $\beta_q = \alpha_q + k$ if $\alpha_q \leq k$, and $\beta_q = \alpha_q - k$ if $\alpha_q > k$. If $v_{\beta_q} = \tau_2$ then the algorithm stops, thus verifying all required properties. If $v_{\beta_1} \neq \tau_2$, since $N(v_{\beta_q})$ is even if $v_{\beta_q} \neq \tau_1$, and is odd if $v_{\beta_q} = \tau_1$, there exists an index $\alpha_{q+1} \neq \alpha_1,\dots,\alpha_q,\beta_1,\dots,\beta_q$ such that $v_{\alpha_{q+1}} = v_{\beta_q}$. The procedure stops after h steps, for some integer h.

Note that the indices $\alpha_1,\dots,\alpha_h,\beta_1,\dots,\beta_h$ are not necessarily ordered.

Let $\theta_1,\theta_2 \in \mathbf{R}/2\pi$; we set $d(\theta_1,\theta_2) = \min(|\theta_1-\theta_2|, 2\pi - |\theta_1-\theta_2|)$. Then $0 \leq d(\theta_1,\theta_2) \leq \pi$, and by (c) of Lemma 3.1 we get

$$d(\Theta(s-),\Theta(s+)) = |\Theta(s+) - \Theta(s-)| \qquad \text{for any } s \in S_\Theta. \tag{9.11}$$

Furthermore, set $\delta(\theta_1,\theta_2) = \min\{d(\theta_1,\theta_2), d(\theta_1,\theta_2+\pi)\}$. Then

$$\delta(\theta_1,\theta_2) = \delta(\theta_1+\pi,\theta_2) = \delta(\theta_1,\theta_2+\pi) = \delta(\theta_1+\pi,\theta_2+\pi), \tag{9.12}$$

and

$$\delta(\theta_1,\theta_2) \leq d(\theta_1,\theta_2). \tag{9.13}$$

It is readily verified that δ is a distance on $\mathbf{R}/2\pi$. If n_1,n_2 are unit vectors, by identifying n_i with the corresponding angle θ_i (hence $-n_i$ corresponds to $\theta_i+\pi$), we shall use the notation $d(n_1,n_2)$ and $\delta(n_1,n_2)$ in place of $d(\theta_1,\theta_2)$ and $\delta(\theta_1,\theta_2)$. In particular

$$\delta(n_1,n_2) = \delta(-n_1,n_2) = \delta(n_1,-n_2). \tag{9.14}$$

Observe that

$$\delta(\tau_1,\tau_2) = \min(\alpha, 2\pi-\alpha, |\pi-\alpha|). \tag{9.15}$$

Then, using the fact that $\{v_{\alpha_1},\dots,v_{\alpha_h},v_{\beta_1},\dots,v_{\beta_h}\} \subseteq T_{(\gamma)}(p)$, recalling (9.10) we deduce

$$\begin{aligned} c_2^{-1}F_\psi(\gamma,\{p\}) &= \sum_{s\in\gamma^{-1}(p)} |\Theta(s+)-\Theta(s-)| = \\ &\sum_{i=1}^{k} |\Theta(s_i+)-\Theta(s_i-)| \geq \sum_{q=1}^{h} d(v_{\alpha_q},-v_{\beta_q}). \end{aligned} \tag{9.16}$$

Consequently, using (9.13), (9.14), the triangular property of δ, and (9.15), we have

$$c_2^{-1}F_\psi(\gamma,\{p\}) \geq \sum_{q=1}^{h} \delta(v_{\alpha_q},-v_{\beta_q}) = \sum_{q=1}^{h} \delta(v_{\alpha_q},v_{\beta_q}) \geq \delta(\tau_1,\tau_2). \tag{9.17}$$

This, in view of (9.15), shows (9.8) (recall that $m = 1$), and concludes the proof of the theorem. □

Note that, if in the proof of Theorem 9.1 one has $T_{(\gamma)}(p) \subseteq \{\tau_1, \tau_2\}$ then

$$F_\psi(\gamma, \{p\}) = c_2 \sum_{s \in \gamma^{-1}(p)} |\Theta(s+) - \Theta(s-)| \geq c_2 d(\tau_1, -\tau_2) = c_2|\pi - \alpha|. \tag{9.18}$$

COROLLARY 9.1. *Let ψ be defined as in (9.1), let E be the set of Figure 1.1, and assume that*

$$\frac{\pi}{2} \leq \alpha \leq \frac{3\pi}{2}.$$

Then

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2|\pi - \alpha|.$$

COROLLARY 9.2. *Let ψ be defined as in (9.1), let E be the disjoint union of a finite number of curvilinear polygons $P_1, \ldots, P_q$. For any $i = 1, \ldots, q$ denote by $\alpha_i^1, \ldots, \alpha_i^{h(i)}$ the angles of the polygon P_i. Then*

$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q} \sum_{j=i}^{h(i)} \min(\alpha_i^j, 2\pi - \alpha_i^j, |\pi - \alpha_i^j|) \leq \overline{F_\psi}(E) \leq$$

$$\leq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q} \sum_{j=i}^{h(i)} |\pi - \alpha_i^j|,$$

where $\partial_r E$ denotes ∂E without the vertices, and κ denotes the curvature of $\partial_r E$. If in particular $\frac{\pi}{2} \leq \alpha_i^j \leq \frac{3\pi}{2}$ for any $i = 1, \ldots, q$ and any $j = 1, \ldots, h(i)$, then

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q} \sum_{j=i}^{h(i)} |\pi - \alpha_i^j|. \tag{9.19}$$

Reasoning as in Theorem 9.1 one can prove the following result.

PROPOSITION 9.1. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set of class C^2 up to a countable set of points $\{p_i\}_{i=1}^\infty$. Assume that each p_i is a corner point of ∂E with angle α_i such that $0 < c \leq \alpha_i \leq 2\pi - c < 2\pi$ for any i. Then $\overline{F_\psi}(E) = +\infty$.*

Let us prove now the following result, which shows that equality (9.19) does not hold in general if $\alpha_i^j < \frac{\pi}{2}$ or $\alpha_i^j > \frac{3\pi}{2}$ for some i, j, and shows also that the functional $\overline{F_\psi}(E, \cdot)$, if considered as a set function, is not a measure and, in particular, cannot be represented as an integral of the form (6.1) (see [3]).

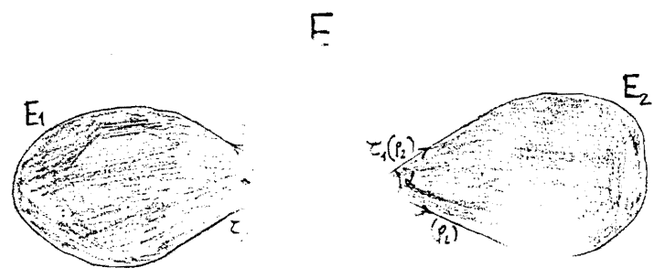

Fig. 9.

THEOREM 9.2. *Let $E = E_1 \cup E_2$ be* [illegible] *that*

$$0 < [illegible] < [illegible].$$

Then if the distance D betweeen p_1 and [illegible] *as*

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2D + 2c_2\alpha,$$

where $\partial_r E = \partial E \setminus \{p_1, p_2\}$.

In particular, if D is sufficiently small,

$$\overline{F_\psi}(E) < \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2c_2(\pi - \alpha).$$

PROOF. It is not difficult to show that the sequence $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ of Figure 9.2 converges to E in $L^1(\mathbf{R}^2)$ and

$$\lim_{h\to+\infty} F_\psi(E_h) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2D + 2c_2\alpha.$$

Consequently

$$\overline{F_\psi}(E) \le \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2D + 2c_2\alpha. \tag{9.20}$$

Let us prove the opposite inequality. Let $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ be a sequence which converges to E in $L^1(\mathbf{R}^2)$ and such that $\lim_{h\to+\infty} F_\psi(E_h) = \overline{F_\psi}(E) < +\infty$. Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$

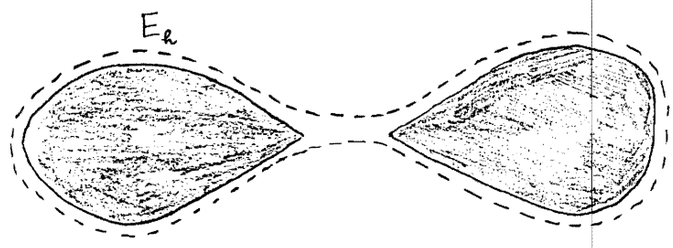

Fig. 9.2: An approximation of E.

be the limit system of curves of class $\mathcal{B}$ given by Theorem 6.2. Reasoning as in (9.4) and (9.5) we deduce

$$\overline{F_\psi}(E) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + l(\Gamma_E) + K^a_\psi(\Gamma_E) + K^s_\psi(\Gamma), \tag{9.21}$$

where $\Gamma_E = \{z \in \mathbf{R}^2 : z \notin \partial E, z \in (\Gamma)\}$. It will be enough to show that, if D is sufficiently small, then

$$l(\Gamma_E) + K^a_\psi(\Gamma_E) + K^s_\psi(\Gamma) \geq 2D + 2c_2\alpha. \tag{9.22}$$

Let $\tau_1(p_i)$, $\tau_2(p_i)$ be the two tangent unit vectors of ∂E at p_i, $i = 1, 2$, chosen as in Figure 9.1. We claim that there is no loss of generality if we suppose that $T_{(\Gamma)}(p_i)$ strictly contains $\{\tau_1(p_i), \tau_2(p_i)\}$, for $i = 1, 2$. Assume by contradiction that $\{\tau_1(p_1), \tau_2(p_1)\} \supseteq T_{(\Gamma)}(p_1)$. Then by (9.18) we have

$$F_\psi(\Gamma, \{p_1\}) \geq c_2|\pi - \alpha| = c_2(\pi - \alpha).$$

Moreover (9.17) yields $F_\psi(\Gamma, \{p_2\}) \geq c_2\alpha$, so that $K^s_\psi(\Gamma) \geq c_2\pi$. Hence, from (9.21) we deduce

$$\overline{F_\psi}(E) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + K^s_\psi(\Gamma) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2\pi.$$

Then (9.20) yields $2D + 2c_2\alpha \geq c_2\pi$, which is a contradiction, provided D is sufficiently small (recall that $0 < \alpha < \frac{\pi}{2}$). Obviously, the same argument can be applied by interchanging p_1 with p_2, and the claim is proven. Therefore, using the parity of the elements of $T_{(\Gamma)}(p_i)$ (see the proof of Theorem 9.1) there are at least two vectors belonging to $T_{(\Gamma)}(p_i)$ which do not belong to $\{\tau_1(p_i), \tau_2(p_i)\}$ for $i = 1, 2$.

Let us show that at least two of the elements of $T_{(\Gamma)}(p_i)$ are contained in a cone C having direction $p_2 - p_1$ with vertex at p_i and of amplitude $\alpha + o(D)$, for $i = 1, 2$. Indeed by (9.17) we have $F_\psi(\Gamma, \{p_2\}) \geq c_2\alpha$ (respectively $F_\psi(\Gamma, \{p_1\}) \geq c_2\alpha$), so that (9.20) forces $F_\psi(\Gamma, \{p_1\}) \leq c_2(\alpha + 2D)$ (respectively $F_\psi(\Gamma, \{p_2\}) \leq c_2(\alpha + 2D)$). The assertion follows from (9.16) with γ replaced by Γ.

Let $v \in T_{(\Gamma)}(p_1)$, $v \neq \tau_1(p_1), \tau_2(p_1)$, be one element of $T_{(\Gamma)}(p_1)$ which is contained in the cone C. Without loss of generality, we can assume that v points towards E_2. Let $i \in \{1, \ldots, m\}$ and $s_0 \in [0, l(\gamma^i)]$ be such that $v = \dot{\gamma}^i(s_0+)$. There are two possibilities, either $(\gamma^i) \cap \partial E_2 = \emptyset$, or $(\gamma^i) \cap \partial E_2 \neq \emptyset$. Suppose that $(\gamma^i) \cap \partial E_2 = \emptyset$, and denote by γ_1^i the x-component of the curve γ^i. Let $\sigma \in [0, l(\gamma^i)]$ be such that $\gamma_1^i(\sigma) = \max\{\gamma_1^i(s) : s \in [0, l(\gamma^i)],\ \gamma^i(]s_0, \sigma]) \cap \partial E_1 = \emptyset\}$. If $\dot{\gamma}_1^i(\sigma+) = \dot{\gamma}_1^i(\sigma-)$, then $\dot{\gamma}^i(\sigma)$ is vertical. Therefore

$$\begin{aligned} l(\Gamma_E) + K_\psi(\Gamma_E) &\geq l({\gamma^i}_{|]s_0,\sigma[}) + K_\psi({\gamma^i}_{|]s_0,\sigma[}) \\ &\geq \omega \int_{s_0}^{\sigma} |\dot{\Theta}_i| \geq \omega |\Theta_i(s_0+) - \Theta_i(\sigma)| > 2D, \end{aligned}$$

where ω is defined in (9.9), and D is sufficiently small (recall that v is contained in the cone C).

If $\dot{\gamma}_1^i(\sigma+) \neq \dot{\gamma}_1^i(\sigma-)$, then using the same arguments we have

$$l(\Gamma_E) + K_\psi(\Gamma_E) \geq \omega |\Theta_i(s_0+) - \Theta_i(\sigma+)| > 2D.$$

Assertion (9.22) then follows by recalling that $K_\psi^s(\Gamma) \geq F_\psi(\Gamma, \{p_1\}) + F_\psi(\Gamma, \{p_2\}) \geq 2c_2\alpha$.

On the other hand, if $(\gamma^i) \cap \partial E_2 \neq \emptyset$, then $F_\psi(\Gamma_E) \geq 2D$, and this concludes the proof. □

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