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TWO-DIMENSIONAL MATRIX FORMALISM

by

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ABSTRACT

A two dimensional matrix formalism is constructed to describe the partition function of a scalar field theory defined over a two-dimensional finite rectangular lattice. The case of 2×2 matrices with parity conservation (8 independent parameters) is presented: it is shown that one relation between the parameters generates factorization properties in the eigenvalues of the transfer matrices. These eigenvalues and the corresponding partition function are explicitly obtained.

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0. INTRODUCTION AND MOTIVATION

Generating functional is a powerful and elegant formalism which proved itself to be most useful to the understanding of quantum field theory. The perturbative expansion and the Feynman diagrams algorithm is a good example of the efficiency of this functional. However, the general non perturbative structure of the theory is far from being understood and for instance general features like renormalization properties (although much progress has been performed there) and phase transitions are difficultly obtained from the functional.

One reason for this situation is that the generating functional appears more like a "formal" entity than a well-defined mathematical object. Two important steps exist between the generating functional in continuous space-time and a well-defined mathematical object like the generating functional over a finite lattice of sites. The first step is the infinite lattice of sites, the so-called thermodynamic limit with the appearance of phase transitions; the second step is the so-called continuous limit where the infinite discrete number of sites become infinite continuous with the appearance of ultraviolet divergences when the lattice spacing goes to zero.

The general non-perturbative structure of field theory must first be understood at the level of a finite lattice of sites where we stand on strong mathematical ground. Let us consider the generating functional over a d-dimensional rectangular lattice with $N_1 \times N_2 \times \cdots \times N_d$ sites, for a scalar field theory consisting of a nearest neighbour interaction and a potential on each site

$$Z(J) = \int_{-\infty}^{+\infty} \prod_i \mathrm{d}\phi_i \exp\left(-\left[\lambda \sum_{\langle ij\rangle} (\phi_i - \phi_j)^2 + \sum_i V(\phi_i)\right] + \sum_i J_i \phi_i\right) \tag{0.1}$$

The potential V is such that the integrals exist at infinity and $\langle ij\rangle$ means that sites i and j are neighbours.

In one dimension, the general non perturbative structure of these integrals is well-known thanks to the existence of a transfer matrix. The interaction between two neighbouring sites i and $i+1$ is described by a matrix $T_{i,i+1}$ so that, assuming periodic boundary conditions, the partition function is obtained as

$$Z = Z(J)|_{J'_S=0} = \mathrm{Trace}\ T^N \tag{0.2}$$

This is what I call a one-dimensional matrix formalism. This formalism is based on the existence of eigenvectors $|\psi_p\rangle$ and eigenvalues r_p of T. Although the numerical problem of finding the ψ'_S and the r'_S from the potential V is highly difficult, the general structure

of the Green functions is always the same for any potential V. For instance, the partition function is

$$Z_N = \sum_p r_p^N \tag{0.3}$$

and the two point Green function between site i and site j is

$$G_{ij} = \sum_{pq} \sigma_{pq} r_q^{|i-j|} \sigma_{qp} r_p^{N-|i-j|} \tag{0.4}$$

where the matrix σ is defined as

$$\sigma_{pq} = (\psi_p|\phi|\psi_q) \tag{0.5}$$

Such general forms allow, for instance, the understanding of the thermodynamic limit in a potential independent way. Froebenius-Perron theorem says that the largest eigenvalue r_0 is never degenerate (absence of phase transition in one dimension) provided that all elements of the transfer matrix is positive; then, when $N \longrightarrow \infty$

$$Z_N \longrightarrow r_0^N \tag{0.6a}$$

$$G_{ij} \longrightarrow r_0^N \sum_q \sigma_{0q} \left(\frac{r_0}{r_q}\right)^{-|i-j|} \sigma_{q0} \tag{0.6b}$$

Such general forms show explicitly the exponential decrease of the two-point Green function when $|i-j| \longrightarrow \infty$ or equivalently, it says that in one-dimensional continuous field theory, the propagator in momentum space is a sum of single poles. Let us remind here that all Feynman graphs can be analytically calculated in one dimension and that the propagator in momentum space is a sum of products of multiple poles; it is interesting to know from the existence of a transfer matrix that the sum of the perturbative series transforms these infinitely many multiple poles into a sum of single poles.

This publication is a first step towards the two-dimensional structure of the partition function. Unfortunately, as soon as two-dimensional lattices are considered, such general structure is far from being known. We do also have a transfer matrix formalism between two neighbouring lines of sites but clearly there is a much richer structure because there is in fact two such transfer matrices: a vertical and a horizontal tranfer matrix. Necessarily, the eigenvectors and eigenvalues of one transfer matrix are very much constrained by the existence of the other transfer matrix. This is for instance well observed when considering the eigenvalues of the Ising model and their characteristic factorization properties. Another unsatisfactory feature when using a unique transfer matrix is for instance the expression obtained for the Green function between two points of coordinates i, k and j, ℓ : similarly

to (0.4), the translational invariant $|i-j|$ dependence appears but the analog translational invariant $|k-\ell|$ dependence is completely destroyed (without speaking of the rotational invariance of the continuous limit).

It looks evident to us that the elementary description of two-dimensional interactions lies in the existence of a plaquette operator S. I call two-dimensional matrix formalism the procedures which permit from S to construct the horizontal and vertical transfer matrices and to construct the two-dimensional partition function and Green functions.

In Section I, we describe the plaquette operator S and the two-dimensional matrix formalism. At the end of this section (and for the following sections) we restrict ourselves to the case of 2×2 matrices. In Section II we describe the vector space. In Sections III-A to III-D we give the solution for the eigenvalues of the transfer matrix of four models of increasing difficulty (models III-D contain the Ising model as a very special case). Finally in Section IV we give the expression for the partition function of these models.

Let us mention here that our main result concerning the 2×2 matrices is stated at the end of Section I, from (I.58) to (I.61).

I. TWO DIMENSIONAL MATRIX FORMALISM

We define a plaquette operator $S = S_{11}$ as a sum

$$S_{11} = \sum_i A_i \cdot B_i \tag{I.1}$$

where A_i and B_i are square matrices defined in two different vector spaces and where $\cdot$ means nothing but a separation between the A'_S which operate vertically while the B'_S operate horizontally.

For instance, the two-horizontal plaquettes operator is

$$S_{12} = \sum_{i,j} [A_i \otimes A_j] \cdot B_i B_j \tag{I.2}$$

where the B'_S multiply by usual matrix product while the A'_S operate by tensorial products. Similarly the two-vertical plaquettes operator is

$$S_{21} = \sum_{i,j} A_i A_j . [B_i \otimes B_j] \tag{I.3}$$

By generalization, the N horizontal plaquettes operator reads

$$S_{1N} = \sum_{\{i_1,\dots,i_N\}} [A_{i_1} \otimes A_{i_2} \otimes \dots \otimes A_{i_N}] \cdot B_{i_1} B_{i_2} \dots B_{i_N} \tag{I.4}$$

while the M vertical plaquettes operator reads

$$S_{M1} = \sum_{\{j_1,\dots,j_M\}} A_{j_1} A_{j_2} \dots A_{j_M} \cdot [B_{j_1} \otimes B_{j_2} \otimes \dots \otimes B_{j_M}] \tag{I.5}$$

Using the commutativity of matrix product with tensorial product

$$AB \otimes CD = (A \otimes C)(B \otimes D) \tag{I.6}$$

we get the $N \times M$ plaquettes operator as a sum over any choices of a tableau of indices i_{jk} (j from 1 to M, k from 1 to N)

$$\begin{aligned} S_{MN} &= \sum_{\{i_{jk}\}} \{[A_{i_{11}} A_{i_{21}} \dots A_{i_{M1}}] \otimes [A_{i_{12}} A_{i_{22}} \dots A_{i_{M2}}] \otimes \dots \otimes [A_{i_{1N}} A_{i_{2N}} \dots A_{i_{MN}}]\} \\ &\quad \cdot \{[B_{i_{11}} B_{i_{12}} \dots B_{i_{1N}}] \otimes [B_{i_{21}} B_{i_{22}} \dots B_{i_{2N}}] \otimes \dots \otimes [B_{i_{M1}} B_{i_{M2}} \dots B_{i_{MN}}]\} \end{aligned} \tag{I.7}$$

This terrible result is the two-dimensional equivalent of T^M in one dimension. The two dimensional equivalent of the partition function $Z_M = \text{Trace}\, T^M$ becomes (with periodic boundaries)

$$Z_{MN} = \sum_{\{i_{jk}\}} \left\{ \begin{array}{l} \text{Trace}\,[A_{i_{11}} \dots A_{i_{M1}}] \cdot \text{Trace}\,[A_{i_{12}} \dots A_{i_{M2}}] \cdot \dots \cdot \text{Trace}\,[A_{i_{1N}} \dots A_{i_{MN}}] \cdot \\ \text{Trace}\,[B_{i_{11}} \dots B_{i_{1N}}] \cdot \text{Trace}\,[B_{i_{21}} \dots B_{i_{2N}}] \cdot \dots \cdot \text{Trace}\,[B_{i_{M1}} \dots B_{i_{MN}}] \end{array} \right\} \tag{I.8}$$

Clearly Z_{MN} is unchanged if we change the vertical and horizontal basis: $A'_{ij} = U A_{ij} U^{-1}$, $B'_{ij} = V B_{ij} V^{-1}$. Such a structure is clearly generalizable to a lattice of any dimension.

However, this result seems completely useless (except maybe for numerical calculations) unless we can generalize the formalism of eigenvectors and eigenvalues in some powerful way. Such a way may arise from the understanding of the vertical or horizontal transfer matrix. The vertical transfer matrix is obtained as

$$T_N \uparrow = \sum_{\{i_1 \dots i_N\}} \text{Trace}(B_{i_1} \dots B_{i_N})\, A_{i_1} \otimes A_{i_2} \otimes \dots \otimes A_{i_N} \tag{I.9}$$

and similarly the horizontal one is

$$\vec{T}_M = \sum_{\{j_1 \dots j_M\}} \text{Trace}(A_{j_1} \dots A_{j_M})\, B_{j_1} \otimes B_{j_2} \otimes \dots \otimes B_{j_M} \tag{I.10}$$

Clearly, a change of horizontal basis $B'_{ij} = V B_{ij} V^{-1}$ does not change $T_N \uparrow$ while a change of vertical basis expresses $T_N \uparrow$ in another basis (without changing its eigenvalues).

Let us mention an important property of these transfer matrices. The property of the trace to be invariant by cyclic permutation creates automatically in $T_N \uparrow$ partial sums of the type

$$[A_{i_1} \otimes A_{i_2} \otimes A_{i_3} \otimes ... \otimes A_{i_N} + A_{i_2} \otimes A_{i_3} \otimes ... \otimes A_{i_N} \otimes A_{i_1} + A_{i_3} \otimes ... \otimes A_{i_N} \otimes A_{i_1} \otimes A_{i_2} + ...] \tag{I.11}$$

Let us define the unitary operator C_N, called cyclic permutation operator, as

$$C_N\,[A_{i_1} \otimes A_{i_2} \otimes ... \otimes A_{i_N}]\,C_N^{+} = A_{i_2} \otimes ... \otimes A_{i_N} \otimes A_{i_1} \tag{I.12}$$

for any choice of A'_S; then,

$$C_N T_N C_N^{+} = T_N \tag{I.13}$$

or

$$[C_N, T_N] = 0 \tag{I.14}$$

Of course $C_N^N = I$ so that the eigenvalues of C_N are N^{th}-root of unity but the degeneracy of the eigenvectors is tremendous. This operator C_N is studied in more details in Section III-B.

To give a better understanding of the situation, we now give one trivial and one easy example which never appears in field theory. First, there is only one A and B matrix

$$S_{11} = A \cdot B \tag{I.15}$$

Then,

$$S_{MN} = [A^M]^{\otimes N} \cdot [B^N]^{\otimes M} \tag{I.16}$$

so that if the eigenvalues of A are r_p and the eigenvalues of B are ρ_q

$$Z_{MN} = \left[\sum_p r_p^M\right]^N \left[\sum_q \rho_q^N\right]^M \tag{I.17}$$

which is a trivial generalization of (0.3). In this case, if r_0 and ρ_0 respectively are the largest eigenvalues, the thermodynamic limit is

$$\text{limit}(M \text{ and } N \longrightarrow \infty) Z_{MN} \sim (r_0\rho_0)^{MN} \tag{I.18}$$

The second example is when all A'_S on one side and all B'_S on the other side commute. Then, we may choose a common set of eigenvectors on each side; there exists U and V matrices such that

$$U^{-1} A_i U = D_i \tag{I.19a}$$

$$V^{-1} B_j V = E_j \tag{I.19b}$$

where D_i and E_j are diagonal matrices. From this transformation, we obtain a new (but equivalent) plaquette operator:

$$\tilde{S}_{11} = \sum_i D_i \cdot E_i \tag{I.20}$$

If we decompose the matrices D_i and E_j over their orthogonal projectors

$$D_i = \sum_p r_{ip} P_p \tag{I.21a}$$

$$E_j = \sum_q \rho_{jq} Q_q \tag{I.21b}$$

We get the system

$$\tilde{S}_{11} = \sum_{pq} a_{pq} P_p \cdot Q_q \tag{I.22}$$

where

$$a_{pq} = \sum_i r_{ip} \rho_{iq} \tag{I.23}$$

Clearly enough

$$S_{1N} = \sum_{\{p_1 ... p_N\}} \sum_q a_{p_1 q} a_{p_2 q} ... a_{p_N q} (P_{p_1} \otimes P_{p_2} \otimes ... \otimes P_{p_N}) \cdot Q_q \tag{I.24}$$

and the vertical transfer matrices

$$T_N \uparrow = \sum_{\{p_1 ... p_N\}} \left(\sum_q a_{p_1 q} ... a_{p_N q}\right) P_{p_1} \otimes P_{p_2} \otimes ... \otimes P_{p_N} \tag{I.25}$$

so that

$$Z_{MN} = \sum_{\{p_1 ... p_N\}} \left(\sum_q a_{p_1 q} ... a_{p_N q}\right)^M \tag{I.26}$$

Similarly

$$S_{M1} = \sum_{\{q_1 ... q_M\}} \sum_p a_{pq_1} a_{pq_2} ... a_{pq_M} P_p \cdot (Q_{q_1} \otimes Q_{q_2} \otimes ... \otimes Q_{q_M}) \tag{I.27}$$

and the horizontal transfer matrices

$$\vec{T}_M = \sum_{\{q_1 ... q_M\}} \left(\sum_p a_{pq_1} ... a_{pq_M}\right) Q_{q_1} \otimes Q_{q_2} \otimes ... \otimes Q_{q_M} \tag{I.28}$$

so that

$$Z_{MN} = \sum_{\{q_1 \dots q_M\}} \left(\sum_p a_{pq_1} \dots a_{pq_M} \right)^N \tag{I.29}$$

A symmetric way of writing the partition function is

$$Z_{MN} = \sum_{\{p_1 \dots p_N\}} \sum_{\{q_1 \dots q_M\}} \prod_{i=1}^{N} \prod_{j=1}^{M} a_{p_i q_j} \tag{I.30}$$

Of course a tremendous degeneracy occur in the eigenvalues of the transfer matrices; for instance in (I.25-26) we may write

$$Z_{MN} = \sum_{\{r'_S\}} \frac{N!}{r_1! \dots r_n!} \left(\sum_q \prod_{i=1}^{n} a_{i_q}^{r_i} \right)^M \tag{I.31}$$

where the sum runs over all r'_S such that $\sum_{i=1}^n r_i = N$ and where n is the dimension of the matrices D_i and E_j. Also, in a symmetric way (I.31) can be written

$$Z_{MN} = \sum_{\{t'_S\}} \frac{M!}{t_1! \dots t_n!} \left(\sum_p \prod_{j=1}^{n} a_{p_j}^{t_j} \right)^N \tag{I.32}$$

where the sum over $\{t'_S\}$ is such that $\sum_{j=1}^n t_j = M$. A completely symmetric result gives

$$Z_{MN} = \sum_{\{r'_S\}} \sum_{\{t'_S\}} \frac{N!}{r_1! \dots r_n!} \frac{M!}{t_1! \dots t_n!} \prod_{i=1}^{n} \prod_{j=1}^{n} a_{ij}^{r_i t_j} \tag{I.33}$$

We now give an expression of the plaquette operator in terms of matrices, the element of which are matrices themselves.

Suppose that we decompose the matrices B_i as

$$B_i = \sum_{jk} b_{jk}^i P^{jk} \tag{I.34}$$

where the matrices P_{jk} are

$$[P_{jk}]_{\ell m} = \delta_{j\ell} \delta_{km} \tag{I.35}$$

Then, we define the matrices

$$A_{jk} = \sum_i b_{jk}^i A_i \tag{I.36}$$

The plaquette operator S_{11} is now written as

$$S_{11} = \sum_{jk} A_{jk} \cdot P^{jk} \tag{I.37}$$

which we denote by a matrix S_{11}^h of matrices A_{jk}

$$S_{11}^h = \begin{bmatrix} A_{00} & A_{01} & \cdots & A_{0p} \\ \vdots & & & \vdots \\ A_{p0} & \cdots & \cdots & A_{pp} \end{bmatrix} \tag{I.38}$$

where p is the dimension of the matrices B_i.

Of course we could have defined similarly an operator S_{11}^v the elements of which are linear combinations of the matrices B; S_{11}^h and S_{11}^v are necessarily correlated.

With this representation, the elements of S_{12}^h are obtained as:

$$[S_{12}^h]_{j\ell} = \sum_k A_{jk} \otimes A_{k\ell} \tag{I.39}$$

which means that we combine the A'_S like in a usual matrix product but multiplications of elements A are tensorial products.

The transfer matrix

$$T_N = \text{Trace } S_{1N}^h = \sum_p [S_{1N}^h]_{pp} \tag{I.40}$$

To illustrate this step, we give the example of the Ising model. The plaquette operator for the Ising model can be obtained under the form

$$S_{11} = \begin{pmatrix} \text{ch}\lambda & 0 \\ 0 & \text{sh}\lambda \end{pmatrix} \cdot \begin{pmatrix} \text{ch}\mu & 0 \\ 0 & \text{sh}\mu \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{\text{sh}\lambda\text{ch}\lambda} \\ \sqrt{\text{sh}\lambda\text{ch}\lambda} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \sqrt{\text{sh}\mu\text{ch}\mu} \\ \sqrt{\text{sh}\mu\text{ch}\mu} & 0 \end{pmatrix} \tag{I.41}$$

Clearly

$$T_1 \uparrow = e^{\mu} \begin{pmatrix} \text{ch}\lambda & 0 \\ 0 & \text{sh}\lambda \end{pmatrix} \tag{I.42}$$

and

$$Z_{1M} = e^{M\mu} \left(\text{ch}^M \lambda + \text{sh}^M \lambda \right) \tag{I.43}$$

Similarly

$$T_2 \uparrow = \text{ch}2\mu \begin{pmatrix} \text{ch}\lambda & 0 \\ 0 & \text{sh}\lambda \end{pmatrix}^{\otimes 2} + \text{sh}2\mu \begin{pmatrix} 0 & \sqrt{\text{sh}\lambda\text{ch}\lambda} \\ \sqrt{\text{sh}\lambda\text{ch}\lambda} & 0 \end{pmatrix}^{\otimes 2} \tag{I.44}$$

etc...

Now

$$S_{11}^{h} = \begin{bmatrix} \mathrm{ch}\mu\,(\mathrm{ch}\lambda P_0 + \mathrm{sh}\lambda P_1) & \sqrt{\mathrm{sh}\mu\mathrm{ch}\mu}\sqrt{\mathrm{sh}\lambda\mathrm{ch}\lambda}S \\ \sqrt{\mathrm{sh}\mu\mathrm{ch}\mu}\sqrt{\mathrm{sh}\lambda\mathrm{ch}\lambda}S & \mathrm{sh}\mu\,(\mathrm{ch}\lambda P_0 + \mathrm{sh}\lambda P_1) \end{bmatrix} \tag{I.45}$$

where the well known matrices P_0, P_1 and S are given in (II.3,5). Then,

$$S_{12}^{h} = \begin{bmatrix} \mathrm{ch}^2\mu\,(\mathrm{ch}\lambda P_0 + \mathrm{sh}\lambda P_1)^{\otimes 2} + x\, S\otimes S & \sqrt{x}\begin{pmatrix} \mathrm{ch}\mu\,(\mathrm{ch}\lambda P_0 + \mathrm{sh}\lambda P_1)\otimes S \\ +\mathrm{sh}\mu S\otimes(\mathrm{ch}\lambda P_0 + \mathrm{sh}\lambda P_1) \end{pmatrix} \\ \sqrt{x}\begin{pmatrix} \mathrm{ch}\mu S\otimes(\mathrm{ch}\lambda P_0 + \mathrm{sh}\lambda P_1) \\ +\mathrm{sh}\mu\,(\mathrm{ch}\lambda P_0 + \mathrm{sh}\lambda P_1)\otimes S \end{pmatrix} & \mathrm{sh}^2\mu\,(\mathrm{ch}\lambda P_0 + \mathrm{sh}\lambda P_1)^{\otimes 2} + x\, S\otimes S \end{bmatrix} \tag{I.46}$$

where $x = \mathrm{sh}\mu\mathrm{ch}\mu\mathrm{sh}\lambda\mathrm{ch}\lambda$. The traces of S_{11}^{h} and of S_{12}^{h} give back $T_1\uparrow$ and $T_2\uparrow$.
The end of this publication is devoted to two by two matrices. We postpone to a later time the general structure of $n\times n$ matrices. A plaquette operator built with two by two matrices depends upon 16 parameters. We shall restrict ourselves to those plaquette operators which have the property of parity conservation.

The property of parity conservation is the following: if we look at (I.41), we may write

$$S_{11} = \sum_{pkk'} \sqrt{\chi_{k+p}(\lambda)\chi_k(\lambda)}\sqrt{\chi_{k'-p}(\mu)\chi_{k'}(\mu)}P_k S^p . P_{k'} S^p \tag{I.47}$$

with

$$\chi_0(\lambda) = \mathrm{ch}\lambda \tag{I.48}$$

$$\chi_1(\lambda) = \mathrm{sh}\lambda \tag{I.49}$$

If we calculate the matrix transfer $T_N\uparrow$, due to the trace property we have

$$\mathrm{Trace}\left\{P_{k_1'} S^{p_1} P_{k_2'} S^{p_2} \ldots P_{k_N'} S^{p_N}\right\} = 0 \quad \text{if} \quad p_1 + p_2 + \ldots p_N \text{ is odd} \tag{I.50}$$

Then $T_N\uparrow$ is a sum of terms of the type

$$P_{k_1} S^{p_1} \otimes P_{k_2} S^{p_2} \otimes \ldots \otimes P_{k_N} S^{p_N} \tag{I.51}$$

with $p_1 + p_2 + \ldots + p_N$ even.
Consequently, if

$$\sigma = P_0 - P_1 \tag{I.52}$$

then,

$$\left[\underbrace{\sigma\otimes\sigma\otimes\ldots\otimes\sigma}_{N}, T_N\uparrow\right] = 0 \tag{I.53}$$

and the orthogonal projectors

$$\Pi_{\pm} = \frac{1}{2}[I\otimes I\otimes I \ldots \otimes I \pm \sigma\otimes\sigma\otimes\sigma\ldots\otimes\sigma] \tag{I.54}$$

define two orthogonal sectors conserved by $T_N\uparrow$. This property is generalizable to $n\times n$ matrices with cyclic interaction between nearest neighbour.
We shall restrict our investigation to the subclass of plaquette operators which conserves parity

$$S_{11} = \sum_{pkk'=0}^{1} a_{pkk'}\, P_k S^p . P_{k'} S^p \tag{I.55}$$

and this reduces to eight the number of free parameters. Such a plaquette operator can be written as

$$\begin{aligned} S_{11} = {} & \alpha P_0.P_0 + \varepsilon' P_1.P_0 + \varepsilon P_0.P_1 + \alpha' P_1.P_1 \\ & + \beta P_0 S.P_0 S + \phi' P_1 S.P_0 S + \phi P_0 S.P_1 S + \beta' P_1 S.P_1 S \end{aligned} \tag{I.56}$$

Vertical and horizontal developments of the plaquettes and corresponding transfer matrices are related by the exchange

$$\begin{aligned} \varepsilon &\longleftrightarrow \varepsilon' \\ \phi &\longleftrightarrow \phi' \end{aligned} \tag{I.57}$$

This plaquette operator can also be written as

$$S_{11}^{h} = \begin{bmatrix} \alpha P_0 + \varepsilon' P_1 & \beta P_0 S + \phi' P_1 S \\ \phi P_0 S + \beta' P_1 S & \varepsilon P_0 + \alpha' P_1 \end{bmatrix} \tag{I.58}$$

We now state our result:

If we impose to the eight parameters the following condition

$$\alpha\alpha' + \varepsilon\varepsilon' = \phi\phi' + \beta\beta' \tag{I.59}$$

then,

i) the even and odd parity transfer matrices separately are tensorial products of 2×2 matrices $\forall N$ (or M)

ii) the corresponding eigenvalues have a characteristic factorization property[1,2]

iii) the partition function is the sum of four quantities (due to the mixing of parity in the horizontal and vertical directions). Each quantity is the product of elementary expressions over $N.M$ plaquettes, in Fourier space.

Namely, we get from the plaquette operator S_{11}^{h} in (I.58) the partition function

$$Z_{NM} = Z_{11} + Z_{10} + Z_{01} - Z_{00} \tag{I.60}$$

with

$$Z_{\delta_1\delta_2} = \prod_{S=0}^{N-1}\prod_{S'=0}^{M-1}\begin{bmatrix} \alpha^2+\alpha'^2+\varepsilon^2+\varepsilon'^2-2(\alpha\varepsilon-\alpha'\varepsilon')\cos\left[(2S+\delta_1)\frac{\pi}{N}\right] \\ -2(\alpha\varepsilon'-\varepsilon\alpha')\cos\left[(2S'+\delta_2)\frac{\pi}{M}\right] \\ -2(\alpha\alpha'-\varepsilon\varepsilon')\cos\left[(2S+\delta_1)\frac{\pi}{N}\right]\cos\left[(2S'+\delta_2)\frac{\pi}{M}\right] \\ -2(\phi\phi'-\beta\beta')\sin\left[(2S+\delta_1)\frac{\pi}{N}\right]\sin\left[(2S'+\delta_2)\frac{\pi}{M}\right] \end{bmatrix}^{1/2} \qquad (I.61)$$

This result is completely symmetric in the exchange $\varepsilon \longleftrightarrow \varepsilon'$, $\phi \longleftrightarrow \phi'$, $N \longleftrightarrow M$, $\delta_1 \longleftrightarrow \delta_2$ and $S \longleftrightarrow S'$ so that Z_{11}, Z_{00} are self symmetric while $Z_{10} \longleftrightarrow Z_{01}$.

The simplicity of S^h_{11} and Z_{NM} hides a very cumbersome demonstration which we perform in Section III with four models of increasing difficulty. This fact means that the mathematics of the two-dimensional matrix formalism is far from understood. Also the condition (I.59) remains obscure for the present time.

In the Ising model, the condition (I.59) is automatically satisfied since

$$\alpha\alpha' = \varepsilon\varepsilon' = \beta\beta' = \phi\phi' = \mathrm{sh}\mu\mathrm{ch}\mu\mathrm{sh}\lambda\mathrm{ch}\lambda \qquad (I.62)$$

Also, a simplification occur in $Z_{\delta_1\delta_2}$ since the products of vertical-horizontal cosines and sines disappear to give the well known result[3] obtained from the statistic of dimers.

We wish finally to indicate that the thermodynamic limit of the functions $Z_{\delta_1\delta_2}$ are easily obtained as

$$\lim_{N,M\longrightarrow\infty} Z_{\delta_1\delta_2} = \exp\frac{NM}{8\pi^2}\int_0^{2\pi}\mathrm{d}x\int_0^{2\pi}\mathrm{d}y\ \mathrm{Log}\ z(x,y) \qquad (I.63)$$

with

$$\begin{aligned} z(x,y) = {} & \alpha^2+\alpha'^2+\varepsilon^2+\varepsilon'^2-2(\alpha\varepsilon-\alpha'\varepsilon')\cos x-2(\alpha\varepsilon'-\alpha'\varepsilon)\cos y \\ & -2(\alpha\alpha'-\varepsilon\varepsilon')\cos x\cos y-2(\phi\phi'-\beta\beta')\sin x\sin y \end{aligned} \qquad (I.64)$$

II. THE STATES

a) Definition of the states on one site

We calculate the transfer matrices $T_1 \uparrow$ and $\vec{T}_1$ and we choose the horizontal and vertical basis such that $T_1 \uparrow$ and $\vec{T}_1$ are diagonal matrices. Then, two by two matrices are defined in a two dimensional vector space. We define conventionally the vacuum state as

$$\begin{pmatrix}1\\0\end{pmatrix} = \mid \Omega > \qquad (II.1)$$

and the excited state as

$$\begin{pmatrix}0\\1\end{pmatrix} = \mid \phi > \qquad (II.2)$$

The projectors

$$\begin{aligned} P_0 &= \frac{1}{2}(I+\sigma) \qquad & I &= \begin{pmatrix}1&0\\0&1\end{pmatrix} \\ P_1 &= \frac{1}{2}(I-\sigma) \qquad & \sigma &= \begin{pmatrix}1&0\\0&-1\end{pmatrix} \end{aligned} \qquad (II.3)$$

are defined in such a way that

$$\begin{cases} P_0 \mid \Omega > &= \mid \Omega > \\ P_1 \mid \phi > &= \mid \phi > \\ P_0 \mid \phi > &= P_1 \mid \Omega > = 0 \end{cases} \qquad (II.4)$$

Moreover the Pauli matrix

$$S = \begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad (II.5)$$

is such that

$$\begin{cases} S \mid \Omega > &= \mid \phi > \\ S \mid \phi > &= \mid \Omega > \end{cases} \qquad (II.6)$$

b) The states in position space

We now define a 2^N dimensional vector space corresponding to all possible states over N sites $0, 1, ..., N-1$. The vacuum is a direct product of each vacuum

$$\mid \Omega >_N = \mid \Omega > \otimes \mid \Omega > \otimes ... \otimes \mid \Omega > \qquad (II.7)$$

For simplicity we also write $\mid \Omega >_N$ as $\mid \Omega >$ without any ambiguity in the understanding. A state corresponding to excited states in the sites $k_1, k_2, ..., k_p$ is denoted by

$$\left|\phi_{k_1,k_2,...,k_p}\right\rangle = \mid \Omega > \otimes ... \otimes \underset{\substack{\uparrow \\ k_1}}{\mid \phi >} \otimes \mid \Omega > \otimes ... \otimes \underset{\substack{\uparrow \\ k_2}}{\mid \phi >} \otimes ... \otimes \underset{\substack{\uparrow \\ k_p}}{\mid \phi >} \otimes \mid \Omega > \otimes ... \qquad (II.8)$$

where all ... on the right hand side means a vacuum state $\mid \Omega >$ except in $k_3, k_4, ..., k_{p-1}$ which have an excited state $\mid \phi >$. It is important to note that our notation implies necessarily $\left|\phi_{k_1<k_2<k_3<...<k_p}\right\rangle$ with $k_1 \geq 0$ and $k_p \leq N-1$. We denote by Π^+ and Π^- the subspaces or sectors of even and odd number of excited states.

By definition, the cyclic permutation operator C_N is defined as

$$\begin{cases} C_N|\Omega\rangle = |\Omega\rangle \\ C_N\left|\phi_{k_1,k_2,...,k_p}\right\rangle = \left|\phi_{k_1-1,k_2-1,...,k_p-1}\right\rangle \quad \text{for} \quad k_1 > 0 \\ C_N\left|\phi_{0,k_2,...,k_p}\right\rangle = \left|\phi_{k_2-1,k_3-1,...,k_p-1,N-1}\right\rangle \end{cases} \qquad (II.9)$$

Of course

$$C_N^N = I \qquad (II.10)$$

so that the eigenvalues of C_N are $q^k = e^{\frac{2ik\pi}{N}}$; $k = 0, ..., N-1$. Since the vector space is of dimension 2^N, the degeneracy of the eigenvectors of C_N is enormous. We show in Section III-B that these eigenvalues can be constructed in Π^+ and Π^- respectively, by factorization of N values since C_N, in each sector, is a tensorial product of 2×2 matrices.

Since the degeneracy is so large, the eigenvectors of C_N are of little help to diagonalize the transfer matrices T_N; however eigenvectors of T_N are necessarily linear combinations of eigenvectors of C_N with the same eigenvalue. Let us organize somewhat the eigenvectors of C_N.

c) The Fourier states

By Fourier states, we mean a Fourier expansion of the states over the N sites. Let us consider the state $|\phi_{0,k_2,...,k_p}\rangle$; then, we define

$$|\mathcal{F}_{s;k_2,...,k_p}\rangle = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} q^{-st} C_N^t \, |\phi_{0,k_2,...,k_p}\rangle, \quad s = 0, ..., N-1. \qquad (II.11)$$

The Fourier expansion conserve the number of excited states. Clearly

$$C_N \, |\mathcal{F}_{s;k_2,...,k_p}\rangle = q^s \, |\mathcal{F}_{s;k_2,...,k_p}\rangle \qquad (II.12)$$

and the Fourier states are eigenvectors of C_N. By inverse Fourier transform we have

$$|\phi_{k_1,k_2,...,k_p}\rangle = \frac{1}{\sqrt{N}} \sum_{s=0}^{N-1} q^{-sk_1} \, |\mathcal{F}_{s;k_2-k_1,...,k_p-k_1}\rangle \qquad (II.13)$$

Note that our notation (II.11) for Fourier states might be misleading since different notations can represent the same Fourier state.

d) The Jordan-Wigner states

The Fourier states are eigenvectors of C_N. However, for a given eigenvalue s, the eigensubspace constructed from the Fourier states is very large. For instance, as soon as $N = 4$, in the even sector, for $s = 0$, we have two Fourier states

$$\begin{aligned} |\mathcal{F}_{0;1}\rangle &= \frac{1}{2} |\phi_{01} + \phi_{12} + \phi_{23} + \phi_{03}\rangle \\ |\mathcal{F}_{0;2}\rangle &= |\phi_{02} + \phi_{13}\rangle \end{aligned} \qquad (II.14)$$

(Note that $|\mathcal{F}_{0;3}\rangle = |\mathcal{F}_{0;1}\rangle$).

We wish to organize further this large degeneracy and we use the fact, proved in Section III-B, that C_N in each parity sector is a tensorial product of 2×2 matrices.

For a matrix, to be a tensorial product is an intrinsec property, but this property appears clearly in a certain subclass of basis. The Fourier states does not define such a privileged basis; the Jordan-Wigner states does. It uses creation and annihilation operators.

The excited state is defined as

$$|\phi\rangle = c^+ |\Omega\rangle \qquad (II.15)$$

and because in each site there is only one excited state, we impose

$$(c^+)^2 = 0. \qquad (II.16)$$

Also, we wish to have an annihilation operator

$$\begin{cases} c|\phi\rangle &= |\Omega\rangle \\ c|\Omega\rangle &= 0 \end{cases} \qquad (II.17)$$

Such a realization can be achieved by defining

$$c^+ = P_1 S \qquad (II.18)$$

The conditions (II.16) and (II.18) show clearly the fermionic character of the creation and annihilation operators since we have

$$cc^+ + c^+c = I \qquad (II.19)$$

The generalization of this construction to several sites has been achieved by the so-called Jordan-Wigner construction[4]. We refer the reader to the corresponding references[2,4] and simply specify the relations needed for our purpose. There exists a creation operator in each site satisfying

$$c_i^+ c_j^+ = -c_j^+ c_i^+ \quad \forall i, j \qquad (II.20)$$

An excited state is defined as

$$|\phi_{k_1,k_2,...,k_p}\rangle = \prod_{i=2}^{p} \theta(k_i - k_{i-1}) \, c_{k_1}^+ c_{k_2}^+ ... c_{k_p}^+ |\Omega\rangle \qquad (II.21)$$

The action of the operator C_N is

$$\begin{cases} C_N c_k^+ C_N^+ &= c_{k-1}^+ . (I \otimes I \otimes ... \otimes I \otimes \sigma) \quad \text{for } k > 0 \\ C_N c_0^+ C_N^+ &= c_{N-1}^+ . (\sigma \otimes \sigma \otimes ... \otimes \sigma \otimes I) \end{cases} \qquad (II.22)$$

so that

$$\begin{cases} C_N c^+_{k_1}...c^+_{k_p} C^+_N & = c^+_{k_1-1}...c^+_{k_p-1}\,(I\otimes I\otimes ...\otimes I\otimes \sigma^p) \quad \text{if all } k'_{i_s}\neq 0 \\ C_N c^+_{k_1}...c^+_0...c^+_{k_p} C^+_N & = (-)^{p-1} c^+_{k_1-1}...c^+_{N-1}...c^+_{k_p-1}\,(\sigma\otimes\sigma\otimes ...\otimes\sigma\otimes\sigma^{p-1}) \end{cases} \qquad (II.23)$$

Equations (II.23) are compatible with (II.9), (II.21) since

$$C_N c^+_0 c^+_{k_2}...c^+_{k_p} C^+_N = c^+_{k_2-1}...c^+_{k_p-1} c^+_{N-1}\,(\sigma\otimes\sigma\otimes ...\otimes\sigma\otimes\sigma^{p-1}) \qquad (II.24)$$

In order to take care of the factor $(-)^{p-1}$ in (II.23), we introduce the Fourier transform of the creation operators as

$$d^+_s = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} q'^{(2s+\delta)k} c^+_k \qquad (II.25)$$

where $q' = e^{i\pi/N}$ and $\delta = 0$ in the odd parity subspace while $\delta = 1$ in the even parity subspace. The factor δ takes care of the periodic boundary conditions so that the Jordan-Wigner states defined below (II.27) are eigenstates of C_N. Consequently,

$$\left|\phi_{k_1,k_2,...,k_p}\right\rangle = \prod_{i=2}^{p}\theta(k_i-k_{i-1})\frac{1}{N^{p/2}}\sum_{s_1...s_p}^{N-1}\prod_{i=1}^{p}\left[q'^{-(2s_i+\delta)k_i} d^+_{s_i}\right]|\Omega\rangle \qquad (II.26)$$

and conversely, the Jordan-Wigner states are defined as

$$d^+_{s_1}...d^+_{s_p}|\Omega\rangle = \frac{1}{N^{p/2}}\sum_{k_1...k_p}\mathcal{A}\left\{\prod_{i=1}^{p} q'^{(2s_i+\delta)k_i}\right\}\left|\phi_{k_1...k_p}\right\rangle \qquad (II.27)$$

where by definition $\left|\phi_{k_1,...,k_p}\right\rangle$ is non zero provided $k_1 < k_2 < ... < k_p$ and where $\mathcal{A}$ is the complete antisymmetrization operator: for instance,

$$\begin{aligned}\mathcal{A}\, f(k_1,k_2,k_3) &= f(k_1,k_2,k_3)+f(k_2,k_3,k_1)+f(k_3,k_1,k_2)\\ &\quad -f(k_3,k_2,k_1)-f(k_2,k_1,k_3)-f(k_1,k_3,k_2)\end{aligned} \qquad (II.28)$$

$\mathcal{A}$ must be introduced to get rid of the θ'_s in (II.26). We use in Section III-B the following example with $N = 4$

$$\begin{cases} d^+_0 d^+_3|\Omega\rangle &= \dfrac{-i}{2\sqrt{2}}\left|\phi_{01}+\phi_{12}+\phi_{23}+\phi_{03}\right\rangle - \dfrac{i}{2}\left|\phi_{02}+\phi_{13}\right\rangle \\ d^+_1 d^+_2|\Omega\rangle &= \dfrac{-i}{2\sqrt{2}}\left|\phi_{01}+\phi_{12}+\phi_{23}+\phi_{03}\right\rangle + \dfrac{i}{2}\left|\phi_{02}+\phi_{13}\right\rangle \end{cases} \qquad (II.29)$$

From (II.25) and (II.23) we get

$$C_N d^+_{s_1}...d^+_{s_p}|\Omega\rangle = q'^{\sum_{i=1}^{p}(2s_i+\delta)} d^+_{s_1}...d^+_{s_p}|\Omega\rangle \qquad (II.30)$$

where we have used the property

$$q'^{-N\delta}(-)^{p-1} = 1 \quad \text{for} \quad \begin{cases}\delta = 0 & \text{if} \quad p \text{ odd}\\ \delta = 1 & \text{if} \quad p \text{ even}\end{cases} \qquad (II.31)$$

We may define the Jordan-Wigner space as the space of states

$$\left|\tilde{\phi}_{s_1 s_2...s_p}\right\rangle = \prod_{i=2}^{p}\theta(s_i-s_{i-1})\, d^+_{s_1}...d^+_{s_p}|\Omega\rangle \qquad (II.32)$$

and clearly, these states are eigenstates of C_N

$$C_N\left|\tilde{\phi}_{s_1 s_2...s_p}\right\rangle = q'^{\sum_{i=1}^{p}(2s_i+\delta)}\left|\tilde{\phi}_{s_1...s_p}\right\rangle. \qquad (II.33)$$

We now come back to a tensorial product representation as

$$\left|\tilde{\phi}_{s_1...s_p}\right\rangle = |\Omega\rangle\otimes ...\otimes \underset{\substack{\uparrow\\ s_1}}{\left|\tilde{\phi}\right\rangle}\otimes|\Omega\rangle...\otimes|\Omega\rangle\otimes \underset{\substack{\uparrow\\ s_p}}{\left|\tilde{\phi}\right\rangle}\otimes|\Omega\rangle... \qquad (II.34)$$

where ... means only vacuum state $|\,\Omega>$ except in $s_2, s_3...s_{p-1}$ which have a state $\left|\tilde{\phi}\right\rangle$. In each Fourier site we may define two orthogonal projectors such that

$$\begin{aligned}\tilde{P}_0|\Omega\rangle &= |\Omega\rangle\\ \tilde{P}_1\left|\tilde{\phi}\right\rangle &= \left|\tilde{\phi}\right\rangle\end{aligned} \qquad (II.35)$$

Then

$$\left|\tilde{\phi}_{s_1...s_p}\right\rangle = \left[\tilde{P}_0\otimes ...\otimes \underset{\substack{\uparrow\\ s_1}}{\tilde{P}_1}\otimes ...\otimes\tilde{P}_0\otimes \underset{\substack{\uparrow\\ s_p}}{\tilde{P}_1}\otimes\tilde{P}_0...\right]|\Omega\rangle \qquad (II.36)$$

where $\tilde{P}_1$ operators occur in $s_1, s_2, ..., s_{p-1}, s_p$.

Clearly, the transformation from position space to Jordan-Wigner space is different in the even and in the odd sector ($\delta = 0$ or 1) so that, for each δ, half of the corresponding Jordan-Wigner space is acceptable (the other half being only useful to formulate the property of tensorial products; this is the source of the constraint upon the number of $\pm$ signs in the eigenvalues of the Ising model[2]).

e) The plaquette operator in position space

The action of the plaquette operator, as developed in Section III, mainly uses the position space (it is only with the transfer matrix that we work in Fourier or Jordan-Wigner spaces). The plaquette operator (I.58) can be decomposed into two parts

$$S_{11}^{h} = S_0 + S_1 \qquad (II.37)$$

with

$$\begin{aligned} S_0|\phi\rangle &= 0 \\ S_1|\Omega\rangle &= 0 \end{aligned} \qquad (II.38)$$

We write

$$S_0|\Omega\rangle = \begin{bmatrix} \alpha|\Omega\rangle & \phi'|\phi\rangle \\ \beta'|\phi\rangle & \varepsilon|\Omega\rangle \end{bmatrix} \qquad (II.39)$$

$$S_1|\phi\rangle = \begin{bmatrix} \varepsilon'|\phi\rangle & \beta|\Omega\rangle \\ \phi|\Omega\rangle & \alpha'|\phi\rangle \end{bmatrix} \qquad (II.40)$$

where the matrices in (II.39), (II.40) are matrices of vectors. Let us describe by the same symbol $\otimes$ the same operation for matrices of matrices and for matrices of vectors; then the operation (I.39) generates four relations

$$S_0^{\otimes 2}|\Omega\rangle = \begin{bmatrix} \alpha^2|\Omega\rangle + \beta'\phi'|\phi_{01}\rangle & \alpha\phi'|\phi_1\rangle + \varepsilon\phi'|\phi_0\rangle \\ \alpha\beta'|\phi_0\rangle + \varepsilon\beta'|\phi_1\rangle & \beta'\phi'|\phi_{01}\rangle + \varepsilon^2|\Omega\rangle \end{bmatrix} \qquad (II.41)$$

where, here $|\Omega>$ means $|\Omega\rangle \otimes |\Omega\rangle$;

$$S_0 \otimes S_1|\phi_1\rangle = \begin{bmatrix} \alpha\varepsilon'|\phi_1\rangle + \phi\phi'|\phi_0\rangle & \alpha\beta|\Omega\rangle + \alpha'\phi'|\phi_{01}\rangle \\ \beta'\varepsilon'|\phi_{01}\rangle + \varepsilon\phi|\Omega\rangle & \beta\beta'|\phi_0\rangle + \alpha'\varepsilon|\phi_1\rangle \end{bmatrix} \qquad (II.42)$$

$$S_1 \otimes S_0|\phi_0\rangle = \begin{bmatrix} \varepsilon'\alpha|\phi_0\rangle + \beta\beta'|\phi_1\rangle & \varepsilon'\phi'|\phi_{01}\rangle + \varepsilon\beta|\Omega\rangle \\ \alpha\phi|\Omega\rangle + \alpha'\beta'|\phi_{01}\rangle & \phi\phi'|\phi_1\rangle + \alpha'\varepsilon|\phi_0\rangle \end{bmatrix} \qquad (II.43)$$

$$S_1 \otimes S_1|\phi_{01}\rangle = \begin{bmatrix} \varepsilon'^2|\phi_{01}\rangle + \beta\phi|\Omega\rangle & \varepsilon'\beta|\phi_0\rangle + \beta\alpha'|\phi_1\rangle \\ \phi\varepsilon'|\phi_1\rangle + \alpha'\phi|\phi_0\rangle & \alpha'^2|\phi_{01}\rangle + \beta\phi|\Omega\rangle \end{bmatrix} \qquad (II.44)$$

The transfer matrix in position space follows as

$$\begin{aligned} T_2|\Omega\rangle &= (\alpha^2 + \varepsilon^2)|\Omega\rangle + 2\beta'\phi'|\phi_{01}\rangle \\ T_2|\phi_1\rangle &= (\alpha\varepsilon' + \alpha'\varepsilon)|\phi_1\rangle + (\beta\beta' + \phi\phi')|\phi_0\rangle \\ T_2|\phi_0\rangle &= (\alpha\varepsilon' + \alpha'\varepsilon)|\phi_0\rangle + (\beta\beta' + \phi\phi')|\phi_1\rangle \\ T_2|\phi_{01}\rangle &= (\alpha'^2 + \varepsilon'^2)|\phi_{01}\rangle + 2\beta\phi|\Omega\rangle \end{aligned} \qquad (II.45)$$

In Section III, an intensive use of this construction is constantly performed.

III-A. MODELS OF TYPE I

These models are defined from the following plaquette operator

$$S_{11}^{h} = \begin{bmatrix} \alpha P_0 + \varepsilon' P_1 & 0 \\ 0 & \varepsilon P_0 + \alpha' P_1 \end{bmatrix} \qquad (III-A.1)$$

These models belong to the class of the commuting matrices described in Section I (I.19-33). Clearly

$$S_{1N}^{h} = \begin{bmatrix} (\alpha P_0 + \varepsilon' P_1)^{\otimes N} & 0 \\ 0 & (\varepsilon P_0 + \alpha' P_1)^{\otimes N} \end{bmatrix} \qquad (III-A.2)$$

and the vertical transfer matrix is

$$T_N \uparrow = (\alpha P_0 + \varepsilon' P_1)^{\otimes N} + (\varepsilon P_0 + \alpha' P_1)^{\otimes N} \qquad (III-A.3)$$

This matrix is diagonal in the position space defined in (II.8) and the eigenvalues are of the type

$$\alpha^k \varepsilon'^{N-k} + \varepsilon^k \alpha'^{N-k}, \qquad k = 0, ..., N \qquad (III-A.4)$$

with the degeneracy $\frac{N!}{k!(N-k)!}$. The partition function is

$$Z_{NM} = \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \left(\alpha^k \varepsilon'^{N-k} + \varepsilon^k \alpha'^{N-k}\right)^M$$

$$Z_{NM} = \sum_{kk'=0}^{N} \frac{N!}{k!(N-k)!} \frac{M!}{k'!(M-k')!} \alpha^{kk'} \varepsilon'^{(N-k)k'} \varepsilon^{k(M-k')} \alpha'^{(N-k)(M-k')} \qquad (III-A.5)$$

This result is the simplest illustration of (I.33).
From

$$T_N \uparrow |\phi_{k_1,k_2,...,k_p}\rangle = (\alpha^{N-p}\varepsilon'^p + \varepsilon^{N-p}\alpha'^p)|\phi_{k_1,k_2,...,k_p}\rangle \qquad (III-A.6)$$

we may use the degeneracy to change the basis into Fourier states

$$T_N \uparrow |\mathcal{F}_{s;k_2,...,k_p}\rangle = (\alpha^{N-p}\varepsilon'^p + \varepsilon^{N-p}\alpha'^p)|\mathcal{F}_{s;k_2,...,k_p}\rangle \qquad (III-A.7)$$

Moreover, the eigenvalues being independent of $k_1, k_2, ..., k_p$, we may also consider the Jordan-Wigner basis and use transformation (II.25) to write

$$T_N \uparrow d_{s_1}^{+} ... d_{s_p}^{+}|\Omega\rangle = (\alpha^{N-p}\varepsilon'^p + \varepsilon^{N-p}\alpha'^p)\, d_{s_1}^{+} ... d_{s_p}^{+}|\Omega\rangle \qquad (III-A.8)$$

To understand the condition (I.59) at the simplest level, we now impose

$$\alpha\alpha' + \varepsilon\varepsilon' = 0 \qquad (III - A.9)$$

The eigenvalues become

$$\left[\alpha^N - (-)^\delta \varepsilon^N\right] \left(\frac{\varepsilon'}{\alpha}\right)^P \qquad (III - A.10)$$

where $\delta = 1$ for p even (even sector) and $\delta = 0$ for p odd (odd sector). If we define in Jordan-Wigner space the orthogonal projectors (II.35)

$$\begin{aligned} \tilde{P}_0|\Omega\rangle &= |\Omega\rangle \\ \tilde{P}_1 d_s^+|\Omega\rangle &= d_s^+|\Omega\rangle \end{aligned} \qquad (III - A.11)$$

and if we use the well known decomposition

$$\alpha^N - (-)^\delta \varepsilon^N = \prod_{s=0}^{N-1} \left(\alpha - q'^{(2s+\delta)}\varepsilon\right) \qquad (III - A.12)$$

where

$$q' = e^{i\pi/N} \qquad (III - A.13)$$

then, we may write the even and odd transfer matrix separately as

$$T_N(\delta) \uparrow = \bigotimes_{s=0}^{N-1} \left(\alpha - q'^{(2s+\delta)}\varepsilon\right) \left[\tilde{P}_0 + \frac{\varepsilon'}{\alpha}\tilde{P}_1\right] \qquad (III - A.14)$$

or equivalently

$$T_N(\delta) \uparrow = \bigotimes_{s=0}^{N-1} \left[\left(\alpha - q'^{(2s+\delta)}\varepsilon\right)\tilde{P}_0 + \left(\varepsilon' + q'^{(2s+\delta)}\alpha'\right)\tilde{P}_1\right] \qquad (III - A.15)$$

It is that structure which is generalized throughout models of type (III-B-C-D).

We note that in this model, the property of factorization is already valid in position space provided that the condition (III-A.9) is true:

$$T_N(\delta) \uparrow = \bigotimes_{s=0}^{N-1} \left[\left(\alpha - q'^{(2s+\delta)}\varepsilon\right)P_0 + \left(\varepsilon' + q'^{(2s+\delta)}\alpha'\right)P_1\right] \qquad (III - A.16)$$

III-B. MODELS OF TYPE II

These models are defined from the following plaquette operator S

$$S_{11}^h = \begin{bmatrix} \alpha P_0 & \phi' P_1 S \\ \phi P_0 S & \alpha' P_1 \end{bmatrix} \qquad (III - B.1)$$

The cyclic permutation operators C_N defined in (II.10) are the transfer matrices generated from the above plaquette operator in the special case where $\alpha = \alpha' = \phi = \phi' = 1$.
We now show that, under the condition

$$\phi\phi' = \alpha\alpha', \qquad (III - B.2)$$

the even and odd transfer matrices $T_N(\delta)$ generated from S are both tensorial products; this property is explicitly obtained in Jordan-Wigner states.

We introduce the operators S_0 and S_1 defined in (II.39-40)

$$S_{11}^h = S_0 + S_1 \qquad (III - B.3)$$

with

$$S_0|\phi\rangle = S_1|\Omega\rangle = 0 \qquad (III - B.4)$$

Here,

$$S_0|\Omega\rangle = \begin{bmatrix} \alpha|\Omega\rangle & \phi'|\phi\rangle \\ 0 & 0 \end{bmatrix} \qquad (III - B.5a)$$

$$S_1|\phi\rangle = \begin{bmatrix} 0 & 0 \\ \phi|\Omega\rangle & \alpha'|\phi\rangle \end{bmatrix} \qquad (III - B.5b)$$

From (III-B.4-5) we get

$$S_0^{\otimes k}|\Omega\rangle = \alpha^{k-1} \begin{bmatrix} \alpha|\Omega\rangle & \theta_k\phi'|\phi_{k-1}\rangle \\ 0 & \delta_k\alpha|\Omega\rangle \end{bmatrix} \qquad (III - B.6)$$

with the conventions

$$\begin{cases} \delta_k = 0 \quad \text{for} \quad k \neq 0, \ \delta_0 = 1 \\ \delta_k + \theta_k = 1 \quad \forall k \geq 0 \\ S_0^{\otimes 0} = \begin{bmatrix} |\Omega\rangle & 0 \\ 0 & |\Omega\rangle \end{bmatrix} \end{cases} \qquad (III - B.7)$$

We get successively

$$S_0^{\otimes k} \otimes S_1 |\phi_k\rangle = \alpha^{k-1} \begin{bmatrix} \theta_k\phi\phi'|\phi_{k-1}\rangle & \theta_k\alpha'\phi'|\phi_{k-1,k}\rangle \\ \delta_k\alpha\phi|\Omega\rangle & \delta_k\alpha\alpha'|\phi_0\rangle \end{bmatrix} \qquad (III - B.8)$$

$$S_1 \otimes S_0^{\otimes k} |\phi_0\rangle = \alpha^{k-1} \begin{bmatrix} 0 & 0 \\ \alpha\phi|\Omega\rangle & \theta_k\phi\phi'|\phi_k\rangle + \delta_k\alpha\alpha'|\phi_0\rangle \end{bmatrix} \qquad (III - B.9)$$

$$S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} |\phi_{k_2}\rangle = \alpha^{k_1+k_2-2} \begin{bmatrix} \theta_{k_2}\alpha\phi\phi' |\phi_{k_2-1}\rangle & \theta_{k_1}\theta_{k_2}\phi\phi'^2 |\phi_{k_2-1,k_2+k_1}\rangle \\ & +\delta_{k_1}\theta_{k_2}\alpha\alpha'\phi' |\phi_{k_2-1,k_2}\rangle \\ \delta_{k_2}\alpha^2\phi|\Omega\rangle & \delta_{k_2}\theta_{k_1}\alpha\phi\phi' |\phi_{k_1}\rangle \\ & +\delta_{k_2}\delta_{k_1}\alpha^2\alpha' |\phi_0\rangle \end{bmatrix} \qquad (III-B.10)$$

If we calculate the transfer matrix at this level by setting $k_1 + k_2 + 1 = N$, we get

$$T_N \uparrow |\phi_k\rangle = \theta_{N-1}\alpha^{N-2}\phi\phi' C_N |\phi_k\rangle + \delta_{N-1}\alpha' |\phi_0\rangle \qquad (III-B.11)$$

which shows that the corresponding Fourier (or Jordan-Wigner) eigenstate of one excited state has for eigenvalue $\alpha^{N-2}\phi\phi'$ for $N > 1$ and α' for $N = 1$. This discrepancy disappear if condition (III-B.2) is satisfied. We now show, on the example of $T_4 \uparrow$, that this discrepancy propagates and that although Fourier states are always eigenstates of the system, condition (III-B.2) makes Jordan-Wigner states to be eigenstates and makes the evensector of $T_4 \uparrow$ to be a tensorial product of two by two matrices.
If we calculate

$$S_0^{\otimes k_3} \otimes S_1 \otimes S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} |\phi_{k_3,k_3+k_2+1}\rangle = \alpha^{k_1+k_2+k_3-3} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (III-B.12)$$

with

$$\begin{cases} A &= \theta_{k_3}\theta_{k_2}\alpha(\phi\phi')^2 |\phi_{k_3-1,k_3+k_2}\rangle + \theta_{k_3}\delta_{k_2}\alpha^2\alpha'\phi\phi' |\phi_{k_3-1,k_3}\rangle \\ D &= \delta_{k_3} \begin{bmatrix} \theta_{k_1}\theta_{k_2}\alpha(\phi\phi')^2 |\phi_{k_2,k_2+k_1+1}\rangle + \delta_{k_1}\theta_{k_2}\alpha^2\alpha'\phi\phi' |\phi_{k_2,k_2+1}\rangle \\ +\delta_{k_2}\theta_{k_1}\alpha^2\alpha'\phi\phi' |\phi_{0,k_1+1}\rangle + \delta_{k_2}\delta_{k_1}\alpha^3\alpha'^2 |\phi_{01}\rangle \end{bmatrix} \end{cases} \qquad (III-B.13)$$

The transfer matrix reads

$$T_N \uparrow |\phi_{k_3,k_3+k_2+1}\rangle = \theta_{N-2}\alpha^{N-4} \left[\theta_{k_2}\theta_{k_3+k_1}(\phi\phi')^2 + (\delta_{k_2}\theta_{k_3+k_1} + \theta_{k_2}\delta_{k_3+k_1})\alpha\alpha'\phi\phi'\right] . \\ .C_N |\phi_{k_3,k_3+k_2+1}\rangle + \delta_{N-2}\alpha'^2 |\phi_{01}\rangle \qquad (III-B.14)$$

If we apply (III-B.14) to $T_4 \uparrow$ we get

$$\begin{cases} T_4 \uparrow |\phi_{k,k+1}\rangle &= \alpha\alpha'\phi\phi' |\phi_{k-1,k}\rangle \qquad \text{for } k = 1,2 \\ T_4 \uparrow |\phi_{01}\rangle &= \alpha\alpha'\phi\phi' |\phi_{03}\rangle \\ T_4 \uparrow |\phi_{03}\rangle &= \alpha\alpha'\phi\phi' |\phi_{23}\rangle \end{cases} \qquad (III-B.15)$$

but

$$\begin{cases} T_4 \uparrow |\phi_{02}\rangle &= (\phi\phi')^2 |\phi_{13}\rangle \\ T_4 \uparrow |\phi_{13}\rangle &= (\phi\phi')^2 |\phi_{02}\rangle \end{cases} \qquad (III-B.16)$$

Then, the Fourier states $|\mathcal{F}_{s;1}\rangle = \frac{1}{2}\left|\phi_{01} + i^s\phi_{12} + i^{2s}\phi_{23} + i^{3s}\phi_{03}\right\rangle$ are eigenstates of $T_4 \uparrow$ with eigenvalue $i^s\alpha\alpha'\phi\phi'$ while the Fourier states $|\mathcal{F}_{\pm;2}\rangle = |\phi_{02} \pm \phi_{13}\rangle$ are eigenstates

of $T_4 \uparrow$ with eigenvalues $\pm(\phi\phi')^2$. Clearly, the Jordan-Wigner states (II.29) cannot be eigenstates of $T_4 \uparrow$ unless $\alpha\alpha' = \phi\phi'$. In that case, taking into account the fact that

$$\begin{aligned} T_4 \uparrow |\Omega\rangle &= \alpha^4|\Omega\rangle \\ T_4 \uparrow d_0^+ d_1^+ d_2^+ d_3^+ |\Omega\rangle &= \alpha'^4 d_0^+ d_1^+ d_2^+ d_3^+ |\Omega\rangle \end{aligned} \qquad (III-B.17)$$

and, since $T_4 \uparrow = (\alpha\alpha')^2 C_4$ for two particle states, we may use (II.33) and write

$$T_4 \uparrow = \bigotimes_{s=0}^{3} \left[\alpha\tilde{P}_0 + q'^{2s+1}\alpha'\tilde{P}_1\right] \qquad (III-B.18)$$

with $q' = e^{i\pi/4}$.

We now consider (III-B.2) as true and prove for all N the tensorial product structure of $T_N \uparrow$ (even or odd) in Jordan-Wigner space. Equation (III-B.10) is now rewritten as

$$S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} |\phi_{k_2}\rangle = \alpha^{k_1+k_2-1} \begin{bmatrix} \theta_{k_2}\alpha\alpha' |\phi_{k_2-1}\rangle & \theta_{k_2}\alpha'\phi' |\phi_{k_2-1,k_2+k_1}\rangle \\ \delta_{k_2}\alpha\phi|\Omega\rangle & \delta_{k_2}\alpha\alpha' |\phi_{k_1}\rangle \end{bmatrix} \qquad (III-B.19)$$

This result can now be easily generalized into

$$S_0^{\otimes k_{j+1}} \otimes S_1 \otimes S_0^{\otimes k_j} \otimes \ldots \otimes S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} \left|\phi_{k_j+1,k_{j+1}+k_j+1,\ldots,\sum_{i=2}^{j+1} k_i+j-1}\right\rangle \\ = \alpha^{\sum_{i=1}^{j+1} k_i - j} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (III-B.20)$$

with

$$\begin{aligned} A &= \theta_{k_{j+1}}(\alpha\alpha')^j \left|\phi_{k_{j+1}-1,k_{j+1}+k_j,\ldots,\sum_{i=2}^{j+1} k_i+j-2}\right\rangle \\ B &= \theta_{k_{j+1}}(\alpha\alpha')^{j-1}\alpha'\phi' \left|\phi_{k_{j+1}-1,k_{j+1}+k_j,\ldots,\sum_{i=1}^{j+1} k_i+j-1}\right\rangle \\ C &= \delta_{k_{j+1}}(\alpha\alpha')^{j-1}\alpha\phi \left|\phi_{k_j,k_j+k_{j-1}+1,\ldots,\sum_{i=2}^{j} k_i+j-2}\right\rangle \\ D &= \delta_{k_{j+1}}(\alpha\alpha')^j \left|\phi_{k_j,k_j+k_{j-1}+1,\ldots,\sum_{i=1}^{j} k_i+j-1}\right\rangle \end{aligned} \qquad (III-B.21)$$

Taking the trace of the matrix (III-B.20) and using

$$\sum_{i=1}^{j+1} k_i + j = N \qquad (III-B.22)$$

With the appropriate change of variables, we get

$$T_N \left|\phi_{k_1,k_2,\ldots,k_j}\right\rangle = \alpha^{N-j}\alpha'^j C_N \left|\phi_{k_1,k_2,\ldots,k_j}\right\rangle \qquad (III-B.23)$$

with C_N defined in (II.9). As in Section III-A, the eigenvalues are only j dependent; Fourier states and Jordan-Wigner states are eigenstates of T_N since from (II.33)

$$T_N d^+_{s_1} \ldots d^+_{s_p} |\Omega\rangle = \alpha^{N-p} \alpha'^p q'^{\sum_{i=1}^{p}(2s_i+\delta)} d^+_{s_1} \ldots d^+_{s_p} |\Omega\rangle \qquad (III-B.24)$$

Again, the factorization of the eigenvalues is such that the even and odd part of the transfer matrix can be written as the tensorial product of 2×2 matrices

$$T_N(\delta) = \bigotimes_{s=0}^{N-1} \left[\alpha \tilde{P}_0 + q'^{2s+\delta} \alpha' \tilde{P}_1\right] \qquad (III-B.25)$$

with $q' = e^{i\pi/N}$.

It is interesting to see that the cyclic permutation operator C_N, the eigenvalues of which are N^{th} square root of unity with a tremendously large degeneracy, can be organized into a factorized structure

$$C_N(\delta) = \bigotimes_{s=0}^{N-1} \left[\tilde{P}_0 + q'^{2s+\delta} \tilde{P}_1\right] \qquad (III-B.26)$$

III-C. MODELS OF TYPE III

These models are defined from the following plaquette operator S

$$S = \begin{bmatrix} \alpha P_0 + \varepsilon' P_1 & \phi' P_1 S \\ \phi P_0 S & \varepsilon P_0 + \alpha' P_1 \end{bmatrix} \qquad (III-C.1)$$

These models contain models of type I and II; they have the property of conserving the number of excited states. It is easy to convince oneself that the transfer matrix T_N is, by construction, a polynomial of degree $E\left(\frac{N}{2}\right)$ in $\phi\phi'$. We wish to show that, under the condition

$$\phi\phi' = \alpha\alpha' + \varepsilon\varepsilon' \qquad (III-C.2)$$

the even and odd transfer matrices $T_N^{\pm}$ generated from S are both tensorial products of 2×2 matrices; this property is explicitly obtained in Jordan-Wigner states.

Here,

$$S_0|\Omega\rangle = \begin{bmatrix} \alpha|\Omega\rangle & \phi'|\phi\rangle \\ 0 & \varepsilon|\Omega\rangle \end{bmatrix} \qquad (III-C.3)$$

$$S_1|\phi\rangle = \begin{bmatrix} \varepsilon'|\phi\rangle & 0 \\ \phi|\Omega\rangle & \alpha'|\phi\rangle \end{bmatrix} \qquad (III-C.4)$$

From (III-C.3), we get

$$S_0^{\otimes k}|\Omega\rangle = \begin{bmatrix} \alpha^k|\Omega\rangle & \phi'\left|\chi_{0k-1}\right\rangle \\ 0 & \varepsilon^k|\Omega\rangle \end{bmatrix} \qquad (III-C.5)$$

with

$$|X_{j,k+j}\rangle = \sum_{p=j}^{k+j} \alpha^{p-j} \varepsilon^{k+j-p} |\phi_p\rangle \qquad (III-C.6)$$

$$|X_{k_1 k_2}\rangle = 0 \quad \text{for} \quad k_2 < k_1 \qquad (III-C.7)$$

we get successively

$$S_0^{\otimes k} \otimes S_1 |\phi_k\rangle = \begin{bmatrix} \alpha^k \varepsilon' |\phi_k\rangle & \alpha'\phi' \left|\chi_{0k-1}\phi_k\right\rangle \\ +\phi\phi' \left|\chi_{0k-1}\right\rangle & \\ \varepsilon^k \phi|\Omega\rangle & \varepsilon^k \alpha' |\phi_k\rangle \end{bmatrix} \qquad (III-C.8)$$

$$S_1 \otimes S_0^{\otimes k} |\phi_0\rangle = \begin{bmatrix} \alpha^k \varepsilon' |\phi_0\rangle & \varepsilon'\phi' \left|\phi_0 \chi_{1k}\right\rangle \\ \alpha^k \phi|\Omega\rangle & \alpha'\varepsilon^k |\phi_0\rangle + \phi\phi' \left|\chi_{1k}\right\rangle \end{bmatrix} \qquad (III-C.9)$$

and finally

$$S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} |\phi_{k_2}\rangle = \begin{bmatrix} \alpha^{k_1+k_2} \varepsilon' |\phi_{k_2}\rangle & \alpha^{k_2} \varepsilon'\phi' \left|\phi_{k_2} \chi_{k_2+1,k_2+k_1}\right\rangle \\ +\alpha^{k_1} \phi\phi' \left|\chi_{0k_2-1}\right\rangle & +\varepsilon^{k_1} \alpha'\phi' \left|\chi_{0k_2-1}\phi_{k_2}\right\rangle \\ & +\phi\phi'^2 \left|\chi_{0k_2-1}\chi_{k_2+1,k_2+k_1}\right\rangle \\ \alpha^{k_1} \varepsilon^{k_2} \phi|\Omega\rangle & \varepsilon^{k_1+k_2} \alpha' |\phi_{k_2}\rangle \\ & +\varepsilon^{k_2} \phi\phi' \left|\chi_{k_2+1,k_2+k_1}\right\rangle \end{bmatrix} \qquad (III-C.10)$$

It is now simple, if operating one more time by $S_1 \otimes$, to find the action of T_4^+ upon two excited states.

$$\begin{aligned} T_4^+ |\phi_{01}\rangle &= \left(\alpha^2\varepsilon'^2 + \varepsilon^2\alpha'^2\right) |\phi_{01}\rangle + \varepsilon\varepsilon'\phi\phi' |\phi_{12}\rangle + \alpha\alpha'\phi\phi' |\phi_{03}\rangle \\ &\quad + \alpha'\varepsilon\phi\phi' |\phi_{02}\rangle + \alpha\varepsilon'\phi\phi' |\phi_{13}\rangle \end{aligned} \qquad (III-C.11)$$

$$\begin{aligned} T_4^+ |\phi_{02}\rangle &= \left(\alpha^2\varepsilon'^2 + \varepsilon^2\alpha'^2\right) |\phi_{02}\rangle + \alpha\varepsilon'\phi\phi' |\phi_{01} + \phi_{23}\rangle \\ &\quad + \phi^2\phi'^2 |\phi_{13}\rangle + \varepsilon\alpha'\phi\phi' |\phi_{12} + \phi_{03}\rangle \end{aligned} \qquad (III-C.12)$$

The action of T_4^+ over the other two excited states can be easily deduced from (III-C.11) by action of the cyclic permutation operator C_4.

Then, we get for T_4^+ over the Fourier states

$$\begin{aligned} T_4^+ |\mathcal{F}_{1;1}\rangle &= \left[\alpha^2\varepsilon'^2 + \varepsilon^2\alpha'^2 + i\left(\alpha\alpha' - \varepsilon\varepsilon'\right)\phi\phi'\right] |\mathcal{F}_{1;1}\rangle \\ T_4^+ |\mathcal{F}_{3;1}\rangle &= \left[\alpha^2\varepsilon'^2 + \varepsilon^2\alpha'^2 - i\left(\alpha\alpha' - \varepsilon\varepsilon'\right)\phi\phi'\right] |\mathcal{F}_{3;1}\rangle \\ T_4^+ |\mathcal{F}_{2;1}\rangle &= \left[\alpha^2\varepsilon'^2 + \varepsilon^2\alpha'^2 - \left(\alpha\alpha' + \varepsilon\varepsilon'\right)\phi\phi'\right] |\mathcal{F}_{2;1}\rangle \\ &\quad + \left(\alpha'\varepsilon - \alpha\varepsilon'\right)\phi\phi' |\mathcal{F}_{2;2}\rangle \\ T_4^+ |\mathcal{F}_{0;1}\rangle &= \left[\alpha^2\varepsilon'^2 + \varepsilon^2\alpha'^2 + \left(\alpha\alpha' + \varepsilon\varepsilon'\right)\phi\phi'\right] |\mathcal{F}_{0;1}\rangle \\ &\quad + \left(\alpha'\varepsilon + \alpha\varepsilon'\right)\phi\phi' |\mathcal{F}_{0;2}\rangle \end{aligned}$$

$$
\begin{aligned}
T_4^+ \left|\mathcal{F}_{2;2}\right\rangle &= \left(\alpha^2\varepsilon'^2 + \varepsilon^2\alpha'^2 - \phi^2\phi'^2\right)\left|\mathcal{F}_{2;2}\right\rangle \\
&\quad +2\left(\alpha\varepsilon' - \varepsilon\alpha'\right)\phi\phi'\left|\mathcal{F}_{2;1}\right\rangle \\
T_4^+ \left|\mathcal{F}_{0;2}\right\rangle &= \left(\alpha^2\varepsilon'^2 + \varepsilon^2\alpha'^2 + \phi^2\phi'^2\right)\left|\mathcal{F}_{0;2}\right\rangle \\
&\quad +2\left(\alpha\varepsilon' + \varepsilon\alpha'\right)\phi\phi'\left|\mathcal{F}_{0;1}\right\rangle
\end{aligned}
\qquad (III-C.13)
$$

Clearly, Fourier states for $s = 0$ and 2 are not eigenstates of T_4^+ and in general, the eigenstates are non trivial combinations of Fourier states. However, if (III-C.2) is satisfied, Jordan-Wigner states become eigenstates since

$$
\begin{aligned}
T_4^+ d_1^+ d_3^+ |\Omega\rangle &= \left(\alpha^2 - i\varepsilon^2\right)\left(i\alpha'^2 + \varepsilon'^2\right) d_1^+ d_3^+ |\Omega\rangle \\
T_4^+ d_0^+ d_2^+ |\Omega\rangle &= \left(\alpha^2 + i\varepsilon^2\right)\left(-i\alpha'^2 + \varepsilon'^2\right) d_0^+ d_2^+ |\Omega\rangle \\
T_4^+ d_0^+ d_3^+ |\Omega\rangle &= \left(\alpha^2 + \sqrt{2}\alpha\varepsilon + \varepsilon^2\right)\left(\alpha'^2 + \sqrt{2}\alpha'\varepsilon' + \varepsilon'^2\right) d_0^+ d_3^+ |\Omega\rangle \\
T_4^+ d_1^+ d_2^+ |\Omega\rangle &= \left(\alpha^2 - \sqrt{2}\alpha\varepsilon + \varepsilon^2\right)\left(\alpha'^2 - \sqrt{2}\alpha'\varepsilon' + \varepsilon'^2\right) d_1^+ d_2^+ |\Omega\rangle
\end{aligned}
$$

$$
\begin{aligned}
T_4^+ d_0^+ d_1^+ |\Omega\rangle &= -\left(\alpha^2 + i\sqrt{2}\alpha\varepsilon - \varepsilon^2\right)\left(\alpha'^2 - i\sqrt{2}\alpha'\varepsilon' - \varepsilon'^2\right) d_0^+ d_1^+ |\Omega\rangle \\
T_4^+ d_2^+ d_3^+ |\Omega\rangle &= -\left(\alpha^2 - i\sqrt{2}\alpha\varepsilon - \varepsilon^2\right)\left(\alpha'^2 + i\sqrt{2}\alpha'\varepsilon' - \varepsilon'^2\right) d_2^+ d_3^+ |\Omega\rangle
\end{aligned}
\qquad (III-C.14)
$$

The factorization of the eigenvalues is characteristic; this structure is compatible with the tensorial product

$$
T_4^+ = \bigotimes_{s=0}^{3}\left[\left(\alpha - q'^{2s+1}\varepsilon\right)\tilde{P}_0 + \left(\varepsilon' + q'^{2s+1}\alpha'\right)\tilde{P}_1\right] \qquad (III-C.15)
$$

where $q' = e^{i\pi/4}$.

From now on, we consider the condition (III-C.2) satisfied and prove in general that T_N^+ and T_N^- are tensorial products. From (III-C.10) and (III-C.2) we have

$$
S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} |\phi_{k_2}\rangle = \begin{bmatrix} \alpha^{k_1+1}\alpha' \left|\chi_{0k_2-1}\right\rangle + \alpha^{k_1}\varepsilon' \left|\chi_{0k_2}\right\rangle & \alpha'\phi' \left|\chi_{0k_2-1}\chi_{k_2,k_2+k_1}\right\rangle + \varepsilon'\phi' \left|\chi_{0k_2}\chi_{k_2+1,k_2+k_1}\right\rangle \\ \alpha^{k_1+1}\varepsilon^{k_2}\dfrac{\alpha'}{\phi'}|\Omega\rangle + \alpha^{k_1}\varepsilon^{k_2+1}\dfrac{\varepsilon'}{\phi'}|\Omega\rangle & \varepsilon^{k_2}\alpha' \left|\chi_{k_2,k_2+k_1}\right\rangle + \varepsilon^{k_2+1}\varepsilon' \left|\chi_{k_2+1,k_2+k_1}\right\rangle \end{bmatrix} \qquad (III-C.16)
$$

In order to generalize this result we use the following property

$$
S_1|\phi\rangle = [S_0|\Omega\rangle]\begin{bmatrix} 0 & 0 \\ \varepsilon'/\phi' & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha'/\phi' & 0 \end{bmatrix}[S_0|\Omega\rangle] \qquad (III-C.17)
$$

provided that condition (III-C.2) is satisfied.

Then,

$$
S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} |\phi_{k_2}\rangle = \sum_{\delta=0,1} M_{0,k_2-1+\delta} R_\delta M_{k_2+\delta,k_2+k_1} \qquad (III-C.18)
$$

where

$$
\begin{cases} S_0^{\otimes k}|\Omega\rangle = M_{0,k-1} \quad \text{and} \quad M_{0,-1} = M_{k,k-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ R_\delta = \begin{bmatrix} 0 & 0 \\ \dfrac{\alpha'^{1-\delta}\varepsilon'^\delta}{\phi'} & 0 \end{bmatrix} \end{cases} \qquad (III-C.19)
$$

By iteration, we may now write

$$
\begin{aligned}
&S_0^{\otimes k_j+1} \otimes S_1 \otimes S_0^{\otimes k_j} \otimes S_1 \otimes \ldots \otimes S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} \left|\phi_{k_{j+1},k_{j+1}+k_j+1,\ldots,\sum_{i=2}^{j+1} k_i+j-1}\right\rangle = \\
&\sum_{\{\delta_i=0;1\}} M_{0,k_{j+1}-1+\delta_j} R_{\delta_j} M_{k_{j+1}+\delta_j,k_{j+1}+k_j+\delta_{j-1}} R_{\delta_{j-1}} \cdot \\
&\cdot M_{k_{j+1}+k_j+\delta_{j-1}+1,k_{j+1}+k_j+k_{j-1}+\delta_{j-2}+1} \ldots R_{\delta_1} M_{\sum_{i=2}^{j+1} k_i+\delta_1+j-1,\sum_{i=1}^{j+1} k_i+j-1}
\end{aligned}
\qquad (III-C.20)
$$

This equation which is generally true for all values of k_s', can be transformed to calculate the transfer matrix over N sites. With the appropriate change of variables, we obtain

$$
\begin{aligned}
&T_N \left|\phi_{k_1,k_2,\ldots,k_j}\right\rangle = \theta(N-k_j)\prod_{i=2}^{j}\theta(k_i-k_{i-1})\theta(k_1+1)\cdot \\
&\cdot \sum_{\{\delta_i=0,1\}} \mathrm{Tr}\left\{M_{0,k_1+\delta_j-1} R_{\delta_j} M_{k_1+\delta_j,k_2+\delta_{j-1}-1} R_{\delta_{j-1}} M_{k_2+\delta_{j-1},k_3+\delta_{j-2}-1} \ldots R_{\delta_1} M_{k_j+\delta_1,N-1}\right\}
\end{aligned}
\qquad (III-C.21)
$$

which is simply

$$
\begin{aligned}
&T_N \left|\phi_{k_1,k_2,\ldots,k_j}\right\rangle = \theta(N-k_j)\prod_{i=2}^{j}\theta(k_i-k_{i-1})\theta(k_1+1) \\
&\sum_{\{\delta_i=0,1\}} \mathrm{Tr}\left\{\begin{bmatrix} \alpha^{k_1+\delta_j}|\Omega\rangle & \phi'\left|\chi_{0,k_1+\delta_j-1}\right\rangle \\ 0 & \varepsilon^{k_1+\delta_j}|\Omega\rangle \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ \dfrac{\alpha'^j}{\phi'}\left(\dfrac{\varepsilon'}{\alpha'}\right)^{\sum_{i=1}^{j}\delta_i}\left|Z_{k_1+\delta_j,k_j+\delta_1-1}\right\rangle & 0 \end{bmatrix} \otimes \right. \\
&\left. \otimes \begin{bmatrix} \alpha^{N-k_j-\delta_1}|\Omega\rangle & \phi'\left|\chi_{k_j+\delta_1,N-1}\right\rangle \\ 0 & \varepsilon^{N-k_j-\delta_1}|\Omega\rangle \end{bmatrix}\right\}
\end{aligned}
\qquad (III-C.22)
$$

where

$$
\left|Z_{k_1+\delta_j,k_j+\delta_1-1}\right\rangle = \left|\chi_{k_1+\delta_j,k_2+\delta_{j-1}-1}\chi_{k_2+\delta_{j-1},k_3+\delta_{j-2}-1}\cdots\chi_{k_{j-1}+\delta_2,k_j+\delta_1-1}\right\rangle \qquad (III-C.23)
$$

Finally,

$$
\begin{aligned}
&T_N \left|\phi_{k_1,k_2,\ldots,k_j}\right\rangle = \theta(N-k_j)\prod_{i=2}^{j}\theta(k_i-k_{i-1})\theta(k_1+1) \\
&\sum_{\{\delta_i=0,1\}} \alpha'^j\left(\frac{\varepsilon'}{\alpha'}\right)^{\sum_{i=1}^{j}\delta_i}\left[\alpha^{N-k_j-\delta_1}\left|\chi_{0,k_1+\delta_j-1}Z_{k_1+\delta_j,k_j+\delta_1-1}\right\rangle + \right. \\
&\left. +\varepsilon^{k_1+\delta_j}\left|Z_{k_1+\delta_j,k_j+\delta_1-1}\chi_{k_j+\delta_1,N-1}\right\rangle\right]
\end{aligned}
\qquad (III-C.24)
$$

Using the result of Appendix A which transforms the right hand side of equation (III-C.24) over Jordan-Wigner states, we may sum over all variables $\delta_i = 0$ or 1. We get

$$T_N \left|\phi_{k_1,k_2,\ldots,k_j}\right\rangle = \theta(N-k_j)\prod_{i=2}^{j}\theta(k_i-k_{i-1})\,\theta(k_1+1)\,.$$
$$\left[\alpha^N-(-)^\delta\varepsilon^N\right]\frac{1}{N^{j/2}}\sum_{s_1\ldots s_j}\prod_{i=1}^{j}\left[q'^{-(2s_i+\delta)k_i}\quad\frac{q'^{-(2s_i+\delta)}\varepsilon'+\alpha'}{q'^{-(2s_i+\delta)}\alpha-\varepsilon}\right]d^+_{s_1}\ldots d^+_{s_j}|\Omega\rangle \qquad (III-C.25)$$

where $\delta = 1$ for j even and 0 otherwise. It is now clear that Jordan- Wigner states are eigenvectors of the transfer matrix: using the definition (II.27), we naturally obtain the eigenvectors of T_N with the following eigenvalues.

$$T_N d^+_{s_1}\ldots d^+_{s_j}|\Omega\rangle = \left[\alpha^N-(-)^\delta\varepsilon^N\right]\prod_{i=1}^{j}\left[\frac{q'^{-(2s_i+\delta)}\varepsilon'+\alpha'}{q'^{-(2s_i+\delta)}\alpha-\varepsilon}\right]d^+_{s_1}d^+_{s_2}\ldots d^+_{s_j}|\Omega\rangle \qquad (III-C.26)$$

where δ is zero in the odd sector and one in the even sector. This result shows clearly that T_N in any of the two sectors can be written as a tensorial product

$$T_N(\delta) = \bigotimes_{s=0}^{N-1}\left[\left(\alpha-q'^{(2s+\delta)}\varepsilon\right)\tilde{P}_0+\left(\varepsilon'+q'^{(2s+\delta)}\alpha'\right)\tilde{P}_1\right] \qquad (III-C.27)$$

and that the eigenvectors are independent of the parameters ϕ, ϕ' provided condition (III-C.2) is satisfied.

III-D. MODELS OF TYPE IV

These models are defined from the following plaquette operator

$$S = \begin{bmatrix}\alpha P_0+\varepsilon' P_1 & \beta P_0 S+\phi' P_1 S\\ \phi P_0 S+\beta' P_1 S & \varepsilon P_0+\alpha' P_1\end{bmatrix} \qquad (III-D.1)$$

These models contain all models of type I, II and III; they do not conserve the number of excited states but only the parity of this number.

We wish to show that under the condition

$$\beta\beta'+\phi\phi' = \alpha\alpha'+\varepsilon\varepsilon' \qquad (III-D.2)$$

the even and odd transfer matrices $T_N(\delta)$ generated from S are both tensorial products of 2×2 matrices; this property is explicitly obtained in Jordan-Wigner states. Let us remind here that Ising model belongs to this type with the special property that $\alpha\alpha' = \varepsilon\varepsilon' = \beta\beta' = \phi\phi'$.

As usual, we define S_0 and S_1 such that

$$S_0|\phi\rangle = S_1|\Omega\rangle = 0 \qquad (III-D.3)$$

and

$$S_0|\Omega\rangle = \begin{bmatrix}\alpha|\Omega\rangle & \phi'|\phi\rangle\\ \beta'|\phi\rangle & \varepsilon|\Omega\rangle\end{bmatrix} \qquad (III-D.4)$$
$$S_1|\phi\rangle = \begin{bmatrix}\varepsilon'|\phi\rangle & \beta|\Omega\rangle\\ \phi|\Omega\rangle & \alpha'|\phi\rangle\end{bmatrix} \qquad (III-D.5)$$

Clearly, $S_0^{\otimes k}$ has its elements which are polynomials of degree $E(k/2)$ in $\beta'\phi'$. We write

$$S_0^{\otimes k}|\Omega\rangle = \begin{bmatrix}\left|Z'_{0k-1}\right\rangle & \phi'\left|Y_{0k-1}\right\rangle\\ \beta'\left|Y'_{0k-1}\right\rangle & \left|Z_{0k-1}\right\rangle\end{bmatrix} \qquad (III-D.6)$$

where

$$\begin{aligned}|Z_{0k}\rangle &= \varepsilon^{k+1}\Bigg[|\Omega\rangle+\sum_{i=1}^{E(\frac{k+1}{2})}\left(\frac{\beta'\phi'}{\alpha\varepsilon}\right)^i\sum_{p_1=0}^{k}\sum_{p_2=0}^{k}\ldots\sum_{p_{2i}=0}^{k}\\ &\quad\prod_{j=2}^{2i}\theta(p_j-p_{j-1})\left(\frac{\alpha}{\varepsilon}\right)^{\sum_{j\ \mathrm{even}}p_j-\sum_{j\ \mathrm{odd}}p_j}|\phi_{p_1p_2\ldots p_{2i}}\rangle\Bigg]\\ |Y_{0k}\rangle &= \varepsilon^k\sum_{i=0}^{E(\frac{k}{2})}\left(\frac{\beta'\phi'}{\alpha\varepsilon}\right)^i\sum_{p_0=0}^{k}\sum_{p_1=0}^{k}\ldots\sum_{p_{2i}=0}^{k}\\ &\quad\prod_{j=1}^{2i}\theta(p_j-p_{j-1})\left(\frac{\alpha}{\varepsilon}\right)^{\sum_{j\ \mathrm{even}}p_j-\sum_{j\ \mathrm{odd}}p_j}|\phi_{p_0p_1\ldots p_{2i}}\rangle\end{aligned} \qquad (III-D.7)$$

and $|Z'_{0k}\rangle$, $|Y'_{0k}\rangle$ are obtained from $|Z_{0k}\rangle$, $|Y_{0k}\rangle$ by exchanging α and ε. We use the convention $|Z_{0-1}\rangle = \left|Z'_{0-1}\right\rangle = |\Omega\rangle$ and $|Y_{0-1}\rangle = \left|Y'_{0-1}\right\rangle = 0$.

From (III-D.6) we obtain

$$T_N|\Omega\rangle = |Z_{0N-1}\rangle+\left|Z'_{0N-1}\right\rangle \qquad (III-D.8)$$

This calculation is performed in Appendix C (C.17-19) in terms of Jordan-Wigner creation operators and factorizes. We find

$$T_N|\Omega\rangle = \left(\alpha^N+\varepsilon^N\right)\prod_{s=0}^{E(\frac{N-2}{2})}\left[1+\frac{2i\beta'\phi'\sin(2s+1)\frac{\pi}{N}}{\alpha^2-2\alpha\varepsilon\cos(2s+1)\frac{\pi}{N}+\varepsilon^2}d^+_s d^+_{N-s-1}\right]|\Omega\rangle \qquad (III-D.9)$$

This is the first sign of the tensorial factorization in terms of 4×4 matrices ($|\Omega\rangle$, $d_s^+|\Omega\rangle$, $d_{N-s-1}^+|\Omega\rangle$, $d_s^+ d_{N-s-1}^+|\Omega\rangle$).

Using (III-D.5) and (III-D.6), we may now write

$$S_0^{\otimes k} \otimes S_1 |\phi_k\rangle = \begin{bmatrix} \varepsilon' |Z'_{0k-1}\phi_k\rangle + \phi\phi' |Y_{0k-1}\rangle & \beta |Z'_{0k-1}\rangle + \alpha'\phi' |Y_{0k-1}\phi_k\rangle \\ \beta'\varepsilon' |Y'_{0k-1}\phi_k\rangle + \phi |Z_{0k-1}\rangle & \beta\beta' |Y'_{0k-1}\rangle + \alpha' |Z_{0k-1}\phi_k\rangle \end{bmatrix} \qquad (III-D.10)$$

$$S_1 \otimes S_0^{\otimes k} |\phi_0\rangle = \begin{bmatrix} \varepsilon' |\phi_0 Z'_{1k}\rangle + \beta\beta' |Y'_{1k}\rangle & \varepsilon'\phi' |\phi_0 Y_{1k}\rangle + \beta |Z_{1k}\rangle \\ \phi |Z'_{1k}\rangle + \alpha'\beta' |\phi_0 Y'_{1k}\rangle & \phi\phi' |Y_{1k}\rangle + \alpha' |\phi_0 Z_{1k}\rangle \end{bmatrix} \qquad (III-D.11)$$

and finally

$$S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1} |\phi_{k_2}\rangle = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (III-D.12)$$

with

$$\begin{aligned}
A &= \varepsilon' |Z'_{0k_2-1}\phi_{k_2} Z'_{k_2+1,k_2+k_1}\rangle + \alpha'\beta'\phi' |Y_{0k_2-1}\phi_{k_2} Y'_{k_2+1,k_2+k_1}\rangle \\
&\quad + \beta\beta' |Z'_{0k_2-1} Y'_{k_2+1,k_2+k_1}\rangle + \phi\phi' |Y_{0k_2-1} Z'_{k_2+1,k_2+k_1}\rangle \\
B &= \varepsilon'\phi' |Z'_{0k_2-1}\phi_{k_2} Y_{k_2+1,k_2+k_1}\rangle + \alpha'\phi' |Y_{0k_2-1}\phi_{k_2} Z_{k_2+1,k_2+k_1}\rangle \\
&\quad + \beta |Z'_{0k_2-1} Z_{k_2+1,k_2+k_1}\rangle + \phi\phi'^2 |Y_{0k_2-1} Y_{k_2+1,k_2+k_1}\rangle \\
C &= \varepsilon'\beta' |Y'_{0k_2-1}\phi_{k_2} Z'_{k_2+1,k_2+k_1}\rangle + \alpha'\beta' |Z_{0k_2-1}\phi_{k_2} Y'_{k_2+1,k_2+k_1}\rangle \\
&\quad + \beta\beta'^2 |Y'_{0k_2-1} Y'_{k_2+1,k_2+k_1}\rangle + \phi |Z_{0k_2-1} Z'_{k_2+1,k_2+k_1}\rangle \\
D &= \varepsilon'\beta'\phi' |Y'_{0k_2-1}\phi_{k_2} Y_{k_2+1,k_2+k_1}\rangle + \alpha' |Z_{0k_2-1}\phi_{k_2} Z_{k_2+1,k_2+k_1}\rangle \\
&\quad + \beta\beta' |Y'_{0k_2-1} Z_{k_2+1,k_2+k_1}\rangle + \phi\phi' |Z_{0k_2-1} Y_{k_2+1,k_2+k_1}\rangle
\end{aligned} \qquad (III-D.13)$$

Various forms can be given to these expressions using the relations of Appendix B (B.3). We choose

$$\begin{aligned}
A &= \alpha' \left[\frac{1+\xi}{2} |Y_{0k_2-1} Z'_{k_2,k_2+k_1}\rangle + \frac{1-\xi}{2} |Z'_{0k_2} Y'_{k_2+1,k_2+k_1}\rangle \right] \\
&\quad + \varepsilon' \left[\frac{1+\xi}{2} |Z'_{0k_2-1} Y'_{k_2,k_2+k_1}\rangle + \frac{1-\xi}{2} |Y_{0k_2} Z'_{k_2+1,k_2+k_1}\rangle \right] \\
&\quad + \frac{1}{2} (\phi\phi' + \beta\beta' - \alpha\alpha' - \varepsilon\varepsilon') \left(|Z'_{0k_2-1} Y'_{k_2+1,k_2+k_1} + Y_{0k_2-1} Z'_{k_2+1,k_2+k_1}\rangle \right) \\
D &= \alpha' \left(\frac{1+\xi}{2} |Z_{0k_2-1} Y_{k_2,k_2+k_1}\rangle + \frac{1-\xi}{2} |Y'_{0k_2} Z_{k_2+1,k_2+k_1}\rangle \right) \\
&\quad + \varepsilon' \left(\frac{1+\xi}{2} |Y'_{0k_2-1} Z_{k_2,k_2+k_1}\rangle + \frac{1-\xi}{2} |Z_{0k_2} Y_{k_2+1,k_2+k_1}\rangle \right) \\
&\quad + \frac{1}{2} (\phi\phi' + \beta\beta' - \alpha\alpha' - \varepsilon\varepsilon') \left(|Z_{0k_2-1} Y_{k_2+1,k_2+k_1} + Y'_{0k_2-1} Z_{k_2+1,k_2+k_1}\rangle \right)
\end{aligned} \qquad (III-D.14)$$

where

$$\xi = \frac{\phi\phi' - \beta\beta'}{\alpha\alpha' - \varepsilon\varepsilon'} \qquad (III-D.15)$$

Similar expressions can be obtained for B and C. Of course condition (III-D.2) is a great simplification. Let us give another approach to this result which will be used in the general proof: we look for two matrices such that

$$S_1 |\phi\rangle = \begin{bmatrix} \alpha|\Omega\rangle & \phi'|\phi\rangle \\ \beta'|\phi\rangle & \varepsilon|\Omega\rangle \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} + \begin{bmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{bmatrix} \begin{bmatrix} \alpha|\Omega\rangle & \phi'|\phi\rangle \\ \beta'|\phi\rangle & \varepsilon|\Omega\rangle \end{bmatrix} \qquad (III-D.16)$$

Two such matrices exist only if

$$\alpha\alpha' + \varepsilon\varepsilon' = \phi\phi' + \beta\beta' \qquad (III-D.17)$$

Then, for any x and y,

$$\begin{aligned}
\begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} &= xI - y \begin{bmatrix} 0 & \varepsilon/\beta' \\ \alpha/\phi' & 0 \end{bmatrix} + \frac{1-\xi}{2} \begin{bmatrix} 0 & \alpha'/\beta' \\ \varepsilon'/\phi' & 0 \end{bmatrix} \\
\begin{bmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{bmatrix} &= -xI + y \begin{bmatrix} 0 & \alpha/\beta' \\ \varepsilon/\phi' & 0 \end{bmatrix} + \frac{1+\xi}{2} \begin{bmatrix} 0 & \varepsilon'/\beta' \\ \alpha'/\phi' & 0 \end{bmatrix}
\end{aligned} \qquad (III-D.18)$$

In the calculation of $S_0^{\otimes k_2} \otimes S_1 \otimes S_0^{\otimes k_1}$, using (III-D.16) and (III-D.18), x and y disappear (Appendix B.5-6) and we get (III-D.14) without the term vanishing with (III-D.17).

Then,

$$\begin{aligned}
T_N |\phi_k\rangle &= \alpha' \left[\frac{1+\xi}{2} |Y_{0k-1} Z'_{k,N-1} + Z_{0k-1} Y_{k,N-1}\rangle + \frac{1-\xi}{2} |Z'_{0k} Y'_{k+1,N-1} + Y'_{0k} Z_{k+1,N-1}\rangle \right] \\
&\quad + \varepsilon' \left[\frac{1+\xi}{2} |Z'_{0k-1} Y'_{k,N-1} + Y'_{0k-1} Z_{k,N-1}\rangle + \frac{1-\xi}{2} |Y_{0k} Z'_{k+1,N-1} + Z_{0k} Y_{k+1,N-1}\rangle \right]
\end{aligned} \qquad (III-D.19)$$

We now use relations (B.13) to show that

$$T_N |\phi_k\rangle = U_k |Z'_{0N-1}\rangle + U_k^* |Z_{0N-1}\rangle \qquad (III-D.20)$$

where

$$U_k = \frac{\varepsilon'}{\alpha} c_k^+ + \frac{\varepsilon^{k-1}}{\alpha^{k+1}} \phi\phi' \sum_{p=0}^{k-1} \left(\frac{\alpha}{\varepsilon}\right)^p c_p^+ + \frac{\alpha^{k-1}}{\varepsilon^{k+1}} \beta\beta' \sum_{p=k+1}^{N-1} \left(\frac{\varepsilon}{\alpha}\right)^p c_p^+ \qquad (III-D.21)$$

and U_k^* is U_k with $\alpha \longleftrightarrow \varepsilon$, $\alpha' \longleftrightarrow \varepsilon'$, $\phi \longleftrightarrow \beta$, $\phi' \longleftrightarrow \beta'$. We transform (III-D.20) into Jordan-Wigner space

$$T_N d_s^+ |\Omega\rangle = U_s |Z'_{0N-1}\rangle + U_s^* |Z_{0N-1}\rangle \qquad (III-D.22)$$

with

$$\begin{aligned}
U_s &= \frac{\eta_s d_s^+}{\alpha^2 - 2\alpha\varepsilon \cos 2s\frac{\pi}{N} + \varepsilon^2} \\
&\quad - \frac{\left(\frac{\varepsilon}{\alpha}\right)^N - 1}{\sqrt{N}} \sum_{s'} \left[\frac{\phi\phi'}{(q^{-s'}\alpha - \varepsilon)(q^s\varepsilon - \alpha)} + \frac{\beta\beta'}{(q^{-s'}\varepsilon - \alpha)(q^s\alpha - \varepsilon)} \right] d_{s'}^+
\end{aligned} \qquad (III-D.23)$$

$$U_s^* = U_s(\alpha \longleftrightarrow \varepsilon, \alpha' \longleftrightarrow \varepsilon', \phi \longleftrightarrow \beta \quad \text{and} \quad \phi' \longleftrightarrow \beta'),$$

$q = e^{2i\pi/N}$ and

$$\eta_s = \alpha\varepsilon' - \varepsilon\alpha' + (\alpha\alpha' - \varepsilon\varepsilon')\cos\frac{2s\pi}{N} + i(\phi\phi' - \beta\beta')\sin\frac{2s\pi}{N} \qquad (III-D.24)$$

Finally in Appendix C (C.22) we prove that the sum over s' in (III-D.23) cancel by symmetry in (III-D.22) so that

$$T_N d_s^+|\Omega\rangle = \frac{\eta_s d_s^+}{\alpha^2 - 2\alpha\varepsilon\cos\frac{2s\pi}{N} + \varepsilon^2}\left|Z'_{0N-1} - Z_{0N-1}\right\rangle \qquad (III-D.25)$$

The state $\left|Z'_{0N-1} - Z_{0N-1}\right\rangle$ is calculated in the odd sector in Appendix C. We obtain by (C.17) and (C.19)

$$T_N d_s^+|\Omega\rangle = \frac{(\alpha^N - \varepsilon^N)\,\eta_s d_s^+}{\alpha^2 - 2\alpha\varepsilon\cos\frac{2s\pi}{N} + \varepsilon^2}\prod_{s'=0}^{E(\frac{N-2}{2})}\left[1 + \frac{2i\beta'\phi'\sin\frac{2s'\pi}{N}}{\alpha^2 - 2\alpha\varepsilon\cos\frac{2s'\pi}{N} + \varepsilon^2}d_{s'}^+ d_{N-s'}^+\right]|\Omega\rangle \qquad (III-D.26)$$

which is compatible with the 4×4 tensorial factorization since the operator in s operates only on the $(s, N-s)$ part of the product.

We are now ready to give the general proof for the factorization in terms of 4×4 matrices. Using (III-D.16) and (III-D.18) we may write

$$S_0^{\otimes k_2}\otimes S_1\otimes S_0^{\otimes k_1}|\phi_{k_2}\rangle = \sum_{\delta=0,1} M_{0,k_2-1+\delta}R_\delta M_{k_2+\delta,k_2+k_1} \qquad (III-D.27)$$

with

$$\begin{cases} M_{0,k-1} &= S_0^{\otimes k}|\Omega\rangle \\ M_{0,-1} &= M_{k,k-1} = I \quad \text{by convention} \end{cases} \qquad (III-D.28)$$

and

$$R_\delta = \frac{1+(-)^\delta\xi}{2}\begin{bmatrix} 0 & \dfrac{\alpha'^\delta\varepsilon'^{1-\delta}}{\beta'} \\ \dfrac{\alpha'^{1-\delta}\varepsilon'^\delta}{\phi'} & 0 \end{bmatrix} \qquad (III-D.29)$$

We generalize (III-D.27) in a similar way as for the models of type III and equations (III-C.20) and (III-C.21) are unchanged. We now calculate the right hand side of (III-C.21)

$$M_{0k-1+\delta}R_\delta = \xi_\delta\begin{bmatrix} \alpha'^{1-\delta}\varepsilon'^\delta\left|Y_{0k-1+\delta}\right\rangle & \dfrac{\alpha'^\delta\varepsilon'^{1-\delta}}{\beta'}\left|Z'_{0k-1+\delta}\right\rangle \\ \dfrac{\alpha'^{1-\delta}\varepsilon'^\delta}{\phi'}\left|Z_{0k-1+\delta}\right\rangle & \alpha'^\delta\varepsilon'^{1-\delta}\left|Y'_{0k-1+\delta}\right\rangle \end{bmatrix} \qquad (III-D.30)$$

with $\xi_\delta = \frac{1+(-)^\delta\xi}{2}$. Then,

$$M_{0,k_1+\delta_2-1}R_{\delta_2}M_{k_1+\delta_2,k_2+\delta_1-1}R_{\delta_1} = \xi_{\delta_2}\xi_{\delta_1}\begin{bmatrix} A & B/\beta' \\ C/\phi' & D \end{bmatrix} \qquad (III-D.31)$$

with

$$\begin{aligned} A &= \alpha'^{2-\delta_1-\delta_2}\varepsilon'^{\delta_1+\delta_2}\left|Y_{0k_1-1+\delta_2}Y_{k_1+\delta_2,k_2-1+\delta_1}\right\rangle \\ &\quad + \frac{\alpha'^{1-\delta_1+\delta_2}\varepsilon'^{1+\delta_1-\delta_2}}{\beta'\phi'}\left|Z'_{0k_1-1+\delta_2}Z_{k_1+\delta_2,k_2-1+\delta_1}\right\rangle \\ B &= \alpha'^{1-\delta_2+\delta_1}\varepsilon'^{1-\delta_1+\delta_2}\left|Y_{0k_1-1+\delta_2}Z'_{k_1+\delta_2,k_2-1+\delta_1}\right\rangle \\ &\quad + \alpha'^{\delta_2+\delta_1}\varepsilon'^{2-\delta_1-\delta_2}\left|Z'_{0k_1-1+\delta_2}Y'_{k_1+\delta_2,k_2-1+\delta_1}\right\rangle \end{aligned} \qquad (III-D.32)$$

and C and D are obtained from B and A respectively by the exchange $\alpha \longleftrightarrow \varepsilon$, $\alpha' \longleftrightarrow \varepsilon'$, $\phi \longleftrightarrow \beta$, $\phi' \longleftrightarrow \beta'$, $Y \longleftrightarrow Y'$, $Z \longleftrightarrow Z'$. Now, we use Appendix B, (B.13), (B.23) to show that

$$\begin{aligned} A &= U_{k_1}(\delta_2)\alpha'^{1-\delta_1}\varepsilon'^{\delta_1}\left|Y_{0k_2-1+\delta_1}\right\rangle \\ B &= U_{k_1}(\delta_2)\alpha'^{\delta_1}\varepsilon'^{1-\delta_1}\left|Z'_{0k_2-1+\delta_1}\right\rangle \end{aligned} \qquad (III-D.33)$$

where

$$\begin{aligned} U_k(\delta) &= \frac{\alpha'^\delta\varepsilon'^{1-\delta}}{\beta'\phi'}\left[\theta_{k+\delta}\alpha\tilde{c}_{k-1+\delta} + \delta_{k+\delta}\varepsilon\tilde{c}_0\right] \\ &+ \frac{\alpha'^\delta\varepsilon'^{1-\delta}}{\alpha}\left(\frac{\alpha}{\varepsilon}\right)^{k+\delta}\sum_{p=k+\delta}^{N-1}\left(\frac{\varepsilon}{\alpha}\right)^p c_p^+ + \frac{\alpha'^{1-\delta}\varepsilon'^\delta}{\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k+\delta}\sum_{p=0}^{k+\delta-1}\left(\frac{\alpha}{\varepsilon}\right)^p c_p^+ \end{aligned} \qquad (III-D.34)$$

and

$$\tilde{c}_k = c_k - \frac{\beta'\phi'}{\alpha\varepsilon}\left[\left(\frac{\alpha}{\varepsilon}\right)^k\sum_{p=k+1}^{N-1}\left(\frac{\varepsilon}{\alpha}\right)^p c_p^+ - \left(\frac{\varepsilon}{\alpha}\right)^k\sum_{p=0}^{k-1}\left(\frac{\alpha}{\varepsilon}\right)^p c_p^+\right] \qquad (III-D.35)$$

Consequently,

$$\begin{aligned} &M_{0,k_1+\delta_2-1}R_{\delta_2}M_{k_1+\delta_2,k_2+\delta_1-1}R_{\delta_1} = \\ &= \xi_{\delta_2}\xi_{\delta_1}\begin{bmatrix} U_{k_1}(\delta_2)\alpha'^{1-\delta_1}\varepsilon'^{\delta_1}\left|Y_{0k_2-1+\delta_1}\right\rangle & U_{k_1}(\delta_2)\dfrac{\alpha'^{\delta_1}\varepsilon'^{1-\delta_1}}{\beta'}\left|Z'_{0k_2-1+\delta_1}\right\rangle \\ U^*_{k_1}(\delta_2)\dfrac{\alpha'^{1-\delta_1}\varepsilon'^{\delta_1}}{\phi'}\left|Z_{0k_2-1+\delta_1}\right\rangle & U^*_{k_1}(\delta_2)\alpha'^{\delta_1}\varepsilon'^{1-\delta_1}\left|Y'_{0k_2-1+\delta_1}\right\rangle \end{bmatrix} \end{aligned} \qquad (III-D.36)$$

where U^* is equal to U with $\alpha \longleftrightarrow \varepsilon$, $\alpha' \longleftrightarrow \varepsilon'$. Equation (III-D.35) may now be applied recurrently as many times as wishes

$$\begin{aligned} &M_{0,k_1+\delta_j-1}R_{\delta_j}\ldots M_{k_{j-1}+\delta_2,k_j+\delta_1-1}R_{\delta_1} = \xi_{\delta_j}\ldots\xi_{\delta_1}\cdot \\ &\cdot\begin{bmatrix} U_{k_1}(\delta_j)\ldots U_{k_{j-1}}(\delta_2)\alpha'^{1-\delta_1}\varepsilon'^{\delta_1}\left|Y_{0k_j-1+\delta_1}\right\rangle & U_{k_1}(\delta_j)\ldots U_{k_{j-1}}(\delta_2)\dfrac{\alpha'^{\delta_1}\varepsilon'^{1-\delta_1}}{\beta'}\left|Z'_{0k_j-1+\delta_1}\right\rangle \\ U^*_{k_1}(\delta_j)\ldots U^*_{k_{j-1}}(\delta_2)\dfrac{\alpha'^{1-\delta_1}\varepsilon'^{\delta_1}}{\phi'}\left|Z_{0k_j-1+\delta_1}\right\rangle & U^*_{k_1}(\delta_j)\ldots U^*_{k_{j-1}}(\delta_2)\alpha'^{\delta_1}\varepsilon'^{1-\delta_1}\left|Y'_{0k_j-1+\delta_1}\right\rangle \end{bmatrix} \end{aligned} \qquad (III-D.37)$$

We now perform the last operation $\otimes$ with $M_{k_j+\delta_1,N-1}$. The same kind of operator U appears again as seen from Appendix B, (B.13), (B.23), we finally get

$$M_{0,k_1+\delta_j-1}R_{\delta_j}...R_{\delta_1}M_{k_j+\delta_1,N-1} = \\ = \xi_{\delta_j}...\xi_{\delta_1}\begin{bmatrix} U_{k_1}(\delta_j)...U_{k_j}(\delta_1)\left|Z'_{0N-1}\right\rangle & U_{k_1}(\delta_j)...U_{k_j}(\delta_1)\phi'\left|Y_{0N-1}\right\rangle \\ U^*_{k_1}(\delta_j)...U^*_{k_j}(\delta_1)\beta'\left|Y'_{0N-1}\right\rangle & U^*_{k_1}(\delta_j)...U^*_{k_j}(\delta_1)\left|Z_{0N-1}\right\rangle \end{bmatrix} \qquad (III-D.38)$$

In this last proof, we used the relation (proved in Appendix B (B.22))

$$\tilde{c}_k\left|Z'_{0N-1}\right\rangle = 0 \qquad (III-D.39)$$

It is now easy to sum over all $\delta'_s = 0,1$; we have

$$\sum_{\delta=0,1}\xi_\delta U_k(\delta) = \tilde{U}_k = U_k + \frac{\beta}{\phi'}\tilde{c}_k \qquad (III-D.40)$$

where U_k is defined in (III-D.21). In the proof of (III-D.40), we use the fact that $(\alpha\tilde{c}_{k-1} - \varepsilon\tilde{c}_k)$ anticommutes with $U_{k'>k}$, with $\tilde{c}_{k'}$, and finally gives zero over $\left|Z'_{0N-1}\right\rangle$ by (III-D.39). The generalization of (III-D.20) is

$$T_N\left|\phi_{k_1,k_2,...,k_j}\right\rangle = \theta(N-k_j)\prod_{i=2}^{j}\theta(k_i-k_{i-1})\theta(k_1+1) \\ \left[\tilde{U}_{k_1}...\tilde{U}_{k_j}\left|Z'_{0N-1}\right\rangle + \tilde{U}^*_{k_1}...\tilde{U}^*_{k_j}\left|Z_{0N-1}\right\rangle\right] \qquad (III-D.41)$$

Note that for one excited state, the difference between U_k and $\tilde{U}_k$ is zero upon $\left|Z'_{0N-1}\right\rangle$ by (III-D.39). The new property in these models of type IV is the presence of the annihilation operator c_k in (III-D.40). In Appendix D we calculate (III-D.41) and find that

$$\tilde{U}_{k_1}...\tilde{U}_{k_j}\left|Z'_{0N-1}\right\rangle + \tilde{U}^*_{k_1}...\tilde{U}^*_{k_j}\left|Z_{0N-1}\right\rangle = \left[A_{k_1}...A_{k_j} + \text{c.d.p.}\right]\left|Z_{0N-1}\right\rangle \qquad (III-D.42)$$

where $\left|Z_{0N-1}\right\rangle$ is defined in (C.17), where

$$A_k = \frac{\alpha^N U_k + (-)^\delta\varepsilon^N U^*_k}{\alpha^N - (-)^\delta\varepsilon^N} \qquad (III-D.43)$$

and where c.d.p. means the sum over all possible contractions of dispoint pairs $k_{i_1}, k_{i_2}; k_{i_3}, k_{i_4}; ...: k_{i_{2q-1}}, k_{i_{2q}}$ with $k_{i_1} < k_{i_2} < ... < k_{i_{2q}}$ and with the contribution

$$A_{k_1}...\underbrace{\hat{A}_{k_{i_1}}...\hat{A}_{k_{i_2}}}...\underbrace{\hat{A}_{k_{i_{2q-1}}}...\hat{A}_{k_{i_{2q}}}}...A_{k_j}(-)^{s(I_q)}\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^q \cdot \\ \cdot\frac{\alpha^N\left(\frac{\varepsilon}{\alpha}\right)^{\sum_{t=1}^{q}(k_{i_{2t}}-k_{i_{2t-1}})} - (-)^\delta\varepsilon^N\left(\frac{\alpha}{\varepsilon}\right)^{\sum_{t=1}^{q}(k_{i_{2t}}-k_{i_{2t-1}})}}{\alpha^N - (-)^\delta\varepsilon^N} \qquad (III-D.44)$$

In (III-D.44), $\hat{A}_k$ means that this term is ommitted to be part of a contracted pair, and $s(I_q)$ is zero or one depending of the number of anticommutations to perform when taking all $\hat{A}'s$ to the right of A_{k_j}. Now, we wish to transform (III-D.41-42) over Jordan-Wigner states. Using the definition (II.27) of the Jordan-Wigner states and the anticommutativity of the operator A_k, we get

$$T_N d^+_{s_1}...d^+_{s_j}|\Omega\rangle = \left[A_{s_1}...A_{s_j} + \text{c.d.p.}\right]\left|Z_{0N-1}\right\rangle \qquad (III-D.45)$$

where

$$A_s = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} q'^{(2s+\delta)k}A_k = \frac{\eta_s d^+_s}{\alpha^2 - 2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N} + \varepsilon^2} \qquad (III-D.46)$$

with η_s given in (III-D.24) and where c.d.p. means now the following: we sum over all possible contractions of pairs $s_{i_1}s_{i_2}; s_{i_3}s_{i_4}$ etc... such that

$$s_{i_{2k-1}} + s_{i_{2k}} = N - \delta. \qquad (III-D.47)$$

Since the operators A_s anticommute, we can choose in (III-D.45) the order $s_1 < s_2 < ... < s_j$. Then the contribution for the contractions is now

$$A_{s_1}...\underbrace{\hat{A}_{s_{i_1}}...\underbrace{\hat{A}_{s_{i_{2q-1}}}...\hat{A}_{s_{i_{2q}}}}...\hat{A}_{s_{i_2}}}...A_{s_j}\tilde{b}_{s_{i_1}s_{i_2}}\tilde{b}_{s_{i_3}s_{i_4}}...\tilde{b}_{s_{i_{2q-1}}s_{i_{2q}}} \qquad (III-D.48)$$

with

$$\tilde{b}_{s_1s_2} = \frac{2i\beta\phi\sin(2s_1+\delta)\frac{\pi}{N}}{\alpha^2 - 2\alpha\varepsilon\cos(2s_1+\delta)\frac{\pi}{N} + \varepsilon^2} \qquad (III-D.49)$$

The result we have obtained shows that the transfer matrix expressed in Jordan-Wigner states can be written as a tensorial product of 4×4 matrices each matrix being attached to states $|\Omega\rangle$, $d^+_s|\Omega\rangle$, $d^+_{N-s-\delta}|\Omega\rangle$, $d^+_s d^+_{N-s-\delta}|\Omega\rangle$. We define the matrix $\tilde{T}_s$ as

$$\tilde{T}_s = \begin{bmatrix} \alpha^2 - 2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N} + \varepsilon^2 & 0 & 0 & 2i\beta'\phi'\sin(2s+\delta)\frac{\pi}{N} \\ 0 & \eta_s & 0 & 0 \\ 0 & 0 & \eta_{N-s-\delta} & 0 \\ -2i\beta\phi\sin(2s+\delta)\frac{\pi}{N} & 0 & 0 & \alpha'^2 + 2\alpha'\varepsilon'\cos(2s+\delta)\frac{\pi}{N} + \varepsilon'^2 \end{bmatrix} \qquad (III-D.50)$$

where

$$\eta_s = \alpha\varepsilon' - \varepsilon\alpha' + (\alpha\alpha' - \varepsilon\varepsilon')\cos(2s+\delta)\frac{\pi}{N} + i(\phi\phi' - \beta\beta')\sin(2s+\delta)\frac{\pi}{N} \qquad (III-D.51)$$

If $s = N - s - \delta$ modulo N, $\tilde{T}_s$ reduces to a 2×2 matrix

$$\begin{bmatrix} |\Omega\rangle \\ |d_s^+\rangle \end{bmatrix} = \begin{bmatrix} \alpha \pm \varepsilon & 0 \\ 0 & \varepsilon' \mp \alpha' \end{bmatrix} \begin{bmatrix} |\Omega\rangle \\ |d_s^+\rangle \end{bmatrix} \qquad (III - D.52)$$

Now, the eigenvalues of the 4×4 matrices are

$$\sqrt{\eta_s \eta_{N-s-\delta}} e^{\pm 2\gamma_s}, \qquad \eta_s, \eta_{N-s-\delta} \qquad (III - D.53)$$

with

$$\mathrm{ch}2\gamma_s = \mathrm{ch}2\gamma_{N-s-\delta} = \frac{\alpha^2 + \alpha'^2 + \varepsilon^2 + \varepsilon'^2 - 2(\alpha\varepsilon - \alpha'\varepsilon')\cos(2s+\delta)\frac{\pi}{N}}{2\sqrt{\eta_s \eta_{N-s-\delta}}} \qquad (III - D.54)$$

Consequently the 4×4 matrices are themselves tensorial products of 2×2 matrices which we denote by

$$\left(\sqrt{\eta_{N-s-\delta}} e^{\gamma_s} \tilde{P}_0 + \sqrt{\eta_s} e^{-\gamma_s} \tilde{P}_1\right) \otimes \left(\sqrt{\eta_s} e^{\gamma_s} \tilde{P}_0 + \sqrt{\eta_{N-s-\delta}} e^{-\gamma_s} \tilde{P}_1\right) \qquad (III - D.55)$$

In the case where $s = N - s - \delta$, the two terms in (III-D.55) reduce to one term.

Finally the transfer matrices $T_N(\delta)$ are

$$T_N(\delta) = \bigotimes_{s=0}^{N-1} \left(\sqrt{\eta_{N-s-\delta}} e^{\gamma_s} \tilde{P}_0 + \sqrt{\eta_s} e^{-\gamma_s} \tilde{P}_1\right) \qquad (III - D.56)$$

Another form can be given to $T_N(\delta)$; we write

$$\eta_s = \sqrt{\eta_s \eta_{N-s-\delta}} e^{2i\theta_s} = \sqrt{\chi_s \chi_{N-s-\delta}} e^{2i\theta_s} \qquad (III - D.57)$$

with $\chi_s = \frac{\alpha - \varepsilon q'^{2s+\delta}}{\alpha - \varepsilon q'^{-(2s+\delta)}} \eta_s$,

$$\cos 2\theta_s = \frac{\alpha\varepsilon' - \varepsilon\alpha' + (\alpha\alpha' - \varepsilon\varepsilon')\cos(2s+\delta)\frac{\pi}{N}}{\sqrt{\eta_s \eta_{N-s-\delta}}}$$

$$\sin 2\theta_s = \frac{(\phi\phi' - \beta\beta')\sin(2s+\delta)\frac{\pi}{N}}{\sqrt{\eta_s \eta_{N-s-\delta}}} \qquad (III - D.58)$$

Then,

$$T_N(\delta) = \bigotimes_{s=0}^{N-1} \left[\sqrt{\chi_s}\left(e^{\gamma_s - i\theta_s} \tilde{P}_0 + e^{-\gamma_s + i\theta_s} \tilde{P}_1\right)\right] \qquad (III - D.59)$$

We remind that the eigenvalues of $T_N(\delta)$ correspond to a number of operators $\tilde{P}_1$ with the same parity as $\delta + 1$. These eigenvalues possess the announced factorization properties. If

If we set in (III-D.59) $\beta = \beta' = 0$ and $\phi\phi' = \alpha\alpha' + \varepsilon\varepsilon'$, we recover the result (III-C.27) with the choice

$$e^{2\gamma_s} = \sqrt{\frac{\alpha^2 - 2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N} + \varepsilon^2}{\alpha'^2 + 2\alpha'\varepsilon'\cos(2s+\delta)\frac{\pi}{N} + \varepsilon'^2}} \qquad (III - D.60)$$

We finally note that in the Ising model where

$$\alpha\alpha' = \varepsilon\varepsilon' = \phi\phi' = \beta\beta' = \frac{1}{4}\mathrm{sh}2\mu\,\mathrm{sh}2\lambda \qquad (III - D.61)$$

the angle θ_s is nul and all η_s are s independent so that

$$T_N(\delta) = \eta^{N/2} \bigotimes_{s=0}^{N-1} \left(e^{\gamma_s} \tilde{P}_0 + e^{-\gamma_s} \tilde{P}_1\right) \qquad (III - D.62)$$

with

$$\eta = \frac{1}{2}\mathrm{sh}2\lambda$$

$$\mathrm{ch}2\gamma_s = \frac{\mathrm{ch}2\mu\,\mathrm{ch}2\lambda - \mathrm{sh}2\mu\cos(2s+\delta)\frac{\pi}{N}}{\mathrm{sh}2\lambda} \qquad (III - D.63)$$

and with the well known parity condition over $\sum_s (\pm\gamma_s)^{[2]}$.

IV. THE PARTITION FUNCTION

Although the eigenvalues might appear to be intricate, their factorization properties permit to reconstruct a completely vertical-horizontal symmetric partition function as the sum of four products of NM relatively simple expressions, each expression being characteristic of a plaquette in the Fourier space.

If we take for $T_N(\delta)$ the expression (III-D.56), we obtain

$$[T_N(\delta)]^M = \bigotimes_{s=0}^{N-1} \left(\left[\sqrt{\eta_{N-s-\delta}} e^{\gamma_s}\right]^M \tilde{P}_0 + \left[\sqrt{\eta_s} e^{-\gamma_s}\right]^M \tilde{P}_1\right) \qquad (IV.1)$$

and

$$\mathrm{Trace}\,[T_N(\delta)]^M = \prod_{s=0}^{N-1} \left(\eta_{N-s-\delta}^{M/2} e^{M\gamma_s} + \eta_s^{M/2} e^{-M\gamma_s}\right) \qquad (IV.2)$$

Of course, we have to take into account the parity of the eigenvalues. If we define

$$Z_{\delta_1\delta_2} = \prod_{s=0}^{N-1} \left[\eta_{N-s-\delta_1}^{M/2} e^{M\gamma_s} - (-)^{\delta_2} \eta_s^{M/2} e^{-M\gamma_s}\right] \qquad (IV.3)$$

then,

$$\begin{aligned}\mathrm{Trace}\,[T_N]^M_{\mathrm{even}} &= Z_{11} + Z_{10}\\ \mathrm{Trace}\,[T_N]^M_{\mathrm{odd}} &= Z_{01} - Z_{00}\end{aligned} \qquad (IV.4)$$

We now calculate the functions Z :

$$Z_{\delta_1\delta_2} = \prod_{s=0}^{N-1}\prod_{s'=0}^{M-1}\left[\eta_s e^{2\gamma_s} + \eta_{N-s-\delta_1} e^{-2\gamma_s} - 2\sqrt{\eta_s\eta_{N-s-\delta_1}}\cos(2s'+\delta_2)\frac{\pi}{M}\right]^{1/2} \qquad (IV.5)$$

We have successively, using symmetries

$$\begin{aligned}Z_{\delta_1\delta_2} = \prod_{s=0}^{N-1}\prod_{s'=0}^{M-1}&\Big[\eta_s\eta_{N-s-\delta_1}\left(e^{4\gamma_s}+e^{-4\gamma_s}\right) + \eta_s^2 + \eta_{N-s-\delta_1}^2 + 4\eta_s\eta_{N-s-\delta_1}\cos^2(2s'+\delta_2)\frac{\pi}{M}\\ &-2\sqrt{\eta_s\eta_{N-s-\delta_1}}\left(\eta_s+\eta_{N-s-\delta_1}\right)\left(e^{2\gamma_s}+e^{-2\gamma_s}\right)\cos(2s'+\delta_2)\frac{\pi}{M}\Big]^{1/4}\end{aligned} \qquad (IV.6)$$

$$\begin{aligned}Z_{\delta_1\delta_2} = \prod_{s=0}^{N-1}\prod_{s'=0}^{M-1}&\Big[z_s^2 + (\eta_s - \eta_{N-s-\delta_1})^2 - 2(\eta_s+\eta_{N-s-\delta_1})\,z_s\cos(2s'+\delta_2)\frac{\pi}{M}\\ &+4\eta_s\eta_{N-s-\delta_1}\cos^2(2s'+\delta_2)\frac{\pi}{M}\Big]^{1/4}\end{aligned} \qquad (IV.7)$$

with

$$z_s = 2\sqrt{\eta_s\eta_{N-s-\delta_1}}\,\mathrm{ch}2\gamma_s = \alpha^2 + \alpha'^2 + \varepsilon^2 + \varepsilon'^2 - 2(\alpha\varepsilon - \alpha'\varepsilon')\cos(2s+\delta_1)\frac{\pi}{N} \qquad (IV.8)$$

Then, using symmetry again

$$Z_{\delta_1\delta_2} = \prod_{s=0}^{N-1}\sum_{s'=0}^{M-1}\left[z_s - (\eta_s+\eta_{N-s-\delta})\cos(2s'+\delta_2)\frac{\pi}{M} + i(\eta_s - \eta_{N-s-\delta_1})\sin(2s'+\delta_2)\frac{\pi}{M}\right]^{1/2} \qquad (IV.9)$$

The partition function can now be written

$$Z = Z_{11} + Z_{10} + Z_{01} - Z_{00} \qquad (IV.10)$$

with

$$Z_{\delta_1\delta_2} = \prod_{s=0}^{N-1}\prod_{s'=0}^{M-1}\begin{bmatrix}\alpha^2+\alpha'^2+\varepsilon^2+\varepsilon'^2 - 2(\alpha\varepsilon-\alpha'\varepsilon')\cos\left[(2s+\delta_1)\frac{\pi}{N}\right]\\ -2(\alpha\varepsilon'-\alpha'\varepsilon)\cos\left[(2s'+\delta_2)\frac{\pi}{M}\right]\\ -2(\alpha\alpha'-\varepsilon\varepsilon')\cos\left[(2s+\delta_1)\frac{\pi}{N}\right]\cos\left[(2s'+\delta_2)\frac{\pi}{M}\right]\\ -2(\phi\phi'-\beta\beta')\sin\left[(2s+\delta_1)\frac{\pi}{N}\right]\sin\left[(2s'+\delta_2)\frac{\pi}{M}\right]\end{bmatrix}^{1/2} \qquad (IV.11)$$

This result is completely symmetric in horizontal and vertical directions, that is by exchanging N and M, s and s', ε and ε', ϕ and ϕ' (Z_{11} and Z_{00} are symmetric and Z_{10} exchanges with Z_{01}). We obtain four expressions characteristic of the mixing of the two parities in the horizontal direction and of the two parities in the vertical one. Finally, in the Ising model, the products of horizontal-vertical cosines and sines are absent and we reproduce a well-known result[3] obtained from a statistic of dimers.

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APPENDIX A

RELATIONS IN THE MODEL OF TYPE III.C

We now use the decomposition over Jordan-Wigner states

$$\left|\chi_{k_1 k_2 - 1}\right\rangle = \frac{1}{\sqrt{N}} \sum_{s=0}^{N-1} \frac{q'^{-(2s+\delta)k_2}\alpha^{k_2-k_1} - q'^{-(2s+\delta)k_1}\varepsilon^{k_2-k_1}}{q'^{-(2s+\delta)}\alpha - \varepsilon} d_s^+ |\Omega\rangle \qquad (A.1)$$

where $\delta = 0$ or 1 depending of the sector, we are in.

Consequently, we have

$$\alpha^{N-k_j-\delta_1}\left|\chi_{0,k_1+\delta_j-1} Z_{k_1+\delta_j,k_j+\delta_1-1}\right\rangle + \varepsilon^{k_1+\delta_j}\left|Z_{k_1+\delta_j,k_j+\delta_1-1}\chi_{k_j+\delta_1,N-1}\right\rangle =$$
$$\frac{1}{\sqrt{N}}\sum_{s=0}^{N-1}\frac{q'^{-(2s+\delta)(k_1+\delta_j)}\alpha^{N+k_1+\delta_j-\delta_1} - \varepsilon^{k_1+\delta_j}\alpha^{N-k_j-\delta_1}}{q'^{-(2s+\delta)}\alpha-\varepsilon}\left|d_s^+ Z_{k_1+\delta_j,k_j+\delta_1-1}\right\rangle$$
$$+\frac{1}{\sqrt{N}}\sum_{s=0}^{N-1}\frac{(-)^\delta\varepsilon^{k_1+\delta_j}\alpha^{N-k_j-\delta_1} - q'^{-(2s+\delta)(k_j+\delta_1)}\varepsilon^{N+k_1+\delta_j-k_j-\delta_1}}{q'^{-(2s+\delta)}\alpha-\varepsilon}\left|Z_{k_1+\delta_j,k_j+\delta_1-1}d_s^+\right|\Omega\rangle \qquad (A.2)$$

Clearly, the commutation between d^+ and Z depends upon the number of anticommuting variables in Z : we have

$$Z\, d_s^+ = (-)^\delta d_s^+ Z \qquad (A.3)$$

The above expression becomes, in a first step,

$$\frac{1}{\sqrt{N}}\sum_{s=0}^{N-1}\frac{q'^{-(2s+\delta)(k_1+\delta_j)}\alpha^{N+k_1+\delta_j-k_j-\delta_1} - (-)^\delta q'^{-(2s+\delta)(k_j+\delta_1)}\varepsilon^{N+k_1+\delta_j-k_j-\delta_1}}{q'^{-(2s+\delta)}\alpha-\varepsilon}\left|d_s^+ Z_{k_1+\delta_j,k_j+\delta_1}\right\rangle \qquad (A.4)$$

Now we must proceed recurrently with the transformation of Z into Jordan-Wigner states using the relation

$$\left|Z_{k_1+\delta_j,k_j+\delta_1-1}\right\rangle = \left|\chi_{k_1+\delta_j,k_2+\delta_{j-1}-1}Z_{k_2+\delta_{j-1},k_j+\delta_1-1}\right\rangle = \left|Z_{k_1+\delta_j,k_{j-1}+\delta_2-1}\chi_{k_{j-1}+\delta_2,k_j+\delta_1-1}\right\rangle \qquad (A.5)$$

Using the antisymmetry properties of the operators d^+, we may write as a second step,

$$\frac{1}{N}\sum_{s_1 s_2=0}^{N-1}\frac{q'^{-(2s_1+\delta)(k_1+\delta_j)}q'^{-(2s_2+\delta)(k_2+\delta_{j-1})}\alpha^{N+k_2+\delta_{j-1}-k_j-\delta_1}}{\left(q'^{-(2s_1+\delta)}\alpha-\varepsilon\right)\left(q'^{-(2s_2+\delta)}\alpha-\varepsilon\right)}d_{s_1}^+ d_{s_2}^+\left|Z_{k_2+\delta_{j-1},k_j+\delta_1-1}\right\rangle$$
$$+\frac{1}{N}\sum_{s_1 s_2=0}^{N-1}\frac{q'^{-(2s_1+\delta)(k_j+\delta_1)}q'^{-(2s_2+\delta)(k_{j-1}+\delta_2)}\varepsilon^{N+k_1+\delta_j-k_{j-1}-\delta_2}}{\left(q'^{-(2s_1+\delta)}\alpha-\varepsilon\right)\left(q'^{-(2s_2+\delta)}\alpha-\varepsilon\right)}\left|Z_{k_1+\delta_j,k_{j-1}+\delta_2-1}d_{s_2}^+ d_{s_1}^+\right\rangle \qquad (A.6)$$

The step before the last one reads

$$\frac{1}{N^{\frac{j-1}{2}}}\sum_{s_1\ldots s_{j-1}}\prod_{i=1}^{j-1}\left[\frac{q^{t-(2s_i+\delta)(k_i+\delta_{j-i+1})}}{q^{t-(2s_i+\delta)}\alpha-\varepsilon}\right]\alpha^{N+k_{j-1}+\delta_2-k_j-\delta_1}d^+_{s_1}\ldots d^+_{s_{j-1}}\left|\chi_{k_{j-1}+\delta_2,k_j+\delta_1-1}\right\rangle$$
$$+(-)^{j-1}\frac{1}{N^{\frac{j-1}{2}}}\sum_{s_1\ldots s_{j-1}}\prod_{i=1}^{j-1}\left[\frac{q^{t-(2s_i+\delta)(k_{j-i+1}+\delta_j)}}{q^{t-(2s_i+\delta)}\alpha-\varepsilon}\right]\varepsilon^{N+k_1+\delta_j-k_2-\delta_{j-1}}\left|\chi_{k_1+\delta_j,k_2+\delta_{j-1}-1}d^+_{s_{j-1}}\ldots d^+_{s_1}\right\rangle \tag{A.7}$$

Finally, the last step gives

$$\left[\alpha^N-(-)^\delta\varepsilon^N\right]\frac{1}{N^{j/2}}\sum_{s_1\ldots s_j}\prod_{i=1}^{j}\frac{q^{t-(2s_i+\delta)(k_i+\delta_{j-i+1})}}{q^{t-(2s_i+\delta)}\alpha-\varepsilon}d^+_{s_1}\ldots d^+_{s_j}|\Omega\rangle \tag{A.8}$$

APPENDIX B

VARIOUS RELATIONS ABOUT THE MODEL OF TYPE (III-D)

From (III-D.4) and (III-D.6) we obtain when operating S_0 on the right

$$\begin{aligned}
\alpha\left|Z'_{0k-1}\right\rangle+\beta'\phi'\left|Y_{0k-1}\phi_k\right\rangle &= \left|Z'_{0k}\right\rangle\\
\varepsilon\left|Z_{0k-1}\right\rangle+\beta'\phi'\left|Y'_{0k-1}\phi_k\right\rangle &= \left|Z_{0k}\right\rangle\\
\varepsilon\left|Y_{0k-1}\right\rangle+\left|Z'_{0k-1}\phi_k\right\rangle &= \left|Y_{0k}\right\rangle\\
\alpha\left|Y'_{0k-1}\right\rangle+\left|Z_{0k-1}\phi_k\right\rangle &= \left|Y'_{0k}\right\rangle
\end{aligned} \tag{B.1}$$

Similarly, when operating S_0 on the left,

$$\begin{aligned}
\alpha\left|Z'_{1k}\right\rangle+\beta'\phi'\left|\phi_0Y'_{1k}\right\rangle &= \left|Z'_{0k}\right\rangle\\
\varepsilon\left|Z_{1k}\right\rangle+\beta'\phi'\left|\phi_0Y_{1k}\right\rangle &= \left|Z_{0k}\right\rangle\\
\alpha\left|Y_{1k}\right\rangle+\left|\phi_0Z_{1k}\right\rangle &= \left|Y_{0k}\right\rangle\\
\varepsilon\left|Y'_{1k}\right\rangle+\left|\phi_0Z'_{1k}\right\rangle &= \left|Y'_{0k}\right\rangle
\end{aligned} \tag{B.2}$$

From (B.1) and (B.2) we may write

$$\begin{aligned}
\left|Z'_{0k-1}\phi_kZ'_{k+1\ k'}\right\rangle &= \left|Y_{0k}Z'_{k+1\ k'}\right\rangle-\varepsilon\left|Y_{0k-1}Z'_{k+1\ k'}\right\rangle\\
&= \left|Z'_{0k-1}Y'_{kk'}\right\rangle-\varepsilon\left|Z'_{0k-1}Y'_{k+1\ k'}\right\rangle\\
\beta'\phi'\left|Y_{0k-1}\phi_kY'_{k+1\ k'}\right\rangle &= \left|Z'_{0k}Y'_{k+1\ k'}\right\rangle-\alpha\left|Z'_{0k-1}Y'_{k+1\ k'}\right\rangle\\
&= \left|Y_{0k-1}Z'_{kk'}\right\rangle-\alpha\left|Y_{0k-1}Z'_{k+1\ k'}\right\rangle\\
\left|Z'_{0k-1}\phi_kY_{k+1\ k'}\right\rangle &= \left|Y_{0k}Y_{k+1\ k'}\right\rangle-\varepsilon\left|Y_{0k-1}Y_{k+1\ k'}\right\rangle\\
&= \frac{1}{\beta'\phi'}\left\{\left|Z'_{0k-1}Z_{kk'}\right\rangle-\varepsilon\left|Z'_{0k-1}Z_{k+1\ k'}\right\rangle\right\}\\
\left|Y_{0k-1}\phi_kZ_{k+1\ k'}\right\rangle &= \frac{1}{\beta'\phi'}\left\{\left|Z'_{0k}Z_{k+1\ k'}\right\rangle-\alpha\left|Z'_{0k-1}Z_{k+1\ k'}\right\rangle\right\}\\
&= \left|Y_{0k-1}Y_{kk'}\right\rangle-\alpha\left|Y_{0k-1}Y_{k+1\ k'}\right\rangle
\end{aligned} \tag{B.3}$$

The other four relations are obtained by the exchange $Y\longleftrightarrow Y'$, $Z\longleftrightarrow Z'$, $\alpha\longleftrightarrow\varepsilon$.

From (B.1) and (B.2), we also have

$$\begin{aligned}
\left|Y_{0k-1}\phi_kZ'_{k+1\ k'}\right\rangle &= \frac{1}{\beta'\phi'}\left\{\left|Z'_{0k}Z'_{k+1\ k'}\right\rangle-\alpha\left|Z'_{0k-1}Z'_{k+1\ k'}\right\rangle\right\}\\
&= \left|Y_{0k-1}Y'_{kk'}\right\rangle-\varepsilon\left|Y_{0k-1}Y'_{k+1\ k'}\right\rangle\\
\left|Z'_{0k-1}\phi_kY'_{k+1\ k'}\right\rangle &= \left|Y_{0k}Y'_{k+1\ k'}\right\rangle-\varepsilon\left|Y_{0k-1}Y'_{k+1\ k'}\right\rangle\\
&= \frac{1}{\beta'\phi'}\left\{\left|Z'_{0k-1}Z'_{kk'}\right\rangle-\alpha\left|Z'_{0k-1}Z'_{k+1\ k'}\right\rangle\right\}\\
\beta'\phi'\left|Y_{0k-1}\phi_kY_{k+1\ k'}\right\rangle &= \left|Z'_{0k}Y_{k+1\ k'}\right\rangle-\alpha\left|Z'_{0k-1}Y_{k+1\ k'}\right\rangle\\
&= \left|Y_{0k-1}Z_{kk'}\right\rangle-\varepsilon\left|Y_{0k-1}Z_{k+1\ k'}\right\rangle
\end{aligned} \tag{B.4}$$

$$\begin{aligned} \left|Z'_{0k-1}\phi_k Z_{k+1\ k'}\right\rangle &= \left|Y_{0k}Z_{k+1k'}\right\rangle - \varepsilon\left|Y_{0k-1}Z_{k+1\ k'}\right\rangle \\ &= \left|Z'_{0k-1}Y_{kk'}\right\rangle - \alpha\left|Z'_{0k-1}Y_{k+1\ k'}\right\rangle \end{aligned}$$

The other four relations are again obtained by the exchange $Y \longleftrightarrow Y'$, $Z \longleftrightarrow Z'$, $\alpha \longleftrightarrow \varepsilon$.

From (B.3) we get

$$\begin{aligned} \alpha\left|Y_{0k}Z'_{k+1\ k'}\right\rangle + \varepsilon\left|Z'_{0k}Y'_{k+1\ k'}\right\rangle &= \varepsilon\left|Y_{0k-1}Z'_{kk'}\right\rangle + \alpha\left|Z'_{0k-1}Y'_{kk'}\right\rangle \\ \alpha\beta'\phi'\left|Y_{0k}Y_{k+1\ k'}\right\rangle + \varepsilon\left|Z'_{0k}Z_{k+1\ k'}\right\rangle &= \varepsilon\beta'\phi'\left|Y_{0k-1}Y_{kk'}\right\rangle + \alpha\left|Z'_{0k-1}Z_{kk'}\right\rangle \end{aligned} \tag{B.5}$$

and two other relations with the exchange $Y \longleftrightarrow Y'$, $Z \longleftrightarrow Z'$, $\alpha \longleftrightarrow \varepsilon$.

From (B.4) we finally obtain

$$\begin{aligned} \left|Z'_{0k}Z'_{k+1\ k'}\right\rangle + \beta'\phi'\left|Y_{0k}Y'_{k+1\ k'}\right\rangle &= \left|Z'_{0k-1}Z'_{kk'}\right\rangle + \beta'\phi'\left|Y_{0k-1}Y'_{kk'}\right\rangle \\ \left|Z'_{0k}Y_{k+1\ k'}\right\rangle + \left|Y_{0k}Z_{k+1\ k'}\right\rangle &= \left|Z'_{0k-1}Y_{kk'}\right\rangle + \left|Y_{0k-1}Z_{kk'}\right\rangle \end{aligned} \tag{B.6}$$

and two other relations with the exchange $Y \longleftrightarrow Y'$, $Z \longleftrightarrow Z'$, $\alpha \longleftrightarrow \varepsilon$.

We consider the operator $S_0^{\otimes k}$ given in (III-D.16) and we write

$$S_0^{\otimes k}|\Omega\rangle = \begin{bmatrix} \left|Z'_{0p-1}\right\rangle & \phi'\left|Y_{0p-1}\right\rangle \\ \beta'\left|Y'_{0p-1}\right\rangle & \left|Z_{0p-1}\right\rangle \end{bmatrix} \otimes \begin{bmatrix} \alpha|\Omega\rangle & \phi'|\phi_p\rangle \\ \beta'|\phi_p\rangle & \varepsilon|\Omega\rangle \end{bmatrix} \otimes \begin{bmatrix} \left|Z'_{p+1\ k-1}\right\rangle & \phi'\left|Y_{p+1\ k-1}\right\rangle \\ \beta'\left|Y'_{p+1\ k-1}\right\rangle & \left|Z_{p+1\ k-1}\right\rangle \end{bmatrix} \tag{B.7}$$

We now operate the creation operator c_p^+ upon (B.7) to get

$$\begin{aligned} \begin{bmatrix} c_p^+\left|Z'_{0k-1}\right\rangle & \phi' c_p^+\left|Y_{0k-1}\right\rangle \\ \beta' c_p^+\left|Y'_{0k-1}\right\rangle & c_p^+\left|Z_{0k-1}\right\rangle \end{bmatrix} &= \begin{bmatrix} \left|Z'_{0p-1}\right\rangle & -\phi'\left|Y_{0p-1}\right\rangle \\ -\beta'\left|Y'_{0p-1}\right\rangle & \left|Z_{0p-1}\right\rangle \end{bmatrix} \otimes \begin{bmatrix} \alpha\left|\phi_p\right\rangle & 0 \\ 0 & \varepsilon\left|\phi_p\right\rangle \end{bmatrix} \\ &\otimes \begin{bmatrix} \left|Z'_{p+1\ k-1}\right\rangle & \phi'\left|Y_{p+1\ k-1}\right\rangle \\ \beta'\left|Y'_{p+1\ k-1}\right\rangle & \left|Z_{p+1\ k-1}\right\rangle \end{bmatrix} \end{aligned} \tag{B.8}$$

where we have used the anticommutation properties of c^+.

From (B.8), we may write

$$\begin{aligned} c_p^+\left|Z'_{0k-1}\right\rangle &= \alpha\left|Z'_{0p-1}\phi_p Z'_{p+1\ k-1}\right\rangle - \beta'\phi'\varepsilon\left|Y_{0p-1}\phi_p Y'_{p+1\ k-1}\right\rangle \\ c_p^+\left|Y_{0k-1}\right\rangle &= \alpha\left|Z'_{0p-1}\phi_p Y_{p+1\ k-1}\right\rangle - \varepsilon\left|Y_{0p-1}\phi_p Z_{p+1\ k-1}\right\rangle \end{aligned} \tag{B.9}$$

and similar expressions for $\left|Z_{0k-1}\right\rangle$ and $\left|Y'_{0k-1}\right\rangle$.

Using (B.3) we get

$$\begin{cases} c_p^+\left|Z'_{0k-1}\right\rangle &= \alpha\left|Y_{0p}Z'_{p+1\ k-1}\right\rangle - \varepsilon\left|Y_{0p-1}Z'_{p\ k-1}\right\rangle \\ c_p^+\left|Z'_{0k-1}\right\rangle &= -\varepsilon\left|Z'_{0p}Y'_{p+1\ k-1}\right\rangle + \alpha\left|Z'_{0p-1}Y'_{p\ k-1}\right\rangle \end{cases} \tag{B.10}$$

$$\begin{cases} c_p^+\left|Y_{0k-1}\right\rangle &= \alpha\left|Y_{0p}Y_{p+1\ k-1}\right\rangle - \varepsilon\left|Y_{0p-1}Y_{p\ k-1}\right\rangle \\ c_p^+\left|Y_{0k-1}\right\rangle &= -\dfrac{\varepsilon}{\beta'\phi'}\left|Z'_{0p}Z_{p+1\ k-1}\right\rangle + \dfrac{\alpha}{\beta'\phi'}\left|Z'_{0p-1}Z_{p\ k-1}\right\rangle \end{cases} \tag{B.11}$$

with the special cases

$$\begin{cases} c_0^+\left|Z'_{0k-1}\right\rangle &= \alpha\left|\phi_0 Z'_{1k-1}\right\rangle &= \alpha\left|Y'_{0k-1}\right\rangle - \alpha\varepsilon\left|Y'_{1k-1}\right\rangle \\ c_{k-1}^+\left|Z'_{0k-1}\right\rangle &= \alpha\left|Z'_{0k-2}\phi_{k-1}\right\rangle &= \alpha\left|Y_{0k-1}\right\rangle - \alpha\varepsilon\left|Y_{0k-2}\right\rangle \end{cases}$$

$$\begin{cases} c_0^+\left|Y_{0k-1}\right\rangle &= \alpha\left|\phi_0 Y_{1k-1}\right\rangle &= \dfrac{\alpha}{\beta'\phi'}\left|Z_{0k-1}\right\rangle - \dfrac{\alpha\varepsilon}{\beta'\phi'}\left|Z_{1k-1}\right\rangle \\ c_{k-1}^+\left|Y_{0k-1}\right\rangle &= -\varepsilon\left|Y_{0k-2}\phi_{k-1}\right\rangle &= -\dfrac{\varepsilon}{\beta'\phi'}\left|Z'_{0k-1}\right\rangle + \dfrac{\alpha\varepsilon}{\beta'\phi'}\left|Z'_{0k-2}\right\rangle \end{cases} \tag{B.12}$$

If we apply (B.11) and (B.12) recursively, we arrive at

$$\begin{aligned} \sum_{q=0}^{p}\left(\frac{\alpha}{\varepsilon}\right)^q c_q^+\left|Z'_{0k-1}\right\rangle &= \frac{\alpha^{p+1}}{\varepsilon^p}\left|Y_{0p}Z'_{p+1\ k-1}\right\rangle, & 0 \le p \le k-1 \\ \sum_{q=p+1}^{k-1}\left(\frac{\varepsilon}{\alpha}\right)^q c_q^+\left|Z'_{0k-1}\right\rangle &= \frac{\varepsilon^{p+1}}{\alpha^p}\left|Z'_{0p}Y'_{p+1\ k-1}\right\rangle, & -1 \le p \le k-2 \\ \sum_{q=0}^{p}\left(\frac{\alpha}{\varepsilon}\right)^q c_q^+\left|Y_{0k-1}\right\rangle &= \frac{\alpha^{p+1}}{\varepsilon^p}\left|Y_{0p}Y_{p+1\ k-1}\right\rangle, & 0 \le p \le k-1 \\ \sum_{q=p+1}^{k-1}\left(\frac{\varepsilon}{\alpha}\right)^q c_q^+\left|Y_{0k-1}\right\rangle &= -\frac{\varepsilon^k}{\alpha^{k-1}}\frac{\left|Z'_{0k-1}\right\rangle}{\beta'\phi'} + \frac{\varepsilon^{p+1}}{\alpha^p\beta'\phi'}\left|Z'_{0p}Z_{p+1\ k-1}\right\rangle, & -1 \le p \le k-2 \end{aligned} \tag{B.13}$$

In a similar fashion

$$\begin{aligned} \begin{bmatrix} c_p\left|Z'_{0k-1}\right\rangle & \phi' c_p\left|Y_{0k-1}\right\rangle \\ \beta' c_p\left|Y'_{0k-1}\right\rangle & c_p\left|Z_{0k-1}\right\rangle \end{bmatrix} &= \begin{bmatrix} \left|Z'_{0p-1}\right\rangle & -\phi'\left|Y_{0p-1}\right\rangle \\ -\beta'\left|Y'_{0p-1}\right\rangle & \left|Z_{0p-1}\right\rangle \end{bmatrix} \otimes \begin{bmatrix} 0 & \phi'|\Omega\rangle \\ \beta'|\Omega\rangle & 0 \end{bmatrix} \\ &\otimes \begin{bmatrix} \left|Z'_{p+1\ k-1}\right\rangle & \phi\left|Y_{p+1\ k-1}\right\rangle \\ \beta'\left|Y'_{p+1\ k-1}\right\rangle & \left|Z_{p+1\ k-1}\right\rangle \end{bmatrix} \end{aligned} \tag{B.14}$$

which means that

$$\begin{cases} c_p\left|Z'_{0k-1}\right\rangle &= -\beta'\phi'\left(\left|Y_{0p-1}Z'_{p+1\ k-1}\right\rangle - \left|Z'_{0p-1}Y'_{p+1\ k-1}\right\rangle\right) \\ c_p\left|Y_{0k-1}\right\rangle &= -\beta'\phi'\left|Y_{0p-1}Y_{p+1\ k-1}\right\rangle + \left|Z'_{0p-1}Z_{p+1\ k-1}\right\rangle \end{cases} \tag{B.15}$$

Using (B.3) we may write

$$\begin{cases} c_p\left|Z'_{0k-1}\right\rangle &= \dfrac{\beta'\phi'}{\varepsilon}\left(\left|Z'_{0p-1}Y'_{p\ k-1}\right\rangle - \left|Y_{0p}Z'_{p+1\ k-1}\right\rangle\right) \\ c_p\left|Z'_{0k-1}\right\rangle &= \dfrac{\beta'\phi'}{\alpha}\left(\left|Z'_{0p}Y'_{p+1\ k-1}\right\rangle - \left|Y_{0p-1}Z'_{p\ k-1}\right\rangle\right) \end{cases} \tag{B.16}$$

and finally

$$c_p\left|Z'_{0k-1}\right\rangle = \beta'\phi'\left[\frac{\alpha^{p-1}}{\varepsilon^{p+1}}\sum_{q=p+1}^{k-1}\left(\frac{\varepsilon}{\alpha}\right)^q c_q^+ - \frac{\varepsilon^{p-1}}{\alpha^{p+1}}\sum_{q=0}^{p-1}\left(\frac{\alpha}{\varepsilon}\right)^q c_q^+\right]\left|Z'_{0k-1}\right\rangle, \text{ for } 0 \le p \le k-1 \tag{B.17}$$

Similarly,

$$\begin{cases} c_p\,|Y_{0k-1}\rangle &= -\dfrac{\beta'\phi'}{\varepsilon}\,|Y_{0p}Y_{p+1\ k-1}\rangle + \dfrac{1}{\varepsilon}\,|Z'_{0p-1}Z_{p\ k-1}\rangle \\ c_p\,|Y_{0k-1}\rangle &= -\dfrac{\beta'\phi'}{\alpha}\,|Y_{0p-1}Y_{p\ k-1}\rangle + \dfrac{1}{\alpha}\,|Z'_{0p}Z_{p+1\ k-1}\rangle \end{cases} \qquad (B.18)$$

which gives

$$c_p\,|Y_{0k-1}\rangle = \frac{\varepsilon^{k-p-1}}{\alpha^{k-p}}\,|Z'_{0k-1}\rangle + \beta'\phi'\left[\frac{\alpha^{p-1}}{\varepsilon^{p+1}}\sum_{q=p+1}^{k-1}\left(\frac{\varepsilon}{\alpha}\right)^q c_q^+ - \frac{\varepsilon^{p-1}}{\alpha^{p+1}}\sum_{q=0}^{p-1}\left(\frac{\alpha}{\varepsilon}\right)^q c_q^+\right]|Y_{0k-1}\rangle\ ,$$
$$\text{for}\quad 0 \le p \le k-1 \qquad (B.19)$$

As a result of (B.17) and (B.19), we can eliminate the sum over q from $p+1$ to $k-1$ in (B.13) by writing

$$|Z'_{0p}Y'_{p+1\ k-1}\rangle = \left[\frac{\alpha}{\beta'\phi'}c_p + \frac{\varepsilon^{p-1}}{\alpha^p}\sum_{q=0}^{p-1}\left(\frac{\alpha}{\varepsilon}\right)^q c_q^+\right]|Z'_{0k-1}\rangle\ ,\quad 0 \le p \le k-1$$

$$|Z'_{0p}Y'_{p+1\ k-1}\rangle = \left[\frac{\varepsilon}{\beta'\phi'}c_{p+1} + \frac{\varepsilon^{p+1}}{\alpha^{p+2}}\sum_{q=0}^{p+1}\left(\frac{\alpha}{\varepsilon}\right)^q c_q^+\right]|Z'_{0k-1}\rangle\ ,\ 0 \le p+1 \le k-1 \qquad (B.20)$$

$$|Z'_{0p}Z_{p+1\ k-1}\rangle = \left[\alpha\ c_p + \beta'\phi'\frac{\varepsilon^{p-1}}{\alpha^p}\sum_{q=0}^{p-1}\left(\frac{\alpha}{\varepsilon}\right)^q c_q^+\right]|Y_{0k-1}\rangle\ ,\quad 0 \le p \le k-1$$

$$|Z'_{0p}Z_{p+1\ k-1}\rangle = \left[\varepsilon\ c_{p+1} + \beta'\phi'\frac{\varepsilon^{p+1}}{\alpha^{p+2}}\sum_{q=0}^{p+1}\left(\frac{\alpha}{\varepsilon}\right)^q c_q^+\right]|Y_{0k-1}\rangle\ ,0 \le p+1 \le k-1 \qquad (B.21)$$

Finally, we introduce the operator $\tilde{c}_p$ defined in (III-D.35) and which satisfies (by III.D.35 and B.17)

$$\tilde{c}_p\,|Z'_{0N-1}\rangle = 0\quad,\quad \forall p \le N-1 \qquad (B.22)$$

Then, equations (B.20)-(B.21) may be rewritten as

$$\begin{aligned}
|Z'_{0p}Y'_{p+1\ k-1}\rangle &= \left[\frac{\alpha}{\beta'\phi'}\tilde{c}_p + \frac{\alpha^p}{\varepsilon^{p+1}}\sum_{q=p+1}^{N-1}\left(\frac{\varepsilon}{\alpha}\right)^q c_q^+\right]|Z'_{0k-1}\rangle\ , && 0 \le p \le k-1 \\
|Z'_{0p}Y'_{p+1\ k-1}\rangle &= \left[\frac{\varepsilon}{\beta'\phi'}\tilde{c}_{p+1} + \frac{\alpha^p}{\varepsilon^{p+1}}\sum_{q=p+1}^{N-1}\left(\frac{\varepsilon}{\alpha}\right)^q c_q^+\right]|Z'_{0k-1}\rangle\ , && 0 \le p+1 \le k-1 \\
|Z'_{0p}Z_{p+1\ k-1}\rangle &= \left[\alpha\ \tilde{c}_p + \beta'\phi'\frac{\alpha^p}{\varepsilon^{p+1}}\sum_{q=p+1}^{N-1}\left(\frac{\varepsilon}{\alpha}\right)^q c_q^+\right]|Y_{0k-1}\rangle\ , && 0 \le p \le k-1 \\
|Z'_{0p}Z_{p+1\ k-1}\rangle &= \left[\varepsilon\ \tilde{c}_{p+1} + \beta'\phi'\frac{\alpha^p}{\varepsilon^{p+1}}\sum_{q=p+1}^{N-1}\left(\frac{\varepsilon}{\alpha}\right)^q c_q^+\right]|Y_{0k-1}\rangle\ , && 0 \le p+1 \le k-1
\end{aligned} \qquad (B.23)$$

APPENDIX C

CALCULATION OF $|Z_{0N-1}\rangle$ AND $|Z'_{0N-1}\rangle$

First we calculate

$$\varepsilon^N \sum_{p_1=0}^{N-1}\cdots\sum_{p_{2i}=0}^{N-1}\prod_{j=2}^{2i}\theta(p_j - p_{j-1})\left(\frac{\alpha}{\varepsilon}\right)^{\sum_{j\ \mathrm{even}} p_j}\left(\frac{\varepsilon}{\alpha}\right)^{\sum_{j\ \mathrm{odd}} p_j}|\phi_{p_1\ldots p_{2i}}\rangle \qquad (C.1)$$

We introduce the Jordan-Wigner states from (II.26)

$$|\phi_{p_1\ldots p_{2i}}\rangle = \frac{1}{N^i}\sum_{s_1\ldots s_{2i}} q'^{-(2s_1+\delta)p_1}\ldots q'^{-(2s_{2i}+\delta)p_{2i}}d^+_{s_1}\ldots d^+_{s_{2i}}|\Omega\rangle \qquad (C.2)$$

and we first sum over indices p_j with j odd using the fact that

$$\sum_{p_{2j+1}=0}^{N-1}\theta(p_{2j+2}-p_{2j+1})\,\theta(p_{2j+1}-p_{2j}) = \theta(p_{2j+2}-p_{2j})\sum_{p_{2j+1}=p_{2j}}^{p_{2j+2}-1} \qquad (C.3)$$

where the equality $p_{2j+1} = p_{2j}$ is added by antisymmetry of $d^+_{s_{2j}}d^+_{s_{2j+1}}$. We obtain

$$\begin{aligned}
&\frac{\varepsilon^N}{N^i}\sum_{p_2=0}^{N-1}\sum_{p_4=0}^{N-1}\sum_{p_{2i}=0}^{N-1}\prod_{j=2}^{i}\theta(p_{2j}-p_{2j-2})\left(\frac{\alpha}{\varepsilon}\right)^{\sum_{j\ \mathrm{even}} p_j}\sum_{s_1\ldots s_{2i}}\left[\frac{q'^{-(2s_1+\delta)p_2}\left(\frac{\varepsilon}{\alpha}\right)^{p_2}-1}{q'^{-(2s_1+\delta)}\frac{\varepsilon}{\alpha}-1}\right]q'^{-(2s_2+\delta)p_2} \\
&\prod_{j=1}^{i-1}\left[\frac{q'^{-(2s_{2j+1}+\delta)p_{2j+2}}\left(\frac{\varepsilon}{\alpha}\right)^{p_{2j+2}} - q'^{-(2s_{2j+1}+\delta)p_{2j}}\left(\frac{\varepsilon}{\alpha}\right)^{p_{2j}}}{\left(q'^{-(2s_{2j+1}+\delta)}\frac{\varepsilon}{\alpha}-1\right)}\right]q'^{-(2s_{2j+2}+\delta)p_{2j+2}}d^+_{s_1}\ldots d^+_{s_{2i}}|\Omega\rangle
\end{aligned} \qquad (C.4)$$

We note that

$$\begin{aligned}
&\frac{q'^{-(2s_1+\delta)p_2}\left(\frac{\varepsilon}{\alpha}\right)^{p_2}-1}{q'^{-(2s_1+\delta)}\frac{\varepsilon}{\alpha}-1}\cdot\frac{q'^{-(2s_3+\delta)p_4}\left(\frac{\varepsilon}{\alpha}\right)^{p_4}-q'^{-(2s_3+\delta)p_2}\left(\frac{\varepsilon}{\alpha}\right)^{p_2}}{q'^{-(2s_3+\delta)}\frac{\varepsilon}{\alpha}-1}\cdot \\
&\cdot\frac{q'^{-(2s_5+\delta)p_6}\left(\frac{\varepsilon}{\alpha}\right)^{p_6}-q'^{-(2s_5+\delta)p_4}\left(\frac{\varepsilon}{\alpha}\right)^{p_4}}{q'^{-(2s_5+\delta)}\frac{\varepsilon}{\alpha}-1}\ldots
\end{aligned} \qquad (C.5)$$

is equal to (by antisymmetry s_1, s_3)

$$\frac{q'^{-(2s_1+\delta)p_2}\left(\frac{\varepsilon}{\alpha}\right)^{p_2}-1}{q'^{-(2s_1+\delta)}\frac{\varepsilon}{\alpha}-1}\cdot\frac{q'^{-(2s_3+\delta)p_4}\left(\frac{\varepsilon}{\alpha}\right)^{p_4}-1}{q'^{-(2s_3+\delta)}\frac{\varepsilon}{\alpha}-1}\cdot\frac{q'^{-(2s_5+\delta)p_6}\left(\frac{\varepsilon}{\alpha}\right)^{p_6}-q'^{-(2s_5+\delta)p_4}\left(\frac{\varepsilon}{\alpha}\right)^{p_4}}{q'^{-(2s_5+\delta)}\frac{\varepsilon}{\alpha}-1}\ldots \qquad (C.6)$$

and also equal to (by antisymmetry s_3, s_5)

$$\frac{q'^{-(2s_1+\delta)p_2}\left(\frac{\varepsilon}{\alpha}\right)^{p_2}-1}{q'^{-(2s_1+\delta)}\frac{\varepsilon}{\alpha}-1}\cdot\frac{q'^{-(2s_3+\delta)p_4}\left(\frac{\varepsilon}{\alpha}\right)^{p_4}-1}{q'^{-(2s_3+\delta)}\frac{\varepsilon}{\alpha}-1}\cdot\frac{q'^{-(2s_5+\delta)p_6}\left(\frac{\varepsilon}{\alpha}\right)^{p_6}-1}{q'^{-(2s_5+\delta)}\frac{\varepsilon}{\alpha}-1}\ldots \qquad (C.7)$$

and so on ... . Consequently, we get

$$\frac{\varepsilon^N}{N^i}\sum_{s_1\ldots s_{2i}}\sum_{p_2=0}^{N-1}\sum_{p_4=0}^{N-1}\sum_{p_{2i}=0}^{N-1}\prod_{j=2}^{i}\theta\left(p_{2j}-p_{2j-2}\right)$$
$$\prod_{j=0}^{i-1}\left[\frac{q'^{-(2s_{2j+1}+\delta)p_{2j+2}}\left(\frac{\varepsilon}{\alpha}\right)^{p_{2j+2}}-1}{q'^{-(2s_{2j+1}+\delta)}\frac{\varepsilon}{\alpha}-1}\left(q'^{-(2s_{2j+2}+\delta)}\frac{\alpha}{\varepsilon}\right)^{p_{2j+2}}\right]d^+_{s_1}\ldots d^+_{s_{2i}}|\Omega\rangle \tag{C.8}$$

Now, the blocs $\left[d^+_{s_{2j+1}}d^+_{s_{2j+2}}\right]$ are commutative. As usual with commutative variables, we can eliminate the constraints $\theta\left(p_{2j}-p_{2j-2}\right)$; we note that the cases of equality $p_{2j+2}=p_{2j'+2}$ can be added in the construction since they contribute nothing by antisymmetry $s_{2j+2},s_{2j'+2}$. We get

$$\frac{\varepsilon^N}{i!N^i}\sum_{s_1\ldots s_{2i}}\prod_{j=0}^{i-1}\left[\frac{N\delta_{s_{2j+1}+s_{2j+2}+\delta}}{\left(q'^{-(2s_{2j+1}+\delta)}\frac{\varepsilon}{\alpha}-1\right)}-\frac{(-)^\delta\left(\frac{\alpha}{\varepsilon}\right)^N-1}{\left(q'^{-(2s_{2j+1}+\delta)}\frac{\varepsilon}{\alpha}-1\right)\left(q'^{-(2s_{2j+2}+\delta)}\frac{\alpha}{\varepsilon}-1\right)}\right]d^+_{s_1}\ldots d^+_{s_{2i}}|\Omega\rangle \tag{C.9}$$

which, by antisymmetry gives

$$\frac{\varepsilon^N}{i!}\sum_{s_1\ldots s_{2i}}\prod_{j=0}^{i-1}\left[\frac{\delta_{s_{2j+1}+s_{2j+2}+\delta}}{\left(q'^{-(2s_{2j+1}+\delta)}\frac{\varepsilon}{\alpha}-1\right)}\right]d^+_{s_1}\ldots d^+_{s_{2i}}|\Omega\rangle$$
$$-(-)^\delta\frac{\left(\alpha^N-(-)^\delta\varepsilon^N\right)}{i!N}\sum_{s_1\ldots s_{2i}}\sum_{j=0}^{i-1}\prod_{\substack{k=0\\k\neq j}}^{i-1}\left[\frac{\delta_{s_{2k+1}+s_{2k+2}+\delta}}{\left(q'^{-(2s_{2k+1}+\delta)}\frac{\varepsilon}{\alpha}-1\right)}\right]\cdot \tag{C.10}$$
$$\cdot\frac{1}{\left(q'^{-(2s_{2j+1}+\delta)}\frac{\varepsilon}{\alpha}-1\right)\left(q'^{-(2s_{2j+2}+\delta)}\frac{\alpha}{\varepsilon}-1\right)}d^+_{s_1}\ldots d^+_{s_{2i}}|\Omega\rangle$$

In (C.9) and (C.10), $\delta_s=1$ if $s=0$ or N and 0 otherwise. We now prove that the second term cancels when we associate the combination $\left[\left|Z'_{0N-1}-(-)^\delta Z_{0N-1}\right\rangle\right]$. First we must note that

$$\sum_{s_1s_2}\frac{\delta_{s_1+s_2+\delta}}{q'^{-(2s_1+\delta)}\frac{\varepsilon}{\alpha}-1}d^+_{s_1}d^+_{s_2}=\sum_{s_1s_2}\frac{\delta_{s_1+s_2+\delta}}{q'^{-(2s_1+\delta)}\frac{\alpha}{\varepsilon}-1}d^+_{s_1}d^+_{s_2} \tag{C.11}$$

The term coming from $\left|Z'_{0N-1}\right\rangle$ is obtained from the term coming from Z_{0N-1} by exchanging α and ε. Thus, by (C.10), we have to sum terms of the type

$$\sum_{s_{2j+1}s_{2j+2}}\left\{\frac{1}{\left(q'^{-(2s_{2j+1}+\delta)}\frac{\varepsilon}{\alpha}-1\right)}\frac{1}{\left(q'^{-(2s_{2j+2}+\delta)}\frac{\alpha}{\varepsilon}-1\right)}+\frac{1}{\left(q'^{-(2s_{2j+1}+\delta)}\frac{\alpha}{\varepsilon}-1\right)}\right.$$
$$\left.\frac{1}{\left(q'^{-(2s_{2j+2}+\delta)}\frac{\varepsilon}{\alpha}-1\right)}\right\}d^+_{s_{2j+1}}d^+_{s_{2j+2}} \tag{C.12}$$

which is zero by antisymmetry s_{2j+1},s_{2j+2}. To proceed further, we note that

$$\sum_{s_1s_2}\frac{\delta_{s_1+s_2+\delta,N}}{q'^{-(2s_1+\delta)}\frac{\varepsilon}{\alpha}-1}d^+_{s_1}d^+_{s_2}=\sum_{s}\frac{i\alpha\varepsilon\sin(2s+\delta)\frac{\pi}{N}}{\alpha^2-2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N}+\varepsilon^2}d^+_sd^+_{N-s-\delta} \tag{C.13}$$

(where $d^+_N=d^+_0$ if $s=\delta=0$). The first line in (C.10) becomes

$$\frac{\varepsilon^N}{i!}\sum_{s_1\ldots s_i}\prod_{j=1}^{i}\left[\frac{i\alpha\varepsilon\sin\left(2s_j+\delta\right)\frac{\pi}{N}}{\alpha^2-2\alpha\varepsilon\cos\left(2s_j+\delta\right)\frac{\pi}{N}+\varepsilon^2}d^+_{s_i}d^+_{N-s_j-\delta}\right]|\Omega\rangle \tag{C.14}$$

In the same way, the second line is

$$-(-)^\delta\frac{\left(\alpha^N-(-)^\delta\varepsilon^N\right)}{i!N}\sum_{ss'}\frac{\frac{1}{2}\left(\alpha^2-\varepsilon^2\right)\alpha\varepsilon\left(q'^{2s+\delta}-q'^{2s'+\delta}\right)d^+_sd^+_{s'}}{\left[\alpha^2-2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N}+\varepsilon^2\right]\left[\alpha^2-2\alpha\varepsilon\cos\left(2s'+\delta\right)\frac{\pi}{N}+\varepsilon^2\right]}$$
$$\sum_{s_1\ldots s_{i-1}}\prod_{j=1}^{i-1}\left[\frac{i\alpha\varepsilon\sin\left(2s_j+\delta\right)\frac{\pi}{N}}{\alpha^2-2\alpha\varepsilon\cos\left(2s_j+\delta\right)\frac{\pi}{N}+\varepsilon^2}d^+_{s_j}d^+_{N-s_j-\delta}\right]|\Omega\rangle \tag{C.15}$$

Finally using the expression of Z_{0N-1} from (III-D.7) we may write

$$\left|Z_{0N-1}\right\rangle=\left[\frac{\varepsilon^N}{\alpha^N-(-)^\delta\varepsilon^N}-\frac{(-)^\delta}{N}\cdot\right.$$
$$\left.\cdot\sum_{ss'}\frac{\frac{1}{2}\left(\alpha^2-\varepsilon^2\right)\beta'\phi'\left(q'^{2s+\delta}-q'^{2s'+\delta}\right)d^+_sd^+_{s'}}{\left[\alpha^2-2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N}+\varepsilon^2\right]\left[\alpha^2-2\alpha\varepsilon\cos\left(2s'+\delta\right)\frac{\pi}{N}+\varepsilon^2\right]}\right]\left|\mathcal{Z}_{0N-1}\right\rangle \tag{C.16}$$

where

$$\left|\mathcal{Z}_{0N-1}\right\rangle=\left[\alpha^N-(-)^\delta\varepsilon^N\right]\prod_{s=0}^{E\left(\frac{N-2}{2}\right)}\left[1+\frac{2i\beta'\phi'\sin(2s+\delta)\frac{\pi}{N}}{\alpha^2-2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N}+\varepsilon^2}d^+_sd^+_{N-s-\delta}\right]|\Omega\rangle \tag{C.17}$$

Using the exchange $\alpha\longleftrightarrow\varepsilon$ we obtain

$$\left|Z'_{0N-1}\right\rangle=\left[\frac{\alpha^N}{\alpha^N-(-)^\delta\varepsilon^N}-\frac{1}{N}\cdot\right.$$
$$\left.\cdot\sum_{ss'}\frac{\frac{1}{2}\left(\alpha^2-\varepsilon^2\right)\beta'\phi'\left(q'^{2s+\delta}-q'^{2s'+\delta}\right)d^+_sd^+_{s'}}{\left[\alpha^2-2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N}+\varepsilon^2\right]\left[\alpha^2-2\alpha\varepsilon\cos\left(2s'+\delta\right)\frac{\pi}{N}+\varepsilon^2\right]}\right]\left|\mathcal{Z}_{0N-1}\right\rangle \tag{C.18}$$

so that

$$\left|Z'_{0N-1}-(-)^\delta Z_{0N-1}\right\rangle=\left|\mathcal{Z}_{0N-1}\right\rangle \tag{C.19}$$

Let us define

$$|\chi_{0N-1}\rangle = \left|\frac{Z'_{0N-1}}{\alpha^N} - \frac{Z_{0N-1}}{\varepsilon^N}\right\rangle \tag{C.20}$$

which is antisymmetric in the exchange $\alpha \longleftrightarrow \varepsilon$. From (C.16), (C.17) we get

$$|\chi_{0N-1}\rangle = \frac{(-)^\delta\left(\alpha^N - (-)^\delta\varepsilon^N\right)}{2N\alpha^N\varepsilon^N}\cdot$$
$$\cdot\sum_{ss'}\frac{\left(\alpha^2-\varepsilon^2\right)\beta'\phi'\left(q'^{2s+\delta}-q'^{2s'+\delta}\right)d_s^+d_{s'}^+}{\left(\alpha^2-2\alpha\varepsilon\cos(2s+\delta)\frac{\pi}{N}+\varepsilon^2\right)\left(\alpha^2-2\alpha\varepsilon\cos(2s'+\delta)\frac{\pi}{N}+\varepsilon^2\right)}|Z_{0N-1}\rangle \tag{C.21}$$

We have the property (by antisymmetry)

$$\sum_s\frac{1}{q'^{-(2s+\delta)}\alpha-\varepsilon}d_s^+|\chi_{0N-1}\rangle = \sum_s\frac{1}{q'^{-(2s+\delta)}\varepsilon-\alpha}d_s^+|\chi_{0N-1}\rangle = 0 \tag{C.22}$$

From (C.19), (C.20) we get

$$|Z'_{0N-1}\rangle = \frac{\alpha^N}{\alpha^N-(-)^\delta\varepsilon^N}|Z_{0N-1}\rangle - \frac{(-)^\delta\alpha^N\varepsilon^N}{\alpha^N-(-)^\delta\varepsilon^N}|\chi_{0N-1}\rangle$$
$$|Z_{0N-1}\rangle = \frac{\varepsilon^N}{\alpha^N-(-)^\delta\varepsilon^N}|Z_{0N-1}\rangle - \frac{\alpha^N\varepsilon^N}{\alpha^N-(-)^\delta\varepsilon^N}|\chi_{0N-1}\rangle \tag{C.23}$$

Relations needed in Appendix D

We define the operator U as

$$U = \left(\sum_{p=0}^{N-1}\left(\frac{\alpha}{\varepsilon}\right)^p c_p^+\right)\left(\sum_{p=0}^{N-1}\left(\frac{\varepsilon}{\alpha}\right)^p c_p^+\right) \tag{C.24}$$

We have, from (II.25)

$$U = -\frac{(-)^\delta\alpha\varepsilon}{N\alpha^N\varepsilon^N}\left(\alpha^N-(-)^\delta\varepsilon^N\right)^2\sum_{ss'}\frac{1}{\left(q'^{-(2s+\delta)}\alpha-\varepsilon\right)\left(q'^{-(2s'+\delta)}\varepsilon-\alpha\right)}d_s^+d_{s'}^+ \tag{C.25}$$

so that

$$U^2 = U|\chi_{0N-1}\rangle = 0 \tag{C.26}$$

We have from (C.16)

$$|Z_{0N-1}\rangle = \frac{\varepsilon^N}{\alpha^N-(-)^\delta\varepsilon^N}\left[1-\frac{\beta'\phi'}{\alpha\varepsilon}\frac{\alpha^N}{\alpha^N-(-)^\delta\varepsilon^N}U\right]|Z_{0N-1}\rangle$$
$$|Z'_{0N-1}\rangle = \frac{\alpha^N}{\alpha^N-(-)^\delta\varepsilon^N}\left[1-(-)^\delta\frac{\beta'\phi'}{\alpha\varepsilon}\frac{\varepsilon^N}{\alpha^N-(-)^\delta\varepsilon^N}U\right]|Z_{0N-1}\rangle \tag{C.27}$$

and from (C.20)

$$|\chi_{0N-1}\rangle = \frac{\beta'\phi'}{\alpha\varepsilon}\frac{1}{\alpha^N-(-)^\delta\varepsilon^N}U|Z_{0N-1}\rangle \tag{C.28}$$

Finally with the operators U_k^+ and U_k^- from (D.1), we have

$$U_k^+U = U_k^-U = 0 \tag{C.29}$$

APPENDIX D

CALCULATION OF $\left[\tilde{U}_{k_1}...\tilde{U}_{k_p}|Z'_{0N-1}\rangle + \tilde{U}^*_{k_1}...\tilde{U}^*_{k_p}|Z_{0N-1}\rangle\right]$

We first define the operators U_k^+ and U_k^- as

$$\begin{cases} U_k + U_k^* = U_k^+ + U_k^- \\ U_k^+ = \dfrac{\phi\phi'}{\alpha\varepsilon}\left(\dfrac{\varepsilon}{\alpha}\right)^k\displaystyle\sum_{p=0}^{N-1}\left(\dfrac{\alpha}{\varepsilon}\right)^p c_p^+ \\ U_k^- = \dfrac{\beta\beta'}{\alpha\varepsilon}\left(\dfrac{\alpha}{\varepsilon}\right)^k\displaystyle\sum_{p=0}^{N-1}\left(\dfrac{\varepsilon}{\alpha}\right)^p c_p^+ \end{cases} \tag{D.1}$$

We note the anticommutators

$$\{\tilde{c}_k, U_{k'}\} = \frac{\phi\phi'}{\alpha\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k'-k} \qquad \forall k' > k$$
$$\{\tilde{c}_k, U_{k'}^+\} = \frac{\phi\phi'}{\alpha\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k'-k} \qquad \forall k, k' \tag{D.2}$$

so that

$$\{\tilde{c}_k, U_{k'} - U_{k'}^+\} = 0 \qquad \forall k' > k \tag{D.3}$$

Consequently, if we write

$$\tilde{U}_{k_j} = U_{k_j} - U_{k_j}^+ + \frac{\beta}{\phi'}\tilde{c}_{k_j} + U_{k_j}^+ \tag{D.4}$$

then, all operators $\left(\frac{\beta}{\phi'}\tilde{c}_k + U_k^+\right)$ can be taken to the right and we obtain

$$\tilde{U}_{k_1}...\tilde{U}_{k_p} = \sum_I(-)^{s(I)}\prod_{k_i\in I}\left(U_{k_i} - U_{k_i}^+\right)\prod_{k_i\in I}\left(\frac{\beta}{\phi'}\tilde{c}_{k_i} + U_{k_i}^+\right) \tag{D.5}$$

where I is an ordered subsequence of $k_1...k_p$, $\bar{I}$ its ordered complement and $s(I)$ is 0 or 1 depending of the number of anticommutations. It is a trivial matter to prove that

$$\prod_{k_i\in I}\left(\frac{\beta}{\phi'}\tilde{c}_{k_i} + U_{k_i}^+\right)|Z'_{0N-1}\rangle = \begin{cases} a_{k_{i_1}k_{i_2}}a_{k_{i_3}k_{i_4}}...a_{k_{i_{2n-1}}k_{i_{2n}}}|Z'_{0N-1}\rangle \\ U_{k_{i_0}}^+a_{k_{i_1}k_{i_2}}...a_{k_{i_{2n-1}}k_{i_{2n}}}|Z'_{0N-1}\rangle \end{cases} \tag{D.6}$$

depending of the parity of $\bar{I}$; in (D.6) we have

$$a_{kk'} = \frac{\beta\phi}{\alpha\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k'-k} \tag{D.7}$$

We sum over all $\bar{I}$ with even parity as

$$\left[\left(U_{k_1} - U^+_{k_1}\right) \dots \left(U_{k_p} - U^+_{k_p}\right) + \text{c.d.p.}\right] \tag{D.8}$$

where c.d.p. means the sum over all possible contractions of disjoint pairs. We have obtained the following result

$$\tilde{U}_{k_1} \dots \tilde{U}_{k_p} \left|Z'_{0N-1}\right\rangle = \left[\left(U_{k_1} - U^+_{k_1}\right) \dots \left(U_{k_p} - U^+_{k_p}\right) + \text{c.d.p.}\right] \left|Z'_{0N-1}\right\rangle$$
$$+ \sum_{i=1}^{p} \left(U_{k_1} - U^+_{k_1}\right) \dots \left(U_{k_{i-1}} - U^+_{k_{i-1}}\right) U^+_{k_i} \left[\left(U_{k_{i+1}} - U^+_{k_{i+1}}\right) \dots \left(U_{k_p} - U^+_{k_p}\right) + \text{c.d.p.}\right] \left|Z'_{0N-1}\right\rangle \tag{D.9}$$

The remaining part of the proof is a recurrence over the number of contractions. We first examine the terms with no contraction. We define

$$A_k = \frac{\alpha^N U_k + (-)^\delta \varepsilon^N U^*_k}{\alpha^N - (-)^\delta \varepsilon^N}$$
$$B_k = \frac{\alpha^N U^+_k + (-)^\delta \varepsilon^N U^-_k}{\alpha^N - (-)^\delta \varepsilon^N} \tag{D.10}$$

which are both antisymmetric in the exchange $\alpha \longleftrightarrow \varepsilon$, $\alpha' \longleftrightarrow \varepsilon'$, $\phi \longleftrightarrow \beta$, $\phi' \longleftrightarrow \beta'$. Then,

$$U_k - U^+_k = A_k - B_k \tag{D.11}$$

We note from (D.10) and (C.27) that

$$\left(U^+_k - B_k\right) \left|Z'_{0N-1}\right\rangle + (-)^\delta \left(U^-_k + B_k\right) \left|Z_{0N-1}\right\rangle = 0 \tag{D.12}$$

because, from (C.29)

$$U^+_k U = U^-_k U = B_k U = 0 \tag{D.13}$$

where U is defined in (C.24). Consequently, when we sum $\tilde{U}_{k_1} \dots \tilde{U}_{k_p} \left|Z'_{0N-1}\right\rangle + \tilde{U}^*_{k_1} \dots \tilde{U}^*_{k_p} \left|Z_{0N-1}\right\rangle$, it remains for the terms with no contractions

$$\left(A_{k_1} - B_{k_1}\right) \dots \left(A_{k_p} - B_{k_p}\right) + \sum_{i=1}^{p} \left(A_{k_1} - B_{k_1}\right) \dots \left(A_{k_{i-1}} - B_{k_{i-1}}\right) B_{k_i} \left(A_{k_{i+1}} - B_{k_{i+1}}\right) \dots$$
$$\dots \left(A_{k_p} - B_{k_p}\right) \tag{D.14}$$

and the corresponding term over $\left|Z_{0N-1}\right\rangle$. Using the fact that

$$B_k B_{k'} B_{k''} = 0 \tag{D.15}$$

we get

$$A_{k_1} \dots A_{k_p} - \sum_{\binom{\text{pairs}}{ij}} A_{k_1} \dots A_{k_{i-1}} B_{k_i} A_{k_{i+1}} \dots A_{k_{j-1}} B_{k_j} A_{k_{j+1}} \dots A_{k_p}. \tag{D.16}$$

We now look for the terms with one contraction. The first term we have is of the form

$$\sum_{\binom{\text{pairs}}{ij}} \left(A_{k_1} - B_{k_1}\right) \dots \left(A_{k_i} \widehat{-} B_{k_i}\right) \dots \left(A_{k_j} \widehat{-} B_{k_j}\right) \dots \left(A_{k_p} - B_{k_p}\right) (-)^{j-i-1} a_{k_i k_j} \tag{D.17}$$

where $\widehat{\ }$ means ommission of the corresponding terms; let us define

$$b_{ij} = \frac{\alpha^N a_{ij} - (-)^\delta \varepsilon^N a^*_{ij}}{\alpha^N - (-)^\delta \varepsilon^N} = b^*_{ij} \tag{D.18}$$

where $*$ means the exchange $\alpha \longleftrightarrow \varepsilon$, $\alpha' \longleftrightarrow \varepsilon'$, $\phi \longleftrightarrow \beta$, $\phi' \longleftrightarrow \beta'$. Then, from (D.18), (D.7), (D.10), (C.24) and (C.27) we have

$$\left(a_{ij} - b_{ij} - B_i B_j\right) \left|Z'_{0N-1}\right\rangle - (-)^\delta \left(a^*_{ij} - b_{ij} - B_i B_j\right) \left|Z_{0N-1}\right\rangle = 0 \tag{D.19}$$

Consequently, the term (D.17) and the corresponding term over $\left|Z_{0N-1}\right\rangle$ gives

$$\sum_{\binom{\text{pairs}}{ij}} \left(A_{k_1} - B_{k_1}\right) \dots \left(A_{k_i} \widehat{-} B_{k_i}\right) \dots \left(A_{k_j} \widehat{-} B_{k_j}\right) \dots \left(A_{k_p} - B_{k_p}\right) (-)^{j-i-1} \left(b_{k_i k_j} + B_{k_i} B_{k_j}\right) \tag{D.20}$$

and the corresponding term over $\left|Z_{0N-1}\right\rangle$. Of course the term $B_{k_i} B_{k_j}$ in (D.20) cancels the last term in (D.16). We now look at the terms

$$\left(A_{k_1} - B_{k_1}\right) \dots \left(A_{k_{i-1}} - B_{k_{i-1}}\right) U^+_{k_i} \left[\left(A_{k_{i+1}} - B_{k_{i+1}}\right) \dots \left(A_{k_r} \widehat{-} B_{k_r}\right)\right.$$
$$\left. \dots \left(A_{k_s} \widehat{-} B_{k_s}\right) \dots \left(A_{k_p} - B_{k_p}\right)\right] (-)^{s-r-1} a_{k_r k_s} \tag{D.21}$$

First we note that

$$a_{k_r k_s} U^+_{k_i} = \frac{\beta\phi}{\alpha\varepsilon} U^+_{k_i + k_s - k_r} \tag{D.22}$$

Then, we use (D.12) for $U^+_{k_i + k_s - k_r}$ to get

$$\left(A_{k_1} - B_{k_1}\right) \dots \left(A_{k_{i-1}} - B_{k_{i-1}}\right) \frac{\beta\phi}{\alpha\varepsilon} B_{k_i + k_s - k_r} \left[\left(A_{k_{i+1}} - B_{k_{i+1}}\right) \dots \left(A_{k_r} \widehat{-} B_{k_r}\right)\right.$$
$$\left. \dots \left(A_{k_s} \widehat{-} B_{k_s}\right) \dots \left(A_{k_p} - B_{k_p}\right)\right] (-)^{s-r-1} \tag{D.23}$$

Let us write

$$\frac{\beta\phi}{\alpha\varepsilon}B_{k_i+k_s-k_r} = B_{k_s}b_{k_ik_r} + B_{k_i}b_{k_rk_s} - B_{k_r}b_{k_ik_s} \tag{D.24}$$

so that (D.23) can be rewritten as

$$\begin{aligned}
&(A_{k_1}-B_{k_1})\dots(A_{k_{i-1}}-B_{k_{i-1}})\,B_{k_i}\,(A_{k_{i+1}}-B_{k_{i+1}})\dots\left(A_{k_r}\widehat{-}B_{k_r}\right)\dots\left(A_{k_s}\widehat{-}B_{k_s}\right)\dots\\
&(A_{k_p}-B_{k_p})(-)^{s-r-1}b_{k_rk_s}+(A_{k_1}-B_{k_1})\dots(A_{k_{i-1}}-B_{k_{i-1}})\left(A_{k_i}\widehat{-}B_{k_i}\right)\\
&(A_{k_{i+1}}-B_{k_{i+1}})\dots\left(A_{k_r}\widehat{-}B_{k_r}\right)\dots B_{k_s}\dots(A_{k_p}-B_{k_p})(-)^{r-i-1}b_{k_ik_r}+(A_{k_1}-B_{k_1})\dots\\
&\quad(A_{k_{i-1}}-B_{k_{i-1}})\left(A_{k_i}\widehat{-}B_{k_i}\right)(A_{k_{i+1}}-B_{k_{i+1}})\dots B_{k_r}\dots\left(A_{k_s}\widehat{-}B_{k_s}\right)\dots\\
&\quad(A_{k_p}-B_{k_p})(-)^{s-i-1}b_{k_ik_s}
\end{aligned} \tag{D.25}$$

Consequently, when we sum (D.23) over all k_i, k_s and k_r, we reconstitute for a given $b_{k_ik_j}$ a situation where B_k is successively at all places $k < k_i < k_j$, at all places $k_i < k < k_j$ and at all places $k_i < k_j < k$. If we sum (D.16), (D.20) and (D.23), we have obtained for the terms without and with one contraction

$$\begin{aligned}
&A_{k_1}\dots A_{k_p} + \sum_{\binom{\text{pairs}}{ij}} A_{k_1}\dots\hat{A}_{k_i}\dots\hat{A}_{k_j}\dots A_{k_p}(-)^{j-i-1}b_{k_ik_j}\\
&- \sum_{\substack{2\text{ pairs}\\(i,j)(m,n)}} A_{k_1}\dots B_{k_m}\dots\hat{A}_{k_i}\dots\hat{A}_{k_j}\dots B_{k_n}\dots A_{k_p}(-)^{j-i-1}b_{k_ik_j}
\end{aligned} \tag{D.26}$$

in all possible positions of i, j, m, n.

We now prove this result from q to $q+1$ number of contractions. We suppose that we have obtained for the sum from no to q contractions

$$\begin{aligned}
&A_{k_1\dots k_p} + \sum_{q'=1}^{q}\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^{q'-1}\sum_{I_{q'}} A_{k_1}\dots\hat{A}_{k_{i_1}}\dots\hat{A}_{k_{i_2}}\dots\hat{A}_{k_{i_{2q'-1}}}\dots\hat{A}_{k_{i_{2q'}}}\dots A_{k_p}(-)^{s(I_{q'})}b_{K^-_{I_{q'}}K^+_{I_{q'}}}\\
&-\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^{q-1}\sum_{I_q\cup(m,n)} A_{k_1}\dots\hat{A}_{k_{i_1}}\dots B_m\dots\hat{A}_{k_{i_2}}\dots B_n\dots\hat{A}_{k_{i_{2q-1}}}\dots\hat{A}_{k_{i_{2q}}}\dots A_{k_p}(-)^{s(I_q)}b_{K^-_{I_q}K^+_{I_q}}
\end{aligned} \tag{D.27}$$

where I_q is a set of q ordered pairs $(i_1i_2; i_3i_4; \dots; i_{2q-1}i_{2q})$, $s(I_q)$ is the signature of I_q

$$s(I_q) = \left(\sum_{i\text{ even }\in I_q} i - \sum_{i\text{ odd }\in I_q} i\right) - q \tag{D.28}$$

and

$$\begin{cases} K^+_{I_q} = \displaystyle\sum_{i\text{ even }\in I_q} k_i \\ K^-_{I_q} = \displaystyle\sum_{i\text{ odd }\in I_q} k_i \end{cases} \tag{D.29}$$

In (D.27) the pair (m, n) may be in any position in comparison with the elements which describes the pairs in I_q.

We first consider in (D.9) a term with $(q+1)$ contractions

$$\begin{aligned}
&(A_{k_1}-B_{k_1})\dots\left(A_{k_{i_1}}\widehat{-}B_{k_{i_1}}\right)\dots\left(A_{k_{i_{2q+2}}}\widehat{-}B_{k_{i_{2q+2}}}\right)\dots(A_{k_p}-B_{k_p})(-)^{s(I_{q+1})}\\
&\times a_{k_{i_1}k_{i_2}}\dots a_{k_{i_{2q+1}}k_{i_{2q+2}}}
\end{aligned} \tag{D.30}$$

and we use the fact that

$$a_{k_1k_2}\dots a_{k_{i_{2q+1}}k_{i_{2q+2}}} = \left(\frac{\beta\phi}{\alpha\varepsilon}\right)^q a_{K^-_{I_{q+1}}K^+_{I_{q+1}}} \tag{D.31}$$

Then, we use (D.19) to write (D.30) as

$$\begin{aligned}
&(A_{k_1}-B_{k_1})\dots\left(A_{k_{i_1}}\widehat{-}B_{k_{i_1}}\right)\dots\left(A_{k_{i_{2q+2}}}\widehat{-}B_{k_{i_{2q+2}}}\right)\dots(A_{k_p}-B_{k_p})(-)^{s(I_{q+1})}\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^q\\
&\times\left[b_{K^-_{I_{q+1}}K^+_{I_{q+1}}} + B_{K^-_{I_{q+1}}}B_{K^+_{I_{q+1}}}\right]
\end{aligned} \tag{D.32}$$

It is proved at the end of this appendix that the above term in $B_{K^-_{I_{q+1}}}B_{K^+_{I_{q+1}}}$ cancels exactly the last term with $B_m\dots B_n$ in (D.27). Now, we consider the terms with $(q+1)$ contractions and one operator U^+_k,

$$\begin{aligned}
&(A_{k_1}-B_{k_1})\dots(A_{k_{i-1}}-B_{k_{i-1}})\,U^+_{k_i}\left[(A_{k_{i+1}}-B_{k_{i+1}})\dots\left(A_{k_{i_1}}\widehat{-}B_{k_{i_1}}\right)\dots\left(A_{k_{i_{2q+2}}}\widehat{-}B_{k_{i_{2q+2}}}\right)\right.\\
&\left.\dots(A_{k_p}-B_{k_p})(-)^{s(I_{q+1})}a_{k_{i_1}k_{i_2}}\dots a_{k_{i_{2q+1}}k_{i_{2q+2}}}\right]
\end{aligned} \tag{D.33}$$

which can be rewritten, using (D.31), (D.22) and the relation (D.12) as

$$\begin{aligned}
&\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^{q+1}(A_{k_1}-B_{k_1})\dots(A_{k_{i-1}}-B_{k_{i-1}})\,B_{k_i+K^+_{I_{q+1}}-K^-_{I_{q+1}}}\,(A_{k_{i+1}}-B_{k_{i+1}})\dots\left(A_{k_{i_1}}\widehat{-}B_{k_{i_1}}\right)\\
&\dots\left(A_{k_{i_{2q+2}}}\widehat{-}B_{k_{i_{2q+2}}}\right)\dots(A_{k_p}-B_{k_p})(-)^{s(I_{q+1})}
\end{aligned} \tag{D.34}$$

Again, at the end of this Appendix, we prove by a generalization of (D.24) that (D.34) can be redistributed into

$$\begin{aligned}
&\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^q(A_{k_1}-B_{k_1})\dots(A_{k_{i-1}}-B_{k_{i-1}})\,B_{k_i}\,(A_{k_{i+1}}-B_{k_{i+1}})\dots\left(A_{k_{i_1}}\widehat{-}B_{k_{i_1}}\right)\dots\\
&\dots\left(A_{k_{i_{2q+2}}}\widehat{-}B_{k_{i_{2q+2}}}\right)\dots(A_{k_p}-B_{k_p})(-)^{s(I_{q+1})}b_{K^-_{I_{q+1}}K^+_{I_{q+1}}}\\
&+\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^q\sum_{j=1}^{2q+2}(A_{k_1}-B_{k_1})\dots(A_{k_{i-1}}-B_{k_{i-1}})\left(A_{k_i}\widehat{-}B_{k_i}\right)(A_{k_{i+1}}-B_{k_{i+1}})\dots\hat{i}_1\dots\hat{i}_{j-1}\\
&B_{k_{i_j}}\hat{i}_{j+1}\dots\hat{i}_{2q+2}\dots(A_{k_p}-B_{k_p})(-)^{s(I'_{q+1})}b_{K^-_{I'_{q+1}}K^+_{I'_{q+1}}}
\end{aligned} \tag{D.35}$$

where $\hat{i}$ means $\left(\widehat{A_i - B_i}\right)$ and $I'_{q+1} = \{k_i, k_{i_1}, ..., k_{i_{j-1}}, k_{i_{j+1}}, ..., k_{i_{2q+2}}\}$. When we sum over k_i, k_{i_1} etc... the terms of (D.35), the terms linear in B cancel with the terms linear in B of (D.32) while the terms quadratic in B change sign in comparison with those of (D.32). Consequently we have proven (D.27) with $q \longrightarrow (q+1)$.

We now explain the end of the recurrence. If p is even, then the end of the recurrence consists of equation (D.32) with all $(A_k - B_k)$ contracted. The terms $B_{K^-_{I_{p/2}}} B_{K^+_{I_{p/2}}}$ cancels with the corresponding terms with $\frac{p-2}{2}$ contractions. Consequently, we get

$$\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^{\frac{p-2}{2}} b_{K^-_{I_{p/2}} K^+_{I_{p/2}}} \qquad (D.36)$$

If p is odd, then the end of the recurrence consists of equations (D.32) and (D.33) with $\frac{p-1}{2}$ contractions. From (D.33-34-35) the terms linear in B cancel in (D.32); the difference now is that in (D.33) there is no $(A_k - B_k)$ and in (D.32) there is only one $(A_k - B_k)$ so that we do not generate terms in $B.B$ and this ends the recurrence.

We have obtained

$$\tilde{U}_{k_1}...\tilde{U}_{k_p} \left|Z'_{0N-1}\right\rangle + \tilde{U}^*_{k_1}...\tilde{U}^*_{k_p} \left|Z_{0N-1}\right\rangle = \left[A_{k_1}...A_{k_p} + \text{c.d.p.}\right] \left|\mathcal{Z}_{0N-1}\right\rangle \qquad (D.37)$$

where c.d.p. means the sum over all possible contractions of disjoint pairs $i_1, i_2; i_3, i_4; ...; i_{2q-1}, i_{2q}$ (with $i_1 < i_2 < i_3... < i_{2q}$) and with the contribution

$$A_{k_1}...\underbrace{\hat{A}_{k_{i_1}}...\hat{A}_{k_{i_2}}}...\underbrace{\hat{A}_{k_{i_{2q-1}}}...\hat{A}_{k_{i_{2q}}}}...A_{k_p}(-)^{s(I_q)}\left(\frac{\beta\phi}{\alpha\varepsilon}\right)^q \times \frac{\alpha^N\left(\frac{\varepsilon}{\alpha}\right)^{\sum_{i=1}^q(k_{i_{2i}}-k_{i_{2i-1}})} - (-)^\delta\varepsilon^N\left(\frac{\alpha}{\varepsilon}\right)^{\sum_{i=1}^q(k_{i_{2i}}-k_{i_{2i-1}})}}{\alpha^N - (-)^\delta\varepsilon^N} \qquad (D.38)$$

Proof of the equality of (D.34) and (D.35)

The operators $(A - B)$ are irrelevant to the proof except for an overall sign which is defined in the signature. Let us consider an odd number of indices $i_1 < i_2 < ... < i_{2q+1}$; one of these indices carry the operator B while the remaining indices define q disjoint pairs. We now concentrate over three consecutive indices $i_{j-1} < i_j < i_{j+1}$. When B is on the index i_{j-1} we have

$$B_{k_{i_{j-1}}} b_{K^-_q K^+_q} = \left[\alpha^N\frac{\phi\phi'}{\alpha\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k_{i_{j-1}}}\sum_p\left(\frac{\alpha}{\varepsilon}\right)^p c^+_p + (-)^\delta\varepsilon^N\frac{\beta\beta'}{\alpha\varepsilon}\left(\frac{\alpha}{\varepsilon}\right)^{k_{i_{j-1}}}\sum_p\left(\frac{\varepsilon}{\alpha}\right)^p c^+_p\right] \cdot\frac{\beta\phi}{\alpha\varepsilon}\left[\alpha^N\left(\frac{\varepsilon}{\alpha}\right)^{...+k_{i_{j+1}}-k_{i_j}+...} - (-)^\delta\varepsilon^N\left(\frac{\alpha}{\varepsilon}\right)^{...+k_{i_{j+1}}-k_{i_j}+...}\right]\left(\alpha^N - (-)^\delta\varepsilon^N\right)^{-2} \qquad (D.39)$$

Similarly, we have

$$B_{k_{i_j}} b_{K^-_q K^+_q} = -\left[\alpha^N\frac{\phi\phi'}{\alpha\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k_{i_j}}\sum_p\left(\frac{\alpha}{\varepsilon}\right)^p c^+_p + (-)^\delta\varepsilon^N\frac{\beta\beta'}{\alpha\varepsilon}\left(\frac{\alpha}{\varepsilon}\right)^{k_{i_j}}\sum_p\left(\frac{\varepsilon}{\alpha}\right)^p c^+_p\right] \cdot\frac{\beta\phi}{\alpha\varepsilon}\left[\alpha^N\left(\frac{\varepsilon}{\alpha}\right)^{...+k_{i_{j+1}}-k_{i_{j-1}}+...} - (-)^\delta\varepsilon^N\left(\frac{\alpha}{\varepsilon}\right)^{...+k_{i_{j+1}}-k_{i_{j-1}}+...}\right]\left(\alpha^N - (-)^\delta\varepsilon^N\right)^{-2} \qquad (D.40)$$

and

$$B_{k_{i_{j+1}}} b_{K^-_q K^+_q} = \left[\alpha^N\frac{\phi\phi'}{\alpha\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k_{i_{j+1}}}\sum_p\left(\frac{\alpha}{\varepsilon}\right)^p c^+_p + (-)^\delta\varepsilon^N\frac{\beta\beta'}{\alpha\varepsilon}\left(\frac{\alpha}{\varepsilon}\right)^{k_{i_{j+1}}}\sum_p\left(\frac{\varepsilon}{\alpha}\right)^p c^+_p\right] \cdot\frac{\beta\phi}{\alpha\varepsilon}\left[\alpha^N\left(\frac{\varepsilon}{\alpha}\right)^{...+k_{i_j}-k_{i_{j-1}}+...} - (-)^\delta\varepsilon^N\left(\frac{\alpha}{\varepsilon}\right)^{...+k_{i_j}-k_{i_{j-1}}+...}\right]\left(\alpha^N - (-)^\delta\varepsilon^N\right)^{-2} \qquad (D.41)$$

In (D.40), the $-$ sign comes from the signature. Clearly, the part of the numerator which contains α^{2N} and ε^{2N} cancels between (D.40) and (D.41); also, the part of the numerator which contains $\alpha^N\varepsilon^N$ cancels between (D.39) and (D.40).

When we sum over all indices from i_1 to i_{2q+1}, everything cancel except the terms in α^{2N} and ε^{2N} in i_1 and the terms in $\alpha^N\varepsilon^N$ in i_{2q+1}; this gives

$$\frac{\beta\phi}{\alpha\varepsilon}\left[\begin{array}{l}\alpha^{2N}\frac{\phi\phi'}{\alpha\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k_{i_1}+k_{i_3}-k_{i_2}+...+k_{i_{2q+1}}-k_{i_{2q}}}\sum_p\left(\frac{\alpha}{\varepsilon}\right)^p c^+_p \\ -\varepsilon^{2N}\frac{\beta\beta'}{\alpha\varepsilon}\left(\frac{\alpha}{\varepsilon}\right)^{k_{i_1}+k_{i_3}-k_{i_2}+...+k_{i_{2q+1}}-k_{i_{2q}}}\sum_p\left(\frac{\varepsilon}{\alpha}\right)^p c^+_p\end{array}\right]\left(\alpha^N - (-)^\delta\varepsilon^N\right)^{-2}$$
$$+\frac{\beta\phi}{\alpha\varepsilon}\left[\begin{array}{l}-(-)^\delta\alpha^N\varepsilon^N\frac{\phi\phi'}{\alpha\varepsilon}\left(\frac{\alpha}{\varepsilon}\right)^{k_{i_2}-k_{i_1}+...+k_{i_{2q}}}\left(\frac{\varepsilon}{\alpha}\right)^{k_{i_{2q+1}}}\sum_p\left(\frac{\alpha}{\varepsilon}\right)^p c^+_p \\ +(-)^\delta\alpha^N\varepsilon^N\frac{\beta\beta'}{\alpha\varepsilon}\left(\frac{\varepsilon}{\alpha}\right)^{k_{i_2}-k_{i_1}+...+k_{i_{2q}}}\left(\frac{\varepsilon}{\alpha}\right)^{k_{i_{2q+1}}}\sum_p\left(\frac{\varepsilon}{\alpha}\right)^p c^+_p\end{array}\right]\left(\alpha^N - (-)^\delta\varepsilon^N\right)^{-2} \qquad (D.42)$$

Now, (D.42) is nothing but

$$\frac{\beta\phi}{\alpha\varepsilon}B_{k_i + K^-_q - K^+_q} \qquad (D.43)$$

which is (D.43). In (D.39-40-41-43), $K^\pm_q$ are defined like in (D.29) over the set of indices which define the q contracted pairs.

Proof of the cancellation of the quadratic terms in B of equation (D.27) with the terms in $B_{K^-_{I_{q+1}}} B_{K^+_{I_{q+1}}}$ in (D.32)

The proof is based on the result (D.43) and on the fact that

$$B_m B_n = \frac{(-)^\delta\alpha^N\varepsilon^N}{(\alpha^N - (-)^\delta\varepsilon^N)^2}\frac{\beta\beta'\phi\phi'}{(\alpha\varepsilon)^2}\left[\left(\frac{\varepsilon}{\alpha}\right)^{m-n} - \left(\frac{\alpha}{\varepsilon}\right)^{m-n}\right]U \qquad (D.44)$$

is function of $m-n$; the operator U in (D.44) is defined in (C.24).

Again the operators A are irrelevant to the proof except for an overall sign. We consider an even number of indices $i_1 < i_2 < ... < i_{2q+2}$ where two of these indices carry an operator B while the remaining indices define q disjoint pairs.

Consider all terms where the index i_1 carries operator B. Then by (D.43), we get

$$\frac{\beta\phi}{\alpha\varepsilon} B_{k_{i_1}} B_{k_{i_2}+K^+-K^-} = \frac{\beta\phi}{\alpha\varepsilon} B_{k_{i_1}+K^-} \quad B_{k_{i_2}+K^+} \tag{D.45}$$

where

$$\begin{aligned} K^+ &= k_{i_4} + k_{i_6} + ... + k_{i_{2q+2}} \\ K^- &= k_{i_3} + k_{i_5} + ... + k_{i_{2q+1}} \end{aligned} \tag{D.46}$$

Now, we consider all terms where the index i_2 carries an operator B. Then, we anticommute $B_{k_{i_2}}$ to the left to take care of the signature and we apply (D.43) on the indices $i_1 < i_3 < i_4 < ... < i_{2q+2}$; we get

$$-\frac{\beta\phi}{\alpha\varepsilon} B_{k_{i_2}} B_{k_{i_1}+K^+-K^-} = -\frac{\beta\phi}{\alpha\varepsilon} B_{k_{i_2}+K^-} \quad B_{k_{i_1}+K^+} \tag{D.47}$$

If we proceed the same way with i_3, we get

$$\frac{\beta\phi}{\alpha\varepsilon} B_{k_{i_3}} B_{k_{i_1}+K^+-k_{i_2}-(K^--k_{i_3})} = \frac{\beta\phi}{\alpha\varepsilon} B_{k_{i_2}+K^-} \quad B_{k_{i_1}+K^+} \tag{D.48}$$

which cancels (D.47). In the same manner the contribution to the indices i_4 and i_5 cancels, etc... up to the contribution of the indices i_{2q} and i_{2q+1} which cancels. Finally the contribution of the index i_{2q+2} is

$$\frac{\beta\phi}{\alpha\varepsilon} B_{k_{i_1}+K^--k_{i_2}-(K^+-k_{i_{2q+2}})} B_{k_{i_{2q+2}}} = \frac{\beta\phi}{\alpha\varepsilon} B_{k_{i_1}+K^-} \quad B_{k_{i_2}+K^+} \tag{D.49}$$

Of course our counting from (D.45) to (D.49) is a double counting of the quadratic terms in (D.27). Consequently, the quadratic terms in B of equation (D.27) are

$$-\frac{\beta\phi}{\alpha\varepsilon} B_{K^-_{i_{q+1}}} \; B_{K^+_{i_{q+1}}} \tag{D.50}$$

which cancels the corresponding term of (D.32).