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Abstract

In the paper Kählerian Killing spinors are defined and their basis properties are investigated. Each Kähler manifold that admits a Kählerian Killing spinor is Einstein of odd complex dimension. Kählerian Killing spinors are a special kind of Kählerian twistor spinors. Real Kählerian Killing spinors appear for example, on closed Kähler manifolds with the smallest possible first eigenvalue of the Dirac operator. For the complex projective spaces $\mathbb{C}P^{2l-1}$ and the complex hyperbolic spaces $\mathbb{C}H^{2l-1}$ with $l > 1$ the dimension of the space of Kählerian Killing spinors is equal to $\binom{2l}{l}$. It is shown that in complex dimension 3 the complex hyperbolic spaces $\mathbb{C}H^3$ is the only simple connected complete spin Kähler manifold admitting an imaginary Kählerian Killing spinor.

Introduction

If (M^n, g, S) is a Riemannian spin manifold with metric g and spinor bundle S, then a section $0 \neq \psi \in \Gamma(S)$ is called a Killing spinor to the Killing number $\kappa \in \mathbb{C}$ $(\kappa \neq 0)$ iff ψ satisfies the differential equation

$$\nabla_X \psi + \kappa X \cdot \psi = 0 \tag{I.1}$$

for all vector fields X. Killing spinors appear in Mathematical Physics as well in purely mathematical context. There are many interesting results concerning the geometrical structure and the classification of Riemannian spin manifolds admitting Killing spinors (see references). For instance, such a manifold must be Einstein with scalar curvature $R = 4n(n-1)\kappa^2$. Moreover, it is known that, on a Kähler manifold, the equation (I.1) has only the trivial solution for $n > 2$. Consequently, for a spin Kähler manifold, equation (I.1) must be modified. This

can be done in the following way: Let (M^{2m}, J, g, S) be a spin Kähler manifold of complex dimension m with complex structure J, Kähler metric g and spinor bundle S. Then S possesses a canonical decomposition $S = S_0 \oplus S_1 \oplus \cdots \oplus S_m$ into holomorphic subbundles S_r with $\mathrm{rank}_{\mathbb{C}} S_r = \binom{m}{r}$. We consider sections $\varphi = \psi_{r-1} + \psi_r \in \Gamma(S_{r-1} \oplus S_r)$ $(1 \leq r \leq m)$ such that for each vector field X the differential equations

$$\begin{aligned} \nabla_X \psi_{r-1} + \tfrac{\kappa}{2}(X + iJX)\psi_r &= 0 \\ \nabla_X \psi_r + \tfrac{\kappa}{2}(X - iJX)\psi_{r-1} &= 0 \end{aligned} \tag{I.2}$$

are satisfied, where $\kappa \neq 0$ is a given complex number. The first result is that the existence of a non-trivial solution of the equations (I.2) implies that M^{2m} is an Einstein space with scalar curvature $R = 4m(m+1)\kappa^2$, where m is odd and $r = (m+1)/2$ (Theorem 3).

If M^{2m} is a spin Kähler-Einstein manifold of complex dimension $m = 2l - 1$ and scalar curvature $R \neq 0$, then $\mathcal{K}_+(M^{2m})$ and $\mathcal{K}_-(M^{2m})$ denote the spaces of solutions of the equations (I.2) for $r = l$ to the Killing numbers $\kappa_+ = \sqrt{R/4m(m+1)}$ and $\kappa_- = -\kappa_+$, respectively. The non-trivial elements of these spaces are called Kählerian Killing spinors. The Killing numbers κ_+ and κ_- can be real $(R > 0)$ or purely imaginary $(R < 0)$. According to this a corresponding Kählerian Killing spinor is called real or imaginary, respectively. We prove

$$\dim_{\mathbb{C}} \mathcal{K}_+(M^{2m}) = \dim_{\mathbb{C}} \mathcal{K}_-(M^{2m}) \leq 2 \cdot \binom{m}{l} \tag{I.3}$$

where $l = (m+1)/2$ (Theorem 4). In general we have the inclusions $\mathcal{K}_+(M^{2m}) \subseteq E^{\lambda_+}(D)$ and $\mathcal{K}_-(M^{2m}) \subseteq E^{\lambda_-}(D)$, where $E^{\lambda_+}(D)$ and $E^{\lambda_-}(D)$ denote the eigenspaces of the Dirac operator D to the eigenvalues $\lambda_\pm = (m+1)\kappa_\pm$. Moreover, if M^{2m} is closed, then $\kappa_+, \kappa_- \in \mathbb{R}$ and $\mathcal{K}_\pm(M^{2m}) = E^{\lambda_\pm}(D)$ (Theorem 5). Hence, Kählerian Killing spinors are related to the limiting

case of the inequality

$$\lambda_1 \geq \frac{1}{2}\sqrt{\frac{m+1}{m}R_0} \tag{I.4}$$

for the first eigenvalue λ_1 of the Dirac operator which holds in case of M^{2m} being closed and $R_0 := \min(R) > 0$. We recall that M^{2m} must be an Einstein space of odd complex dimension if (I.4) is an equality (see [18]). Thus, Kählerian Killing spinors are important for the investigation of the limiting case of the inequality (I.4) and the associated classification problem, which has only been solved for the trivial case $m = 1$ and for $m = 3$ up to now (see [19]). Further, the first component ψ_{l-1} of each Kählerian Killing spinor $\psi = \psi_{l-1} + \psi_l$ $(m = 2l - 1)$ is a Kählerian twistor spinor of type $(l, l-1)$. Hence, Kählerian Killing spinors yield examples of Kählerian twistor spinors of special kind. Moreover, for $m = 2l-1 > 1$, it holds $p_{l-1}\mathcal{K}_+(M^{2m}) = p_{l-1}\mathcal{K}_-(M^{2m}) = \ker \mathcal{D}^{(l,l-1)}$, where p_{l-1} denotes the projection onto the subbundle S_{l-1} and $\ker \mathcal{D}^{(l,l-1)}$ is the space of Kählerian twistor spinors of type $(l, l-1)$ (Theorem 6). The motive for the denomination "Killing spinor" is the fact that the Kählerian Killing spinors are closely related to Killing fields (Theorem 9).

In Section 2 we further investigate the structure of the space $\mathcal{K}_+(M^{2m})$ being isomorphic to $\mathcal{K}_-(M^{2m})$ and we construct Kählerian Killing spinors with special properties in case $\mathcal{K}_+(M^{2m}) \neq 0$ (Theorem 10, 11). By Theorem 12 we determine the action of the curvature tensor on the Killing field associated to a Kählerian Killing spinor.The results mentioned last are used in Section 4, where we treat the problem of classification of all spin Kähler manifolds M^6 admitting a Kählerian Killing spinor. If M^6 is closed, the solution of this problem is already known. The only closed spin Kähler manifolds M^6 possessing Kählerian Killing spinors are the complex projective space $\mathbb{C}P^3$ and the flag manifold $F(1,2)$. Since, in the closed case, each Kählerian Killing spinor is real, $\mathbb{C}P^3$ and $F(1,2)$ are

examples of manifolds with real Kählerian Killing spinors. We prove that each simple connected complete spin Kähler manifold M^6 with imaginary Kählerian Killing spinor is analytically isometric to the complex hyperbolic space $\mathbb{C}H^3$ (Theorem 17). This is one of the main results of the paper. In Section 3, for $m = 2l - 1 > 1$, we compute a basis of the space $\ker \mathcal{D}^{(l,l-1)}$ in case of the Fubini metric on $\mathbb{C}^m$ and in case of $\mathbb{C}H^m$, respectively (Theorem 13). For these two examples the limiting case of the inequality (I.3) is realized and also in case of $\mathbb{C}P^m$ (Theorem 15). Thus, we obtain that (I.3) is a sharp estimation. In Section 1 we define the basic notions appearing in this paper. We describe the structure of the spinor bundle of a spin Kähler manifold and we mention the needed facts of the theory of Kählerian twistor spinors. In more detail, one can find all this in [20] and [22]. Moreover, in Section 1 we prove a result on the description of Kählerian twistor spinors in terms of complex forms (Theorem 1) and we give an estimation of the dimension of the space of Kählerian twistor spinors of type $(r, r-1)$ in case $r \neq 1$, $(m+2)/2$ (Theorem 2). These results are used in Section 3.

1 Some basic notions and facts

Let (M^{2m}, J, g, S) be a spin Kähler manifold of complex dimension m with complex structure J, Kähler metric g and spinor bundle S. We always suppose that M^{2m} is connected. The spin structure of M^{2m} is defined by a holomorphic line bundle L being a square root of the canonical bundle $\Lambda^{m,0}$. There exists a canonical splitting

$$S = S_0 \oplus S_1 \oplus \cdots \oplus S_m \tag{1.1}$$

into holomorphic subbundles $S_r \cong \Lambda^{0,r} \otimes L$ $(r = 0, \cdots, m)$. Let $\iota : S \to S$ be the bundle map $\iota = \sum_{r=0}^{m} i^r p_r$, where $p_r : S \to S$ is the corresponding projection with $p_r S = S_r$. Then we have

$\iota^2 = \sum_{r=0}^{m}(-1)^r p_r$. There is a canonical anto-unitary bundle map $j : S \to S$ having the property $j^2 = (-1)^{m(m+1)/2}$. Hence, for $m = 4k$ and $m = 4k+3$ ($m = 4k+1$ and $m = 4k+2$), j is a real (quaternionic) structure of the spinor bundle. It holds $jS_r = S_{m-r}$, i.e., j determines an anti-unitary isomorphism of the bundles S_r and S_{m-r}. Since $jp_r = p_{m-r}j$, we obtain the relations $j\iota = i^m \iota j$ and $j\iota^2 = (-1)^m \iota^2 j$. j commutes with the Cliffors multiplication by real vectors. Thus, if Z is a complex vector field on M^{2m} and $\psi \in \Gamma(S)$, then we have $jZ\psi = \bar{Z}j\psi$. Using the denotations $p(X) = \frac{1}{2}(X - iJX)$, $\bar{p}(X) = \frac{1}{2}(X + iJX)$ from the real Clifford relation $XY + YX = -2g(X,Y)$ one derives the comples Clifford relations $p(X)p(Y) + p(Y)p(X) = 0$, $\bar{p}(X)\bar{p}(Y) + \bar{p}(Y)\bar{p}(X) = 0$ and $p(X)\bar{p}(Y) + \bar{p}(Y)p(X) = -2g(p(X), \bar{p}(Y)) = -g(X,Y) - i\Omega(X,Y)$, where Ω is the Kähler form defined by $\Omega(X,Y) = g(X, JY)$. Using the convention $S_r = 0$ if $r \notin \{0, 1, \cdots, m\}$, for any $r \in \mathbb{Z}$, each $X \in T(M^{2m})$ and each $\psi \in \Gamma(S_r)$, it holds that $p(X)\psi \in \Gamma(S_{r+1})$ and $\bar{p}(X)\psi \in \Gamma(S_{r-1})$. Moreover, we have the relations $\iota X = J(X)\iota$, $X\iota = -\iota J(X)$ and $X\iota^2 = -\iota^2 X$ for each vector field X. For $\varphi, \psi \in \Gamma(S)$ let $\varphi\psi$ be the complex vector field defined by $g(X, \varphi\psi) = -i\langle X\varphi, \psi\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the usual Hermitian scalar product on S. Then we obtain the formulas

$$\psi\varphi = \overline{\varphi\psi} \quad (\iota\varphi)(\iota\psi) = J(\varphi\psi) \quad (j\varphi)(j\psi) = -\psi\varphi \tag{1.2}$$

$$\varphi(Z\psi) = (\bar{Z}\varphi)\psi - 2i\langle\varphi, \psi\rangle\bar{Z} \tag{1.3}$$

where Z is any complex vector field. Moreover, if $\varphi \in \Gamma(S_{r-1})$ and $\psi \in \Gamma(S_r)$, then $J(\varphi\psi) = -i\varphi\psi$ and $J(\psi\varphi) = i\psi\varphi$. Let ∇ be the covariant derivative defined by the metric g and let ∇ also denote the corresponding covariant derivative on S. Then it holds

$$\nabla_Z(\varphi\psi) = (\nabla_Z\varphi)\psi + \varphi(\nabla_{\bar{Z}}\psi). \tag{1.4}$$

Let X (ξ) be any vector field (covector field or 1-form) then we denote the corresponding 1-form (vector field) by $g(X)$

$(g^{-1}(\xi))$. The Clifford product $\xi\psi \in \Gamma(S)$ of a spinor field $\psi \in \Gamma(S)$ by any 1-form ξ is defined by $\xi\psi = g^{-1}(\xi)\psi$. Moreover, for any r-form ω and $\psi \in \Gamma(S)$, the Clifford product $\omega\psi \in \Gamma(S)$ is defined as follows: Let $(X_1, \ldots, X_{2m})$ be a local orthonormal frame of vector fields and let $(\xi^1, \ldots, \xi^{2m})$ denote the corresponding coframe. Then we have the local representation $\omega = \sum_{1\leq\alpha_1<\cdots<\alpha_r\leq m} \omega_{\alpha_1\cdots\alpha_r}\xi^{\alpha_1}\wedge\ldots\wedge\xi^{\alpha_r}$ and we define locally

$$\omega\psi = \sum_{1\leq\alpha_1<\cdots<\alpha_r\leq m} \omega_{\alpha_1\ldots\alpha_r} X_{\alpha_1}\cdots X_{\alpha_r}\cdot\psi,$$

where $X_{\alpha_1}\cdots X_{\alpha_r}\cdot\psi$ denotes an iterated Clifford product. Clearly, this definition does not depend on the choice of the orthonormal frame. Hence, we obtain a globally defined spinor field $\omega\psi$. In this sense any r-form ω can be considered as an endomorphism of S or $\Gamma(S)$. For example, if we consider the Kähler form Ω as an endomorphism of S, then we find

$$\Omega = \sum_{r=0}^{m} i(m-2r)p_r. \tag{1.5}$$

If ξ is any 1-form and ω any r-form (real or complex), then we have the relation (in the sense of endomorphisms)

$$\xi\omega = \xi\wedge\omega - \xi \lrcorner\, \omega \quad \omega\xi = \omega\wedge\xi + (-1)^r\xi \lrcorner\, \omega. \tag{1.6}$$

Moreover, for any complex r-form ω and any $\varphi, \psi \in \Gamma(S)$, the relation

$$\langle\omega\varphi, \psi\rangle = (-1)^{r(r+1)/2}\langle\varphi, \bar{\omega}\psi\rangle \tag{1.7}$$

is valid. Further, we have the formula

$$\nabla_X(\omega\psi) = (\nabla_X\omega)\psi + \omega(\nabla_X\psi). \tag{1.8}$$

If $\psi \in \Gamma(S_r)$ and $\omega \in \Gamma(\Lambda^{s,0})$, then $\omega\psi \in \Gamma(S_{r-s})$ and $\bar{\omega}\psi \in \Gamma(S_{r+s})$. Let $\psi_0 \in \Gamma(S_0)$ and $\omega, \sigma \in \Gamma(\Lambda^{r,0})$, then

$$\omega\bar{\sigma}\psi_0 = (-1)^{r(r+1)/2}2^r\langle\omega, \sigma\rangle\psi_0, \tag{1.9}$$

where here $\langle\cdot,\cdot\rangle$ is the usual Hermitian scalar product on $\Lambda^{r,0}$. For each $\psi \in \Gamma(S_r)$ this yields the following: Let $P \in M^{2m}$ be

any point and let U be a sufficiently small neighbourhood of P such that there exists a section ψ_0 of $S_0|_U$ which does not vanish at any point of U. Then there is an r-form $\omega \in \Gamma(\Lambda^{r,0}|_U)$ that is uniquely determined by $\psi = \bar{\omega}\psi_0$. We say that ω is a local representation of ψ with respect to ψ_0. According to this a local representation of $\psi \in \Gamma(S)$ with respect to $\psi_0 \in \Gamma(S_0|_U)$ is a set of complex forms $\omega_0, \omega_1, \ldots, \omega_m$ with $\omega_r \in \Gamma(\Lambda^{r,0}|_U)$ such that $p_r\psi = \bar{\omega}_r\psi_0$. Such a local representation is said to be holomorphic iff the corresponding section ψ_0 is holomorphic.
Let $(X_1, \ldots, X_{2m})$ be any local frame of vector fields. We use the denotations $g_{kl} = g(X_k, X_l)$, $(g^{kl}) = (g_{kl})^{-1}$ and $X^k = g^{kl}X_l$ (Einstein's convention of summation). Let $D, \tilde{D} : \Gamma(S) \to \Gamma(S)$ be the differential operators of first order locally defined by $D\psi = X^k \cdot \nabla_{X_k}\psi$ and $\tilde{D}\psi = J(X^k) \cdot \nabla_{X_k}\psi$. Then D is the Dirac operator and $\tilde{D}$ is called the Kähler twist of D. D and $\tilde{D}$ satisfy the relations

$$D\tilde{D} + \tilde{D}D = 0 \quad \tilde{D}^2 = D^2. \tag{1.10}$$

For the operators D_+ and D_- defined by $D_\pm = \frac{1}{2}(D_\mp i\tilde{D})$ this implies the identities

$$D_+^2 = 0 \quad D_-^2 = 0 \quad D_+D_- + D_-D_+ = D^2. \tag{1.11}$$

Moreover, if $\psi \in \Gamma(S_r)$, then $D_\pm\psi \in \Gamma(S_{r\pm1})$. For any r-form ω and any spinor field ψ, the formula

$$D(\omega\psi) = ((d+d^*)\omega)\psi + (-1)^r\omega D\psi - 2(X^k \lrcorner\, \omega)\nabla_{X_k}\psi \tag{1.12}$$

is valid.
Let $\mathcal{D}_X^{(r)}\psi := \nabla_X\psi + \frac{1}{4r}(X \cdot D\psi + J(X) \cdot \tilde{D}\psi)$, where $r \in \{1, \ldots, m\}$. Then the operator $\mathcal{D}^{(r)} : \Gamma(S) \to \Gamma(TM^{2m} \otimes S)$ locally given byn $\mathcal{D}^{(r)}\psi = X^k \otimes \mathcal{D}_{X_k}^{(r)}\psi$ is the Kählerian twistor operator of type r. The elements of the space $\ker \mathcal{D}^{(r)}$ are called (Kählerian) twistor spinors of type r. By definition, we have $\psi \in \ker \mathcal{D}^{(r)}$ if ψ satisfies the differential equation of first order

$$\nabla_X\psi + \frac{1}{4r}(XD\psi + J(X)\tilde{D}\psi) = 0 \tag{1.13}$$

for each vector field X. The operators $\mathcal{D}^{(r)}$ are compatible with the decomposition (1.1), i.e., it holds that $(1 \otimes p_s) \circ \mathcal{D}^{(r)} = \mathcal{D}^{(r)} \circ p_s$ for $r = 1, \ldots, m$ and $s = 0, \ldots, m$. Let $\mathcal{D}^{(r,s)} := \mathcal{D}^{(r)} \circ p_s$, then $\mathcal{D}^{(r)} = \oplus_{s=0}^{m} \mathcal{D}^{(r,s)}$ and, hence, $\ker \mathcal{D}^{(r)} = \oplus_{s=0}^{m} \ker \mathcal{D}^{(r,s)}$. The elements of the space $\ker \mathcal{D}^{(r,s)} = \ker \mathcal{D}^{(r)} \cap \Gamma(S_s)$ are called twistor spinors of type (r, s). It holds that $j \ker \mathcal{D}^{(r,s)} = \ker \mathcal{D}^{(r,m-s)}$. Moreover, if M^{2m} is not Ricci-flat, then $\ker \mathcal{D}^{(r)} = \ker \mathcal{D}^{(r,r-1)} + j \ker \mathcal{D}^{(r,r-1)}$. Thus, in case $\mathrm{Ric} \not\equiv 0$ (Ric denotes the Ricci tensor), it suffices to consider the space $\ker \mathcal{D}^{(r,r-1)}$. $\psi \in \ker \mathcal{D}^{(r)}$ implies

$$D^2\psi = \frac{rR}{4r-2}\psi \tag{1.14}$$

$$\Delta|\psi|^2 = \frac{R}{4r-2}|\psi|^2 - \frac{1}{r}|D\psi|^2. \tag{1.15}$$

For $r \neq (m+2)/2$, we have $\psi \in \ker \mathcal{D}^{(r,r-1)}$ iff $\psi \in \Gamma(S_{r-1})$ and ψ satisfies the equation

$$\nabla_X \psi + \frac{1}{2r}\bar{p}(X)D\psi = 0 \tag{1.16}$$

for each vector field X. This equation implies

$$\bar{p}(\mathrm{Ric}(X) - \frac{R}{4r-2}X)\psi = 0 \tag{1.17}$$

and, moreover, in case $r > 1$,

$$\nabla_X D\psi + \frac{1}{2}p(\mathrm{Ric}(X))\psi = 0. \tag{1.18}$$

From (1.16) and (1.18), respectively, it follows

$$\nabla_{p(X)}\psi = 0 \quad \nabla_{\bar{p}(X)}D\psi = 0, \tag{1.19}$$

i.e., ψ is antiholomorphic and $D\psi$ is holomorphic. The first of the equations (1.19) implies

$$D_-\psi = 0. \tag{1.20}$$

Lemma 1.1: *Let $U \subseteq M^{2m}$ be an open set and let $\psi_0 \in \Gamma(S_0|_U)$ be a holomorphic section that does not vanish at any point of U. Then, for any $\omega \in \Gamma(\Lambda^{r,0}|_U)$, it holds that $D_+(\bar{\omega}\psi_0) = (\bar{\partial}\bar{\omega})\psi_0$ and $D_-(\bar{\omega}\psi_0) = (\bar{\partial}^*\bar{\omega} - 2\bar{p}(X_0) \lrcorner \bar{\omega})\psi_0$, where $X_0 :=$* $\operatorname{grad}\log|\psi_0|^2$.

Proof: Since ψ_0 is holomorphic, the equation

$$\nabla_X\psi_0 = (\partial\log|\psi_0|^2)(X)\psi_0 = g(X, \bar{p}(X_0))\psi_0 \qquad (*)$$

is satisfied for each vector field X. Using this we have

$$D\psi_0 = X^k\nabla_{X_k}\psi_0 = g(X_k, \bar{p}(X_0))X^k\psi_0 = \bar{p}(X_0)\psi_0 = 0.$$

Thus, ψ_0 has the property

$$D\psi_0 = 0. \qquad (2*)$$

Moreover, since $\bar{\omega} \in \Gamma(\Lambda^{0,r}|_U)$, we find

$$\partial^*\bar{\omega} = 0 \quad (\partial\bar{\omega})\psi_0 = 0. \qquad (3*)$$

By (1.12), $(*)$, $(2*)$ and $(3*)$ it holds

$$\begin{aligned} D(\bar{\omega}\psi_0) &= ((\partial + \bar{\partial} + \partial^* + \bar{\partial}^*)\bar{\omega})\psi_0 + (-1)^r\bar{\omega}D\psi_0 - \\ &\quad 2(X^k \lrcorner \bar{\omega})\nabla_{X_k}\psi_0 = \\ &= (\bar{\partial}\bar{\omega})\psi_0 + (\bar{\partial}^*\bar{\omega})\psi_0 - 2((g(X_k, \bar{p}(X_0))X^k) \lrcorner \bar{\omega})\psi_0 = \\ &= (\bar{\partial}\bar{\omega}\psi_0 + (\bar{\partial}^*\bar{\omega})\psi_0 - 2(\bar{p}(X_0) \lrcorner \bar{\omega})\psi_0. \end{aligned}$$

Since $D = D_+ + D_-$, $(\bar{\partial}\bar{\omega})\psi_0 \in \Gamma(S_{r+1}|_U)$ and $(\bar{\partial}^*\bar{\omega})\psi_0 - 2(\bar{p}(X_0) \lrcorner \bar{\omega})\psi_0 \in \Gamma(S_{r-1}|_U)$, this yields $D_+(\bar{\omega}\psi_0) = (\bar{\partial}\bar{\omega})\psi_0$ and $D_-(\bar{\omega}\psi_0) = (\bar{\partial}^*\bar{\omega} - 2\bar{p}(X_0) \lrcorner \bar{\omega})\psi_0$, q.e.d.

Theorem 1: *Let $\psi \in \Gamma(S_{r-1})$ and $r \neq (m+2)/2$, then $\psi \in$* $\ker \mathcal{D}^{(r,r-1)}$ *iff each local holomorphic representation (ω, ψ_0) of ψ satisfies the differential equations*

$$\nabla_{\bar{p}(X)}(|\psi_0|^2\omega) = 0 \quad \nabla_{p(X)}\omega = \frac{1}{r}p(X) \lrcorner \partial\omega \qquad (1.21)$$

for each vector field X.

Proof: Let $\psi \in \ker \mathcal{D}^{(r,r-1)}$ and let (ω, ψ_0) be any local holomorphic representation of ψ. Then we have $\psi = \bar{\omega}\psi_0$. Since ψ_0 is holomorphic, it holds

$$\nabla_X \psi = (\nabla_X \bar{\omega})\psi_0 + \bar{\omega}\nabla_X \psi_0 = (\nabla_X \bar{\omega} + g(X, \bar{p}(X_0))\bar{\omega})\psi_0,$$

where $X_0 = \text{grad } \log |\psi_0|^2$. Using Lemma 1.1 and the equations (1.6) it follows

$$\bar{p}(X)D_+\psi = \bar{p}(X)D_+(\bar{\omega}\psi_0) = \bar{p}(X)(\bar{\partial}\bar{\omega})\psi_0 = -2(\bar{p}(X) \lrcorner \bar{\partial}\bar{\omega})\psi_0.$$

Inserting these results into equation (1.16) we obtain

$$(\nabla_X \bar{\omega} + g(X, \bar{p}(X_0))\bar{\omega} - \frac{1}{r}\bar{p}(X) \lrcorner \bar{\partial}\bar{\omega})\psi_0 = 0$$

and hence the equation

$$\nabla_X \omega + g(X, p(X_0))\omega = \frac{1}{r}p(X) \lrcorner \partial\omega, \tag{1.22}$$

which is equivalent to the equations (1.21). Conversely, it is not hard to see that the equations (1.21) imply the twistor equation (1.16), q.e.d.

Theorem 2: *Let $r \neq 1$, $(m+2)/2$. Then we have the estimation*

$$\dim_{\mathbb{C}} \ker \mathcal{D}^{(r,r-1)} \leq \binom{m+1}{r}. \tag{1.23}$$

Proof: We consider on $S_{r-1} \oplus S_r$ the covariant derivative $\nabla^{(r)}$ defined by

$$\nabla_X^{(r)}(\psi_{r-1}+\psi_r) = \nabla_X(\psi_{r-1}+\psi_r) + \frac{1}{2r}\bar{p}(X)\psi_r + \frac{1}{2}p(\text{Ric}(X))\psi_{r-1}.$$

We see that $\psi_{r-1} + \psi_r \in \Gamma(S_{r-1} \oplus S_r)$ is parallel with respect to $\nabla^{(r)}$ iff the differential equations

$$\nabla_X \psi_{r-1} + \frac{1}{2r}\bar{p}(X)\psi_r = 0 \tag{1.24}$$

$$\nabla_X \psi_r + \frac{1}{2} p(\mathrm{Ric}(X))\psi_{r-1} = 0 \tag{1.25}$$

are satisfied for each vector field X. Equation (1.24) implies the equation

$$\psi_r = D\psi_{r-1}. \tag{1.26}$$

Thus, by (1.24) and (1.26), ψ_{r-1} satisfies (1.16), i.e., we have $\psi_{r-1} \in \ker \mathcal{D}^{(r,r-1)}$. Conversely, we conclude from (1.16) and (1.18) that $\psi \in \ker \mathcal{D}^{(r,r-1)}$ implies $\psi + D\psi \in \ker \nabla^{(r)}$. Hence, we obtain an isomorphism of complex vector spaces

$$\ker \mathcal{D}^{(r,r-1)} \cong \ker \nabla^{(r)} \tag{1.27}$$

defined by $\psi \mapsto \psi + D\psi$. Each $\varphi \in \ker \nabla^{(r)}$ is generated by parallel displacements of a spinor $\varphi_{P_0} \in (S_{r-1} \oplus S_r)_{P_0}$ from any fixed point $P_0 \in M^{2m}$ into all the other points. Thus, we have

$$\dim_{\mathbb{C}} \ker \mathcal{D}^{(r,r-1)} = \dim_{\mathbb{C}} \ker \nabla^{(r)} \leq \dim_{\mathbb{C}} (S_{r-1} \oplus S_r)_{P_0} =$$
$$= \binom{m}{r-1} + \binom{m}{r} = \binom{m+1}{r}$$

and hence (1.23), q.e.d.

2 General properties of Kählerian Killing spinors

Let $\psi = \psi_{r-1} + \psi_r \in \Gamma(S_{r-1} \oplus S_r)$ $(1 \leq r \leq m)$ be a section such that, for each vector field X, the equations

$$\nabla_X \psi_{r-1} + \kappa \bar{p}(X)\psi_r = 0 \quad \nabla_X \psi_r + \kappa p(X)\psi_{r-1} = 0 \tag{2.1}$$

are satisfied, where $\kappa \neq 0$ is a given complex number.

Theorem 3: *Let M^{2m} be a spin Kähler manifold such that the equations (2.1) have a non-trivial solution. Then M^{2m} is an Einstein space of odd complex dimension with scalar curvature*

$R = 4m(m+1)\kappa^2$ *and* $r = (m+1)/2$ *holds.*

Proof: Suppose that $\psi = \psi_{r-1} + \psi_r$ is a non-trivial solution of (2.1). By direct calculations the equations (2.1) imply

$$\begin{aligned} C(X,Y)\psi_{r-1} &= \kappa^2(\bar{p}(Y)p(X) - \bar{p}(X)p(Y))\psi_{r-1} \quad (2.2) \\ C(X,Y)\psi_r &= \kappa^2(p(Y)\bar{p}(X) - p(X)\bar{p}(Y))\psi_r, \quad (2.3) \end{aligned}$$

where C denotes the curvature tensor of the spinor bundle. Using the well-known relations

$$X^k C(X, X_k) = -\frac{1}{2}\mathrm{Ric}\,(X) \tag{2.4}$$

$$X^k \bar{p}(X_k) = -m - i\Omega \quad X^k p(X_k) = -m + i\Omega \tag{2.5}$$

and (1.5) we have by (2.2)

$$\begin{aligned} & -\frac{1}{2}\mathrm{Ric}\,(X)\psi_{r-1} = X^k C(X, X_k)\psi_{r-1} = \\ = \; & \kappa^2(X^k\bar{p}(X_k)p(X) - X^k\bar{p}(X)p(X_k))\psi_{r-1} = \\ = \; & \kappa^2(X^k\bar{p}(X_k)p(X) - X^k(-p(X_k)\bar{p}(X) - \\ & \quad 2g(X_k, \bar{p}(X))))\psi_{r-1} = \\ = \; & \kappa^2((-m - i\Omega)p(X) + (-m + i\Omega)\bar{p}(X) + 2\bar{p}(X))\psi_{r-1} = \\ = \; & \kappa^2(-mp(X) + (m-2r)p(X) - m\bar{p}(X) - \\ & \quad (m - 2(r-2))\bar{p}(X) + 2\bar{p}(X))\psi_{r-1} = \\ = \; & -2\kappa^2(rp(X) + (m - r + 1)\bar{p}(X))\psi_{r-1}. \end{aligned}$$

Hence, we obtain

$$\mathrm{Ric}\,(X)\psi_{r-1} = 4\kappa^2(rp(X) + (m - r + 1)\bar{p}(X))\psi_{r-1} \tag{$*$}$$

and, by a similar calculation,

$$\mathrm{Ric}\,(X)\psi_r = 4\kappa^2(rp(X) + (m - r + 1)\bar{p}(X))\psi_r. \tag{$2*$}$$

Using the general identity

$$X^k \mathrm{Ric}\,(X_k) = -R, \tag{2.6}$$

from $(*)$ and $(2*)$ one derives the relations

$$\begin{aligned}(R - 8\kappa^2(m-r+1)(2r-1))\psi_{r-1} &= 0 \qquad (3*)\\ (R - 8\kappa^2 r(2m-2r+1))\psi_r &= 0. \qquad (4*)\end{aligned}$$

Now, it cannot be that ψ_{r-1} or ψ_r vanishes identically. For example, suppose that $\psi_{r-1} \equiv 0$. Then it follows from (2.1) that ψ_r is a non-vanishing parallel spinor. This implies $\mathrm{Ric} \equiv 0$ and hence $R \equiv 0$. Because of $(4*)$ we obtain a contradiction. The supposition $\psi_r \equiv 0$ contradicts $(3*)$. Thus, the relations $(3*)$ and $(4*)$ yield the equations $R = 8\kappa^2(m-r+1)(2r-1)$ and $R = 8\kappa^2 r(2m-2r+1)$, which imply immediately $r = (m+1)/2$ and $R = 4m(m+1)\kappa^2$. By this, forming the sum of the equations $(*)$ and $(2*)$ we obtain

$$(\mathrm{Ric}(X) - 2(m+1)\kappa^2 X)\psi = 0$$

and hence

$$\mathrm{Ric}(X) = 2(m+1)\kappa^2 X = \frac{R}{2m}X,$$

since $\psi = \psi_{r-1} + \psi_r \not\equiv 0$. Thus, we see that M^{2m} is Einstein, q.e.d.

In the following we use the denotations and definitions mentioned in the introduction, for example, the denotations $\mathcal{K}_\pm(M^{2m})$ and the definition of a Kählerian Killing spinor (real, imaginary).

Theorem 4: *Let M^{2m} be a spin Kähler manifold admitting a Kählerian Killing spinor ψ. Then ψ does not vanish at any point. If ψ is real, then the function $|\psi|^2$ is constant. Moreover, it holds*

$$\imath^2\mathcal{K}_\pm(M^{2m}) = \mathcal{K}_\mp(M^{2m}) \qquad (2.7)$$

and we have the estimation

$$\dim_{\mathbb{C}} \mathcal{K}_\pm(M^{2m}) \leq 2\binom{m}{l}, \qquad (2.8)$$

where $m = 2l - 1$.

Proof: Applying the map ι^2 to the equations (2.1) we obtain (2.7). Now, let us consider the covariant derivatives ∇^+ and ∇^- on the bundle $S_{l-1} \oplus S_l$ $(m = 2l - 1)$ defined by

$$\nabla^{\pm}_X \varphi = \nabla_X \varphi + \kappa_{\pm}(p(X)p_{l-1} + \bar{p}(X)p_l)\varphi.$$

It is easy to see that

$$\mathcal{K}_{\pm}(M^{2m}) = \ker \nabla^{\pm}. \tag{2.9}$$

Hence, each Kählerian Killing spinor ψ is generated by parallel displacements of a spinor $\psi_{P_0} \in (S_{l-1} \oplus S_l)_{P_0}$ with respect to ∇^+ or ∇^-, where $P_0 \in M^{2m}$ is any point. Thus, we have $\psi_P \neq 0$ for each $P \in M^{2m}$ iff $\psi_{P_0} \neq 0$. It follows further

$$\dim_{\mathbb{C}} \mathcal{K}_{\pm}(M^{2m}) \leq \dim_{\mathbb{C}}(S_{l-1} \oplus S_l)_{P_0} = 2\binom{m}{l}.$$

Finally, we remark that for $R > 0$ the covariant derivatives ∇^+ and ∇^- are metric since the corresponding Killing numbers κ_+ and κ_- are real. This implies that, for a real Kählerian Killing spinor ψ, the function $|\psi|^2$ is constant, q.e.d.

Theorem 5: *Let M^{2m} be a closed spin Kähler manifold admitting a Kählerian Killing spinor. Then each Kählerian Killing spinor is real and*

$$\mathcal{K}_{\pm}(M^{2m}) = E^{\lambda_{\pm}}(D). \tag{2.10}$$

Proof: Let $\psi = \psi_{l-1} + \psi_l \in \mathcal{K}_+(M^{2m})$ $(m = 2l - 1)$. Then, by definition, the equations

$$\nabla_X \psi_{l-1} + \kappa_+ \bar{p}(X)\psi_l = 0 \quad \nabla_X \psi_l + \kappa_+ p(X)\psi_{l-1} = 0 \tag{2.11}$$

are satisfied. Using the first equation, by (1.5) and (2.5) we have $D\psi_{l-1} = X^k \nabla_{X_k}\psi_{l-1} = -\kappa_+ X^k \bar{p}(X_k)\psi_l = -\kappa_+(-m - i\Omega)\psi_l = -\kappa_+(-m + (m - 2l))\psi_l = 2l\kappa_+\psi_l = (m + 1)\kappa_+\psi_l =$

$\lambda_+\psi_l$. By a quite similar calculation we obtain $D\psi_l = \lambda_+\psi_{l-1}$. Thus, the equations (2.11) imply

$$D\psi_{l-1} = \lambda_+\psi_l \quad D\psi_l = \lambda_+\psi_{l-1}. \tag{2.12}$$

This yields $D\psi = \lambda_+\psi$ and hence $\mathcal{K}_+(M^{2m}) \subseteq E^{\lambda_+}(D)$. By Theorem 3, M^{2m} is an Einstein space of scalar curvature $R = \frac{4m}{m+1}\lambda_\pm^2$. Hence, we have

$$\lambda_\pm = \pm\frac{1}{2}\sqrt{\frac{m+1}{m}R}. \tag{2.13}$$

Now, let M^{2m} be closed. Since, in the closed case, the Dirac operator D has only real eigenvalues, λ_+ is real and hence $R > 0$. Thus, by (2.13) we see that here the limiting case of the inequality (I.4) is realized and that λ_+ is the first eigenvalue of D. If follows from Section 3 in [18] that each eigenspinor $\psi \in E^{\lambda_+}(D)$ is a section of the bundle $S_{l-1} \oplus S_l$ such that the components φ_{l-1} and φ_l of φ satisfy the equations

$$\nabla_X\varphi_{l-1} + \frac{\lambda_+}{m+1}\bar{p}(X)\varphi_l = 0 \quad \nabla_X\varphi_l + \frac{\lambda_+}{m+1}p(X)\varphi_{l-1} = 0.$$

Because of $\lambda_+ = (m+1)\kappa_+$ this implies $\varphi \in \mathcal{K}_+(M^{2m})$. Hence, we have $\mathcal{K}_+(M^{2m}) = E^{\lambda_+}(D)$. By (2.7) and the general relation

$$\iota^2 E^\lambda(D) = E^{-\lambda}(D) \tag{2.14}$$

this equation is equivalent to $\mathcal{K}_-(M^{2m}) = E^{\lambda_-}(D)$, q.e.d.

Theorem 6: *Let M^{2m} be a spin Kähler-Einstein manifold of complex dimension $m = 2l-1 > 1$ and scalar curvature $R \neq 0$. Then there are isomorphisms*

$$\mathcal{K}_\pm(M^{2m}) \cong \ker \mathcal{D}^{(l,l-1)}. \tag{2.15}$$

Proof: Let $\psi = \psi_{l-1} + \psi_l \in \mathcal{K}_+(M^{2m})$. Then, by (2.12), the first of the equations (2.11) can be written

$$\nabla_X\psi_{l-1} + \frac{1}{2l}\bar{p}(X)D\psi_{l-1} = 0.$$

Hence, by (1.16), it follows that $\psi_{l-1} \in \ker \mathcal{D}^{(l,l-1)}$. Conversely, let $\varphi \in \ker \mathcal{D}^{(l,l-1)}$. Since M^{2m} is Einstein, we obtain by (1.16), (1.18) and (1.20) that

$$\varphi + \frac{1}{\lambda_\pm} D_+ \varphi \in \mathcal{K}_\pm(M^{2m}). \tag{2.16}$$

This yields the isomorphism (2.15), q.e.d.

Theorem 7: *Let $\psi = \psi_{l-1} + \psi_l$ be a Kählerian Killing spinor to the Killing number κ.*

(i) If ψ is real, then

$$\mathrm{grad}|\psi_l|^2 = -\mathrm{grad}|\psi_{l-1}|^2 = \kappa J(\psi\psi) \tag{2.17}$$

$$\Delta|\psi_l|^2 = -\Delta|\psi_{l-1}|^2 = \frac{R}{2m}(|\psi_l|^2 - |\psi_{l-1}|^2). \tag{2.18}$$

(ii) If ψ is imaginary, then

$$\mathrm{grad}|\psi_l|^2 = \mathrm{grad}|\psi_{l-1}|^2 = -i\kappa\psi\psi \tag{2.19}$$

$$\Delta|\psi_l|^2 = \Delta|\psi_{l-1}|^2 = \frac{R}{2m}(|\psi_{l-1}|^2 + |\psi_l|^2). \tag{2.20}$$

Proof: Let $\psi = \psi_{l-1} + \psi_l$ be a real Kählerian Killing spinor to the Killing number κ. Then, by (2.1), it holds

$$\begin{aligned} & g(X, \mathrm{grad}|\psi_l|^2) = X(|\psi_l|^2) = \langle \nabla_X \psi_l, \psi_l \rangle + \langle \psi_l, \nabla_X \psi_l \rangle = \\ = \; & \langle -\kappa p(X)\psi_{l-1}, \psi_l \rangle + \langle \psi_l, -\kappa p(X)\psi_{l-1} \rangle = \\ = \; & -i\kappa(g(X, \psi_{l-1}\psi_l) - g(X, \psi_l\psi_{l-1})) = \\ = \; & \kappa g(X, J(\psi_{l-1}\psi_l) + J(\psi_l\psi_{l-1})) = g(X, \kappa J(\psi\psi)). \end{aligned}$$

This yields $\mathrm{grad}|\psi_l|^2 = \kappa J(\psi\psi)$. By Theorem 4, the function $|\psi|^2 = |\psi_{l-1}|^2 + |\psi_l|^2$ is constant. Thus, it follows that grad $|\psi_{l-1}|^2 = -\kappa J(\psi\psi)$. Moreover, using the general formula

$$\mathrm{div}(\varphi\psi) = -i(\langle D\varphi, \psi \rangle - \langle \varphi, D\psi \rangle), \tag{2.21}$$

by (2.12) and (2.17) we have

$$\begin{aligned}
&\Delta|\psi_l|^2 = -\mathrm{div}\ \mathrm{grad}|\psi_l|^2 = -\kappa\mathrm{div}(J(\psi\psi)) = \\
= \ & -\kappa\mathrm{div}(J(\psi_l\psi_{l-1} + \psi_{l-1}\psi_l)) = -\kappa\mathrm{div}(i\psi_l\psi_{l-1} - i\psi_{l-1}\psi_l) = \\
= \ & -\kappa(\langle D\psi_l, \psi_{l-1}\rangle - \langle \psi_l, D\psi_{l-1}\rangle - \langle D\psi_{l-1}, \psi_l\rangle + \langle \psi_{l-1}, D\psi_l\rangle) = \\
= \ & -\kappa(2(m+1)\kappa|\psi_{l-1}|^2 - 2(m+1)\kappa|\psi_l|^2) = \frac{R}{2m}(|\psi_l|^2 - |\psi_{l-1}|^2).
\end{aligned}$$

In the imaginary case one obtains the equations (2.19) and (2.20) by quite similar calculations, q.e.d.

Corollary 7.1: *Let ψ be any Kählerian Killing spinor.*

(i) If ψ is real, then

$$\Delta\langle\psi, \iota^2\psi\rangle = \frac{R}{m}\langle\psi, \iota^2\psi\rangle. \tag{2.22}$$

(ii) If ψ is imaginary, then

$$\Delta|\psi|^2 = \frac{R}{m}|\psi|^2. \tag{2.23}$$

Corollary 7.2: *If ψ is any imaginary Kählerian Killing spinor, then the function $\langle\psi, \iota^2\psi\rangle$ is constant and the function $|\psi|^2$ does not possess any local maximum.*

Theorem 8: *Let $\psi = \psi_{l-1} + \psi_l$ be any Kählerian Killing spinor to the Killing number κ. Then we have the following:*

(i) It holds

$$\mathrm{grad}\langle\psi_l, j\psi_{l-1}\rangle = \kappa J(\psi j\psi). \tag{2.24}$$

(ii) If l is even, then the function $\langle\psi_l, j\psi_{l-1}\rangle$ is constant.

(iii) If l is odd, then

$$\Delta\langle\psi_l, j\psi_{l-1}\rangle = \frac{R}{m}\langle\psi_l, j\psi_{l-1}\rangle. \tag{2.25}$$

Proof: Using the obvious relation

$$\psi j \psi = \psi_{l-1} j \psi_{l-1} + \psi_l j \psi_l, \tag{2.26}$$

for any vector field X, we have

$$\begin{aligned}
& g(X, \mathrm{grad}\langle \psi_l, j\psi_{l-1}\rangle) = X(\langle \psi_l, j\psi_{l-1}\rangle) = \\
= \; & \langle \nabla_X \psi_l, j\psi_{l-1}\rangle + \langle \psi_l, j\nabla_X \psi_{l-1}\rangle = \\
= \; & -\kappa(\langle p(X)\psi_{l-1}, j\psi_{l-1}\rangle + \langle \psi_l, p(X) j\psi_l\rangle) = \\
= \; & -\kappa(ig(X, \psi_{l-1} j\psi_{l-1}) - ig(X, \psi_l j\psi_l)) = \\
= \; & g(X, \kappa(-i\psi_{l-1} j\psi_{l-1} + i\psi_l j\psi_l)) = g(X, \kappa J(\psi j\psi)).
\end{aligned}$$

This implies (2.24). Because of $m = 2l - 1$ it holds

$$j^2 = (-1)^l. \tag{2.27}$$

Using this and (1.2) it follows that $\psi j\psi = -(j^2\psi) j\psi = (-1)^{l+1}\psi j\psi$. Hence, the vector field $\psi j\psi$ vanishes identically for even l. By (2.24) this implies (ii).
Now, let l be odd. Using (2.12), (2.21) and (2.24) it holds

$$\begin{aligned}
& \Delta\langle \psi_l, j\psi_{l-1}\rangle = -\mathrm{div}\ \mathrm{grad}\langle \psi_l, \psi_{l-1}\rangle = -\kappa \mathrm{div} J(\psi j\psi) = \\
= \; & \kappa \mathrm{div}(i\psi_{l-1} j\psi_{l-1} - i\psi_l j\psi_l) = \\
= \; & \kappa(\langle D\psi_{l-1}, j\psi_{l-1}\rangle - \langle \psi_{l-1}, jD\psi_{l-1}\rangle - \langle D\psi_l, j\psi_l\rangle + \langle \psi_l, jD\psi_l\rangle) = \\
= \; & 2\kappa(\langle D\psi_{l-1}, j\psi_{l-1}\rangle + \langle \psi_l, jD\psi_l\rangle) = \\
= \; & 2\kappa((m+1)\kappa\langle \psi_l, j\psi_{l-1}\rangle + (m+1)\kappa\langle \psi_l, j\psi_{l-1}\rangle) = \\
= \; & 4(m+1)\kappa^2\langle \psi_l, j\psi_{l-1}\rangle = \frac{R}{m}\langle \psi_l, j\psi_{l-1}\rangle.
\end{aligned}$$

This proves (iii), q.e.d.

Theorem 9: *Let $\psi = \psi_{l-1} + \psi_l$ be a Kählerian Killing spinor. Then the following holds:*

(i) If ψ is real, then $\psi\psi$ is a Killing field iff the function $\langle \psi, \iota^2\psi\rangle$ does not vanish identically.

(ii) If ψ is imaginary, then $J(\psi\psi)$ is a Killing field.

(iii) If l is odd, then $\mathrm{Re}(\psi j\psi)$ $(\mathrm{Im}(\psi j\psi))$ *is a Killing field iff the function* $\mathrm{Re}\langle\psi_l, j\psi_{l-1}\rangle$ $(\mathrm{Im}\langle\psi_l, j\psi_{l-1}\rangle)$ *does not vanish identically.*

Proof: Let $\psi = \psi_{l-1} + \psi_l$ be a real Kählerian Killing spinor to the Killing number κ. Then one proves by a straightforward calculation that

$$g(\nabla_X(\psi\psi), Y) = -i\kappa(\langle(\bar{p}(X)p(Y) - \bar{p}(Y)p(X))\psi_{l-1}, \psi_{l-1}\rangle + \\ +\langle(p(X)\bar{p}(Y) - p(Y)\bar{p}(X))\psi_l, \psi_l\rangle) \tag{2.28}$$

for all vector fields X, Y. This shows that the tensor field $\nabla(\psi\psi)$ is skew symmetric. Thus, $\psi\psi$ is a Killing field iff it does not vanish identically. From (2.17) and (2.22) we see that this is the case iff the function $\langle\psi, \iota^2\psi\rangle$ does not vanish identically. If $\psi = \psi_{l-1} + \psi_l$ is imaginary, we find

$$g(\nabla_X(J(\psi\psi)), Y) = \kappa(\langle(\bar{p}(X)p(Y) - \bar{p}(Y)p(X))\psi_{l-1}, \psi_{l-1}\rangle - \\ -\langle(p(X)\bar{p}(Y) - p(Y)\bar{p}(X))\psi_l, \psi_l\rangle). \tag{2.29}$$

Hence, in the imaginary case the tensor field $\nabla(J(\psi\psi))$ is skew symmetric. From (2.19) and (2.23) we see that the vector field $\psi\psi$ can never vanish identically since $R < 0$ is constant and $|\psi|^2$ is a function that does not vanish at any point. Now, let l be odd. Using (1.2) and (2.1) it holds

$$\nabla_X(\psi_{l-1}j\psi_{l-1}) = (\nabla_X\psi_{l-1})j\psi_{l-1} + \psi_{l-1}(j\nabla_X\psi_{l-1}) = \\ = -\kappa(\bar{p}(X)\psi_l)j\psi_{l-1} - \kappa\psi_{l-1}(p(X)j\psi_l) = -2\kappa(\bar{p}(X)\psi_l)j\psi_{l-1}.$$

The second of the equations

$$\begin{aligned} \nabla_X(\psi_{l-1}j\psi_{l-1}) &= -2\kappa(\bar{p}(X)\psi_l)j\psi_{l-1} \\ \nabla_X(\psi_l j\psi_l) &= -2\kappa(p(X)\psi_{l-1})j\psi_l \end{aligned} \tag{2.30}$$

can be derived quite similarly. Using (2.26) and (2.30) we obtain

$$g(\nabla_X(\psi j\psi), Y) = 2i\kappa\langle(p(Y)\bar{p}(X) - p(X)\bar{p}(Y))\psi_l, j\psi_{l-1}\rangle. \tag{2.31}$$

This shows that $\nabla(\psi j\psi)$ is skew symmetric. Thus, our assertion (iii) is completely proved by (2.24) and (2.25), q.e.d.

Theorem 10: *Let M^{2m} be a spin Kähler manifold of complex dimension $m = 2l - 1$ admitting any Kählerian Killing spinor. Then there is an anti-unitary automorphism C of the space $\mathcal{K}_+(M^{2m}) \oplus \mathcal{K}_-(M^{2m})$ such that*

$$C\mathcal{K}_\pm(M^{2m}) = \mathcal{K}_\pm(M^{2m}). \tag{2.32}$$

Moreover, if the scalar curvature R is positive (negative), then it holds

$$C^2 = (-1)^l \quad (C^2 = (-1)^{l+1}). \tag{2.33}$$

Proof: First of all, we remark that for $R > 0$ ($R < 0$) we have

$$j\mathcal{K}_\pm(M^{2m}) = \mathcal{K}_\pm(M^{2m}) \quad (j\mathcal{K}_\pm(M^{2m}) = \mathcal{K}_\mp(M^{2m})). \tag{2.34}$$

Thus, for $R > 0$, we define C by $C\psi = j\psi$. According to (2.7) and (2.35) we define C by $C\psi = \imath^2 j\psi$ if $R < 0$. Because of (2.27) and

$$\imath^2 j = -j\imath^2 \tag{2.35}$$

in this case we have $C^2 = \imath^2 j\imath^2 j = -\imath^2 j^2 \imath^2 = (-1)^{l+1}$, q.e.d.

Corollary 10.1: *Let M^{2m} be any spin Kähler manifold of complex dimension $m = 2l - 1$ admitting a real (imaginary) Kählerian Killing spinor. If l is odd (even), then $\dim_{\mathbb{C}} \mathcal{K}_\pm(M^{2m})$ is even and not less than 2.*

Proof: In the situation considered here we have $C^2 = -1$, i.e., according to (2.32) C determines a quaternionic structure of the spaces $\mathcal{K}_+(M^{2m})$ and $\mathcal{K}_-(M^{2m})$. Since $\mathcal{K}_\pm(M^{2m}) \neq 0$, this implies $\dim_{\mathbb{C}} \mathcal{K}_\pm(M^{2m}) \geq 2$, q.e.d.

Corollary 10.2: *Let M^{2m} be a spin Kähler manifold of complex dimension $m = 2l-1$ admitting a real (imaginary) Kählerian*

Killing spinor. If l is even (odd), then, for each Killing number, there is a Kählerian Killing spinor ψ with $\langle\psi, \iota^2\psi\rangle = 0$.

Proof: Here we have $C^2 = 1$, i.e., C determines a real structure on $\mathcal{K}_+(M^{2m})$ and $\mathcal{K}_-(M^{2m})$. Hence, in each of these spaces there are Kählerian Killing spinors ψ having the property $C\psi = \psi$. In case $R > 0$, we have $C\psi = \psi$ iff $j\psi = \psi$. Hence, it holds that $\langle\psi, \iota^2\psi\rangle = \langle j\iota^2\psi, j\psi\rangle = -\langle\iota^2 j\psi, j\psi\rangle = -\langle\iota^2\psi, \psi\rangle = -\langle\psi, \iota^2\psi\rangle$. This implies $\langle\psi, \iota^2\psi\rangle = 0$. In case $R < 0$, we have $C\psi = \psi$ iff $j\psi = \iota^2\psi$, where $j^2 = -1$. Thus, it holds that $\langle\psi, \iota^2\psi\rangle = \langle\psi, j\psi\rangle = \langle j^2\psi, j\psi\rangle = -\langle\psi, j\psi\rangle$ and, hence, $\langle\psi, \iota^2\psi\rangle = 0$, q.e.d.

Theorem 11: *Let M^{2m} be a spin Kähler manifold of complex dimension $m = 2l - 1$ admitting an imaginary Kählerian Killing spinor. Then we have the following:*

(i) To each Killing number there is a Kählerian Killing spinor ψ with $\langle\psi, \iota^2\psi\rangle = 0$.

(ii) If l is even, then, for each Killing number, there is a Kählerian Killing spinor $\psi = \psi_{l-1} + \psi_l$ with $\langle\psi_{l-1}, j\psi_l\rangle = 0$.

Proof: By Corollary 10.2, it suffices to consider the case where l is even. Let $\psi = \psi_{l-1} + \psi_l$ be any (imaginary) Kählerian Killing spinor. From Corollary 7.2 and Theorem 8 we know that the functions $a := \langle\psi, \iota^2\psi\rangle = |\psi_l|^2 - |\psi_{l-1}|^2$ and $b := \langle\psi_{l-1}, j\psi_l\rangle$ are constant.
Moreover, it follows from Theorem 10 that, for all pairs of complex numbers $(z, w) \neq (0, 0)$, the spinor field $\varphi := z\psi + wC\psi = (z\psi_{l-1} - wj\psi_l) + (z\psi_l + wj\psi_{l-1})$ is a Kählerian Killing spinor to the same Killing number. Using the notations $\varphi_{l-1} = z\psi_{l-1} - wj\psi_l$, $\varphi_l = z\psi_l + wj\psi_{l-1}$ we find

$$|\varphi_l|^2 - |\varphi_{l-1}|^2 = a(|z|^2 - |w|^2) + 4\mathrm{Re}(bz\bar{w}) \qquad (*)$$

$$\langle\varphi_{l-1}, j\varphi_l\rangle = bz^2 - \bar{b}w^2 - azw. \qquad (2*)$$

Let us suppose $b \neq 0$. In this case the equation $bz^2 - \bar{b}w^2 - azw = 0$ possesses the special non-trivial solution $(z_0, w_0) = (a/2b + \sqrt{(a/2b)^2 + \bar{b}/b}, 1)$. By $(2*)$ this proves the assertion (ii). On the other hand, if $\psi = \psi_{l-1} + \psi_l$ has the property $\langle \psi_{l-1}, j\psi_l \rangle = 0$, then it follows from $(*)$ that $\varphi := \psi + C\psi$ has the property $\langle \varphi, \iota^2 \varphi \rangle = 0$, q.e.d.

By Theorem 4 and Theorem 11, (i) we immediately obtain

Corollary 11.1: *Let M^{2m} $(m = 2l-1)$ be a spin Kähler manifold admitting an imaginary Kählerian Killing spinor. Then, for each Killing number, there is a Kählerian Killing spinor $\psi = \psi_{l-1} + \psi_l$ whose components ψ_{l-1} and ψ_l do not vanish at any point.*

Let K be the curvature tensor of M^{2m} defined by

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and let T denote the tensor field given by

$$T(X,Y)Z = 2\Omega(X,Y)J(Z) + Z \lrcorner (Y \wedge X + J(Y) \wedge J(X)),$$

where Ω is the Kähler form.

Theorem 12: *Let $\psi = \psi_{l-1} + \psi_l$ be a Kählerian Killing spinor to the Killing number κ.*

(i) It holds

$$K(X,Y)(\psi\psi) = \kappa^2 T(X,Y)(\psi\psi) \tag{2.36}$$

(ii) If l is odd, then

$$\begin{aligned} K(X,Y)(\psi_{l-1} j\psi_{l-1}) &= 2\kappa^2((\bar{p}(Y)p(X) - \bar{p}(X)p(Y))\psi_{l-1})j\psi_{l-1} \\ K(X,Y)(\psi_l j\psi_l) &= 2\kappa^2((p(Y)\bar{p}(X) - p(X)\bar{p}(Y))\psi_l)j\psi_l. \end{aligned} \tag{2.37}$$

Proof: For example, let $\psi = \psi_{l-1} + \psi_l$ be real ($\bar{\kappa} = \kappa$). Then we have

$$\begin{aligned}\nabla_Y(\psi_{l-1}\psi_l) &= (\nabla_Y\psi_{l-1})\psi_l + \psi_{l-1}(\nabla_Y\psi_l) = \\ &= -\kappa((\bar{p}(Y)\psi_l)\psi_l + \psi_{l-1}(p(Y)\psi_{l-1})).\end{aligned}$$

Using this we obtain the equation

$$\begin{aligned}&\nabla_X\nabla_Y(\psi_{l-1}\psi_l) = -\kappa((\bar{p}(\nabla_X Y)\psi_l)\psi_l + \psi_{l-1}(p(\nabla_X Y)\psi_{l-1})) + \\ &+ \kappa^2((\bar{p}(Y)p(X)\psi_{l-1})\psi_l + \psi_{l-1}(p(Y)\bar{p}(X)\psi_l) + \\ &+ (\bar{p}(Y)\psi_l)(p(X)\psi_{l-1}) + (\bar{p}(X)\psi_l)(p(Y)\psi_{l-1})).\end{aligned}$$

Hence, we find

$$\begin{aligned}K(X,Y)(\psi_{l-1}\psi_l) &= \kappa^2(((\bar{p}(Y)p(X) - \bar{p}(X)p(Y))\psi_{l-1})\psi_l + \\ &+ \psi_{l-1}((p(Y)\bar{p}(X) - p(X)\bar{p}(Y))\psi_l)).\end{aligned} \qquad (*)$$

We obtain the same equation if ψ is imaginary.
Now, by (1.3) we have

$$\begin{aligned}&\psi_{l-1}(p(Y)\bar{p}(X)\psi_l) = (\bar{p}(Y)\psi_{l-1})(\bar{p}(X)\psi_l) - \\ &2i\langle\psi_{l-1}, \bar{p}(X)\psi_l\rangle\bar{p}(Y) = \\ = &(p(X)\bar{p}(Y)\psi_{l-1})\psi_l - 2i\langle\bar{p}(Y)\psi_{l-1}, \psi_l\rangle p(X) - \\ &2i\langle\psi_{l-1}, \bar{p}(X)\psi_l\rangle\bar{p}(Y) = \\ = &(p(X)\bar{p}(Y)\psi_{l-1})\psi_l - 2g(X, \psi_{l-1}\psi_l)\bar{p}(Y)\end{aligned}$$

and, hence,

$$\begin{aligned}&\psi_{l-1}((p(Y)\bar{p}(X) - p(X)\bar{p}(Y))\psi_l) = ((p(X)\bar{p}(Y) - \\ &p(Y)\bar{p}(X))\psi_{l-1})\psi_l - 2g(X, \psi_{l-1}\psi_l)\bar{p}(Y) + \\ &2g(Y, \psi_{l-1}\psi_l)\bar{p}(X).\end{aligned}$$

Inserting this into the equation $(*)$ we obtain

$$\begin{aligned}&K(X,Y)(\psi_{l-1}, \psi_l) = 2\kappa^2(i\Omega(X,Y)\psi_{l-1}\psi_l + g(X, \psi_{l-1}\psi_l)\bar{p}(Y) - \\ &\quad -g(Y, \psi_{l-1}\psi_l)\bar{p}(X)) = \\ = &\kappa^2(2\Omega(X,Y)J(\psi_{l-1}\psi_l) - g(X, \psi_{l-1}\psi_l)Y + g(Y, \psi_{l-1}\psi_l)X - \\ &\quad -g(JX, \psi_{l-1}\psi_l)JY + g(JY, \psi_{l-1}\psi_l)JX).\end{aligned}$$

Because of $\bar{p}(\psi\psi) = \psi_{l-1}\psi_l$ this implies (2.36).
Using the equations (2.30) one obtains the equations (2.37) by similar calculations, q.e.d.

3 The Fubini metric and the complex hyperbolic space

In this section we work with the complex manifolds $\mathbb{C}^m$ and $D^m = \{(z^1, \ldots, z^m) \in \mathbb{C}^m | \sum_{\alpha=1}^m |z^\alpha|^2 < 1\}$ simultaneously, where $m = 2l - 1$ is any odd natural number. On $\mathbb{C}^m(D^m)$ we consider the function $f = 1 + \sum_{\alpha=1}^m |z^\alpha|^2$ ($f = 1 - \sum_{\alpha=1}^m |z^\alpha|^2$) and, for any real number $c > 0$, the function $F = c^2 \log f$ ($F = -c^2 \log f$). Using the denotations $Z_\alpha = \frac{\partial}{\partial z^\alpha}$, $Z_{\bar{\alpha}} = \bar{Z}_\alpha = \frac{\partial}{\partial \bar{z}^\alpha}$ ($\alpha = 1, \ldots, m$) we obtain a Kähler metric $g = g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + g_{\bar{\alpha}\beta} d\bar{z}^\alpha \otimes dz^\beta$ on $\mathbb{C}^m(D^m)$ whose components $g_{\alpha\bar{\beta}} := g(Z_\alpha, Z_{\bar{\beta}}) = \overline{g_{\bar{\alpha}\beta}}$ are defined by

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} \quad (\alpha, \beta = 1, \ldots, m).$$

The explicit expressions are

$$g_{\alpha\bar{\beta}} = \frac{c^2}{f}(\delta_{\alpha\beta} - \frac{\bar{z}^\alpha z^\beta}{f}) \quad (g_{\alpha\bar{\beta}} = \frac{c^2}{f}(\delta_{\alpha\beta} + \frac{\bar{z}^\alpha z^\beta}{f})). \tag{3.1}$$

This yields

$$g^{\alpha\bar{\beta}} = \frac{f}{c^2}(\delta^{\alpha\beta} + z^\alpha \bar{z}^\beta) \quad (g^{\alpha\bar{\beta}} = \frac{f}{c^2}(\delta^{\alpha\beta} - z^\alpha \bar{z}^\beta)). \tag{3.2}$$

This Kähler metric on $\mathbb{C}^m(D^m)$ is called the Fubini metric (metric of the complex huperbolic space $\mathbb{C}H^m$). In general, the corresponding Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$ are given by

$$\Gamma^\gamma_{\alpha\beta} = g^{\gamma\bar{\varepsilon}} \frac{\partial g_{\bar{\varepsilon}\alpha}}{\partial z^\beta}.$$

Using this we obtain by direct calculations

$$\Gamma^\gamma_{\alpha\beta} = -\frac{1}{f}(\delta^\gamma_\alpha \bar{z}^\beta + \delta^\gamma_\beta \bar{z}^\alpha) \quad (\Gamma^\gamma_{\alpha\beta} = \frac{1}{f}(\delta^\gamma_\alpha \bar{z}^\beta + \delta^\gamma_\beta \bar{z}^\alpha)). \tag{3.3}$$

The canonical bundle $\Lambda^{m,0}$ of $\mathbb{C}^m$ is trivial. It is generated by the global holomorphic section $\omega_m = dz^1 \wedge \ldots \wedge dz^m$. Thus, by $\psi_0 := \sqrt{\omega_m}$ we have a global holomorphic section of $S_0 = \sqrt{\Lambda^{m,0}}$. Because of $|\omega_m|^2 = \det(g^{\alpha\bar{\beta}})$ and $|\psi_0|^2 = |\omega_m|$ from (3.2) we obtain

$$|\psi_0|^2 = \frac{f^l}{c^m} \quad (m = 2l - 1). \tag{3.4}$$

For $1 \leq \alpha_1 < \cdots < \alpha_{l-1} \leq m$ let $\xi^{\alpha_1 \ldots \alpha_{l-1}}$ be the $(l-1)$-forms on $\mathbb{C}^m(D^m)$ defined by

$$\xi^{\alpha_1 \ldots \alpha_{l-1}} = \frac{1}{f^l} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}}.$$

Then, by (3.4), it follows immediately that

$$\nabla_{Z_{\bar{\alpha}}}(|\psi_0|^2 \xi^{\alpha_1 \ldots \alpha_{l-1}}) = 0 \tag{3.5}$$

for $\alpha = 1, \ldots, m$. Moreover, we have

Lemma 13.1: *For $\alpha = 1, \ldots, m$, it holds*

$$\nabla_{Z_\alpha} \xi^{\alpha_1 \ldots \alpha_{l-1}} = \frac{1}{l} Z_\alpha \lrcorner\, \partial \xi^{\alpha_1 \ldots \alpha_{l-1}}. \tag{3.6}$$

Proof: For example, let us consider the case of the Fubini metric. By (3.3) we have

$$\nabla_{Z_\alpha} dz^\gamma = -\Gamma^\gamma_{\alpha\beta} dz^\beta = \frac{1}{f}(\bar{z}^\alpha dz^\gamma + \delta^\gamma_\alpha \sum_{\beta=1}^{m} \bar{z}^\beta dz^\beta).$$

Using this it holds

$$\begin{aligned} \nabla_{Z_\alpha} \xi^{\alpha_1 \ldots \alpha_{l-1}} &= \Big(- l \frac{\bar{z}^\alpha}{f^{l+1}} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}} + \\ &\quad + \frac{1}{f^l} \sum_{k=1}^{l-1} dz^{\alpha_1} \wedge \cdots \wedge \nabla_{Z_\alpha} dz^{\alpha_k} \wedge \cdots \wedge dz^{\alpha_{l-1}} \Big) = \\ &= \frac{1}{f^{l+1}} \Big(- \bar{z}^\alpha dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}} + \end{aligned}$$

$$+(\sum_{\beta=1}^{m} \bar{z}^\beta dz^\beta) \wedge (\sum_{k=1}^{l-1}(-1)^{k-1}\delta_\alpha^{\alpha_k} dz^{\alpha_1} \wedge \cdots \wedge \widehat{dz^{\alpha_k}} \wedge \cdots \wedge dz^{\alpha_{l-1}}))$$

$$= \frac{1}{f^{l+1}}(- \bar{z}^\alpha dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}} +$$

$$+(\sum_{\beta=1}^{m} \bar{z}^\beta dz^\beta) \wedge (Z_\alpha \lrcorner (dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}}))) =$$

$$= \frac{1}{l} Z_\alpha \lrcorner (- \frac{l}{f^{l+1}}(\sum_{\beta=1}^{m} \bar{z}^\beta dz^\beta) \wedge dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}}) =$$

$$= \frac{1}{l} Z_\alpha \lrcorner \partial \xi^{\alpha_1 \ldots \alpha_{l-1}}.$$

In case of $\mathbb{C}H^m$ the calculation is quite analogous, q.e.d.

Now, for $1 \leq \beta_1 < \cdots < \beta_l \leq m$, let us consider the $(l-1)$-forms defined on $\mathbb{C}^m(D^m)$

$$\eta^{\beta_1 \ldots \beta_l} = \frac{1}{f^l}(z^\beta Z_\beta) \lrcorner (dz^{\beta_1} \wedge \cdots \wedge dz^{\beta_l}),$$

where Einstein's convention is used in the term $z^\beta Z_\beta$. Clearly, it holds

$$\nabla_{Z_{\bar{\alpha}}}(|\psi_0|^2 \eta^{\beta_1 \cdots \beta_l}) = 0 \quad (\alpha = 1, \ldots, m). \tag{3.7}$$

Lemma 13.2: *Each of the forms $\eta^{\beta_1 \ldots \beta_l}$ satisfies the equations*

$$\nabla_{Z_\alpha} \eta^{\beta_1 \ldots \beta_l} = \frac{1}{l} Z_\alpha \lrcorner \partial \eta^{\beta_1 \ldots \beta_l} \quad (\alpha = 1, \ldots, m). \tag{3.8}$$

Proof: The definition implies the relation

$$\eta^{\beta_1 \ldots \beta_l} = \sum_{k=1}^{l} (-1)^{k-1} z^{\beta_k} \xi^{\beta_1 \cdots \hat{\beta}_k \cdots \beta_l}. \tag{3.9}$$

Using this, (3.6) and the identity

$$(l-1) \sum_{k=1}^{l} (-1)^{k-1} \delta_\alpha^{\beta_k} \xi^{\beta_1 \cdots \hat{\beta}_k \cdots \beta_l} = \sum_{k=1}^{l} (-1)^k dz^{\beta_k} \wedge (Z_\alpha \lrcorner \xi^{\beta_1 \cdots \hat{\beta}_k \cdots \beta_l})$$

we obtain (3.8) by a direct calculation, q.e.d.

We define spinor fields on $\mathbb{C}^{2l-1}(D^{2l-1})$ by

$$\varphi^{\alpha_1\cdots\alpha_{l-1}} = \overline{\xi^{\alpha_1\cdots\alpha_{l-1}}} \cdot \psi_0 \quad \psi^{\beta_1\cdots\beta_l} = \overline{\eta^{\beta_1\cdots\beta_l}} \cdot \psi_0. \tag{3.10}$$

By Theorem 1, (3.5), (3.6), (3.7) and (3.8) we see that each of these spinor fields is a twistor spinor of type $(l, l-1)$. Obviously, the set of all these Kählerian twistor spinors is linearly independent over $\mathbb{C}$. Hence, it generates a complex vector space of dimension $2 \cdot \binom{2l-1}{l} = \binom{2l}{l} = \binom{m+1}{l}$. By Theorem 2, this yields immediately

Theorem 13: *Let $m = 2l - 1 > 1$. Then the set*

$$\{\varphi^{\alpha_1\cdots\alpha_{l-1}}, \psi^{\beta_1\cdots\beta_l} | 1 \leq \alpha_1 < \cdots < \alpha_{l-1} \leq m, 1 \leq \beta_1 < \cdots < \beta_l \leq m\} \tag{3.11}$$

forms a basis of the space $\ker \mathcal{D}^{(l,l-1)}$ *on* $\mathbb{C}^m(D^m)$ *furnished with the Fubini metric (complex hyperbolic metric).*

In particular, Theorem 13 shows that the estimation (1.23) is sharp since (1.23) is an equality in the situation considered here. Now we consider the case of $\mathbb{C}P^m$.

Lemma 14.1: *Let $m = 2l - 1 > 1$. Then, in case of the Fubini metric, we have*

$$|\varphi^{\alpha_1\cdots\alpha_{l-1}}|^2 = \frac{2^{l-1}}{c^{2m-1}} \frac{1}{f} (1 + \sum_{k=1}^{l-1} |z^{\alpha_k}|^2) \tag{3.12}$$

$$|\psi^{\beta_1\cdots\beta_l}|^2 = \frac{2^{l-1}}{c^{2m-1}} \frac{1}{f} \sum_{k=1}^{l} |z^{\beta_k}|^2 \tag{3.13}$$

for $1 \leq \alpha_1 > \cdots < \alpha_{l-1} \leq m$ and $1 \leq \beta_1 < \cdots < \beta_l \leq m$.

Proof: First we remark that the relation

$$|dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}}|^2 = (\frac{f}{c^2})^{l-1} (1 + \sum_{k=1}^{l-1} |z^{\alpha_k}|^2) \tag{$*$}$$

is valid. Moreover, for $i \neq k$, one proves the identity

$$\langle dz^{\beta_1} \wedge \cdots \wedge \widehat{dz}^{\beta_i} \wedge \cdots \wedge dz^{\beta_l}, dz^{\beta_1} \wedge \cdots \wedge \widehat{dz}^{\beta_k} \wedge \cdots \wedge dz^{\beta_l} \rangle =$$
$$= (\frac{f}{c^2})^{l-1}(-1)^{i+k+1}\bar{z}^{\beta_i} z^{\beta_k}. \qquad (2*)$$

Using the formulas (1.7), (1.9), (3.2), (3.4) and $(*)$ we have

$$\begin{aligned}
& |\varphi^{\alpha_1\cdots\alpha_{l-1}}|^2 = \langle \overline{\xi^{\alpha_1\cdots\alpha_{l-1}}} \cdot \psi_0, \overline{\xi^{\alpha_1\cdots\alpha_{l-1}}} \cdot \psi_0 \rangle = \\
= \; & (-1)^{l(l-1)/2} \langle \xi^{\alpha_1\cdots\alpha_{l-1}} \overline{\xi^{\alpha_1\cdots\alpha_{l-1}}} \cdot \psi_0, \psi_0 \rangle = \\
= \; & 2^{l-1} |\xi^{\alpha_1\cdots\alpha_{l-1}}|^2 |\psi_0|^2 = 2^{l-1} \frac{1}{f^{2l}} |dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}}|^2 \cdot \frac{f^l}{c^m} = \\
= \; & \frac{2^{l-1}}{c^m f^l} |dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_{l-1}}|^2 = \frac{2^{l-1}}{c^m f^l} (\frac{f}{c^2})^{l-1} (1 + \sum_{k=1}^{l-1} |z^{\alpha_k}|^2) = \\
= \; & \frac{2^{l-1}}{c^{2m-1}} \frac{1}{f} (1 + \sum_{k=1}^{l-1} |z^{\alpha_k}|^2)
\end{aligned}$$

and, hence, (3.12). Further, by $(*)$ and $(2*)$, it holds

$$\begin{aligned}
& |\psi^{\beta_1\cdots\beta_l}|^2 = \frac{2^{l-1}}{c^m f^l} |(z^\beta Z_\beta) \lrcorner (dz^{\beta_1} \wedge \cdots \wedge dz^{\beta_l}|^2 = \\
= \; & \frac{2^{l-1}}{c^m f^l} \langle \sum_{i=1}^{l} (-1)^{i-1} z^{\beta_i} dz^{\beta_1} \wedge \cdots \wedge \widehat{dz}^{\beta_i} \wedge \cdots \wedge dz^{\beta_l}, \\
& \sum_{k=1}^{l} (-1)^{k-1} z^{\beta_k} dz^{\beta_1} \wedge \cdots \wedge \widehat{dz}^{\beta_k} \wedge \cdots \wedge dz^{\beta_l} \rangle = \\
= \; & \frac{2^{l-1}}{c^m f^l} (\sum_{i=1}^{l} |z^{\beta_i}|^2 |dz^{\beta_1} \wedge \cdots \wedge \widehat{dz}^{\beta_i} \wedge \cdots \wedge dz^{\beta_l}|^2 + \\
+ \; & \sum_{i=1}^{l} \sum_{k \neq i} (-1)^{i+k} z^{\beta_i} \bar{z}^{\beta_k} \langle dz^{\beta_1} \wedge \cdots \wedge \widehat{dz}^{\beta_i} \wedge \cdots \wedge dz^{\beta_l}, \\
& dz^{\beta_1} \wedge \cdots \wedge \widehat{dz}^{\beta_k} \wedge \cdots \wedge dz^{\beta_l} \rangle) = \\
= \; & \frac{2^{l-1}}{c^{2m-1}} \frac{1}{f} (\sum_{i=1}^{l} |z^{\beta_i}|^2 (1 + \sum_{k \neq i} |z^{\beta_k}|^2) - \sum_{i=1}^{l} \sum_{k \neq i} |z^{\beta_i}|^2 |z^{\beta_k}|^2) = \\
= \; & \frac{2^{l-1}}{c^{2m-1}} \frac{1}{f} \sum_{i=1}^{l} |z^{\beta_i}|^2, \qquad \text{q.e.d.}
\end{aligned}$$

Theorem 14: *Let* $m = 2l - 1 > 1$. *Then* $\dim_{\mathbb{C}} \ker \mathcal{D}^{(l,l-1)} = 2\binom{m}{l}$ *holds on* $\mathbb{C}P^m$.

Proof: We show that each element of the basis (3.11) can be extended on $\mathbb{C}P^m$. It suffices to prove that the functions $|\varphi^{\alpha_1\cdots\alpha_{l-1}}|^2$ and $|\psi^{\beta_1\cdots\beta_l}|^2$ are bounded on $\mathbb{C}^m$. By Lemma 14.1 we have

$$|\varphi^{\alpha_1\cdots\alpha_{l-1}}|^2 \leq \frac{2^{l-1}}{c^{2m-1}} \quad \text{and} \quad |\psi^{\beta_1\cdots\beta_l}|^2 \leq \frac{2^{l-1}}{c^{2m-1}},$$

q.e.d.

The Fubini metric (metric of $\mathbb{C}H^m$) given by (3.1) is a Kähler metric of constant holomorphic sectional curvature $2/c^2$ $(-2/c^2)$. In particular it is Einstein with scalar curvature

$$R = \frac{2m(m+1)}{c^2} \quad (R = -\frac{2m(m+1)}{c^2}). \tag{3.14}$$

Hence, by Theorem 6, Theorem 13, and Theorem 14 we immediately obtain

Theorem 15: *Let* $m = 2l - 1 > 1$. *Then*

$$\dim_{\mathbb{C}} \mathcal{K}_{\pm}(\mathbb{C}P^m) = \dim_{\mathbb{C}} \mathcal{K}_{\pm}(\mathbb{C}H^m) = 2 \cdot \binom{m}{l}. \tag{3.15}$$

By Theorem 5, we conclude from (3.15) that for all $\mathbb{C}P^m$ of odd complex dimension m the limiting case of the inequality (I.4) is realized (For $m = 1$ the assertion is trivial.). This is already known from [6], where the spectrum of the Dirac operator on $\mathbb{C}P^{2l-1}$ is computed. Finally, we remark that by (2.16) one can derive the explicit expressions of the Kählerian Killing spinors corresponding to the twistor spinors given by (3.10).

4 Imaginary Kählerian Killing spinors in dimension six

If M^6 is a closed spin Kähler manifold admitting a Kählerian Killing spinor, then M^6 is analytically isometric to the complex projective space $\mathbb{C}P^3$ or to the flag manifold $F(1,2)$, respectively, furnished with their natural Kähler structures (see [19]). $\mathbb{C}P^3$ and $F(1,2)$ are examples of Kähler manifolds with real Killing spinors. In this section we will prove an analogous classification result in case of 6-dimensional Kähler manifolds admitting an imaginary Kählerian Killing spinor.
Let M^6 be a connected spin Kähler manifold with an imaginary Kählerian Killing spinor $\psi = \psi_1 + \psi_2$ to the Killing number κ. Then we know that M^6 is an Einstein space of scalar curvature

$$R = 48\kappa^2 < 0. \tag{4.1}$$

Moreover, we have the relation

$$\varrho = \frac{R}{6}\Omega = 8\kappa^2\Omega \tag{4.2}$$

between the Ricci form ϱ and the Kähler form Ω. The spinor bundle S of M^6 splits in the form

$$S = S_0 \oplus S_1 \oplus S_2 \oplus S_3$$

and we have $\psi_1 \in \Gamma(S_1)$, $\psi_2 \in \Gamma(S_2)$. For each vector field X on M^6, the differential equations

$$\nabla_X\psi_1 + \kappa\bar{p}(X)\psi_2 = 0 \quad \nabla_X\psi_2 + \kappa p(X)\psi_1 = 0 \tag{4.3}$$

are satisfied. By Theorem 11, (ii) we can assume that

$$\langle\psi_1, j\psi_2\rangle = 0. \tag{4.4}$$

Let $X_3 := \psi\psi = \psi_1\psi_2 + \psi_2\psi_1$. Then it holds

$$\bar{p}(X_3) = \psi_1\psi_2 \quad p(X_3) = \psi_2\psi_1. \tag{4.5}$$

Theorem 7, (ii) yields

$$\mathrm{grad}|\psi|^2 = -2i\kappa X_3 \tag{4.6}$$

$$\Delta|\psi|^2 = \frac{R}{3}|\psi|^2 = 16\kappa^2|\psi|^2. \tag{4.7}$$

Further, by Theorem 9, $J(X_3)$ is a Killing field and Theorem 12 implies

$$K(X,Y)X_3 = \kappa^2 T(X,Y)X_3. \tag{4.8}$$

Now, let $P \in U_\psi := \mathrm{supp}(\psi_1\psi_2)$ be any point and let $U \subseteq U_\psi$ be a neighbourhood of P such that there is a holomorphic section $\psi_0 \in \Gamma(S_0|_U)$ that does not vanish at any point of U. Then we define real vector fields X_1, X_2 on U by

$$X_1 = \psi_0\psi_1 + \psi_1\psi_0 \quad X_2 = \psi_0(j\psi_2) + (j\psi_2)\psi_0.$$

By definition, we have the relations

$$\bar{p}(X_1) = \psi_0\psi_1 \quad p(X_1) = \psi_1\psi_0 \tag{4.9}$$

$$\bar{p}(X_2) = \psi_0(j\psi_2) \quad p(X_2) = (j\psi_2)\psi_0. \tag{4.10}$$

Moreover, we find

$$p(X_1)\psi_0 = 2i|\psi_0|^2\psi_1 \quad p(X_2)\psi_0 = 2i|\psi_0|^2 j\psi_2. \tag{4.11}$$

In particular, this implies

$$p(X_1)\psi_1 = 0 \quad \bar{p}(X_2)\psi_2 = 0. \tag{4.12}$$

Lemma 16.1: *For all vector fields X and Y, defined on U the equations*

$$K(X,Y)X_k = \kappa^2 T(X,Y)X_k \quad (k = 1,2) \tag{4.13}$$

are satisfied.

Proof: By (4.3) and because of $\nabla\psi_0 = \partial \log|\psi_0|^2 \otimes \psi_0$ we have

$$\begin{aligned} &\nabla_Y(\psi_0\psi_1) = (\nabla_Y\psi_0)\psi_1 + \psi_0(\nabla_Y\psi_1) = \\ = \; &(\partial \log|\psi_0|^2)(Y)\psi_0\psi_1 + \kappa\psi_0(\bar{p}(Y)\psi_2). \end{aligned}$$

Using this we find

$$\begin{aligned}
& \nabla_X \nabla_Y(\psi_0\psi_1) = X((\partial \log|\psi_0|^2)(Y))\psi_0\psi_1 + \\
+ \; & (\partial \log|\psi_0|^2)(X)(\partial \log|\psi_0|^2)(Y)\psi_0\psi_1 + \\
& \kappa(\partial \log|\psi_0|^2)(Y)\psi_0(\bar{p}(X)\psi_2) + \\
+ \; & \kappa(\partial \log|\psi_0|^2)(X)\psi_0(\bar{p}(Y)\psi_2) + \kappa\psi_0(\bar{p}(\nabla_X Y)\psi_2) + \\
+ \; & \kappa^2\psi_0(\bar{p}(Y)p(X)\psi_1)
\end{aligned}$$

and hence

$$\begin{aligned}
& K(X,Y)(\psi_0\psi_1) = \kappa^2\psi_0((\bar{p}(Y)p(X) - \bar{p}(X)p(Y))\psi_1) + \\
+ \; & (X((\partial \log|\psi_0|^2)(Y)) - Y((\partial \log|\psi_0|^2)(X)) - \\
& -(\partial \log|\psi_0|^2))([X,Y]))\psi_0\psi_1 = \\
= \; & \kappa^2\psi_0((\bar{p}(Y)p(X) - \bar{p}(X)p(Y))\psi_1) + (\bar{\partial}\partial \log|\psi_0|^2)(X,Y)\psi_0\psi_1.
\end{aligned}$$

Since the curvature form of S_0 is equal to $\frac{i}{2}\varrho$ it holds by (4.2)

$$\bar{\partial}\partial \log|\psi_0|^2 = -\frac{i}{2}\varrho = -4i\kappa^2\Omega. \tag{4.14}$$

Moreover, using the formula (1.3) we obtain

$$\begin{aligned}
\psi_0((\bar{p}(Y)p(X) - \bar{p}(X)p(Y))\psi_1) = 2i\Omega(X,Y)\psi_0\psi_1 + \\
+2g(Y,\psi_0\psi_1)\bar{p}(X) - 2g(X,\psi_0\psi_1)\bar{p}(Y).
\end{aligned}$$

Inserting this and (4.14) in the result of the calculation above we find

$$\begin{aligned}
K(X,Y)(\psi_0\psi_1) = -2\kappa^2(i\Omega(X,Y)\psi_0\psi_1 + \\
+g(X,\psi_0\psi_1)\bar{p}(Y) - g(Y,\psi_0\psi_1)\bar{p}(X)).
\end{aligned} \tag{4.15}$$

By similar calculations one obtains

$$\begin{aligned}
K(X,Y)(\psi_o j\psi_2) = -2\kappa^2(i\Omega(X,Y)\psi_0 j\psi_2 + \\
+g(X,\psi_0 j\psi_2)\bar{p}(Y) - g(Y,\psi_0 j\psi_2)\bar{p}(X)).
\end{aligned} \tag{4.16}$$

From (4.9) and (4.10) we see that the equations (4.15) and (4.16) are equivalent to the equations (4.13), q.e.d.

Lemma 16.2: *There are the relations*

$$g(p(X),\bar{p}(X_2)) = 0,\ g(p(X_1),\bar{p}(X_3)) = 0,\ g(p(X_2),\bar{p}(X_3)) = 0$$

and it holds $|X_1|^2 = 4|\psi_0|^2|\psi_1|^2$, $|X_2|^2 = 4|\psi_0|^2|\psi_2|^2$.

Proof: By (4.4) and (4.11) we have

$$\begin{aligned} 0 &= \langle\psi_1, j\psi_2\rangle = \langle -\frac{i}{2|\psi_0|^2}p(X_1)\psi_0, -\frac{i}{2|\psi_0|^2}p(X_2)\psi_0\rangle = \\ &= -\frac{1}{4|\psi_0|^4}\langle\bar{p}(X_2)p(X_1)\psi_0,\psi_1\rangle = \frac{1}{2|\psi_0|^2}g(p(X_1),\bar{p}(X_2)) \end{aligned}$$

and, hence, $g(p(X_1),\bar{p}(X_2)) = 0$. Moreover, using (4.5) and (4.12) it holds

$$\begin{aligned} g(p(X_1),\bar{p}(X_3)) &= g(p(X_1),\psi_1\psi_2) = -i\langle p(X_1)\psi_1,\psi_2\rangle = 0, \\ g(p(X_2),\bar{p}(X_3)) &= g(p(X_2),\psi_1\psi_2) = -i\langle p(X_2)\psi_1,\psi_2\rangle = \\ &= i\langle\psi_1,\bar{p}(X_2)\psi_2\rangle = 0. \end{aligned}$$

Finally, we obtain by (4.9), (4.10) and (4.11)

$$\begin{aligned} & |X_1|^2 = 2|p(X_1)|^2 = 2g(p(X_1),\bar{p}(X_1)) = 2g(p(X_1),\psi_0\psi_1) = \\ = & -2i\langle p(X_1)\psi_0,\psi_1\rangle = -2i\langle 2i|\psi_0|^2\psi_1,\psi_1\rangle = 4|\psi_0|^2|\psi_1|^2, \\ & |X_2|^2 = 2g(p(X_2),\psi_0 j\psi_2) = -2i\langle p(X_2)\psi_0, j\psi_2\rangle = \\ = & -2i\langle 2i|\psi_0|^2 j\psi_2, j\psi_2\rangle = 4|\psi_0|^2|\psi_2|^2, \quad \text{q.e.d.} \end{aligned}$$

Theorem 16: *Let M^6 be a connected spin Kähler manifold admitting an imaginary Kählerian Killing spinor to the Killing number κ. Then M^6 is of constant holomorphic sectional curvature $4\kappa^2$.*

Proof: By (4.8), Lemma 16.1 and Lemma 16.2 we see that $K|_U = \kappa^2 T|_U$ since $(p(X_1),p(X_2),p(X_3))$ is a complex orthogonal frame on $U \subseteq U_\psi$. This implies the identity $K = \kappa^2 T$ on U_ψ. By (4.3) ψ_1 is antiholomorphic and ψ_2 is holomorphic. Hence, $p(X_3) = \psi_2\psi_1$ is holomorphic and $U_\psi = \mathrm{supp}(\psi_2\psi_1)$

is an open dense set in M^6. Thus, we can conclude that the equation

$$K = \kappa^2 T \tag{4.17}$$

is satisfied globally, q.e.d.

By Theorem 15, Theorem 16 and the Theorems 7.8, 7.9 in [21], Chapter IX we immediately obtain

Theorem 17: *The only connected and simple connected complete spin Kähler manifold M^6 admitting an imaginary Kählerian Killing spinor is the complex hyperbolic space $\mathbb{C}H^3$.*

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