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Some properties of viability problems depending on a parameter *

Vladimir V. GONCHAROV
Irkutsk Computer Center, Russian Academy of Science
664033 Irkutsk, Russia

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Abstract

This paper is devoted to the solutions of differential inclusion viable in a compact set and continuously depending on a parameter. It contains the inequality of Filippov-Gronwall type for such solutions and the version of relaxation theorem uniformed by parameter.

Key words: viable solutions; continuous selections; relaxation property; lipschitzean multivalued maps.

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Via Beirut 2-4 - 34013 Trieste - ITALY

1 Introduction

Let Ξ and X be, correspondently, metric separable and Banach spaces, $K : \Xi \to X$ be a multifunction admiting the continuous selection $x^0(\xi)$. Consider the family of Lipschitzean differential inclusions

$$\dot{x} \in \Gamma(t, x, \xi); \tag{1.1}$$

$$x(t) \in K(\xi); \tag{1.2}$$

$$x(0) = x^0(\xi), \tag{1.3}$$

where the right-hand side Γ is defined for all $x \in K(\xi)$, $\xi \in \Xi$, and has closed, bounded, not necessary compact or convex values. The purpose of this paper is to study the solutions of the problem (1.1)-(1.3) which are continuous in ξ as the mappings from Ξ to the space of absolutely continuous functions.

The first results on existence of such solutions without viable constraint (1.2) were obtained in [1-4] when $X = \mathbf{R}^n$ and the right-hand side Γ does not depend on ξ. In [5] they were generalized to the case of a separable Banach space X and the multifunction Γ being lower semicontinuous in ξ. Recently [6] Cellina and Staicu proved the existence of solution to differential inclusion $\dot{x} \in F(t, x)$ viable in a compact subset of finite-dimensional space and continuously depending on initial point from this subset.

Now by combining Fillippov's technique of the successive approximations [7] and some modification of Lyapunov's theorem on convexity of the range of vector measure, we investigate the existence of solution continuously depending on a parameter for the more general problem (1.1)-(1.3). Moreover, we give here the inequality of Filippov-Gronwall type which allows us to obtain the relaxation result for continuous selections from the solution set of the viability problem (1.1)-(1.3) uniformed by the parameter ξ. Remark that for the problem without phase constraints the inequalities of such kind are contained in [4,5] and the continuous version of the relaxation theorem was proved by Fryszkowski and Rzežuchowski in [8].

2 Preliminaries

Denote by T the segment $[0, a]$, $a > 0$, of the number line with the Lebesgue measure $\mu_0(dt)$ and the σ-algebra $\mathcal{L}$ of measurable subsets. Let X be an

arbitrary Banach space with the norm $\|\cdot\|$.

For the main notions and facts of multivalued analysis, differential inclusions and viability theory we refer to [9]. Remind that a multifunction $F : T \to X$ is said to be measurable if the set $F_{-1}(C) = \{t \in T : F(t) \cap C \neq \emptyset\}$ belongs to $\mathcal{L}$ for any closed $C \subset X$. If Y is a metric space then a multivalued map $F : Y \to X$ is called lower semicontinuous (by Vietoris) in the case when the set $F_{-1}(U)$ is open for any open $U \subset X$. Also we say that $F : Y \to X$ is Hausdorff upper semicontinuous (Hausdorff lower semicontinuous) if fixed any point $y_0 \in Y$ and $\varepsilon > 0$ one can find a neibourhood U of y_0 such that $F(y) \subset F(y_0) + \varepsilon\bar{B}$ (correspondently, $F(y_0) \subset F(y) + \varepsilon\bar{B}$) for all $y \in U$. Here $\bar{B}$ means the unique closed ball with the centre in zero from X. Notice that for compact-valued maps the notions of lower semicontinuity by Vietoris and Hausdorff coincide. We denote by $D(\cdot,\cdot)$ the Hausdorff metric on the space of all closed bounded subsets of X.

Define the contingent cone $\mathbf{T}_K(x)$ and Clarke's tangent cone $\mathbf{C}_K(x)$ to a closed set $K \subset X$ in a point $x \in K$ (see [9])

$$\mathbf{T}_K(x) = \bigcap_{\varepsilon>0} \bigcap_{\delta>0} \bigcup_{0<\lambda<\delta} (\lambda^{-1}(K - x) + \varepsilon\bar{B});$$

$$\mathbf{C}_K(x) = \bigcap_{\varepsilon>0} \bigcup_{\delta>0} \bigcap_{0<\lambda<\delta} \bigcap_{\|y-x\|\leq\delta} (\lambda^{-1}(K - y) + \varepsilon\bar{B}).$$

Consider two simple properties of these cones.

Proposition 2.1 *If K is a compact subset of X then there exists a separable Banach subspace $Z \subset X$ with $K \subset Z$ and $\mathbf{T}_K(x) \subset Z$ for all $x \in K$.*

Obviously the separable Banach space Z generated by the set K satisfies this Proposition.

Proposition 2.2 *Let $K \subset X$ be closed and a multifunction $F : K \to X$ be lower semicontinuous. Then the following statements are equivalent:*

(i) $F(x) \subset \mathbf{T}_K(x)$ *for all* $x \in K$;

(ii) $F(x) \subset \mathbf{C}_K(x)$ *for all* $x \in K$.

Proof. Since $\mathbf{C}_K(x) \subset \mathbf{T}_K(x)$, $x \in K$, it is enough to show that (i) implies (ii). Fix $x \in K$ and $u \in F(x)$. By lower semicontinuity for any $\varepsilon > 0$ there exists such $\delta > 0$ that $F(y) \cap (u + \varepsilon \bar{B}) \neq \emptyset$ for each $y \in K$, $\|x - y\| \leq \delta$. Hence $u \in F(y) + \varepsilon \bar{B} \subset \mathbf{T}_K(y) + \varepsilon \bar{B}$. Consequently, $u \in \liminf_{y \to x, y \in K} \mathbf{T}_K(y)$ where by definition

$$\liminf_{y \to x, y \in K} \mathbf{T}_K(y) = \bigcap_{\varepsilon > 0} \bigcup_{\delta > 0} \bigcap_{\|y - x\| \leq \delta, y \in K} (\mathbf{T}_K(y) + \varepsilon \bar{B}).$$

Therefore it follows [10] that $u \in \mathbf{C}_K(x)$ and the proposition is proved.

Denote by $\mathbf{L}_1(T, X)$ ($\mathbf{L}_w(T, X)$) the Banach space of the classes of all Bochner integrable functions $u : T \to X$ with the norm $\|u\|_1 = \int_T \|u(t)\| dt$ (correspondently, $\|u\|_w = \sup_{t \in T} \| \int_0^t u(\tau) d\tau \|$) and by $\mathcal{B}_w$ the open unique ball with the centre in zero from $\mathbf{L}_w(T, X)$.

Remind that a subset $\mathcal{A} \subset \mathbf{L}_1(T, X)$ is called decomposable [11,12] if for any $u, v \in \mathcal{A}$ and $E \in \mathcal{L}$ the cut-function $u\chi_E + v\chi_{T \setminus E}$ belongs to $\mathcal{A}$. Here χ_E stands for the characteristic function of E.

Further we shall use also the space $\mathbf{AC}(T, X)$ of all absolutely continuous functions $x : T \to X$, having almost everywhere derivative $\dot{x}(\cdot) \in \mathbf{L}_1(T, X)$, with the norm $\|x\|_{AC} = \|x(0)\| + \int_T \|\dot{x}(t)\| dt$.

3 Hypotheses and formulation of the result

Consider multivalued maps $K : \Xi \to X$, Ξ is a separable metric space, and $\Gamma : T \times grK \to X$, $grK = \{(x, \xi) : x \in K(\xi)\}$. Assume the following hypotheses:

(H_1) K is compact-valued and lower semicontinuous.

(H_2) Γ has closed, bounded values.

(H_3) The mapping $t \mapsto \Gamma(t, x, \xi)$ is measurable for any $(x, \xi) \in grK$.

(H_4) $(x, \xi) \mapsto \Gamma(t, x, \xi)$ is lower semicontinuous on grK for a.a. $t \in T$.

(H_5) There is a continuous function $k : \Xi \to \mathbf{L}_1(T, \mathbf{R})$ such that

$$D\left(\Gamma(t,x,\xi), \Gamma(t,y,\xi)\right) \leq k(\xi)(t)\|x-y\|$$

for all $\xi \in \Xi$, $x, y \in K(\xi)$, a.a. $t \in T$.

(H_6) $\Gamma(t,x,\xi) \subset \mathbf{T}_{K(\xi)}(x)$, $(x,\xi) \in grK$, a.a. $t \in T$.

We say that a function $y : T \times \Xi \to X$ satisfies the hypothesis (H_Γ) if $\xi \mapsto y(\cdot,\xi)$ is the continuous mapping from Ξ to $\mathbf{AC}(T,X)$ and there exist continuous functions $\omega : \Xi \to \mathbf{R}$ and $\rho : \Xi \to \mathbf{L}_1(T,\mathbf{R})$ with

$$d_{K(\xi)}(y(0,\xi)) \leq \omega(\xi); \tag{3.1}$$

$$d_{\Gamma(t,b(t,\xi),\xi)}(\dot{y}(t,\xi)) \leq \rho(\xi)(t), \tag{3.2}$$

$\xi \in \Xi$, a.a. $t \in T$, where $d_K(\cdot)$ means the distance from a point to the set K in the space X and for any $\xi \in \Xi$ the measurable function $b(\cdot,\xi)$ is such that $b(t,\xi) \in K(\xi)$, $\|y(t,\xi) - b(t,\xi)\| = d_{K(\xi)}(y(t,\xi))$ a.e. in T.

The main result of this paper is the following

Theorem 3.1 *Suppose the hypotheses $(H_1)-(H_6)$ are valid and $y : T \times \Xi \to X$, $y(0,\xi) = y^0(\xi)$, is an arbitrary function satisfying (H_Γ). Then for any continuous function $x^0 : \Xi \to X$, $x^0(\xi) \in K(\xi)$, $\xi \in \Xi$, and continuous $\nu : \Xi \to (0,+\infty)$, there exists a continuous mapping $\xi \mapsto x(\cdot,\xi)$ from Ξ to $\mathbf{AC}(T,X)$ such that $x(\cdot,\xi)$ is the (Carathéodory) solution of the problem (1.1)-(1.3) for every $\xi \in \Xi$ and*

$$\int_0^t \|\dot{x}(\tau,\xi) - \dot{y}(\tau,\xi)\| d\tau \leq \int_0^t \rho(\xi)(\tau)\Phi\left(m(\xi)(t) - m(\xi)(\tau)\right) d\tau +$$

$$\left(\|x^0(\xi) - y^0(\xi)\| + \omega(\xi) + \nu(\xi)\right) \Phi(m(\xi)(t)) \tag{3.3}$$

for all $\xi \in \Xi$, $t \in T$, where $m(\xi)(t) = \int_0^t k(\xi)(\tau)d\tau$ and $\Phi : \mathbf{R}^+ \to \mathbf{R}^+$, $\mathbf{R}^+ = [0,+\infty)$, is some continuous, nondecreasing function does not depending on the conditions of the problem.

The following section contains the main tool of the proving of this theorem.

4 Some properties of measures

The results of this section continue investigations [8,11-15]. They will be true also when T is an arbitrary space with a finite nonnegative nonatomic measure μ_0 and the σ-algebra $\mathcal{L}$ of μ_0-measurable subsets.

For $s = 1, 2, \ldots$ denote by $\mathcal{M}_s$ the Banach space of vector measures $\mu : \mathcal{L} \to \mathbf{R}^s$ of bounded variation, absolutely continuous relative to μ_0 [16], with the norm $|\mu|_s = |\mu|(T)$, where $|\mu| : \mathcal{L} \to \mathbf{R}$ is the variation of μ, $|\mu|(E) = \sup_\Pi \sum_{\Omega\in\Pi} |\mu(\Omega)|$, $E \in \mathcal{L}$, and supremum is taken over all partitions Π of E into a finite number of pairwise disjoint measurable subsets. Let $\bar{\mathbf{B}}_s = \{\mu \in \mathcal{M}_s : |\mu|_s \leq 1\}$. We shall consider the $\sigma-$ algebra $\mathcal{L}$ with the pseudometric $\mu_0(E'\Delta E'')$, E', $E'' \in \mathcal{L}$, where Δ stands for the symmetric difference.

The following statement is some modification of Lemma 1 [12].

Lemma 4.1 *For any sequence of measures $\mu_n \in \mathcal{M}_{s_n}$, $n = 1, 2, \ldots$, where $s_n \geq 1$ are arbitrary integer numbers, there exists a mapping $\Lambda : \mathbf{R}^+ \times [0,1] \to \mathcal{L}$, possessing the properties:*

$$\Lambda(z,\alpha) \subset \Lambda(z,\beta), \quad z \geq 0, \quad \alpha \leq \beta; \tag{4.1}$$

$$\mu_0\left(\Lambda(z,\alpha)\Delta\Lambda(y,\beta)\right) \leq |\alpha - \beta| + 4|z - y|, \quad z, y \geq 0; \tag{4.2}$$

$$\mu_0\left(\Lambda(z,\alpha)\right) = \alpha\mu_0(T), \quad z \geq 0; \tag{4.3}$$

$$\mu_n\left(\Lambda(z,\alpha)\right) = \alpha\mu_n(T), \quad z \geq ns_n, \quad n = 1, 2, \ldots \tag{4.4}$$

Proof. By Radon-Nikodym theorem [16] every measure μ_n may be represented as $\mu_n = (\mu_n^1, \ldots, \mu_n^{s_n})$, where $\mu_n^i(A) = \int_A f_n^i(t)d\mu_0$, $A \in \mathcal{L}$, $i = 1, \ldots, s_n$, for some μ_0-integrable functions $f_n^i : T \to \mathbf{R}$. Denote by $s_0^* = 0$, $s_n^* = \sum_{i=1}^n s_i$ and consider the sequence $g_0(t), g_1(t), \ldots$ of μ_0-integrable functions:

$$g_0(t) = 1, \quad g_{2s_{n-1}^*+2i-1}(t) = f_n^{i+}(t), \quad g_{2s_{n-1}^*+2i}(t) = f_n^{i-}(t),$$

$t \in T$; $i = 1, \ldots, s_n$; $n = 1, 2, \ldots$, where $f_n^{i+}(t) = \max\{f_n^i(t), 0\}$, $f_n^{i-}(t) = -\min\{f_n^i(t), 0\}$. In accordance with Lemma 1 [12] there exists a mapping $\Psi : \mathbf{R}^+ \times [0,1] \to \mathcal{L}$ with:

$$\Psi(z,\alpha) \subset \Psi(z,\beta), \quad z \geq 0, \quad \alpha \leq \beta; \tag{4.5}$$

$$\mu_0\left(\Psi(z,\alpha)\Delta\Psi(y,\beta)\right) \leq |\alpha-\beta| + 2|z-y|, \quad z,y \geq 0; \tag{4.6}$$

$$\int_{\Psi(z,\alpha)} g_s(t)d\mu_0 = \alpha \int_T g_s(t)d\mu_0, \quad z \geq s. \tag{4.7}$$

Put $\Lambda(z,\alpha) = \Psi(2z,\alpha), z \geq 0, \alpha \in [0,1]$. Then (4.1), (4.2), (4.3) follow from (4.5), (4.6) and (4.7), correspondently. Moreover, if $z \geq s_n^*$ then $2z \geq 2s_{n-1}^* + 2s_n$ and for $i = 1, ..., s_n$ we have

$$\int_{\Lambda(z,\alpha)} f_n^{i+}(t)d\mu_0 = \alpha \int_T f_n^{i+}(t)d\mu_0; \quad \int_{\Lambda(z,\alpha)} f_n^{i-}(t)d\mu_0 = \alpha \int_T f_n^{i-}(t)d\mu_0.$$

Hence $\mu_n(\Lambda(z,\alpha)) = \alpha\mu_n(T), \alpha \in [0,1]$. Since without loss of generality one may assume $s_n \geq s_{n-1}$ then $z \geq ns_n$ implies $z \geq s_n^*$ and lemma is proved.

Further we shall study numerical measures depending on a parameter $\xi \in \Xi$. For a pair of sets C and U, $C \subset U \subset \Xi$, C is closed and U is open in Ξ denote by $h[C,U]$ the continuous function from Ξ to $\mathbf{R}$, $0 \leq h[C,U](\xi) \leq 1$, equal to 1 for $\xi \in C$ and 0 for $\xi \notin U$.

Lemma 4.2 *Let $\mathcal{P} : \Xi \to \mathcal{M}_1$ be a Hausdorff upper semicontinuous multifunction with nonempty relatively compact values. Then for any $\varepsilon > 0$ there exists a continuous mapping $A : \Xi \times [0,1] \to \mathcal{L}$ such that*

$$A(\xi,\alpha) \subset A(\xi,\beta), \quad \xi \in \Xi, \quad \alpha \leq \beta; \tag{4.8}$$

$$\mu_0(A(\xi,\alpha)) = \alpha\mu_0(T), \quad \xi \in \Xi, \quad \alpha \in [0,1]; \tag{4.9}$$

$$|\mu(A(\xi,\alpha)) - \alpha\mu(T)| \leq \varepsilon, \quad \xi \in \Xi, \quad \mu \in \mathcal{P}(\xi), \quad \alpha \in [0,1]. \tag{4.10}$$

Proof. Fix $\xi \in \Xi$ and choose a neibourhood $U(\xi)$ of the point ξ so that $\mathcal{P}(\zeta) \subset \mathcal{P}(\xi) + \frac{\varepsilon}{4}\bar{\mathbf{B}}_1$ for all $\zeta \in U(\xi)$. Since Ξ is the metric separable space there exists ([17], p.537) a countable locally finite covering $\{U_n\}_{n=1}^{\infty}$ of Ξ by open sets with the further property that for any $n = 1,2,...$ one can find $\xi_n \in \Xi$, $U_n \subset U(\xi_n)$. Let $\{\mu_n^i\}_{i=1}^{s_n}$ be some finite $\frac{\varepsilon}{4}$ -net of the relatively compact set $\mathcal{P}(\xi_n)$, $n = 1,2,....$ Denote $\mu_n = (\mu_n^1, ..., \mu_n^{s_n}) \in \mathcal{M}_{s_n}$ and by using Lemma 4.1 construct the continuous mapping $\Lambda : \mathbf{R}^+ \times [0,1] \to \mathcal{L}$ satisfying (4.1)-(4.4).

It follows from ([17], p.540), that there exists an open covering $\{V_n\}_{n=1}^{\infty}$ of the space Ξ, $\overline{V_n} \subset U_n, n = 1,2,....$ Then the number function

$$\sigma(\xi) = \sum_{n=1}^{\infty} ns_n h[\overline{V_n}, U_n](\xi), \quad \xi \in \Xi,$$

has the finite values and is continuous. Put $A(\xi,\alpha) = \Lambda(\sigma(\xi),\alpha)$, $\xi \in \Xi$, $\alpha \in [0,1]$. In accordance with (4.1)-(4.3) the map $A : \Xi \times [0,1] \to \mathcal{L}$ is continuous and the properties (4.8), (4.9) are valid.

Let $\xi \in \Xi$ and $\mu \in \mathcal{P}(\xi)$. Find $n = 1,2,...$ so that $\xi \in \overline{V_n} \subset U_n \subset U(\xi_n)$, $\mathcal{P}(\xi) \subset \mathcal{P}(\xi_n) + \frac{\varepsilon}{4}\bar{\mathbf{B}}_1$ and choose $\nu \in \mathcal{P}(\xi_n)$, $|\mu - \nu|_1 \leq \frac{\varepsilon}{4}$. Then $|\nu - \mu_n^i|_1 \leq \frac{\varepsilon}{4}$ for some $i = 1,...,s_n$ and by (4.4) we have

$$|\mu(A(\xi,\alpha)) - \alpha\mu(T)| \leq |\mu - \nu|_1 + |\nu - \mu_n^i|_1 +$$

$$|\mu_n^i(\Lambda(\sigma(\xi),\alpha)) - \alpha\mu_n^i(T)| + \alpha|\mu_n^i(T) - \mu(T)| \leq \varepsilon$$

because $h[\overline{V_n}, U_n](\xi) = 1$ and $\sigma(\xi) \geq ns_n$. Lemma is proved.

Theorem 4.1 *Let $\{U_n\}_{n=1}^{\infty}$ be an open locally finite covering of the space Ξ and $\{e_n(\xi)\}_{n=1}^{\infty}$ be a continuous partition of unity, subordinated to this covering. Assume also that $\mathcal{P}_n : \overline{U}_n \to \mathcal{M}_1$, $n = 1,2,...$, is a sequence of Hausdorff upper semicontinuous multifunctions with nonempty relatively compact values. Then for any $\varepsilon > 0$ there exists a sequence of continuous mappings $E_n : \Xi \to \mathcal{L}, n \geq 1$, satisfying*

(i) *$\{E_n(\xi)\}_{n=1}^{\infty}$ is the measurable partition of T for every $\xi \in \Xi$, i.e the sets $E_n(\xi)$, $n = 1,2,...$, are pairwise disjoint and $\mu_0\left(\bigcup_{n=1}^{\infty} E_n(\xi)\right) = \mu_0(T)$;*

(ii) *$E_n(\xi) = \emptyset$ if and only if $e_n(\xi) = 0$;*

(iii) *$|\mu(E_n(\xi)) - e_n(\xi)\mu(T)| \leq \varepsilon$ for any $n \geq 1$, $\xi \in \overline{U_n}$ and measure $\mu \in \mathcal{P}_n(\xi)$.*

Proof. Consider for each $n = 1,2,...$ the map $h_n(\xi) = h[supp\ e_n, U_n](\xi)$, $0 \leq h_n(\xi) \leq 1$, $\xi \in \Xi$, *supp* $h_n \subset U_n$, where *supp* means support set of the function. Since $I(\xi) = \{n \geq 1 : h_n(\xi) > 0\}$ is finite, the set

$$\mathcal{P}(\xi) = \bigcup_{n \in I(\xi)} \{h_n(\xi)\mu : \mu \in \mathcal{P}_n(\xi)\}$$

is relatively compact for any $\xi \in \Xi$. Prove the Hausdorff upper semicontinuity of the multifunction $\mathcal{P} : \Xi \to \mathcal{M}_1$.

Let $\xi \in \Xi$ and $\delta > 0$. Choose a neibourhood $V(\xi)$ of the point ξ for which the set $J(\xi) = \{n \geq 1 : V(\xi) \cap U_n \neq \emptyset\}$ is finite. Take $M > 0$ from the condition $|\mu|_1 \leq M$, $\mu \in \mathcal{P}_n(\xi)$, $n \in J(\xi)$. By the properties of h_n and $\mathcal{P}_n$ there is a neibourhood $W(\xi)$ of ξ such that $|h_n(\zeta) - h_n(\xi)| \leq \frac{\delta}{2M}$ and $\mathcal{P}_n(\zeta) \subset \mathcal{P}_n(\xi) + \frac{\delta}{2}\bar{\mathbf{B}}_1$, whenever $\zeta \in W(\xi)$, $n \in J(\xi)$. If $\zeta \in W(\xi) \cap V(\xi)$ and $\nu \in \mathcal{P}(\zeta)$ then $\nu = h_n(\zeta)\mu$ for some $n \in I(\xi)$ and $\mu \in \mathcal{P}_n(\zeta)$. Since in this case $n \in J(\xi)$ we have $|h_n(\zeta) - h_n(\xi)| \leq \frac{\delta}{2M}$ and there exists $\mu' \in \mathcal{P}_n(\xi)$, $|\mu - \mu'|_1 \leq \frac{\delta}{2}$. Hence

$$|\nu - h_n(\xi)\mu'|_1 \leq |h_n(\zeta) - h_n(\xi)| \cdot |\mu'|_1 + h_n(\zeta)|\mu - \mu'|_1 \leq \delta.$$

Therefore $\mathcal{P}(\zeta) \subset \mathcal{P}(\xi) + \delta\bar{\mathbf{B}}_1$, $\zeta \in W(\xi) \cap V(\xi)$.

By using Lemma 4.2 find a continuous mapping $A : \Xi \times [0,1] \to \mathcal{L}$ satisfying (4.8)-(4.10) where $\frac{\varepsilon}{2}$ stands instead of ε. Put $z_0(\xi) = 0$, $z_n(\xi) = z_{n-1}(\xi) + e_n(\xi)$, $n \geq 1$, and

$$E_n(\xi) = A(\xi, z_n(\xi)) \backslash A(\xi, z_{n-1}(\xi)), \quad n = 1, 2, ..., \quad \xi \in \Xi.$$

It is easy to verify that the mapping $E_n : \Xi \to \mathcal{L}$ is continuous. Also the sets $E_n(\xi)$, $n = 1, 2, ...$, are pairwise disjoint and

$$\mu_0\left(\bigcup_{n=1}^{\infty} E_n(\xi)\right) = \mu_0\left(\bigcup_{n=1}^{\infty} A(\xi, z_n(\xi))\right) = \lim_{n\to\infty} \mu_0(A(\xi, z_n(\xi))) = \mu_0(T).$$

The property (ii) follows from (4.9).

Take $n \geq 1$, $\xi \in \Xi$ and $\mu \in \mathcal{P}_n(\xi)$. Then $h_n(\xi)\mu \in \mathcal{P}(\xi)$ and from (4.10)

$$|h_n(\xi)\mu(A(\xi, z_n(\xi))) - z_n(\xi)h_n(\xi)\mu(T)| \leq \frac{\varepsilon}{2};$$

$$|h_n(\xi)\mu(A(\xi, z_{n-1}(\xi))) - z_{n-1}(\xi)h_n(\xi)\mu(T)| \leq \frac{\varepsilon}{2}.$$

Therefore $|h_n(\xi)\mu(E_n(\xi)) - e_n(\xi)h_n(\xi)\mu(T)| \leq \varepsilon$ and (iii) follows since always either $e_n(\xi) = 0$ or $h_n(\xi) = 1$. Theorem is proved.

5 Proof of the main theorem

At the beginning we prove one statement generalizing the Proposition 2.1 [8].

Proposition 5.1 *Let X be separable, $\Gamma : T \times grK \to X$ satisfy the hypotheses $(H_2)-(H_4)$ and continuous functions $\gamma : \Xi \to \mathbf{L}_1(T,X)$, $\gamma(\xi)(t) \in K(\xi)$, $l : \Xi \to \mathbf{L}_1(T,\mathbf{R})$ be such that*

$$\Gamma(t,\gamma(\xi)(t),\xi) \cap l(\xi)(t)\bar{B} \neq \emptyset \tag{5.1}$$

for all $\xi \in \Xi$, a.a. $t \in T$. Then the mapping $G_\gamma : \Xi \to \mathbf{L}_1(T,X)$, $G_\gamma(\xi) = \{u(\cdot) \in \mathbf{L}_1(T,X) : u(t) \in \Gamma(t,\gamma(\xi)(t),\xi)$ a.e. in $T\}$, $\xi \in \Xi$, has nonempty, closed, decomposable values and is lower semicontinuous.

Proof. We show that for each $\xi \in \Xi$ the map $t \mapsto \Gamma(t,\gamma(\xi)(t),\xi)$ is measurable. To this purpose take a sequence $\gamma_n : T \to K(\xi)$ of measurable step-functions, converging to $\gamma(\xi)(t)$ almost everywhere in T. By (H_3) the mappings $t \mapsto \Gamma(t,\gamma_n(t),\xi)$, $n \geq 1$, are measurable. Then (see [18], Propositions 2.3, 2.6 and Theorem 3.5) so is the multifunction

$$t \mapsto \mathbf{Ls}\Gamma(t,\gamma_n(t),\xi), \quad \mathbf{Ls}\Gamma(t,\gamma_n(t),\xi) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \Gamma(t,\gamma_m(t),\xi)},$$

where bar over means the closure in X, but from (H_5) and ([19], p.342), it follows that $\mathbf{Ls}\Gamma(t,\gamma_n(t),\xi) = \Gamma(t,\gamma(\xi)(t),\xi)$. Now by Kuratowski and Ryll-Nardzewski theorem on measurable selections (see [18], theorem 5.1) using (H_6) and Proposition 2.1 we obtain $G_\gamma(\xi) \neq \emptyset$. The closeness and decomposability are obvious.

In order to prove the lower semicontinuity of G_γ take a sequence $\xi_n \in \Xi$ converging to ξ and $u(\cdot) \in G_\gamma(\xi)$. By using Theorem 6.4 [18] choose measurable selections $u_n(t)$ of $t \mapsto \Gamma(t,\gamma(\xi_n)(t),\xi_n)$ with

$$\|u_n(t) - u(t)\| \leq d_{\Gamma(t,\gamma(\xi_n)(t),\xi_n)}(u(t)) + \frac{1}{n} \tag{5.2}$$

a.e. in T. This inequality together with (H_4) implies that some subsequence of $\{u_n(\cdot)\}$ converges to $u(\cdot)$ a.e. in T. Without loss of generality one may assume $u_n(t) \to u(t)$ for a.a $t \in T$. By (5.1) and (5.2) we have

$$\|u_n(t) - u(t)\| \leq l(\xi_n)(t) + \|u(t)\| + \frac{1}{n}$$

a.e. in T, $n = 1,2,\ldots$. Since $l(\xi_n) \to l(\xi)$ in $L_1(T,\mathbf{R})$ for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\int_E \left(l(\xi_n)(t) + \|u(t)\| + \frac{1}{n}\right) dt \leq \frac{\varepsilon}{2}$$

whenever $E \in \mathcal{L}$ with $\mu_0(E) \leq \delta$, $n \geq 1$. Choosing a set $T_\delta \subset T$, $\mu_0(T \backslash T_\delta) \leq \delta$, and N so that $\|u_n(t) - u(t)\| \leq \frac{\varepsilon}{2\delta}$ for all $t \in T_\delta$, $n \geq N$, we shall write

$$\int_T \|u_n(t) - u(t)\| dt \leq \int_{T_\delta} \|u_n(t) - u(t)\| dt +$$

$$\int_{T \backslash T_\delta} \left(l(\xi_n)(t) + \|u(t)\| + \frac{1}{n} \right) dt \leq \varepsilon, \quad n \geq N.$$

It means that $u_n(\cdot)$ converges to $u(\cdot)$ in $\mathbf{L}_1(T, X)$. Hence the mapping G_γ is lower semicontinuous. Proposition is proved.

Put $x_0(t, \xi) = y(t, \xi)$, $b_0(t, \xi) = b(t, \xi)$,

$$\eta_0(\xi)(t) = \omega(\xi) + \int_0^t \rho(\xi)(\tau) d\tau, \quad \varphi_0(\xi)(t) = \rho(\xi)(t),$$

$\xi \in \Xi$, $t \in T$, and denote by

$$\Pi_K(x) = \{y \in K : \|y - x\| = d_K(x)\}$$

the projection of a point x to the set K. For fixed $\xi \in \Xi$ the function $f_0(\cdot, \xi)$, $f_0(t, \xi) = d_{K(\xi)}(x_0(t, \xi))$, is absolutely continuous and its derivative by Lemma 1 [20], Proposition 2.2 and (3.2) almost everywhere in T satisfies

$$\frac{d}{dt} f_0(t, \xi) = \lim_{h \to 0+} h^{-1} \left[d_{K(\xi)}(x_0(t, \xi) + h\dot{x}_0(t, \xi)) - d_{K(\xi)}(x_0(t, \xi)) \right] \leq$$

$$\leq \inf_{b \in \Pi_{K(\xi)}(x_0(t, \xi))} d_{\mathbf{C}_{K(\xi)}(b)}(\dot{x}_0(t, \xi)) \leq d_{\Gamma(t, b_0(t, \xi), \xi)}(\dot{x}_0(t, \xi)) \leq \varphi_0(\xi)(t).$$

Hence (see (3.1)) $d_{K(\xi)}(x_0(t, \xi)) \leq \eta_0(\xi)(t)$, $\xi \in \Xi$, $t \in T$.

By considering the continuous functions ψ_0, $\eta_1 : \Xi \to \mathbf{AC}(T, \mathbf{R})$; $\varphi_1 : \Xi \to \mathbf{L}_1(T, \mathbf{R})$,

$$\psi_0(\xi)(t) = \int_0^t [\varphi_0(\xi)(\tau) + 2k(\xi)(\tau)\eta_0(\xi)(\tau)] \, d\tau + 4\nu(\xi)m(\xi)(t) + \nu(\xi); \quad (5.3)$$

$$\eta_1(\xi)(t) = \int_0^t \left[\eta_0(\xi)(\tau) + \psi_0(\xi)(\tau) + \|y^0(\xi) - x^0(\xi)\| + 3\nu(\xi) \right] \times$$
$$\times k(\xi)(\tau) e^{m(\xi)(t) - m(\xi)(\tau)} d\tau; \quad (5.4)$$

$$\varphi_1(\xi)(t) = k(\xi)(t)[\eta_0(\xi)(t) + \eta_1(\xi)(t) + \psi_0(\xi)(t) + \|y^0(\xi) - x^0(\xi)\| + 3\nu(\xi)], \tag{5.5}$$

construct the continuous mapping $\xi \mapsto x_1(\cdot,\xi)$ from Ξ to $\mathbf{AC}(T,X)$, $x_1(0,\xi) = x^0(\xi)$, such that

$$\int_0^t \|\dot{x}_1(\tau,\xi) - \dot{x}_0(\tau,\xi)\| d\tau \le \psi_0(\xi)(t); \tag{5.6}$$

$$d_{K(\xi)}(x_1(t,\xi)) \le \eta_1(\xi)(t); \tag{5.7}$$

$$d_{\Gamma(t,b_1(t,\xi),\xi)}(\dot{x}_1(t,\xi)) \le \varphi_1(\xi)(t), \tag{5.8}$$

where $b_1(\cdot,\xi)$ is some measurable selection of the map $t \mapsto \Pi_{K(\xi)}(x_1(t,\xi))$ for each $\xi \in \Xi$.

Fixed any $\xi \in \Xi$ we find its neibourhood V_ξ^0 such that

$$\|x_0(\cdot,\xi) - x_0(\cdot,\zeta)\|_{AC} < \nu(\zeta); \tag{5.9}$$

$$K(\xi) \subset K(\zeta) + \frac{\nu(\zeta)}{2}\bar{B}; \tag{5.10}$$

$$\sup_{t\in T} |\eta_0(\xi)(t) - \eta_0(\zeta)(t)| < \nu(\zeta) \tag{5.11}$$

for all $\zeta \in V_\xi^0$. Since the space Ξ is metric and separable there exist ([17], p.537, 540) an open locally finite covering $\{U_i^0\}_{i=1}^\infty$ of Ξ and the sequence of points $\xi_i^0 \in \Xi$ with $\overline{U_i^0} \subset V_{\xi_i^0}^0$, $i \ge 1$. Let $\{e_i^0(\xi)\}_{i=1}^\infty$ be a locally finite continuous partition of unity subordinated to $\{U_i^0\}_{i=1}^\infty$.

Denote by $\mathcal{K}(\xi) = \{u(\cdot) \in \mathbf{L}_1(T,X) : u(t) \in K(\xi)$ a.e. in $T\}$ and

$$\mathcal{K}_i^0(\xi) = \{u(\cdot) \in \mathcal{K}(\xi) : \|u(t) - b_0(t,\xi_i^0)\| < \nu(\xi) \ \ a.e. \ in \ T\},$$

$\xi \in \Xi$, $i \ge 1$. If $\xi \in \overline{U_i^0}$ then $d_{K(\xi)}(b_0(t,\xi_i^0)) \le \frac{\nu(\xi)}{2}$ and $\mathcal{K}_i^0(\xi) \ne \emptyset$. Therefore it follows from (H_1), Proposition 4 and Theorem 3 [12] that the multifunction $\xi \mapsto \overline{\mathcal{K}_i^0(\xi)}$, $\xi \in \overline{U_i^0}$ ($\overline{\mathcal{K}_i^0(\xi)}$ means the closure of $\mathcal{K}_i^0(\xi)$ in $\mathbf{L}_1(T,X)$), admits a continuous selection $b_i^0(\xi)$,

$$b_i^0(\xi)(t) \in K(\xi), \quad \|b_i^0(\xi)(t) - b_0(t,\xi_i^0)\| \le \nu(\xi), \tag{5.12}$$

$\xi \in \overline{U_i^0}$, a.a. $t \in T$. Furthermore,

$$d_{\Gamma(t,b_i^0(\xi)(t),\xi)}(\dot{x}_0(t,\xi)) \le d_{\Gamma(t,b_0(t,\xi),\xi)}(\dot{x}_0(t,\xi)) + k(\xi)(t)\|b_i^0(\xi)(t) - b_0(t,\xi)\| \le$$

$$\leq \varphi_0(\xi)(t) + k(\xi)(t)[\|b_i^0(\xi)(t) - b_0(t,\xi_i^0)\| + \|b_0(t,\xi_i^0) - x_0(t,\xi_i^0)\| +$$
$$+\|x_0(t,\xi_i^0) - x_0(t,\xi)\| + \|x_0(t,\xi) - b_0(t,\xi)\|] \leq$$
$$\leq \varphi_0(\xi)(t) + k(\xi)(t)[3\nu(\xi) + 2\eta_0(\xi)(t)], \tag{5.13}$$

$\xi \in \overline{U_i^0}$. By combining Propositions 2.1 and 5.1 we obtain that the multifunction $G_i^0 : \overline{U_i^0} \to \mathrm{L}_1(T,X)$, $G_i^0(\xi) = \{u(\cdot) \in \mathrm{L}_1(T,X) : u(t) \in \Gamma(t, b_i^0(\xi)(t), \xi)$ a.e. in $T\}$, $\xi \in \overline{U_i^0}$, has nonempty, closed, decomposable values and is lower semicontinuous. Consider the multivalued map $\mathcal{G}_i^0 : \overline{U_i^0} \to \mathrm{L}_1(T,X)$,

$$\mathcal{G}_i^0(\xi) = \left\{ u(\cdot) \in G_i^0(\xi) : \|u(t) - \dot{x}_0(t,\xi)\| < \frac{d}{dt}\psi_0(\xi)(t) \ \ a.e. \ in \ T \right\},$$

$\xi \in \overline{U_i^0}$. By (5.13), (5.3) and Proposition 4 [12] $\mathcal{G}_i^0$ is also lower semicontinuous with nonempty values. Then (see theorem 3 [12]) there exists a continuous selection $v_i^1(\xi) \in \overline{\mathcal{G}_i^0(\xi)}$, $v_i^1(\xi) \in \Gamma(t, b_i^0(\xi)(t), \xi)$;

$$\|v_i^1(\xi)(t) - \dot{x}_0(t,\xi)\| \leq \frac{d}{dt}\psi_0(\xi)(t), \tag{5.14}$$

$\xi \in \overline{U_i^0}$, a.a. $t \in T$.

For $\xi \in \overline{U_i^0}$ let $\mathcal{P}_i^0(\xi)$ be the family of all measures

$$E \mapsto \int_E \|v_i^1(\xi)(\tau) - \dot{x}_0(\tau,\xi)\| \chi_{[0,t]}(\tau) d\tau, \quad t \in T,$$

which is relatively compact in $\mathcal{M}_1$. It is clear the multifunction $\mathcal{P}_i^0 : \overline{U_i^0} \to \mathcal{M}_1$ is Hausdorff upper semicontinuous (even continuous). Since the function $r_0(\xi) = \sum_{i=1}^{\infty} h[supp\ e_i^0, U_i^0](\xi)$ is finite, positive and continuous on Ξ applying Theorem 4.1 one can find continuous mappings $E_i^0 : \Xi \to \mathcal{L}$, $i \geq 1$, such that $\{E_i^0(\xi)\}_{i=1}^{\infty}$ is the measurable partition of T for each $\xi \in \Xi$; $E_i^0(\xi) = \emptyset$ if and only if $e_i^0(\xi) = 0$ and

$$|\mu(E_i^0(\xi)) - e_i^0(\xi)\mu(T)| \leq \frac{\nu(\xi)}{r_0(\xi)} \tag{5.15}$$

for all $i \geq 1$, $\xi \in \overline{U_i^0}$, $\mu \in \mathcal{P}_i^0(\xi)$. Finally put

$$x_1(t,\xi) = x^0(\xi) + \int_0^t \sum_{i \in I_0(\xi)} v_i^1(\xi)(\tau) \chi_{E_i^0(\xi)}(\tau) d\tau, \tag{5.16}$$

$t \in T$, $\xi \in \Xi$, where $I_0(\xi) = \{i \geq 1 : \xi \in \overline{U_i^0}\}$.

From (5.16), (5.15), (5.14) and (5.3)

$$\int_0^t \|\dot{x}_1(\tau,\xi) - \dot{x}_0(\tau,\xi)\| d\tau \leq \sum_{i \in I_0(\xi)} \int_{E_i^0(\xi)} \|v_i^1(\xi)(\tau) - \dot{x}_0(\tau,\xi)\| \chi_{[0,t]}(\tau) d\tau \leq$$

$$\nu(\xi) + \sum_{i \in I_0(\xi)} e_i^0(\xi) \int_0^t \|v_i^1(\xi)(\tau) - \dot{x}_0(\tau,\xi)\| d\tau \leq$$

$$\nu(\xi) + \int_0^t \frac{d}{d\tau} \psi_0(\xi)(\tau) d\tau = \psi_0(\xi)(t), \quad \xi \in \Xi, \quad t \in T.$$

Fixed $\xi \in \Xi$ take an arbitrary measurable selection $b_1(\cdot,\xi)$, $b_1(t,\xi) \in \Pi_{K(\xi)}(x_1(t,\xi))$, and consider the function $f_1(\cdot,\xi)$, $f_1(t,\xi) = d_{K(\xi)}(x_1(t,\xi))$, $t \in T$. By Lemma 1 [20] and Proposition 2.2

$$\frac{d}{dt} f_1(t,\xi) \leq d_{\Gamma(t,b_1(t,\xi),\xi)}(\dot{x}_1(t,\xi)) \tag{5.17}$$

a.e. in T. Furthermore for a.a. $t \in T$ and $i \in I_0(\xi)$ such that $t \in E_i^0(\xi)$ we have $\dot{x}_1(t,\xi) = v_i^1(\xi)(t) \in \Gamma(t, b_i^0(\xi)(t), \xi)$. Then in accordance with (H_5), (5.9), (5.11), (5.12) and (5.6)

$$d_{\Gamma(t,b_1(t,\xi),\xi)}(\dot{x}_1(t,\xi)) \leq k(\xi)(t) \|b_i^0(\xi)(t) - b_1(t,\xi)\| \leq$$

$$\leq k(\xi)(t)[\|b_i^0(\xi)(t) - b_0(t,\xi_i^0)\| + \|b_0(t,\xi_i^0) - x_0(t,\xi_i^0)\| +$$

$$+ \|x_0(t,\xi_i^0) - x_0(t,\xi)\| + \|x_0(t,\xi) - x_1(t,\xi)\| + \|x_1(t,\xi) - b_1(t,\xi)\|] \leq$$

$$\leq k(\xi)(t) \left[f_1(t,\xi) + \|y^0(\xi) - x^0(\xi)\| + \psi_0(\xi)(t) + \eta_0(\xi)(t) + 3\nu(\xi) \right]. \tag{5.18}$$

From (5.17), (5.18) applying the Gronwall inequality and (5.4) one immediately obtain (5.7). Finally (5.8) is the consequence of (5.7), (5.18) and (5.5).

Define the sequences of continuous functions $\psi_n, \eta_n : \Xi \to \mathbf{AC}(T, \mathbf{R})$; $\varphi_n : \Xi \to \mathbf{L}_1(T, \mathbf{R})$ by the following recoursive relations:

$$\psi_n(\xi)(t) = 3 \int_0^t \varphi_n(\xi)(\tau) d\tau + \frac{4\nu(\xi)}{2^n} m(\xi)(t) + \frac{\nu(\xi)}{2^n}; \tag{5.19}$$

$$\eta_{n+1}(\xi)(t) = \int_0^t \left[\eta_n(\xi)(\tau) + \psi_n(\xi)(\tau) + \frac{3\nu(\xi)}{2^n} \right] k(\xi)(\tau) e^{m(\xi)(t) - m(\xi)(\tau)} d\tau; \tag{5.20}$$

$$\varphi_{n+1}(\xi)(t) = k(\xi)(t)\left[\eta_n(\xi)(t) + \eta_{n+1}(\xi)(t) + \psi_n(\xi)(t) + \frac{3\nu(\xi)}{2^n}\right], \quad (5.21)$$

$t \in T$, $\xi \in \Xi, n = 1, 2, \ldots$.

We claim that there exist sequences $x_n : T \times \Xi \to X$, $x_n(0, \xi) = x^0(\xi)$, and $b_n : T \times \Xi \to X$ such that $\xi \mapsto x_n(\cdot, \xi)$ is the continuous mapping from Ξ into $\mathbf{AC}(T, X)$;

$$\int_0^t \|\dot{x}_{n+1}(\tau, \xi) - \dot{x}_n(\tau, \xi)\| d\tau \leq \psi_n(\xi)(t); \quad (5.22)$$

$b_n(\cdot, \xi)$ is measurable for each $\xi \in \Xi$, $b_n(t, \xi) \in K(\xi)$;

$$\|x_n(t, \xi) - b_n(t, \xi)\| = d_{K(\xi)}(x_n(t, \xi)) \leq \eta_n(\xi)(t) \quad (5.23)$$

and

$$d_{\Gamma(t, b_n(t,\xi), \xi)}(\dot{x}_n(t, \xi)) \leq \varphi_n(\xi)(t), \quad (5.24)$$

$t \in T$, $\xi \in \Xi$, $n = 1, 2, \ldots$.

Suppose that the functions $x_n(t, \xi)$, $b_n(t, \xi)$ are found for some $n \geq 1$. In order to construct $x_{n+1} : T \times \Xi \to X$ choose for each $\xi \in \Xi$ its neibourhood V_ξ^n such that

$$\|x_n(\cdot, \xi) - x_n(\cdot, \zeta)\|_{AC} < \frac{\nu(\zeta)}{2^n}; \quad (5.25)$$

$$K(\xi) \subset K(\zeta) + \frac{\nu(\zeta)}{2^{n+1}} \bar{B}; \quad (5.26)$$

$$\sup_{t \in T} |\eta_n(\xi)(t) - \eta_n(\zeta)(t)| < \frac{\nu(\zeta)}{2^n} \quad (5.27)$$

whenever $\zeta \in V_\xi^n$. By the same argument as above we find an open locally finite covering $\{U_i^n\}_{i=1}^\infty$ of Ξ; a locally finite continuous partition of unity $\{e_i^n(\xi)\}_{i=1}^\infty$ subordinated to this covering and points $\xi_i^n \in \Xi$ with $\overline{U_i^n} \subset V_{\xi_i^n}^n$, $i \geq 1$. The multivalued map $\mathcal{K}_i^n : \overline{U_i^n} \to \mathrm{L}_1(T, X)$,

$$\mathcal{K}_i^n(\xi) = \left\{u(\cdot) \in \mathcal{K}(\xi) : \|u(t) - b_n(t, \xi_i^n)\| < \frac{\nu(\xi)}{2^n} \ \ a.e. \ in \ T\right\},$$

$\xi \in \Xi, i \geq 1$, has nonempty values by (5.26) and is lower semicontinuous (see Proposition 4 [12]). For the continuous selection $b_i^n(\xi) \in \overline{\mathcal{K}_i^n(\xi)}$, $\xi \in \overline{U_i^n}$, we have

$$b_i^n(\xi)(t) \in K(\xi), \quad \|b_i^n(\xi)(t) - b_n(t, \xi_i^n)\| \leq \frac{\nu(\xi)}{2^n} \quad (5.28)$$

a.e. in T. Then by (H_5), (5.24), (5.28), (5.25), (5.23) and (5.27)

$$d_{\Gamma(t,b_i^n(\xi)(t),\xi)}(\dot{x}_n(t,\xi)) \leq d_{\Gamma(t,b_n(t,\xi),\xi)}(\dot{x}_n(t,\xi)) + k(\xi)(t)\|b_i^n(\xi)(t) - b_n(t,\xi)\|$$

$$\leq \varphi_n(\xi)(t) + k(\xi)(t)[\|b_i^n(\xi)(t) - b_n(t,\xi_i^n)\| + \|b_n(t,\xi_i^n) - x_n(t,\xi_i^n)\| +$$

$$\|x_n(t,\xi_i^n) - x_n(t,\xi)\| + \|x_n(t,\xi) - b_n(t,\xi)\|] \leq \varphi_n(\xi)(t) +$$

$$k(\xi)(t)\left[\frac{3\nu(\xi)}{2^n} + 2\eta_n(\xi)(t)\right] \leq 3\varphi_n(\xi)(t) + \frac{3\nu(\xi)}{2^n}k(\xi)(t), \tag{5.29}$$

$\xi \in \overline{U_i^n}$. The last inequality follows from (5.5) for $n = 1$ and from (5.21) for $n \geq 2$. Denote by $G_i^n(\xi) = \{u(\cdot) \in \mathrm{L}_1(T,X) : u(t) \in \Gamma(t, b_i^n(\xi)(t), \xi)$ a.e. in $T\}$ and

$$\mathcal{G}_i^n(\xi) = \left\{u(\cdot) \in G_i^n(\xi) : \|u(t) - \dot{x}_n(t,\xi)\| < \frac{d}{dt}\psi_n(\xi)(t) \ \ a.e. \ in \ T\right\},$$

$\xi \in \overline{U_i^n}$. In accordance with (5.29), (5.19), Proposition 4 [12], Propositions 2.2 and 5.1 the multifunctions $G_i^n : \overline{U_i^n} \to \mathrm{L}_1(T,X)$ and $\mathcal{G}_i^n : \overline{U_i^n} \to \mathrm{L}_1(T,X)$ have nonempty values and are lower semicontinuous. Take for each $i \geq 1$ a continuous selection $v_i^{n+1}(\xi) \in \overline{\mathcal{G}_i^n(\xi)}$, $v_i^{n+1}(\xi)(t) \in \Gamma(t, b_i^n(\xi)(t), \xi)$;

$$\|v_i^{n+1}(\xi)(t) - \dot{x}_n(t,\xi)\| \leq \frac{d}{dt}\psi_n(\xi)(t), \tag{5.30}$$

$\xi \in \overline{U_i^n}$, a.a. $t \in T$, existing by Theorem 3 [12]. Fixed $\xi \in \overline{U_i^n}$ consider the set $\mathcal{P}_i^n(\xi)$ of all measures

$$E \mapsto \int_E \|v_i^{n+1}(\xi)(\tau) - \dot{x}_n(\tau,\xi)\|\chi_{[0,t]}(\tau)d\tau, \quad t \in T.$$

It is relatively compact in $\mathcal{M}_1$ and the multivalued map $\mathcal{P}_i^n : \overline{U_i^n} \to \mathcal{M}_1$ is Hausdorff upper semicontinuous. Therefore using Theorem 4.1 we may construct continuous mappings $E_i^n : \Xi \to \mathcal{L}$, $i \geq 1$, with the further properties that for any $\xi \in \Xi$ the sets $E_i^n(\xi)$, $i \geq 1$, are pairwise disjoint, $\mu_0(\bigcup_{i=1}^{\infty} E_i^n(\xi)) = \mu_0(T)$; $E_i^n(\xi) = \emptyset$ if and only if $e_i^n(\xi) = 0$ and

$$|\mu(E_i^n(\xi)) - e_i^n(\xi)\mu(T)| \leq \frac{\nu(\xi)}{2^n r_n(\xi)} \tag{5.31}$$

whenever $i \geq 1$, $\xi \in \overline{U_i^n}$, $\mu \in \mathcal{P}_i^n(\xi)$. Here $r_n(\xi) = \sum_{i=1}^{\infty} h[supp\ e_i^n, \overline{U_i^n}](\xi)$, $\xi \in \Xi$. Define the function

$$x_{n+1}(t,\xi) = x^0(\xi) + \int_0^t \sum_{i \in I_n(\xi)} v_i^{n+1}(\xi)(\tau)\chi_{E_i^n(\xi)}(\tau)d\tau, \tag{5.32}$$

$t \in T$, $\xi \in \Xi$, $I_n(\xi) = \{i \geq 1 : \xi \in \overline{U_i^n}\}$. Continuity of $\xi \mapsto x_{n+1}(\cdot,\xi)$ follows from continuity of the mappings $x^0 : \Xi \to X$, $v_i^{n+1} : \Xi \to \mathbf{L}_1(T,X)$ and $E_i^n : \Xi \to \mathcal{L}$, $i \geq 1$.

By (5.32), (5.31), (5.30) and (5.19) we have

$$\int_0^t \|\dot{x}_{n+1}(\tau,\xi) - \dot{x}_n(\tau,\xi)\|d\tau \leq \sum_{i \in I_n(\xi)} \int_{E_i^n(\xi)} \|v_i^{n+1}(\xi)(\tau) - \dot{x}_n(\tau,\xi)\|\chi_{[0,t]}(\tau)d\tau$$

$$\leq \frac{\nu(\xi)}{2^n} + \sum_{i \in I_n(\xi)} e_i^n(\xi) \int_0^t \|v_i^{n+1}(\xi)(\tau) - \dot{x}_n(\tau,\xi)\|d\tau \leq \psi_n(\xi)(t).$$

Let $b_{n+1}(\cdot,\xi)$ be a measurable selection of the map $t \mapsto \Pi_{K(\xi)}(x_{n+1}(t,\xi))$ for any fixed $\xi \in \Xi$ and $f_{n+1}(t,\xi) = d_{K(\xi)}(x_{n+1}(t,\xi))$, $t \in T$. By using Lemma 1 [20] and Proposition 2.2 one can write

$$\frac{d}{dt} f_{n+1}(t,\xi) \leq d_{\Gamma(t,b_{n+1}(t,\xi),\xi)}(\dot{x}_{n+1}(t,\xi)) \tag{5.33}$$

a.e. in T. Take $i \in I_n(\xi)$ such that $t \in E_i^n(\xi)$. In this case $\dot{x}_{n+1}(t,\xi) = v_i^{n+1}(\xi)(t) \in \Gamma(t, b_i^n(\xi)(t), \xi)$ and recalling (5.28), (5.25), (5.23), (5.27), (5.22) we obtain

$$d_{\Gamma(t,b_{n+1}(t,\xi),\xi)}(\dot{x}_{n+1}(t,\xi)) \leq k(\xi)(t)\|b_i^n(\xi)(t) - b_{n+1}(t,\xi)\| \leq$$

$$k(\xi)(t)[\|b_i^n(\xi)(t) - b_n(t,\xi_i^n)\| + \|b_n(t,\xi_i^n) - x_n(t,\xi_i^n)\| +$$

$$\|x_n(t,\xi_i^n) - x_n(t,\xi)\| + \|x_n(t,\xi) - x_{n+1}(t,\xi)\| + \|x_{n+1}(t,\xi) - b_{n+1}(t,\xi)\|] \leq$$

$$k(\xi)(t)\left[f_{n+1}(t,\xi) + \psi_n(\xi)(t) + \eta_n(\xi)(t) + \frac{3\nu(\xi)}{2^n}\right]. \tag{5.34}$$

From (5.33), (5.34), (5.20) and (5.21) it follows now that

$$f_{n+1}(t,\xi) \leq \eta_{n+1}(\xi)(t); \quad d_{\Gamma(t,b_{n+1}(t,\xi),\xi)}(\dot{x}_{n+1}(t,\xi)) \leq \varphi_{n+1}(\xi)(t),$$

$t \in T$, $\xi \in \Xi$. Since $n \geq 1$ is arbitrary and the function $x_1(t,\xi)$ is given by (5.16) the claim is proved.

The equalities (5.7), (5.5) imply

$$\eta_1(\xi)(t) = \int_0^t [\varphi_1(\xi)(\tau) - k(\xi)(\tau)\eta_1(\xi)(\tau)]\, e^{m(\xi)(t)-m(\xi)(\tau)} d\tau.$$

Hence $\varphi_1(\xi)(t) = \frac{d}{dt}\eta_1(\xi)(t)$ a.e. in T, $\xi \in \Xi$. By the similar way from (5.20), (5.21) we obtain $\varphi_n(\xi)(t) = \frac{d}{dt}\eta_n(\xi)(t)$ a.e. in T, $\xi \in \Xi$, for $n \geq 2$. Consequently,

$$\psi_n(\xi)(t) = 3\eta_n(\xi)(t) + \frac{4\nu(\xi)}{2^n} m(\xi)(t) + \frac{\nu(\xi)}{2^n} \tag{5.35}$$

and

$$\eta_{n+1}(\xi)(t) = 4\int_0^t \eta_n(\xi)(\tau)k(\xi)(\tau)e^{m(\xi)(t)-m(\xi)(\tau)}d\tau +$$

$$+\frac{4\nu(\xi)}{2^n}\left[2(e^{m(\xi)(t)} - 1) - m(\xi)(t)\right], \tag{5.36}$$

$t \in T$, $\xi \in \Xi$, $n \geq 1$.

From (5.3), (5.4) it is easy to see that there exists a continuous function $c : \Xi \to \mathbf{R}$, $\eta_1(\xi)(t) \leq c(\xi)$ for all $t \in T$, $\xi \in \Xi$. Then by induction we find

$$\eta_n(\xi)(t) \leq c(\xi)e^{m(\xi)(t)}\frac{(4m(\xi)(t))^{n-1}}{(n-1)!} + \frac{4\nu(\xi)}{2^{n-1}}e^{5m(\xi)(t)}, \tag{5.37}$$

$n \geq 1$, $\xi \in \Xi$, $t \in T$. The estimation (5.37) and (5.35) imply the convergence of the serieses $\sum_{n=1}^{\infty} \eta_n(\xi)$ and $\sum_{n=1}^{\infty} \psi_n(\xi)$ in the space $\mathbf{C}(T,\mathbf{R})$ of all continuous functions with the sup-norm uniformly in each compact subset from Ξ. Recalling (5.22) we conclude that the sequence $\{x_n(\cdot,\xi)\}_{n=1}^{\infty}$ is fundamental in the Banach space $\mathbf{AC}(T,X)$ and therefore it converges to some function $x(\cdot,\xi) \in \mathbf{AC}(T,X)$, $x(0,\xi) = x^0(\xi)$, $\xi \in \Xi$. Moreover the mapping $\xi \mapsto x(\cdot,\xi)$ is continuous from Ξ into $\mathbf{AC}(T,X)$.

In accordance with (5.23) $x(t,\xi) \in K(\xi)$ and $b_n(t,\xi) \to x(t,\xi)$ as $n \to \infty$ for any $\xi \in \Xi$ uniformly by $t \in T$. Using (5.24) we have

$$\int_0^t d_{\Gamma(\tau,x(\tau,\xi),\xi)}(\dot{x}(\tau,\xi))d\tau \leq \int_0^t [d_{\Gamma(\tau,b_n(\tau,\xi),\xi)}(\dot{x}_n(\tau,\xi)) +$$

$$\|\dot{x}_n(\tau,\xi) - \dot{x}(\tau,\xi)\| + k(\xi)(\tau)\|x(\tau,\xi) - b_n(\tau,\xi)\|]d\tau \leq$$

$$\eta_n(\xi)(t) + \|x_n(\cdot,\xi) - x(\cdot,\xi)\|_{AC} + \int_0^t k(\xi)(\tau)\|x(\tau,\xi) - b_n(\tau,\xi)\|d\tau. \quad (5.38)$$

Hence $\dot{x}(t,\xi) \in \Gamma(t, x(t,\xi), \xi)$ for a.a. $t \in T$, $\xi \in \Xi$, because the right-hand side of the inequality (5.38) tends to zero.

Finally from (5.22) and (5.35) we obtain

$$\int_0^t \|\dot{x}(\tau,\xi) - \dot{y}(\tau,\xi)\|d\tau \leq \psi_0(\xi)(t) +$$

$$+3\sum_{n=1}^{\infty} \eta_n(\xi)(t) + 4\nu(\xi)m(\xi)(t) + \nu(\xi). \quad (5.39)$$

By (5.36) the function $S(\xi)(t) = \sum_{n=1}^{\infty} \eta_n(\xi)(t)$ satisfies

$$S(\xi)(t) = \eta_1(\xi)(t) + 4\int_0^t S(\xi)(\tau)k(\xi)(\tau)e^{m(\xi)(t)-m(\xi)(\tau)}d\tau +$$

$$4\nu(\xi)\left[2(e^{m(\xi)(t)} - 1) - m(\xi)(t)\right].$$

Find the solution of this integral equation

$$S(\xi)(t) = 2\int_0^t \rho(\xi)(\tau)\left[e^{5(m(\xi)(t)-m(\xi)(\tau))}\int_0^{m(\xi)(t)-m(\xi)(\tau)}(1+\theta)e^{5\theta}d\theta\right]d\tau$$

$$+\left(2\omega(\xi) + 8\nu(\xi)\right)e^{5m(\xi)(t)}\int_0^{m(\xi)(t)}\theta e^{-5\theta}d\theta$$

$$+\frac{1}{5}\left(\omega(\xi) + \|x^0(\xi) - y^0(\xi)\| + 8\nu(\xi)\right)\left(e^{5m(\xi)(t)} - 1\right), \quad (5.40)$$

$t \in T$, $\xi \in \Xi$. Thus from (5.39), (5.40) and (5.3) the inequality (3.3) follows with some continuous nondecreasing function $\Phi : \mathbf{R}^+ \to \mathbf{R}^+$. Theorem is completely proved.

6 Uniform relaxation property

Assumed the existence of a continuous function $x^0(\xi) \in K(\xi)$, $\xi \in \Xi$, we consider now the convexified problem

$$\dot{x} \in \overline{co}\Gamma(t, x, \xi); \quad (6.1)$$

$$x(t) \in K(\xi); \tag{6.2}$$

$$x(0) = x^0(\xi), \tag{6.3}$$

where $\overline{co}$ stands for the closed convex hull of the set. Applying Theorem 3.1 we shall prove that any solution $z(\cdot,\xi)$ of the problem (6.1)-(6.3) contionuously depending on ξ may be approximated by the similar solutions of the original problem (1.1)-(1.3) uniformly by the parameter.

In the following $\mathcal{H}_\Gamma(\xi)$ ($\mathcal{H}_{\overline{co}\Gamma}(\xi)$) denotes the family of all (Carathéodory) solutions to the problem (1.1)-(1.3) (correspondently, (6.1)-(6.3)), $\xi \in \Xi$.

Theorem 6.1 *Suppose that $K : \Xi \to X$ and $\Gamma : T \times grK \to X$ satisfy the hypotheses $(H_1) - (H_6)$ and there exists a continuous function $l : \Xi \to \mathbf{L}_1(T, \mathbf{R})$ with*

$$\Gamma(t, x^0(\xi), \xi) \cap l(\xi)(t)\bar{B} \neq \emptyset \tag{6.4}$$

whenever $\xi \in \Xi$, a.a. $t \in T$. Then for any continuous selection $z(\cdot,\xi)$ of the mapping $\xi \mapsto \mathcal{H}_{\overline{co}\Gamma}(\xi)$ from Ξ to $\mathbf{AC}(T,X)$ and any continuous function $\varepsilon : \Xi \to \mathbf{R}^+$, $\varepsilon(\xi) > 0$, there exists a continuous mapping $\xi \mapsto x(\cdot,\xi)$ from Ξ to $\mathbf{AC}(T,X)$ such that $x(\cdot,\xi) \in \mathcal{H}_\Gamma(\xi)$, $\xi \in \Xi$, and

$$\|x(t,\xi) - z(t,\xi)\| \leq \varepsilon(\xi)$$

for all $t \in T$, $\xi \in \Xi$.

Proof. Since $z(t,\xi) \in K(\xi)$, $t \in T$, $\xi \in \Xi$, the multifunction $G : \Xi \to \mathbf{L}_1(T,X)$,

$$G(\xi) = \{u(\cdot) \in \mathbf{L}_1(T,X) : u(t) \in \Gamma(t, z(t,\xi), \xi) \ \ a.e. \ in \ T\},$$

$\xi \in \Xi$, is well defined, lower semicontinuous and has nonempty closed decomposable values by (6.4) and Proposition 5.1. Put

$$\nu(\xi) = \frac{\varepsilon(\xi)}{2} \min\left\{\left[\Phi(m(\xi)(a)) + 2\int_0^{m(\xi)(a)} \Phi(\theta)d\theta\right]^{-1}, 1\right\}$$

where $\Phi : \mathbf{R}^+ \to \mathbf{R}^+$ is taken as in Theorem 3.1. Notice that by Corollary 1.1 [21] the set $G(\xi) \cap (\dot{z}(\cdot,\xi) + \nu(\xi)\mathcal{B}_w)$ is not empty for each $\xi \in \Xi$, because $\dot{z}(t,\xi) \in \overline{co}\Gamma(t, z(t,\xi), \xi)$ a.e. in T. Therefore it follows from Theorem 3.1

[22] that there exists a continuous function $v : \Xi \to \mathbf{L}_1(T, X)$, $v(\xi)(t) \in \Gamma(t, z(t,\xi), \xi)$,

$$\| \int_0^t (\dot{z}(\tau,\xi) - v(\xi)(\tau))d\tau \| < \nu(\xi), \tag{6.5}$$

$t \in T$, $\xi \in \Xi$. Denote $y(t,\xi) = x^0(\xi) + \int_0^t v(\xi)(\tau)d\tau$, $t \in T$, $\xi \in \Xi$, and choose for each $\xi \in \Xi$ a measurable function $b(\cdot,\xi)$, $b(t,\xi) \in \Pi_{K(\xi)}(y(t,\xi))$, $t \in T$. By using (6.5) and (H_5) write the inequality

$$d_{\Gamma(t,b(t,\xi),\xi)}(\dot{y}(t,\xi)) \leq k(\xi)(t)\|z(t,\xi) - b(t,\xi)\| \leq$$

$$\leq k(\xi)(t)(\|y(t,\xi) - z(t,\xi)\| + \|y(t,\xi) - b(t,\xi)\|) \leq 2\nu(\xi)k(\xi)(t),$$

$t \in T$, $\xi \in \Xi$. It means that the function $y(t,\xi)$ satisfies the hypothesis (H_Γ). Applying Theorem 3.1 we find the continuous mapping $\xi \mapsto x(\cdot,\xi)$ from Ξ to $\mathbf{AC}(T,X)$ such that $x(\cdot,\xi) \in \mathcal{H}_\Gamma(\xi)$ and

$$\int_0^t \|\dot{x}(\tau,\xi) - \dot{y}(\tau,\xi)\|d\tau \leq 2\nu(\xi)\int_0^t k(\xi)(\tau)\Phi(m(\xi)(t) - m(\xi)(\tau))d\tau +$$

$$\nu(\xi)\Phi(m(\xi)(t)) \leq \nu(\xi)\left[\Phi(m(\xi)(a)) + 2\int_0^{m(\xi)(a)} \Phi(\theta)d\theta\right] \leq \frac{\varepsilon(\xi)}{2},$$

$t \in T$, $\xi \in \Xi$. Finally, recalling (6.5) we obtain $\|x(t,\xi) - z(t,\xi)\| \leq \|x(t,\xi) - y(t,\xi)\| + \|y(t,\xi) - z(t,\xi)\| \leq \varepsilon(\xi)$ for all $t \in T$, $\xi \in \Xi$. Theorem is proved.

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