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A note on the type of local von Neumann algebras in quantum field theory over curved space-time

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Abstract. It is shown that there are relations between Connes invariant for local v.Neumann algebras belonging to different curved space-times provided these local algebras are connected by a certain asymptotic condition. This asymptotic condition includes as special cases the scaling limit and connectedness by large translations.

1 Introduction

It is an important question to study the algebraic structure of the von Neumann algebras which appear in quantum physics. In particular it is interesting to ask for Connes invariant $S(\mathcal{M})$ of the mentioned von Neumann algebras $\mathcal{M}$. In algebraic quantum field theory over Minkowski space-time one finds that the local von Neumann algebras have the Connes invariant $S(\mathcal{M}) = [0, \infty)$, i.e. in the central decomposition of $\mathcal{M}$ only factors of type III_1 can occur (see e.g. [1], [5], [8], [6], [4] or [2]). For the local von Neumann algebras over curved space-times the situation is more difficult. The main techniques using spectrality and existence of groups of automorphisms for the local algebras do not work because a general space-time has no symmetries. A reasonable assumption is that the physical system under consideration - described by the local algebras - is neighbourhed (in some sense) at each space-time point to a corresponding physical system over Minkowski space-time (scaling limit assumption, see [7]). Using such an assumption and generalizing some ideas of Fredenhagen [6] it was proved in [10] that Connes invariant $S(\mathcal{M})$ for the local von Neumann algebras $\mathcal{M}$ over globally hyperbolic space-times - belonging to sufficiently small regions - is $[0, \infty)$.
The aim of this note is to generalize the necessary assumptions and the proof in [10] in such a way that for example we get the mentioned result about the Connes invariant also for local von Neumann algebras belonging to special unbounded regions in curved space-times under the assumption that these local algebras are related with corresponding local von Neumann algebras over Minkowski space-time by large spacelike translations.
In Section 2 the necessary definitions and assumptions are given. In Section 3 a general result about Connes invariant for local algebras is proven. In the last section this result is applied to some situations in quantum field theory over curved space-time.

2 Definitions and Assumptions

We say a C^∞ - manifold $\mathcal{X}$ connected with a map $\perp$, $\perp: \mathcal{X} \supset \mathcal{O} \to \mathcal{O}^\perp \subset \mathcal{X}$, is a *causal manifold* if the following assumptions are fulfilled:
(i) $\mathcal{O} \subseteq (\mathcal{O}^\perp)^\perp$,

(ii) $\mathcal{O} \cap \mathcal{O}^{\perp} = \emptyset$,
(iii) $(\mathcal{O}_1 \cup \mathcal{O}_2)^{\perp} = \mathcal{O}_1^{\perp} \cap \mathcal{O}_2^{\perp}$.

In particular if $\mathcal{X}$ is a space-time then $\mathcal{O}^{\perp}$ is the set of all points which cannot be reached by causal or lightlike curves from $\mathcal{O}$.

If there is given a sequence of continuous transformations $(g^n)_{\mathbf{N}}$ on $\mathcal{X}$ then we say a set $D \subset \mathcal{X}$ is *absorbing* a set $W \subseteq \mathcal{X}$ w.r.t. $(g^n)_{\mathbf{N}}$ if $W \supseteq D, g^n W \subseteq W$ for all $n \in \mathbf{N}$, and for each point $x \in W$ there is $n_x \in \mathbf{N}$ such that $g^n x \in D$ for all $n > n_x$.

Definition 2.1 *Let $(\mathcal{A}, \mathcal{M}(\mathcal{O}), \mathcal{X})$ be a family of von Neumann algebras indexed by open bounded sets $\mathcal{O}$ from a causal manifold $\mathcal{X}$ and acting on a common separable Hilbert space $\mathcal{H}$ with $\mathcal{A} := \cup_{\mathcal{O}} \mathcal{M}(\mathcal{O})$ where $\mathcal{O}$ runs through all open bounded subsets of $\mathcal{X}$. This family is called a* causal net of von Neumann algebras *if it satisfies:*
(a) $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$,
(b) $\mathcal{M}(\mathcal{O}^{\perp}) \subseteq \mathcal{M}(\mathcal{O})'$
where $\mathcal{M}(\mathcal{O}^{\perp}) := (\cup_{D \subset \mathcal{O}^{\perp}} \mathcal{M}(D))''$ and D runs through all open bounded subsets of $\mathcal{O}^{\perp}$.

For open unbounded sets $\mathcal{O} \subset \mathcal{X}$ we define $\mathcal{M}(\mathcal{O})$ similarly as $\mathcal{M}(\mathcal{O}^{\perp})$.

Definition 2.2 *Let $(\mathcal{A}, \mathcal{M}(\mathcal{O}), \mathcal{X})$ be a causal net of von Neumann algebras. Let $D \subset \mathcal{X}$ be a fixed open subset and let $(g^m)_{\mathbf{N}}$ be a family of transformations of D. Then we say $(\mathcal{M}(D), g^m)$ is* asymptotically trivial *if for each open bounded subset $\mathcal{O} \subset D$ it holds $\cap_m \mathcal{M}(g^m(\cup_n g^n \mathcal{O})) = \mathbf{C}1$.*

Now we are considering two causal manifolds $\mathcal{X}_1, \mathcal{X}_2$.
Suppose there is a set of transformations $(g_2^n)_{\mathbf{N}}$ with corresponding inverse transformations $(g_2^n)^{-1} =: g_{2,i}^n$ and a transformation j_2 on $\mathcal{X}_2$ with $g_2^n j_2 = j_2 g_2^n, n \in \mathbf{N}$. Further suppose there is an open set $W \subset \mathcal{X}_2$ such that $g_2^n W = W, g_2^n W^{\perp} = W^{\perp}$ and $j_2 W \subseteq W^{\perp}$. To connect $\mathcal{X}_1$ with $\mathcal{X}_2$ we assume that there are open sets $\mathcal{O}_1 \subseteq \mathcal{X}_1, \mathcal{O}_2 \subseteq \mathcal{X}_2$ and a diffeomorphism β from $\mathcal{O}_2$ onto $\mathcal{O}_1$. We define $W_2 := W \cap \mathcal{O}_2, W_1 := \beta W_2, g_1^n := \beta g_2^n \beta^{-1}$, and $j_1 := \beta j_2 \beta^{-1}$.
To avoid trivial cases we require that W_2 is nonempty, $g_2^n \mathcal{O}_2 \subseteq \mathcal{O}_2$ and $j_2 W_2 \subseteq \mathcal{O}_2$. Further we assume that W_2 is absorbing W w.r.t. (g_2^n). An asymptotic form of this absorbtion assumption and of the connection of j_2 with the causality relation $\perp$ is assumed for the other causal manifold: There is an open set $D \subset W_1$ absorbing W_1 w.r.t. (g_1^n), and for each fixed open bounded set $\mathcal{O} \subset j_1 W_1 = \beta(j_2 W_2) \subset \mathcal{O}_1$ there is $n_{\mathcal{O}}$ such that $g_1^n \mathcal{O} \subseteq D^{\perp}$ for all $n > n_{\mathcal{O}}$.
In particular we think of the following two examples.

Example 2.3 (dilatation transformations) In this example $\mathcal{X}_1$ is a globally hyperbolic space-time and $\mathcal{X}_2 = \mathbf{R}^n, n > 2$, is the Minkowski space-time of the same dimension. Further, $(\mathcal{O}_1, \beta^{-1})$ is a chart with $P \in \mathcal{O}_1$ and $\beta^{-1}(P) = 0$ such that the coordinates of the Minkowski space-time $\mathcal{X}_2 \supseteq \mathcal{O}_2 = \beta^{-1}\mathcal{O}_1$ define us locally flat coordinates in $P \in \mathcal{O}_1$ and the metric $g_{ij}(P)$ goes over in the metric of the Minkowski space-time $\mathcal{X}_2$ via β^{-1}. The region $W \subset \mathcal{X}_2$ is a wedge with the point 0 in its

edge. For simplicity we assume that $W = W_R := \{x \in \mathbf{R}^n; |x_0| < x_1\}$. The transformations (g_2^n) are the dilatations $g_2^n x := n^{-1}x$ in $\mathcal{X}_2$ and j_2 is the transformation $j_2(x_0, x_1, x_2, \ldots, x_n) := (-x_0, -x_1, x_2, \ldots, x_n)$. If $\mathcal{D} \subset W_2 = \mathcal{O}_2 \cap W$ is a double cone ($\mathcal{D} :=$ int $(x^\perp \cap y^\perp)$ for two points $x, y \in \mathcal{X}_2$) with clo $\mathcal{D} \ni 0$ which is sufficiently small then $D := \beta\mathcal{D}$ satisfies the assumptions made above (see e.g. [10]).
Note that in this example asymptotically trivial for a causal net $(\mathcal{A}_1, \mathcal{M}_1(\mathcal{O}), \mathcal{O} \subset \mathcal{X}_1)$ means $\cap_{\mathcal{O} \ni P} \mathcal{M}_1(\mathcal{O}) = \mathbf{C}1$. For a proof and/or a discussion of such property see [7], [2].

Example 2.4 (spacelike transformations) Let $\mathcal{X}_1$ be a space-time and let $\mathcal{X}_2$ be a Minkowski space-time as in Example 2.3. Let $(\mathcal{O}_1, \beta^{-1})$ be a chart with $\mathcal{O}_1 \ni P$ and $\beta(0) = P$. Further let $W = W_R$ be the wedge from $\mathcal{X}_2$ and let j_2 be the transformation as in Example 2.3. Let e be a spacelike unit vector from $\mathcal{X}_2$ in the direction x_2. Define $g_2^m x := x + em$. Then W and W_R are invariant under these transformations. Suppose $\mathcal{O}_2$ is also invariant under these spacelike transformations g_2^m. If we assume that $\mathcal{X}_1$ is asymptotically flat (more precisely the region $\mathcal{O}_1$ for large spacelike distances $n\ e$) then we can find open subsets $D \subset \mathcal{O}_1 \subseteq \mathcal{X}_1$ with the properties described above.
Note that in this example asymptotically trivial follows easily from the assumptions that the algebra $\mathcal{A}_1$ of the causal net $(\mathcal{A}_1, \mathcal{M}_1(\mathcal{O}), \mathcal{O} \subset \mathcal{X}_1)$ is irreducible and for each fixed open bounded regions $D_1 \subset \mathcal{X}_1, D_2 \subset \mathcal{O}_1$, it holds $D_1^\perp \supset g_1^m D_2$ for all m large enough. The second assumption is trivially fulfilled in Minkowski space-time and is a reasonable assumption for space-times which are asymptotically flat i.e. asymptotically like a Minkowski space-time.

Now we can formulate the essential assumption which connects local algebras belonging to two different causal nets.

Definition 2.5 *Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{O}_1, \mathcal{O}_2$ and g^m be as above. Let $(\mathcal{A}_1, \mathcal{M}_1(\mathcal{O}), \mathcal{X}_1)$ and $(\mathcal{A}_2, \mathcal{M}_2(\mathcal{O}), \mathcal{X}_2)$ be causal nets of von Neumann algebras over $\mathcal{X}_1$ and $\mathcal{X}_2$ acting on separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Suppose there is a representation of (g_2^m) by unitary operators $U(g_2^m), m \in \mathbf{N}$, on $\mathcal{H}_2$ such that $U(g_2^m)\mathcal{M}_2(\mathcal{O})U(g_2^m)^* = \mathcal{M}_2(g_2^m\mathcal{O})$ for all $m \in \mathbf{N}$ and $\mathcal{O} \subset \mathcal{X}_2$. Then $\mathcal{M}_1(\mathcal{O}_1)$ is called* asymptotically connected *with $\mathcal{M}_2(\mathcal{O}_2)$ by (g^m) if there are two unit vectors $\Omega_1 \in \mathcal{H}_1$, $\Omega_2 \in \mathcal{H}_2$ with $U(g_2^m)\Omega_2 = \Omega_2$, $m \in \mathbf{N}$, and a family of *-subsets $\mathcal{C}_\epsilon \subset \mathcal{M}_2(\mathcal{O}_2), \epsilon > 0$, with the following properties:*

(i) If $\epsilon > \epsilon'$ then $\mathcal{C}_\epsilon \supseteq \mathcal{C}_{\epsilon'}$.

(ii) (localization) For each $C \in \mathcal{C}_\epsilon$ there is an open bounded subset $\mathcal{O}_C \subseteq \mathcal{O}_2$ such that $C \in \mathcal{M}_2(\mathcal{O}_C)$.

(iii) For each $\epsilon > 0$ the set $\{C\Omega_2; C \in \mathcal{C}_\epsilon\}$ is generating for $\mathcal{M}_2(\mathcal{O}_2)\Omega_2$.

(iv) (approximation) For each $C \in \mathcal{C}_\epsilon$ there is a sequence of uniformly bounded elements $C^m \in \mathcal{M}_1(g_1^m\beta\mathcal{O}_C)$ such that

$$| \, \|C^m\Omega_1\| - \|C\Omega_2\| \, | \leq \epsilon, \quad m > m(C).$$

(v) If $c \in \mathbf{C}$ and $|c|\epsilon' \leq \epsilon$ then $cC \in \mathcal{C}_\epsilon$ for each $C \in \mathcal{C}_{\epsilon'}$. If $X, Y \in \mathcal{C}_{\epsilon/2}$ then $X + Y \in \mathcal{C}_\epsilon$ and for the corresponding sequences we have $(X+Y)^m = X^m + Y^m$.

If we require that for given $\epsilon > 0$, $\mathcal{C}_\epsilon$ is the maximal set satisfying (ii) and (iv) then (i) and (v) are consequences of this maximality. Furthermore we easily see that the maximal sets $\mathcal{C}_\epsilon$ contain with C also $U(g_2^m)CU(g_2^m)^*$.
The following proposition shows that the estimates in Definition 2.5 remain true if we put another unit vector in the place of Ω_1 provided a weak additional assumption is made.

Proposition 2.6 *Suppose the assumptions of Definition 2.5 are fulfilled with the unit vectors Ω_1, Ω_2. Further suppose that $(\mathcal{M}_1(\mathcal{O}_1), g_1^m)$ is asymptotically trivial. Then (iv) of Definition 2.5 remains true (with possibly other sequences) if we replace Ω_1 by an arbitrary other unit vector $u \in \mathcal{H}_1$.*

Proof. Since $(\mathcal{M}_1(\mathcal{O}_1), g_1^m)$ is asymptotically trivial and the sequences $C_\epsilon^m \in \mathcal{M}_1(g_1^m(\mathcal{O}_{C_\epsilon}))$ for $C_\epsilon \in \mathcal{C}_\epsilon$ are uniformly bounded there are subsequences $C_\epsilon^{m_i}$ which weakly converge to c1. Now choose as new sequences for $C_\epsilon \in \mathcal{C}_\epsilon$ and $\epsilon > 0$ the weakly converging subsequences $\tilde{C}_\epsilon^i := C_{\epsilon/2}^{m_i}$ with $C_{\epsilon/2} \in \mathcal{C}_{\epsilon/2}$. Thus we have

$$| \, \|\tilde{C}_\epsilon^i \Omega_1\| - \|C_\epsilon \Omega_2\| \, | < \epsilon/2 \tag{1}$$

for all $i > i(C)$. Now let $u \in \mathcal{H}_1$ be a unit vector. Therefore $u = U\Omega_1$ with U a unitary operator. This implies

$$| \, \|\tilde{C}_\epsilon^i U\Omega_1\| - \|U\tilde{C}_\epsilon^i \Omega_1\| \, | \leq \|[\tilde{C}_\epsilon^i, U]\Omega_1)\| \leq \epsilon/2$$

for $i > i(m)$. Using this estimate together with (1) we obtain

$$| \, \|\tilde{C}_\epsilon^i u\| - \|C_\epsilon \Omega_2\| \, | < \epsilon$$

for $i > i(U)$ proving this proposition.□

3 Connes invariant for local algebras

For each normal state ϕ on a von Neumann algebra $\mathcal{M}$ there is a minimal projection $E \in \mathcal{M}$ with $\phi(E) = 1$. The restriction ϕ_E of ϕ to the algebra $\mathcal{M}_E := E\mathcal{M}E$ is faithful. Thus one defines the modular operator Δ_ϕ for $(\mathcal{M}, \phi)$ as the modular operator for the pair $(\mathcal{M}_E, \phi_E)$. Then the (generalized) Connes invariant for $\mathcal{M}$ is defined by

$$S(\mathcal{M}) := \cap_\phi \operatorname{spec} \Delta_\phi$$

where ϕ runs through the set of all normal states of $\mathcal{M}$ (see e.g. [9]).

We want to compare the Connes invariant for local algebras $\mathcal{M}_1(D), \mathcal{M}_2(W)$ of causal nets of von Neumann algebras $(\mathcal{A}_1, \mathcal{M}_1(\mathcal{O}), \mathcal{X}_1), (\mathcal{A}_2, \mathcal{M}_2(\mathcal{O}), \mathcal{X}_2)$ which are asymptotically connected for special regions $\mathcal{O}_1 \subset \mathcal{X}_1, \mathcal{O}_2 \subset \mathcal{X}_2$ and transformations (g^m). Recall the notations and definitions of Section 2 for these regions and transformations. We use the notation *generalized core* $\mathcal{D}$ for an operator T in the sense that the (not necessarily linear) set $\{[u, Tu]; u \in \mathcal{D}\}$ is dense in the graph $\{[v, v]; v \in \operatorname{dom} T\}$

Theorem 3.1 *Let $\mathcal{M}_1(\mathcal{O}_1)$ be asymptotically connected with $\mathcal{M}_2(\mathcal{O}_2)$ by (g^m) and let $\Omega_1, \Omega_2, \mathcal{C}_\epsilon$ be the corresponding objects. Suppose that:*

(i) Ω_2 is cyclic and separating for $\mathcal{M}_2(W)$ and $U(g_2^m)\mathcal{M}(W)U(g_2^m)^* = \mathcal{M}(W), m \in \mathbf{N}$. Denote by S, Δ, J the corresponding modular objects (Tomita operator, modular operator, modular conjugation) for $(\mathcal{M}_2(W), \Omega_2)$.

(ii) $J\mathcal{M}_2(W_2)J \subseteq \mathcal{M}_2(j_2W_2)$ and $J\mathcal{C}_\epsilon J \subseteq \mathcal{C}_\epsilon$, for $\epsilon > 0$.

(iii) For each $\epsilon > 0$ there is a set $\mathcal{C}_{W,\epsilon} \subseteq \mathcal{M}_2(W) \cap \mathcal{A}_2$ such that $\mathcal{C}_{W,\epsilon}\Omega_2$ is a 'generalized' core for $\Delta^{1/2}$, $U(g_2^m)\mathcal{C}_{W,\epsilon}U(g_2^m)^* \subseteq \mathcal{C}_{W,\epsilon}$ and $\mathcal{C}_\epsilon \cap \mathcal{M}(W_2) = \mathcal{C}_{W,\epsilon} \cap \mathcal{M}_2(\mathcal{O}_2)$.

(iv) $(\mathcal{M}_1(\mathcal{O}_1), g_1^m)$ is asymptotically trivial.

Then $S(\mathcal{M}_1(D)) \supseteq$ spec $\Delta \supseteq S(\mathcal{M}_2(W))$.

If $(\mathcal{M}_2(\mathcal{O}_2), g_2^m)$ is also asymptotically trivial then the inequality on the right hand side is an equality.

Proof. 1. Suppose that for some $\lambda > 0$ and each $\epsilon > 0$ there exist sequences $A_m \in \mathcal{M}_1(D), C_m \in \mathcal{M}_1(D)'$ such that

$$\|A_m\Omega_1\| \geq 1, \|(\lambda^{1/2}C_m^* - A_m^*)\Omega_1\| \leq \epsilon, \|(C_m - \lambda^{1/2}A_m)\Omega_1\| \leq \epsilon$$

for all $m \in \mathbf{N}$ and $|C_m|^2, |A_m|^2, C_m^*A_m$ weakly converge to multiples of 1.
Let ω be a normal state on $\mathcal{M}_1(D)$. For simplicity we assume that ω is generated by a vector u (normal states generated by trace class operators can be treated in a similar way using absolutely converging sums of vector states), i.e. $\omega(.) = (u, .u)$ on $\mathcal{M}_1(D)$. Writing $u = U\Omega_1$ where U is a unitary operator and using that $|A_m|^2, |C_m|^2, C_m^*A_m$ weakly commute with U asymptotically, we easily obtain the estimates

$$\|A_m u\| \geq 1 - \epsilon, \|(C_m - \lambda^{1/2}A_m)u\| \leq 2\epsilon, \|(\lambda^{1/2}C_m^* - A_m^*)u\| \leq 2\epsilon$$

for all $m > m(\epsilon)$. This implies for all $B \in \mathcal{M}_1(D)$ the estimate

$$|\omega(A_mB) - \lambda\omega(BA_m)| = |((A_m^* - \lambda^{1/2}C_m^*)u, Bu) +$$

$$+\lambda^{1/2}(C_m^*u, Bu) - \lambda^{1/2}(B^*u, (\lambda^{1/2}A_m - C_m)u) - \lambda^{1/2}(B^*u, C_mu)|$$

$$\leq 2\epsilon(\|Bu\| + \lambda^{1/2}\|B^*u\|) \leq 2\sqrt{2}\epsilon(\omega(B^*B) + \lambda\omega(BB^*))^{1/2}$$

where we have used that $C_m \in \mathcal{M}_1(D)'$ commutes with $B \in \mathcal{M}_1(D)$. But this estimate says that $\lambda \in$ spec Δ_ω (see [6] or [2,Prop.16.2.5]). Because ω was an arbitrary state we find that $\lambda \in S(\mathcal{M}_1(D))$.
2. Because $\mathcal{C}_{W,\epsilon}\Omega_2$ is a core for $\Delta^{1/2}$ we can find for each $\lambda \in$ spec Δ an element $A \in \mathcal{C}_{W,\epsilon}$ such that $\|A\Omega_2\| \geq 1$ and $\|(\Delta^{1/2} - \lambda^{1/2})A\Omega_2\| \leq \epsilon$.
Since $ad\ U(g_2^m) =: \alpha_m$ are automorphisms of $\mathcal{M}_2(W), \mathcal{C}_{W,\epsilon}$ and Ω_1 are invariant under α_m and $U(g_2^m)$, respectively, we have

$$\|(\Delta^{1/2} - \lambda^{1/2})\alpha_m(A)\Omega_2\| \leq \epsilon$$

and $\alpha_m(A) \in \mathcal{C}_{W,\epsilon}$ for all m. But for all sufficiently large m, $\alpha_m(A) \in \mathcal{M}_2(W_2)$ because W_2 absorbs W and $A \in \mathcal{A}_2$. Thus $\tilde{A} := \alpha_m(A) \in \mathcal{C}_\epsilon$ for $m > m(A)$.Set $\tilde{C} := J\tilde{A}^*J$. Then we get (using $S = J\Delta^{1/2}$ and the action of S)

$$\|(\Delta^{1/2} - \lambda^{1/2})\tilde{A}\Omega_2\| = \|(\tilde{C} - \lambda^{1/2}\tilde{A})\Omega_2\| \leq \epsilon. \tag{2}$$

Further

$$\begin{aligned} \|(\tilde{C} - \lambda^{1/2}\tilde{A})\Omega_2\| &= \|J\tilde{C}\Omega_2 - \lambda^{1/2}J\tilde{A}\Omega_2\| \\ = \|\tilde{A}^*\Omega_2 - \lambda^{1/2}S^*JS\tilde{A}\Omega_2\| &= \|\tilde{A}^*\Omega_2 - \lambda^{1/2}S^*J\tilde{A}^*\Omega_2\| \\ &= \|\tilde{A}^*\Omega_2 - \lambda^{1/2}\tilde{C}^*\Omega_2\| \leq \epsilon \end{aligned} \tag{3}$$

with $\tilde{C} \in \mathcal{M}_2(j_2W_2) \cap \mathcal{C}_\epsilon \subset \mathcal{M}_2(W_2)' \cap \mathcal{C}_\epsilon$.

3. Next we use the asymptotic connectedness. This property gives that for $\tilde{A}, \tilde{C}$ there exist sequences $\tilde{A}_m, \tilde{C}_m$ localized in $\mathcal{M}_1(g_1^m\mathcal{O}_{\tilde{C}}), \mathcal{M}_1(g_1^m\mathcal{O}_{\tilde{A}})$, respectively. By assumption on the geometry of the regions and transformations we obtain that for $m > m(\tilde{C}, \tilde{A})$,

$$\tilde{A}_m \in \mathcal{M}_1(D), \tilde{C}_m \in \mathcal{M}_1(D^\perp) \subset \mathcal{M}_1(D)'.$$

From the estimates for $\tilde{A}, \tilde{C}$ we find the corresponding estimates (using the properties of $\mathcal{C}_\epsilon$) for $\tilde{A}_m, \tilde{C}_m$,

$$\|(\tilde{A}_m^* - \lambda^{1/2}\tilde{C}_m^*)\Omega_1\| \leq 2\epsilon + \lambda^{1/2}\epsilon, \tag{4}$$

$$\|(\tilde{C}_m - \lambda^{1/2}\tilde{A}_m)\Omega_1\| \leq 2\epsilon + \lambda^{1/2}\epsilon, \tag{5}$$

$$\|\tilde{A}_m^*\Omega_1\| \geq 1 - \epsilon. \tag{6}$$

So we have found sequences $\tilde{A}_m, \tilde{C}_m$ satisfying the desired estimates in 1 (choosing ϵ small enough in dependence of the fixed λ). Since $|\tilde{A}_m|^2, |\tilde{C}_m|^2, \tilde{C}_m^*\tilde{A}_m$ are from $\mathcal{M}_1(\mathcal{O}_1) \cap \mathcal{A}_1$ and (iii) holds we find (as in Proposition 2.6) that subsequences exist which converge weakly to multiples of 1. Thus we can apply the arguments in 1. and obtain $\lambda \in S(\mathcal{M}_1(D))$. The last assertion can be proved in the same way or use directly that $(\mathcal{M}(W), \alpha_m)$ is asymptotic abelian (see e.g. [4]) .□

Remark 3.2 (a) A consequence of this theorem is that $\mathcal{M}_1(D)$ is of type III if $\mathcal{M}_2(W)$ is of type III provided the assumptions (i) - (iv) are fulfilled. If spec $\Delta = [0, \infty)$ then $S(\mathcal{M}_1(D)) = [0, \infty)$ and thus in the type decomposition of $\mathcal{M}_1(D)$ only factors of type III_1 can occur (see the arguments in [6]).

(b) Assumption (ii) says that J acts geometrically in a simple way. In case that $W = W_2$ the relation $J\mathcal{M}_2(W_2)J \subseteq \mathcal{M}_2(j_2W_2)$ follows from duality for the local algebra $\mathcal{M}_2(W_2)$. Assumption (iii) says that the subsets $\mathcal{C}_\epsilon$ are not too small so that the information about the spectrum of Δ is contained in $\mathcal{C}_\epsilon$.

(c) From the proof we see that the assumption (iii) in Definition 2.5 is not necessary for the theorem.

4 Applications and Discussion

First we note that the assumptions of Theorem 3.1 are wide enough to include an essential part of the results of [4]. Namely let $\mathcal{X}_1 = \mathcal{X}_2$ be the Minkowski space-time and let W, j_2 be as in Example 2.4. Further, let β be the identity map. Let $(\mathcal{A}_2, \mathcal{M}_2(\mathcal{O}), \mathcal{X}_2)$ be a vacuum representation of a causal net satisfying the usual assumptions and duality. Suppose $\mathcal{A}_1 = \pi(\mathcal{A}_2), \mathcal{M}_1(\mathcal{O}) = \pi(\mathcal{M}_2(\mathcal{O}))''$is another representation of the same net such that this new representation is asymptotically connected with the vacuum representation by the transformations g^m (in the sense described in [4]). Set $\mathcal{C}_\epsilon = \mathcal{M}_2(\mathcal{O}_2) \cap \mathcal{A}_2$, $\mathcal{C}_{W,\epsilon} = \mathcal{M}_2(W) \cap \mathcal{A}_2$, independent of ϵ, and $C_m = \pi(\alpha_{g_2^m}(C))$. Then we can easily calculate that the assumptions of Theorem 3.1 are satisfied. In this case $\mathcal{C}_\epsilon$ is even an algebra.

In other applications it is often easier to prove a kind of asymptotic connectedness for quantum fields which are affiliated to the given causal nets.
Let $\Phi_j(.)$ be operator-valued distributions over $C_o^\infty(\mathcal{X}_j)$ with a common invariant domain $\mathcal{D}_j, \mathcal{H}_j \supset \mathcal{D}_j \ni \Omega_j$, for all $\Phi_j(f), j = 1, 2$ and $f \in C_o^\infty(\mathcal{X}_j)$.

Definition 4.1 *We say $\Phi_1(.)|C_o^\infty(\mathcal{O}_1)$ is asymptotically connected with $\Phi_2(.)|C_o^\infty(\mathcal{O}_2)$ by $(g^m)_{\mathbf{N}}$ if (I) there is a representation of g_2^m by unitary operators $U(g_2^m)$ on $\mathcal{H}_2$ and unit vectors $\Omega_1 \in \mathcal{H}_1, \Omega_2 \in \mathcal{H}_2$ such that $U(g_2^m)\mathcal{D}_2 \subseteq \mathcal{D}_2, U(g_2^m)\Phi_2(f(.))U(g_2^m)^* = \Phi_2(f(g_2^m.))$, $U(g_2^m)\Omega_2 = \Omega_2, m \in \mathbf{N}$, the set $\{\Phi_2(f_1)\cdots\Phi_2(f_s)\Omega_2; f_j \in C_o^\infty(\mathcal{O}), s \in \mathbf{N}\}$ is generating in $\mathcal{H}_2$ and (II) there is a monotone function $N(m)$ from $\mathbf{N}$ into $(0, \infty)$ such that*

$$\lim_{m\to\infty} N(m)^s(\Omega_1, \Phi_1(f_1(g_{2,i}^m\beta^{-1}.))\cdots\Phi_1(f_s(g_{2,i}^m\beta^{-1}.))\Omega_1)$$

$$= (\Omega_2, \Phi_2(f_1)\cdots\Phi_2(f_s)\Omega_2)$$

for all $s \in \mathbf{N}$ and $f_j \in C_o^\infty(\mathcal{O}_2)$.

For the set of polynomials $P(\Phi_j)$ in the field operators $\Phi_j(f_s)$ with $f_s \in C_o^\infty(\mathcal{O})$ we write $P(\Phi_j)(\mathcal{O})$. We say $\Phi_j(.)$ is *strongly affiliated* with the causal net $(\mathcal{A}_j, \mathcal{M}_j(\mathcal{O}), \mathcal{X}_j)$ if for each $\mathcal{O}$ and $A \in P(\Phi_j)(\mathcal{O})$ the bounded functions of A and A^* belong to $\mathcal{M}_2(\mathcal{O})$.
Now we come to the relations with causal nets.

Proposition 4.2 *Suppose $\Phi_1(.)|C_o^\infty(\mathcal{O}_1)$ is asymptotically connected with $\Phi_2(.)|C_o^\infty(\mathcal{O}_2)$ by $(g^m)_{\mathbf{N}}$. Suppose that $\Phi_j(.)$ are strongly affiliated with causal nets $(\mathcal{A}_j, \mathcal{M}_j(\mathcal{O}), \mathcal{X}_j), j = 1, 2$. Further suppose that the causal net $(\mathcal{A}_2, \mathcal{M}_2(\mathcal{O}), \mathcal{X}_2)$ transforms covariantly with respect to $U(g_2^m)$, i.e. $U(g_2^m)\mathcal{M}_2(\mathcal{O})U(g_2^m)^* = \mathcal{M}_2(g_2^m\mathcal{O}), \mathcal{O} \subset \mathcal{X}_2$. Then $\mathcal{M}_1(\mathcal{O}_1)$ is asymptotically connected with $\mathcal{M}_2(\mathcal{O}_2)$ by $(g^m)_{\mathbf{N}}$.*

Proof. First we construct subsets of $\mathcal{M}_2(\mathcal{O}_2)$ from $\Phi_j(.)$.
We choose the following simple definition (a slight generalization from [6])

$$X_c^j(P) := (1 + cP(\Phi_j)^*P(\Phi_j)^{**})^{-1}P(\Phi_j)^{**}, c > 0, \tag{7}$$

where $P(\Phi_j)$ is a polynomial in $\Phi_j(f_s)$ with $f_s \in C_o^\infty(\mathcal{X}_j)$. Then we obtain the estimate

$$\|(X_c^j(P) - P(\Phi_j))u\| \leq (c)^{1/2}\|P(\Phi_j)^* P(\Phi_j)u\| \tag{8}$$

(see e.g. [6] or [2,]) for $u \in \mathcal{D}_j$. Then we define $D_\epsilon^j(\mathcal{O})$ as the set of all $X_c^j(P)$ such that the support of the functions f_s appearing in the polynomial $P(\Phi_j)$ lies in $\mathcal{O}$ and the right hand side of (8) is smaller than $\epsilon/4$ (By choosing c small enough this can be shown for each fixed polynomial).
Now let $C_\epsilon \in D_\epsilon^2(\mathcal{O}_2)$ with support $\mathcal{O}_C$ and corresponding polynomial $P(\Phi_2)$. We consider the sequence of polynomials $P^m(\Phi_1)$ which we obtain from $P(\Phi_2)$ by replacing the terms $\Phi_2(f_l)$ by $N(m)\Phi_1(f_l(g_{2,i}^m\beta^{-1}.))$. For these polynomials we have the corresponding elements $X_c^1(P^m)$ from (7) and set $C_\epsilon^m := X_{c_\epsilon}^1(P^m)$ where c_ϵ is so chosen that on the right hand side of (8) we can set $\epsilon/4$.
Then we obtain the estimate

$$\begin{aligned}
&|\ \|C_\epsilon^m\Omega_1\| - \|C_\epsilon\Omega_2\|\ | \\
&= |\ \|(C_\epsilon^m - P^m(\Phi_1))\Omega_1 + P^m(\Phi_1)\Omega_1\| - \\
&\qquad \|(C_\epsilon - P(\Phi_2))\Omega_2 + P(\Phi_2)\Omega_2\|\ | \\
&\leq \epsilon/2 + |\ \|P^m(\Phi_1)\Omega_1\| - \|P(\Phi_2)\Omega_2\|\ |.
\end{aligned}$$

The second term on the right hand side can be estimated by a finite sum of terms

$$\begin{aligned}
a_s|N(m)^s(\Omega_1, \Phi_1(f_1(g_{2,i}^m\beta^{-1}.))\cdots\Phi_1(f_s(g_{2,i}^m\beta^{-1}.))\Omega_1) - \\
-(\Omega_2, \Phi_2(f_1)\cdots\Phi_2(f_s)\Omega_2)|.
\end{aligned}$$

Thus for sufficiently large m we obtain the desired estimate from the asymptotic connectedness of the quantum fields. As $\mathcal{C}_\epsilon, \epsilon > 0$, we choose the maximal sets satisfying (i), (ii) and (iv) of Definition 2.5 and containing $D_\epsilon^2(\mathcal{O}_2)$. From the assumption that $P(\Phi_2)(\mathcal{O}_2)\Omega_2$ is generating in $\mathcal{H}_2$ and (7) and (8) we obtain that $D_\epsilon^2(\mathcal{O}_2)\Omega_2$ is generating in $\mathcal{H}_2$ proving (iii) of Definition 2.5. Using the remarks after Definition 2.5 we see that all assumptions of this definition are fulfilled. □
Note that for causal nets generated by linear quantum fields over globally hyperbolic space-times on Fock spaces the asymptotic connectedness can be shown by proving this property only for two-point functions (see e.g. the arguments in [10]).
The other assumptions needed in Theorem 3.1 are assumptions on (a) the geometric and causal structure of $\mathcal{X}_1$ and $\mathcal{X}_2$ and (b) the modular objects for $(\mathcal{M}_2(W), \Omega_2)$. We discuss these assumptions for two examples.

Example 4.3 Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{O}_1, \mathcal{O}_2, W, \beta, g^m$ be as in Example 2.3 . Let $(\mathcal{A}_1, \mathcal{M}_1(\mathcal{O}), \mathcal{X}_1)$, $(\mathcal{A}_2, \mathcal{M}_2(\mathcal{O}), \mathcal{X}_2)$ be causal nets of algebras. Suppose we have operator-valued distributions (quantum fields) $\Phi_1(.), \Phi_2(.)$ such that $\Phi_1(.)|C_o^\infty(\mathcal{O}_1)$ is asymptotically connected with $\Phi_2(.)|C_o^\infty(\mathcal{O}_2)$ by $(g^m)_{\mathbf{N}}$ and the assumptions of Proposition 4.2 are fulfilled. As

subsets $\mathcal{C}_{\epsilon,W}$ we take maximal sets satisfying (i), (ii) and (iv) of Definition 2.5 with $\mathcal{C}_\epsilon, \mathcal{O}_2$ replaced by $\mathcal{C}_{\epsilon,W}, W$ and containing $D^2_\epsilon(W)$, defined in the proof of Proposition 4.2 . From the Bisognano-Wichmann result [3] we know that the modular objects of $(\mathcal{M}_2(W), \Omega_2)$ satisfy the assumptions of Theorem 3.1 and $S(\mathcal{M}_2(W)) = [0, \infty)$. Thus we have $S(\mathcal{M}_1(D)) = [0, \infty)$. This case was treated in [10] as scaling limit.

Example 4.4 Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{O}_1, \mathcal{O}_2, W, \beta, g^m$ be as in Example 2.4 . Let $(\mathcal{A}_1, \mathcal{M}_1(\mathcal{O}), \mathcal{X}_1)$, $(\mathcal{A}_2, \mathcal{M}_2(\mathcal{O}), \mathcal{X}_2)$ be causal nets of algebras. Suppose we have operator-valued distributions (quantum fields) $\Phi_1(.), \Phi_2(.)$ such that $\Phi_1(.)|C_o^\infty(\mathcal{O}_1)$ is asymptotically connected with $\Phi_2(.)|C_o^\infty(\mathcal{O}_2)$ by $(g^m)_{\mathbf{N}}$and the assumptions of Proposition 4.2 are fulfilled with $N(m) \equiv 1$. As subset $\mathcal{C}_{\epsilon,W}$ we take the set as in Example 4.3. From the Bisognano-Wichmann result (see [3] or [10]) we know that the modular objects of $(\mathcal{M}_2(W), \Omega_2)$ satisfy the assumptions of Theorem 3.1 and $S(\mathcal{M}_2(W)) = [0, \infty)$. Thus we have $S(\mathcal{M}_1(D)) = [0, \infty)$. This example describes causal nets connected by large spacelike translations with a causal net over a Minkowski space-time. If for example $\Phi_1(.)$ is a linear quantum field over an asymptotically flat space-time $\mathcal{X}_1$ acting on a Fock space and its two-point function $(\Omega_1, \Phi_1(f_1(g^m_{2,i}\beta^{-1}.))\Phi_1(f_2(g^m_{2,i}\beta^{-1}.))\Omega_1)$ tends to $(\Omega_2, \Phi_2(f_1)\Phi_2(f_2)\Omega_2)$ then the assumptions are fulfilled.

Note that if we have a hermitian scalar quantum field $\Phi_2(.)$ over Minkowski space-time $\mathcal{X}_2$ satisfying the Wightman axioms then we know from the Bisognano/Wichmann paper [3] that there are operators S, Δ, J for the polynomial algebra in the field operators $\Phi_2(f), f \in C_o^\infty(W_R)$, and the vacuum vector Ω_2 with the following properties:

$$S\Phi_2(f(.))\Omega_2 = \Phi_2(\overline{f(.)})\Omega_2,$$

$$J\Phi_2(f(.))\Omega_2 = \Phi_2(\overline{f(j_2.)})\Omega_2,$$

$$\text{spec} \quad \Delta = [0, \infty),$$

For each $\lambda > 0$ and $\epsilon > 0$ there is $f \in C_o^\infty(W_R)$ such that

$$\|\Phi_2(f)\Omega_2\| \geq 1, \quad \|(\Delta^{1/2} - \lambda^{1/2})\Phi_2(f)\Omega_2\| \leq \epsilon.$$

Assuming now that $\Phi_1(.)|C_o^\infty(\mathcal{O}_1)$ is asymptotically connected with $\Phi_2(.)|C_o^\infty(\mathcal{O}_2)$ by (g^m) (we assume that $W = W_R, \mathcal{O}_1, \mathcal{O}_2, \beta, j_2$ satisfy the assumptions of Section 2) we can prove for a causal net of algebras $(\mathcal{A}_1, \mathcal{M}_1(\mathcal{O}), \mathcal{X}_1)$ which is strongly affiliated with $\Phi_1(.)$ and for which $(\mathcal{M}_1(\mathcal{O}_1), g^m_1)$ is asymptotically trivial that $S(\mathcal{M}_1(D)) = [0, \infty)$, without assuming the existence of a causal net $(\mathcal{A}_2, \mathcal{M}_2(\mathcal{O}), \mathcal{X}_2)$. This can be done in a similar way as in [10] replacing the dilatations δ_λ by g^m and using some ideas of the proof of Theorem 3.1.

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