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Comments on the Thermodynamics of Reissner-Nordström Black Holes in Anti-de Sitter Space

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ABSTRACT

Thermodynamical properties of Reissner-Nordström black holes are described in anti-de Sitter space. It is shown that the black hole solution exists at any finite temperature and satisfies the so-called area theorem of the horizon. It is also argued that there is no limiting mass beyond which microcanonical equilibrium configuration is not materialized.

It has recently been argued [1 – 3] that the $N = 4$ super Yang-Mills theory exhibits a large N phase structure related to the thermodynamics of Schwarzschild black holes in anti-de Sitter [AdS] space. It is worthwhile mentioning that the existence of the phase transition of a AdS Schwarzschild black hole has been long substantiated by Hawking and Page [4] in the quantum gravity approach developed by Hawking and his collaborators [5]. In the present communication, the thermodynamics of AdS Reissner-Nordström black holes is described *à la* ref. [4] in sharp contrast to statistical mechanics of black holes in asymptotically flat space [6 – 8].

The AdS Reissner-Nordström metric of a cosmological constant Λ can be expressed as

$$ds^2 = \lambda^2 dt^2 - \lambda^{-2} dr^2 - r^2 d\Omega^2 \tag{1}$$

in natural units, *i.e.* $\hbar = c = G = 1$ as well as $k = 1$, where

$$\lambda^2 = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3} r^2 \tag{2}$$

and $d\Omega^2$ reads the line element on a round two-sphere. Here m and q denote the mass and charge of the black hole, respectively, and the radius b of the AdS curvature is described as $b = \sqrt{-3/\Lambda}$; $\Lambda < 0$ [4]. The horizon of the black hole appears at $r = r_+$, where

$$r_+ \left(1 + \frac{q^2}{r_+^2} + \frac{r_+^2}{b^2}\right) = 2m \quad . \tag{3}$$

The surface area A_+ of the horizon evidently reads $A_+ = 4\pi r_+^2$. The inverse Hawking temperature β_H on the horizon r_+ is determined by

$$\beta_H = \frac{2\pi}{\lambda \frac{d\lambda}{dr}|_{r=r_+}} = \frac{4\pi b^2 r_+}{b^2 - b^2 q^2/r_+^2 + 3r_+^2} \quad . \tag{4}$$

Accordingly the extremal black hole has the horizon at $r = r_E$, where

$$3r_E^4 + b^2 r_E^2 - b^2 q^2 = 0 \tag{5}$$

on which $\beta_H^{-1} = 0$.

Let us suppose the case of $-b < q < b$, hereafter, since our principal purpose is to clarify physical consequences of the AdS curvature on the Reissner-Nordström black hole in asymptotically flat space. We then obtain

$$r_E \simeq q\left(1 - \frac{3}{2}\frac{q^2}{b^2}\right) \quad . \tag{6}$$

Thus the mass m_E of the extremal black hole reads

$$m_E \simeq q\left(1 + \frac{1}{2}\frac{q^2}{b^2}\right) \quad . \tag{7}$$

A nonextremal black hole exhibits the concavo-convex structure of β_H as a funciton of r_+ for $|q| \lesssim r_+ \lesssim b$. It is indeed straightforward to confirm that β_H decreases monotonically from $\beta_H = \infty$ at $r = r_E$ and has a minimum value $\beta_{H;m}$ which is given by

$$\beta_{H;m} \simeq 6\sqrt{3}\,\pi q\left(1 - \frac{27}{2}\frac{q^2}{b^2}\right) \tag{8}$$

at

$$r_m \simeq \sqrt{3}\,q\left(1 + \frac{9}{2}\frac{q^2}{b^2}\right) \quad , \tag{9}$$

and then shows a maximum value $\beta_{H;M}$ which is expressed as

$$\beta_{H;M} \simeq \frac{2}{\sqrt{3}}\,\pi b\left(1 + \frac{3}{2}\frac{q^2}{b^2}\right) \tag{10}$$

at

$$r_M \simeq \frac{1}{\sqrt{3}}\,b\left(1 - \frac{9}{2}\frac{q^2}{b^2}\right) \quad . \tag{11}$$

At $r = r_m\ [r_M]$, in addition, the mass is described as

$$m_m \simeq \frac{4}{\sqrt{3}}\,q\left(1 + \frac{9}{2}\frac{q^2}{b^2}\right)\left[m_M \simeq \frac{4}{3\sqrt{3}}\,b\left(1 - \frac{9}{2}\frac{q^2}{b^2}\right)\right] \quad . \tag{12}$$

For sufficiently large values of r_+ , *i.e.* $r_+ \gg b; |q|$, on the other hand, β_H decreases indefinitely with the mass $m \approx r_+^3/2b^2$. As a consequence, the AdS Reissner-Nordström solution exists at any finite temperature in sharp contrast to the AdS Schwarzschild solution [4] which is assured only if $\beta_H \leq 2\pi b/\sqrt{3}$.

The Euclidean action for the AdS Reissner-Nordström black hole of period β yields the contribution of thermal radiation to the partition function Ξ of the grand canonical ensemble as follows:

$$\ln \Xi = -\frac{\pi r_+^2(b^2 + 3b^2q^2/r_+^2 - r_+^2)}{b^2 - b^2q^2/r_+^2 + 3r_+^2} \tag{13}$$

which is reduced to

$$\ln \Xi \simeq -\frac{\pi r_0^2(b^2 - r_0^2)}{b^2 + 3r_0^2} + O\left(\frac{b^2q^2}{r_0^2}\right) \ ; \qquad -b < q < b \quad , \tag{14}$$

where

$$r_0\left(1 + \frac{r_0^2}{b^2}\right) = 2m \quad . \tag{15}$$

The expectation value $\langle E\rangle$ of the energy is written in the form

$$\langle E\rangle - q\langle\Phi_+\rangle = -\frac{\partial}{\partial\beta}\ln\Xi = \frac{r_+}{2}\left(1 + \frac{q^2}{r_+^2} + \frac{r_+^2}{b^2}\right) = m \quad , \tag{16}$$

where $\Phi_+ = q/r_+$ reads the electric potential of the black hole on the horizon r_+ . Equation (16) turns out to be

$$\langle E\rangle \simeq \frac{r_0}{2}\left(1 + \frac{r_0^2}{b^2}\right) + O\left(\frac{q^2}{r_0}\right) \ ; \qquad -b < q < b \quad . \tag{17}$$

The Bekenstein-Hawking entropy S is then determined by

$$S = \ln\Xi + \beta\langle E\rangle - \beta q\langle\Phi_+\rangle = \pi r_+^2 = \frac{1}{4}A_+ \tag{18}$$

which eventuates in

$$S \simeq \pi r_0^2 + O\left(\frac{b^2q^2}{r_0^2}\right) \ ; \qquad -b < q < b\,. \tag{19}$$

Thus the so-called area theorem of the horizon is confirmed for the AdS Reissner-Nordström black hole in the same way as for the AdS Schwarzschild black hole [4]. The specific heat C_V is determined by

$$C_V = -\beta^2 \frac{dm}{d\beta} = -2\pi r_+^2 \cdot \frac{b^2 - b^2q^2/r_+^2 + 3r_+^2}{b^2 - 3b^2q^2/r_+^2 - 3r_+^2} \tag{20}$$

which is reduced to

$$C_V \simeq 2\pi r_0^2 + O\left(\frac{b^2q^2}{r_0^2}\right) \; ; \qquad -b < q < b \tag{21}$$

at sufficiently large values of r_+ , *i.e.* $r_+ \gg b\,;|q|$. Consequently the specific heat C_V is positive not only at $r_+ > r_M$ but also at $r_E < r_+ < r_m$. At $r_m < r_+ < r_M$, on the other hand, C_V is negative. The fact mentioned above will be paraphrased as follows: The AdS Reissner-Nordström black hole has positive specific heat C_V and is therefore at least locally stable not only at $m > m_M \sim 4b/3\sqrt{3}$ but also at $q \sim m_E < m < m_m \sim 4q/\sqrt{3}$ in sharp contrast to the pathological situation [6 – 8] for black holes in asymptotically flat space. At $4q/\sqrt{3} \sim m_m < m < m_M \sim 4b/3\sqrt{3}$, on the other hand, the AdS Reissner-Nordström black hole has negative specific heat C_V and is therefore unstable to decay either into pure thermal radiation or to the stable black hole in the same fashion as the AdS Schwarzschild black hole [4] in equilibrium with thermal radiation.

Let us turn our attention to the quantum degeneracy ρ of the AdS Reissner-Nordström black hole. As a consequence of Eqs. (18) and (19), we then obtain

$$\rho \sim \check{c}\, \exp\left[\frac{1}{4}\, A_+\right] = \check{c}\, \exp\left[\pi r_+^2\right] \tag{22}$$

which is reduced to

$$\rho \sim \check{c}\, \exp\left[\pi r_0^2\right] \sim \check{c}\, \exp\left[\pi (2b^2 m)^{2/3}\right] \tag{23}$$

for sufficiently large values of m . Thus the grand canonical ensemble of AdS Reissner-Nordström black holes is always well defined in full agreement with the canonical ensemble

of AdS Schwarzschild black holes [4] and yields sharp contrast to the pathological situation [6 – 8] for Reissner-Nordström as well as Schwarzschild black holes in asymptotically flat space in which the (grand) canonical partition function diverges at any temperature higher than $(4\pi m)^{-1}$ since the exponential factor of

$$\rho \sim \check{c}\, \exp\left[\pi \left(m + \sqrt{m^2 - q^2}\right)^2\right] \sim \check{c}\, \exp\left[4\pi m^2\right] \; ; \qquad |q| \leq m \tag{24}$$

dominates over the Boltzmann factor $\exp[-\beta m]$ at $0 \leq \beta \lesssim 4\pi m$ in the statistical sum.

Let us now discuss a dilute gas of AdS Reissner-Nordström black holes. The microcanonical density Ω_{RN} of such states is then described as

$$\Omega_{RN}(E, Q\,; \Lambda) = \sum_{n=1}^{\infty} \Omega_n(E, Q\,; \Lambda) \quad , \tag{25}$$

where the density Ω_n of the n black hole configuration is asymptotically expressed as [6 – 8]

$$\begin{aligned} \Omega_n(M, Q\,; \Lambda) \;&=\; \frac{1}{n!} \prod_{i=1}^{n} \int_{m_0}^{\infty} dm_i \int_{-b}^{b} dq_i\; \rho(m_i, q_i\,; \Lambda) \\ &\quad \times \delta(M - \sum_{i=1}^{n} m_i)\; \delta(Q - \sum_{i=1}^{n} q_i) \end{aligned} \tag{26}$$

at high energy. Here we have set up $E = M$ and m_0 reads the minimum allowed mass for each black hole. The Bekenstein-Hawking entropy S_n for such a gas of n AdS Reissner-Nordström black holes is written in the form

$$S_n(M, Q\,; \Lambda) = \sum_{i=1}^{n} S(m_i, q_i\,; \Lambda)\; ; \; \sum_{i=1}^{n} m_i = M\; ; \; \sum_{i=1}^{n} q_i = Q \quad , \tag{27}$$

where $S(m, q\,; \Lambda) = 1/4 \cdot A_+(m, q\,; \Lambda)$ and the corresponding Hawking temperature β_H^{-1} is in turn expressed as

$$\beta_H(m, q\,; \Lambda) = \frac{dS(m, q\,; \Lambda)}{dm} = \frac{4\pi b^2 r_+}{b^2 - b^2q^2/r_+^2 + 3r_+^2} \quad . \tag{28}$$

in full consonance with Eq. (4) as well as Eq. (3). As an inevitable consequence of the asymptotic behaviour of $A_+ \sim 4\pi(2b^2m)^{2/3}$ at large m , the most probable configuration in association with the maximum of the entropy S_n is brought to realization *à la* Frautschi [6 (1992)] at thermal equilibrium, *i.e.* the equipartition configuration with $m_i = M/n$ and $q_i = Q/n$; $i = 1, \cdots, n$, in sharp contrast to the single-massive-mode dominance scenario [6 - 10] for black holes in asymptotically flat space. Accordingly microcanonical equilibrium configuration of AdS Reissner-Nordström black holes is always guaranteed for any finite value of energy M at least such as $M \gg b\,;|Q|$ which yields sharp contrast to the microcanonical ensemble [6 (1992), 8] of black holes in asymptotically flat space. It is parenthetically mentioned that the large $\bar{N}$ behaviour of the most probable microcanonical configuration $\Omega_{\bar{N}}$ is asymptotically described by the self-consistency condition $\psi(\bar{N}+1) \simeq \ln \bar{c} \gg 1$, *i.e.* $\bar{N} \simeq \bar{c} \gg 1$.

Thermodyncamical properties of AdS Reissner-Nordström black holes have been studied in proper reference to ref. [4] and ref. [6 (1992)] as well as ref. [8]. Principal observations are as follows: Firstly, the black hole solution exists at any finite temperature and satisfies the so-called area theorem of the horizon. Secondly, the black hole is at least locally stable if the radius of the horizon is either not less than that of the AdS curvature or of the order of that of the extremal horizon in asymptotically flat space. Thirdly, there appears no limiting mass beyond which microcanonical equilibirum configuration of black holes can never be materialized. The fruitful thermodynamical investigation of string excitations yet remains to be elaborated *à la* ref. [11], however, within the general framework of Matrix theory thermodynamics.

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Comments on the Thermodynamics of Kerr-Newman Black Holes in Anti-de Sitter Space

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ABSTRACT

Thermodynamical properties of Kerr-Newman black holes are described in anti-de Sitter space. It is shown that the black hole solution satisfies the so-called Bekenstein-Hawking relation between entropy and surface area of the horizon if and only if angular momentum per unit mass is sufficiently small in natural units as compared with the radius of anti-de Sitter curvature.

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In a previous paper [1] of one of the present authors [H. F.], the thermodynamics of Reissner-Nordström black holes has been studied *à la* classic work [2] of Hawking and Page in anti-de Sitter [AdS] space. It has been shown that the black hole solution exists at any finite temperature and satisfies the Bekenstein-Hawking relation [3] between the entropy and the surface area of the horizon without loss of generality. In the present communication, the Kerr-Newman metric is derived in AdS space through direct application of pioneering works [4, 5] of Newman and his collaborators in asymptotically flat space, and then such thermodynamical properties as mentioned above are examined for AdS Kerr-Newman black holes in proper reference to the angular momentum of the black hole.

Let us start with the AdS Reissner-Nordström solution of a cosmological constant Λ, written in standard coordinates, as follows [1]:

$$ds^2 = \lambda^2 dt^2 - \lambda^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

in natural units, *i.e.* $\hbar = c = G = 1$ as well as $k = 1$, where

$$\lambda^2 = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3} r^2 \quad , \tag{2}$$

m and q are the mass and charge of the black hole, respectively, and the radius b of the AdS curvature is described as $b = \sqrt{-3/\Lambda}$; $\Lambda < 0$. Let us perform the coordinate transformation

$$du = dt - \lambda^{-2} dr \quad , \tag{3}$$

where u labels the null surfaces [4, 5]. Equation (1) is then rewritten in "null" coordinates as

$$ds^2 = \lambda^2 du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad . \tag{4}$$

The contravariant components of the metric [Eq. (4)] can be paraphrased into the alternate form [4, 5]

$$g^{\mu\nu} = \ell^\mu n^\nu + \ell^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu \quad , \tag{5}$$

where a pair of real null tetrads ℓ^μ and n^μ are expressed as

$$\ell^\mu = \delta_1^\mu \; ; \; n^\mu = \delta_0^\mu - \frac{1}{2}\left\{1 - m\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) + \frac{q^2}{r\bar{r}} + \frac{r\bar{r}}{b^2}\right\}\delta_1^\mu \tag{6}$$

and a pair of complex-conjugate null tetrads m^μ and $\bar{m}^\mu$ are described as

$$m^\mu = \frac{1}{\sqrt{2}\,\bar{r}}\left(\delta_2^\mu + \frac{i}{\sin\theta}\,\delta_3^\mu\right) \; ; \; \bar{m}^\mu = \frac{1}{\sqrt{2}\,r}\left(\delta_2^\mu - \frac{i}{\sin\theta}\,\delta_3^\mu\right) \quad . \tag{7}$$

Here the radial coordinate r is allowed to take complex values and $\bar{r}$ reads the complex conjugate of r.

Let us now introduce the complex coordinate transformation [4, 5]

$$u' = u - ia\cos\theta \; ; \; r' = r + ia\cos\theta \quad , \tag{8}$$

where u' and r' are allowed to be real. We then obtain the contravariant form of the metric

$$g'^{\mu\nu} = \ell'^\mu n'^\nu + \ell'^\nu n'^\mu - m'^\mu \bar{m}'^\nu - m'^\nu \bar{m}'^\mu \quad , \tag{9}$$

where

$$\begin{aligned} \ell'^\mu &= \delta_1^\mu \; ; \\ m'^\mu &= \frac{1}{\sqrt{2}(r' + ia\cos\theta)}\left\{ia\sin\theta\,(\delta_0^\mu - \delta_1^\mu) + \delta_2^\mu + \frac{i}{\sin\theta}\,\delta_3^\mu\right\} \; ; \\ n'^\mu &= \delta_0^\mu - \frac{1}{2}\left\{1 - \frac{2mr' - q^2}{r'^2 + a^2\cos^2\theta} + \frac{r'^2 + a^2\cos^2\theta}{b^2}\right\}\delta_1^\mu \quad . \end{aligned} \tag{10}$$

Let us next carry out the coordinate transformation

$$d\hat{t} = du - Adr \; ; \; d\hat{\varphi} = d\varphi - Bdr \quad , \tag{11}$$

where

$$A = V^{-1}\cdot(r^2 + a^2) \quad ; \quad B = -V^{-1}\cdot a \tag{12}$$

and

$$V = r^2 - 2mr + a^2 + q^2 + b^{-2}\cdot(r^2 + a^2\cos^2\theta)^2 \quad . \tag{13}$$

Here r' has been replaced with r. We are then eventually led to the AdS Kerr-Newman solution in Boyer-Lindquist coordinates [6] as follows:

$$\begin{aligned} ds^2 &= U^2dt^2 - V^{-1}\cdot(r^2 + a^2\cos^2\theta)dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 \\ &\quad - \left\{r^2 + a^2 + \left(\frac{2mr - q^2}{r^2 + a^2\cos^2\theta} - \frac{r^2 + a^2\cos^2\theta}{b^2}\right)a^2\sin^2\theta\right\}\sin^2\theta\,d\varphi^2 \\ &\quad + 2a\left(\frac{2mr - q^2}{r^2 + a^2\cos^2\theta} - \frac{r^2 + a^2\cos^2\theta}{b^2}\right)\sin^2\theta\,dtd\varphi \quad . \end{aligned} \tag{14}$$

where

$$U^2 = 1 - \frac{2mr - q^2}{r^2 + a^2\cos^2\theta} + \frac{r^2 + a^2\cos^2\theta}{b^2} \quad . \tag{15}$$

Here $\hat{t}$ and $\hat{\varphi}$ have been replaced with t and φ, respectively. It is of crucial importance to note that a plays the role of a measure of the angular momentum per unit mass, *i.e.* $a = j/m$ of the black hole and consequently the "geometrical" angular momentum is described as ma $[= Gj/c^3]$.

The horizon of the AdS Kerr-Newman black hole appears at $r = r_+$, where

$$2m = r_+\left\{1 + \frac{a^2 + q^2}{r_+^2} + \frac{(r_+^2 + a^2)^2}{b^2 r_+^2}\right\} \quad . \tag{16}$$

The inverse Hawking temperature β_H on the horizon r_+ is determined by

$$\begin{aligned} \beta_H &= 2\pi\left[U\frac{dU}{dr}\bigg|_{r=r_+}\right]^{-1} \\ &= \frac{2\pi(r_+^2 + a^2)^2}{m(r_+^2 - a^2) - q^2 r_+ + b^{-2}\cdot r_+(r_+^2 + a^2)^2} \quad . \end{aligned} \tag{17}$$

Accordingly the extremal black hole has the horizon at $r = r_E$, where

$$m(r_E^2 - a^2) - q^2 r_E + b^{-2} \cdot r_E (r_E^2 + a^2)^2 = 0 \tag{18}$$

on which $\beta_H^{-1} = 0$. Equations (17) and (18) eventually turn out to be

$$\beta_H = \frac{2\pi(r_+^2 + a^2)}{r_+ - m + 2b^{-2} \cdot r_+ (r_+^2 + a^2)} \tag{19}$$

and

$$r_E - m + 2b^{-2} \cdot r_E (r_E^2 + a^2) = 0 \quad , \tag{20}$$

respectively, after straightforward but rather tedious arithmetic. Let us suppose the case of $0 \lesssim a < b$; $0 \lesssim |q| < b$, hereafter, since our principal purpose is to clarify physical consequences of the AdS curvature on Kerr-Newman black holes in asymptotically flat space. We then obtain $r_E \sim m$. A nonextremal black hole will exhibit the concavo-convex structure of β_H as a function of r_+ for $m \lesssim r_+ \lesssim b$. For sufficiently large values of r_+, *i.e.* $r_+ \gg b$; $|q|$; a, β_H decreases indefinitely with the mass $m \approx r_+^3/2b^2$. Thus the AdS Kerr-Newman solution will exist at any finite temperature in full consonance with the AdS Reissner-Nordström solution [1] which in turn gives sharp contrast to the AdS Schwarzschild solution [2].

The Bekenstein-Hawking entropy S is associated with β_H through $\beta_H = dS/dm$ which inevitably yields

$$S = \pi(r_+^2 + a^2) \quad , \tag{21}$$

where the full use has been made of

$$\begin{aligned} \frac{dr_+}{dm} &= 2\left\{1 + \frac{a^2}{mr_+} + 2\frac{a^2(r_+^2 + a^2)}{b^2 m r_+}\right\} \\ &\quad \times \left\{1 - \frac{a^2 + q^2}{r_+^2} + 4\frac{r_+^2 + a^2}{b^2} - \frac{(r_+^2 + a^2)^2}{b^2 r_+^2}\right\}^{-1} \quad . \end{aligned} \tag{22}$$

The surface area A_+ of the horizon is written in the form

$$A_+ = \iint |g_{\theta\theta} \cdot g_{\varphi\varphi}|_{r=r_+}^{1/2} d\theta d\varphi \tag{23}$$

which eventuates in

$$A_+ = 2\pi \int_{-1}^{1} Z^{1/2} dz \quad , \tag{24}$$

where

$$Z = \left\{r_+^2 + a^2 + \frac{a^4}{b^2}(z^2 - 1)^2\right\}^2 + \frac{a^6}{b^2}(z^2 - 1)^3 - \frac{a^8}{b^4}(z^2 - 1)^4 \quad . \tag{25}$$

Thus we are inevitably led to

$$A_+ \simeq 4\pi(r_+^2 + a^2) + O\left(\frac{a^4}{b^2}\right) \gtrsim 4S \quad , \tag{26}$$

or equivalently $S \leqq 1/4 \cdot A_+$, where equality holds if and only if the angular momentum a per unit mass is negligibly small in comparison with the radius b of the AdS curvature. In striking contrast to the AdS Reissner-Nordström black hole [1] as well as the AdS Schwarzschild black hole [2], therefore, the so-called Bekenstein-Hawking relation, *i.e.* $S = 1/4 \cdot A_+$, is broken for the AdS Kerr-Newman black hole unless the angular momentum of the black hole is sufficiently small. It is parenthetically mentioned that Eqs. (21) and (26) are asymptotically reduced to

$$S \simeq \pi r_0^2 \simeq \frac{1}{4} A_+ \tag{27}$$

at large values of r_+, *i.e.* $r_+ \gg |q|$; a, where [1]

$$r_0 \left(1 + \frac{r_0^2}{b^2}\right) = 2m \quad . \tag{28}$$

The specific heat C_V is determined by

$$\begin{aligned} C_V &= -\beta^2 \left(\frac{d\beta}{dm}\right)^{-1} \\ &= -2\pi(r_+^2 + a^2) \cdot \left\{1 - \frac{a^2 + q^2}{r_+^2} + 4\frac{r_+^2 + a^2}{b^2} - \frac{(r_+^2 + a^2)^2}{b^2 r_+^2}\right\} \\ &\quad \times \left[\left(3 + 4\frac{a^2 r_+}{b^2 m}\right)\left\{1 - \frac{a^2 + q^2}{r_+^2} + 4\frac{r_+^2 + a^2}{b^2} - \frac{(r_+^2 + a^2)^2}{b^2 r_+^2}\right\}\right. \\ &\quad \left. -2\left\{1 + \frac{2}{b^2}(3r_+^2 + a^2)\right\}\left\{1 + \frac{a^2}{mr_+} + 2\frac{a^2(r_+^2 + a^2)}{b^2 m r_+}\right\}\right]^{-1} \end{aligned} \tag{29}$$

which is asymptotically reduced to

$$C_V \simeq 2\pi r_0^2 \tag{30}$$

at sufficiently large values of r_+, *i.e.* $r_+ \gg b$; $|q|$; a in full agreement with AdS Reissner-Nordström black holes [1] as well as AdS Schwarzschild black holes [2]. We are now in a position to touch upon the quantum degeneracy ρ of the AdS Kerr-Newman black hole. We then obtain

$$\rho \sim \tilde{c}\ \exp[\pi(r_+^2 + a^2)] \tag{31}$$

which is reduced to

$$\rho \sim \tilde{c}\ \exp[\pi r_0^2] \sim \tilde{c}\ \exp[\pi(2b^2 m)^{2/3}] \tag{32}$$

at sufficiently large values of m. Consequently the grand canonical ensemble of AdS Kerr-Newman black holes is well defined in harmony with AdS Reissner-Nordström black holes [1] in the same fashion as the canonical ensemble of AdS Schwarzschild black holes [2]. It is, by way of parenthesis, noted that microcanonical equilibrium configuration of AdS Kerr-Newman black holes is brought to realization in the sense of the equipartition configuration in consistency with AdS Reissner-Nordström black holes [1] as well as AdS Schwarzschild black holes [2].

The AdS Kerr-Newman solution has been derived on the basis of the AdS Reissner-Nordström solution [1] with the aid of the null tetrad formalism *à la* Newman and others [4, 5] in asymptotically flat space. Thermodynamical properties of AdS Kerr-Newman black holes have then been described in proper reference to the angular momentum of the black hole. The most salient observation is as follows: In striking contrast to AdS non-rotational black holes [1, 2], the AdS Kerr-Newman black hole satisfies the Bekenstein-Hawking relation between the entropy and the surface area of the horizon if and only if the angular momentum per unit mass is sufficiently small in natural units as compared with the radius of the AdS curvature. Let us conclude by emphasizing that the elaborate investigation of the thermal string ensemble in various D-brane backgrounds [7, 8] will be

prerequisite for building up the fruitful AdS thermodynamics or equivalently for clarifying physics underlying the gravitational dynamics on AdS-type spaces.

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