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Microcanonical Algorithm of Charged Rotating Dilatonic Black Holes

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ABSTRACT

Microcanonical ensemble paradigm is described in proper reference to the thermal aspect of the extremal state for a dilute gas of charged rotating dilatonic black holes on the basis of an exact solution of the Kaluza-Klein theory.

The microcanonical ensemble paradigm of primordial black holes has gradually been investigated in black hole physics [1 - 8]. Their scenario has been based upon the ansatz that primordial black holes are fundamental particle states in the sense of quantum extended objects. It has long been recognized that black holes coupled to a nonvanishing dilaton field have properties qualitatively different from those in ordinary Einstein gravity [9 - 15]. Indeed the charged unrotating black hole solution is substantially different from the standard Reissner-Nordström black hole. Moreover extremal charged black holes of small angular momentum are quite remote from the unrotating black holes. In the present communication, the microcanonical ensemble algorithm is commentarially applied *àla* refs. [3 (1992)], [4], [7] and [8] to a gas of charged rotating dilatonic black holes on the basis of an exact solution in the 5-dimensional theory of Kaluza and Klein. The most probable microcanonical configuration of such black holes is then self-consistently explained in proper regard to the thermal aspect of the extremal state.

Let us start with the stationary, axisymmetric, asymptotically flat solution of the Kaluza-Klein theory. When reinterpreted in four dimensions, the solution describes the charged rotating black hole coupled to a nonvanishing dilaton field. The resulting 4-dimensional metric is written in Boyer-Lindquist coordinates as follows [14, 16] :

$$\begin{aligned} \mathrm{d}s^2 \;=\; & -\frac{1-Z}{B}\,\mathrm{d}t^2 \;+\; B\,\frac{\rho}{\Delta}\,\mathrm{d}r^2 \;+\; B\rho\,\mathrm{d}\theta^2 \\ & +\left(B\,(r^2+a^2) \;+\; \frac{Z}{B}\,a^2\sin^2\theta\right)\,\sin^2\theta\,\mathrm{d}\varphi^2 \\ & -\frac{2Za}{B\sqrt{1-v^2}}\,\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\varphi \end{aligned} \tag{1}$$

in natural units, *i.e.* $\hbar = c = G = 1$ as well as $k = 1$, where

$$B \;=\; \left(1 \;+\; \frac{Zv^2}{1-v^2}\right)^{1/2} \quad ; \quad Z = \frac{2mr}{\rho} \tag{2}$$

and

$$\rho^2 \;=\; r^2 + a^2\,\cos^2\theta \quad ; \quad \Delta \;=\; r^2 + a^2 - 2m_0 r \quad ; \quad a \;=\; j_0/m_0\;. \tag{3}$$

Here the bare parameters m_0 and j_0 read the mass and angular momentum of the original Kerr solution, respectively, and v denotes the velocity of the boost in the fifth direction.

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The physical mass m, charge q and angular momentum j are described as

$$m = m_0 \left(1 + \frac{v^2}{2(1-v^2)}\right) \quad ; \quad q = \frac{m_0 v}{1-v^2} \quad ; \quad j = \frac{j_0}{\sqrt{1-v^2}} \ . \tag{4}$$

The detailed expression of a vector potential as well as a dilaton field is merely referred to ref. [14]. For $v = 0$, of course, the boosted line element (1) is reduced to the standard Kerr solution of mass m_0 and angular momentum j_0. The Kaluza-Klein solution yields a magnetic dipole moment

$$\mu \equiv g\,\frac{qj}{2m} = \frac{m_0 a v}{\sqrt{1-v^2}} \quad ; \quad g = 2 - v^2 \tag{5}$$

to a distant observer. It is almost needless to mention that the gyromagnetic ratio g approaches the Dirac value at no boost $v = 0$ and reduces to the classical value in the ultrarelativistic limit $|v| \to 1$ [16]. Furthermore, for $a = 0$, the boosted black hole metric (1) is physically equivalent to the charged unrotating dilatonic black hole solution [8 (2001) - 14] if the dimensionless parameter α governinig the coupling strength of the dilaton is set up at $\alpha = \sqrt{3}$. The detailed procedure is left out of consideration in the present context but merely referred to refs. [14] and [16].

As can transparently be seen from eq. (1), the causal structure of the charged rotating dilatonic black hole is resemblant to the original Kerr black hole. Thus we obtain both outer and inner horizons $r_\pm$ which read

$$r_\pm = m_0 \pm \sqrt{m_0^2 - a^2} \ . \tag{6}$$

The extremal black hole has the horizon at $r = r_{\rm E} = m_{\rm E} = |a|$. The surface area of the event horizon r_+ is expressed as

$$A_+ = \int\!\!\int |g_{\theta\theta} \cdot g_{\varphi\varphi}|^{1/2}_{r=r_+} {\rm d}\theta {\rm d}\varphi = \frac{8\pi m_0 \left(m_0 + \sqrt{m_0^2 - a^2}\right)}{\sqrt{1-v^2}} \ . \tag{7}$$

The inverse Hawking temperature $\beta_{\rm H} = T_{\rm H}{}^{-1}$ on the horizon r_+ is determined through the postulate of Eucleadian regularity as

$$\beta_{\rm H} = \frac{4\pi m_0 \left(m_0 + \sqrt{m_0^2 - a^2}\right)}{\sqrt{(1-v^2)(m_0^2 - a^2)}} \ . \tag{8}$$

The Bekenstein-Hawking entropy S is associated with $\beta_{\rm H}$ through

$$\beta_{\rm H} = \left(\frac{\partial S}{\partial m}\right)_{j,\,q} \tag{9}$$

at constant angular momentum j and charge q, which eventually yields

$$S = \frac{2\pi m_0 \left(m_0 + \sqrt{m_0^2 - a^2}\right)}{\sqrt{1-v^2}} = \frac{1}{4} A_+ \ . \tag{10}$$

Thus the so-called area theorem of the horizon is generically confirmed for the black hole solution under consideration. On the extremal horizon $r_{\rm E}$, the Hawking temperature $T_{\rm H}$ vanishes or diverges in accordance with $j \neq 0$ or $j = 0$, while the entropy $S_{\rm E}$ depends only on the angular momentum :

$$S_{\rm E} = 2\pi|j| = \frac{2\pi a^2}{\sqrt{1-v^2}} \quad ; \quad m_{\rm E} = |a|, \tag{11}$$

i.e. $S_{\rm E} = 0$ or $S_{\rm E} \neq 0$ according to $j = 0$ or $j \neq 0$. The Hawking radiation will then end up at extremality, irrespective of whether $j = 0$ or $j \neq 0$ [8 (2001), 14, 15].

The angular velocity Ω of the horizon is described as

$$\Omega = \left(\frac{\partial m}{\partial j}\right)_{S,\,q} = -\,T_{\rm H} \left(\frac{\partial S}{\partial j}\right)_{m,\,q} \tag{12}$$

at constant mass m and charge q, which yields

$$\Omega = \frac{a\sqrt{1-v^2}}{2m_0 r_+} \tag{13}$$

and turns out to be

$$\Omega_{\rm E} = \frac{\sqrt{1-v^2}}{2m_{\rm E}}\,{\rm sgn}(j) \tag{14}$$

at extremality. Thus the extremal angular velocity $\Omega_{\rm E}$ diverges as the rotational parameter a goes to zero. The electromagnetic potential Φ is prescribed as

$$\Phi = \left(\frac{\partial m}{\partial q}\right)_{S,\,j} = -\,T_{\rm H} \left(\frac{\partial S}{\partial q}\right)_{m,\,j} \tag{15}$$

at constant mass m and angular momentum j, which brings forth

$$\Phi = \frac{r_+ q\,(1-v^2)}{{r_+}^2 + a^2} = \frac{v}{2} \ . \tag{16}$$

Thus the electromagnetic potential Φ is literally related to the velocity v of the boost in the fifth direction. As a simple and natural consequence, we are self-consistently led to the first law of thermodynamics

$$\mathrm{d}m = T_{\mathrm{H}}\,\mathrm{d}S + \Omega\,\mathrm{d}j + \Phi\,\mathrm{d}q \tag{17}$$

for the black hole system, where $\beta_{\mathrm{H}} = {T_{\mathrm{H}}}^{-1}$, S, Ω and Φ are expressed as eqs. (8), (10), (13) and (16), respectively. The specific heat $C_{j,\,q}$ of the black hole is represented as

$$\begin{aligned} C_{j,\,q} &= -\,{\beta_{\mathrm{H}}}^2 \left(\frac{\partial m}{\partial \beta}\right)_{j,\,q} \\ &= -\,\frac{4\pi\, r_+}{\sqrt{1-v^2}}\,\frac{m_0^2}{2m_0 - (m_0^2+a^2)/\sqrt{m_0^2-a^2}} \end{aligned} \tag{18}$$

at constant angular momentum j and charge q. On the extremal horizon r_{E}, then, the specific heat (18) vanishes, irrespective of whether $j=0$ or $j \neq 0$, which yields sharp contrast to the charged rotating dilatonic black hole on the basis of an exact solution of Sen [16 as well as 8 (2001)] in 4-dimensional heterotic string theory. Irrespective of the value of a, furthermore, the specific heat $C_{j,\,q}$ is negative so long as $m_0^2 > (1+2/\sqrt{3})a^2$ and eventually turns out to be $C_{j,\,q} \simeq -8\pi m_0^2/\sqrt{1-v^2}$ at the asymptopia $m_0 \gg |a|$ and $m_0 \gg |q|$ at which we also obtain asymptotically $r_+ \simeq 2m_0$, $\beta_{\mathrm{H}} \simeq 8\pi m_0/\sqrt{1-v^2}$, $S \simeq 4\pi m_0^2/\sqrt{1-v^2}$, $\Omega \simeq a\sqrt{1-v^2}/r_+^2 \sim 0$ and $\Phi \simeq q(1-v^2)/r_+ \sim 0$. Accordingly the highly massive, nonextremal black hole solution is thermodynamically unstable and asymptotically bears a qualitative resemblance to the familiar Schwarzschild solution.

The quantum degeneracy ρ of the charged rotating dilatonic black hole is described as

$$\rho\,(m,\,j,\,q\,;\,v) \sim \bar{c}\,\exp\left[\,S(m,\,j,\,q\,;\,v)\,\right] \ , \tag{19}$$

where S reads the Bekenstein-Hawking entropy (10) and $\bar{c}$ remains to be self-consistently determined. The degeneracy (19) dominates over the Boltzmann factor $\exp[-\beta m]$ at $0 \leqq \beta < 4\pi m$ for sufficiently large values of mass. Consequently the (grand) canonical partition function of the black hole system diverges at any temperature higher than $(4\pi m)^{-1}$. The microcanonical ensemble algorithm is then applied to a dilute gas of charged rotating dilatonic black holes. The microcanonical density Ω_n of the n black hole configuration is asymptotically written in the form

$$\begin{aligned} \Omega_n(M,J,Q\,;\,v) &= \frac{1}{n!}\prod_{i=1}^{n}\int_{\tilde{m}_0}^{\infty}\mathrm{d}m_i \int_{-\tilde{m}_i^2/\sqrt{1-v^2}}^{\tilde{m}_i^2/\sqrt{1-v^2}}\mathrm{d}j_i \int_{-2m_i}^{2m_i}\mathrm{d}q_i\ \rho\,(m_i,\,j_i,\,q_i\,;\,v) \\ &\quad\times\ \delta(M-\sum_{i=1}^{n} m_i)\,\delta(J-\sum_{i=1}^{n} j_i)\,\delta(Q-\sum_{i=1}^{n} q_i) \end{aligned} \tag{20}$$

at high energy, where

$$\tilde{m}_i = m_i\left(1+\frac{v^2}{2(1-v^2)}\right)^{-1} \quad ; \quad \tilde{m}_0 = m_{\mathrm{E}}\left(1+\frac{v^2}{2(1-v^2)}\right) \ . \tag{21}$$

The Bekenstein-Hawking entropy S_n for such a gas of n black holes is represented as

$$S_n\,(M,\,J,\,Q\,;\,v) = \sum_{i=1}^{n} S\,(m_i,\,j_i,\,q_i\,;\,v) \ , \tag{22}$$

where $\sum_{i=1}^{n} m_i = M$, $\sum_{i=1}^{n} j_i = J$, $\sum_{i=1}^{n} q_i = Q$, $\tilde{m}_i^2 \geqq a_i^2$, $a_i = j_i\sqrt{1-v^2}/\tilde{m}_i$; $i = 1,\,2,\,\ldots,\,n$. The extrema of the entropy S_n are obtained by solving

$$\begin{pmatrix} \beta_{\mathrm{H}}(m_i,\,j_i,\,q_i\,;\,v) \\ \beta_{\mathrm{H}}(m_i,\,j_i,\,q_i\,;\,v)\Omega(m_i,\,j_i,\,q_i\,;\,v) \\ \beta_{\mathrm{H}}(m_i,\,j_i,\,q_i\,;\,v)\Phi(m_i,\,j_i,\,q_i\,;\,v) \end{pmatrix} = \begin{pmatrix} \lambda_M \\ -\lambda_J \\ -\lambda_Q \end{pmatrix} \ ;\ i = 1,\,2,\,\ldots,\,n \ , \tag{23}$$

where the Lagrange multipliers λ_M, λ_J and λ_Q are associated with the mass, angular momentum and charge constraints, respectively. Here β_{H}, Ω and Φ read the inverse Hawking temperature (8), the angular velocity (13) and the electromagnetic potential (16), respectively, of course.

The equipartition configuration S_n^{E} of n black holes immediately yields

$$S_n^{\mathrm{E}}(M,\,J,\,Q\,;\,v) \sim \frac{1}{n}\,S(M,\,J,\,Q\,;\,v) \quad ; \quad n \lesssim M^2/J \tag{24}$$

at the asymptopia. Thus the most probable distribution in association with the maximum of the entropy S_n is very far from thermal equilibrium. We are rather led *à la* refs. [3 (1992)], [4], [7] and [8 (2001)] to the most probable configuration in which a single massive, charged rotating dilatonic black hole occupies most of the mass $M-(n-1)\tilde{m}_0$, most of the charge $Q-(n-1)q$ and almost all the angular momentum J. All the resultant black holes are of the highly light, extremal charged unrotating dilatonic black holes of mass $\tilde{m}_0 = |q|/2$ and charge q, and consequently have vanishing entropy [8 (2001)]. Equation (20) will then turn out to be

$$\Omega_n(M,\ J,\ Q\ ;\ v)\ \simeq\ \frac{\bar{c}^{\,n}}{n!}\ \exp\left[\,S(M-(n-1)|q|/2\,,\ J\,,\ Q-(n-1)q\ ;\ v)\,\right] \quad (25)$$

at the asymptopia. The most probable configuration $\Omega_{\bar{N}}$ of black holes is obtained from the extremum condition

$$\left.\frac{\mathrm{d}\Omega_n(M,\ J,\ Q\ ;\ v)}{\mathrm{d}n}\right|_{n=\bar{N}(M,\,J,\,Q\,;\,v)} = 0\ . \quad (26)$$

Let us suppose the asymptopia $M \gg |Q|$ and $M \gg (\bar{N}-1)|q|/2$. As a simple and natural consequence of the microcanonical equilibrium condition (26), then, we are self-consistently led to the maximum energy M_c beyond which no microcanonical equilibrium configuration is brought to realization, where

$$M_c\ \sim\ \frac{1}{8\pi|q|}\ \{\ln\ \bar{c}\ -\psi(\bar{N}+1)\}\ ;\ \bar{N}\ =\ 1 \quad (27)$$

and $\psi(x)$ reads the psi function. Thus eq. (27) plays the role of the self-consistency condition on the microcanonical density at $M \simeq M_c$. The present observation is paraphrased as follows: The gas is composed of the only one highly massive, charged rotating dilatonic black hole at $M \simeq M_c$ whose spatial extension is of the order of the Planck volume. The emergence of the maximum energy M_c will be inherent to the fact that the (highly) nonextremal charged rotating dilatonic black hole is thermodynamically unstable and asymptotically resemblant to the Schwarzschild black hole. It may be of interest to call to remembrance the recent stimulating argument of Reall [17] on the so-called Gregory-Laflamme instability [18] of nonextremal black branes in the sense that

the highly nonextremal black brane shows a close similarity to the uncharged black brane of negative specific heat and will eventually turn out to be classically unstable. Physics underlying the critical mass M_c still remains to be settled, anyhow, in the microcanonical black hole ensemble. The large $\bar{N}$ behaviour is described in the same manner as for a dilute gas of black holes in ordinary Einstein gravity. The detailed discussion is wholly left out of consideration in the present context but merely referred to refs. [3 (1992)], [4], [7] and [8 (2001)].

The microcanonical ensemble algorithm has been discussed in proper reference to the thermal aspect of the extremal state for a dilute gas of charged rotating black holes coupled to a nontrivial dilaton field on the basis of an exact solution of the Kaluza-Klein theory. The principal observation is as follows : The most probable microcanonical configuration is self-consistently substantialized at high energy in asymptotically flat space through the so-called single-massive-mode dominance scenario [1 - 3 (1992)] in the sense that most of the mass, most of the charge and almost all of the angular momentum of the whole system are asymptotically associated with a single energetic, charged rotating dilatonic black hole. Consequently it may be possible to claim that we have succeeded in shedding some light upon statistical mechanical description of microcanonical black hole ensemble beyond the scope of ordinary Einstein gravity so long as the demise of the Hawking radiation is actually brought to realization at extremality.

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