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Comments on Microcanonical Algorithm of Kerr-Newman Dilatonic Black Holes

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ABSTRACT

Statistical mechanics of a dilute gas of Kerr-Newman dilatonic black holes is commentarially discussed in the microcanonical ensemble paradigm. The most probable microcanonical configuration of black holes is then self-consistently described within the framework of the small charge-to-mass approximation.

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The microcanonical description of a gas of primordial black holes has gradually turned out to be a real solid subject of black hole physics [1 - 8]. Their paradigm is based upon the ansatz that primordial black holes are fundamental particle states in the sense of quantum extended objects. In the present communication, the microcanonical ensemble algorithm is commentarially applied *à la* refs. [3] (1992), [4], [7] and [8] (2001) to a gas of Kerr-Newman (KN) dilatonic black holes within the framework of the small charge-to-mass approximation [9, 10]. The most probable microcanonical distribution of KN dilatonic black holes is then self-consistently explained in the single-massive-mode dominance scenario [1, 2, 3 (1992)].

Let us consider a charged rotating black hole coupled to a nonvanishing dilaton field in four dimensions. It has long been recognized that the uncharged rotating black hole solution is identical to the familiar Kerr solution of ordinary Einstein gravity with a constant dilaton field and that the charged unrotating black hole solution is significantly different at nonzero dilaton coupling strength from the standard Reissner-Nordstöm (RN) solution of the Einsein-Maxwell field [11 - 17]. Moreover it has been shown that extremal rotating black holes with small charge will resemble the uncharged black holes, while an arbitrary small amount of angular momentum can substantially affect the extremal unrotating black holes coupled to a dilaton [16, 17]. It still remains to be settled which of these two altrenatives is likely to bear qualitatve resemblance to the charged rotating black hole in general coupled to a dilaton field. Our principal concern is now confined to newfashioned perturbative solutions [9, 10] of the Einstein-Maxwell dilatonic equations for arbitrary angular momentum in the small charge-to-mass approximation. The corresponding metric in their solutions can be simplified by suitably shifting the radial coordinate r and substituting for the bare parameters of mass m, charge q and angular momentum $j \equiv \alpha m$ the corresponding physical quantities $\tilde{m}$, $\tilde{q}$ and $\tilde{j} \equiv \alpha\tilde{m}$, respectively, which determine the geodesic motion in the asymptotically flat region. We are then led to the standard

KN metric up to order $(q/m)^2$ which is written in Boyer-Lindquist coordinates as follows:

$$\begin{aligned} \mathrm{d}s^2 &= \left(1-\frac{2\tilde{m}r-\tilde{q}^2}{\rho^2}\right)\mathrm{d}t^2-\frac{\rho^2}{\Delta}\,\mathrm{d}r^2-\rho^2\,\mathrm{d}\theta^2 \\ &- \left(r^2+\alpha^2+\frac{(2\tilde{m}r-\tilde{q}^2)\,\alpha^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2 \\ &- \frac{2\alpha\,(2\tilde{m}r-\tilde{q}^2)}{\rho^2}\sin^2\theta\,\mathrm{d}\varphi\,\mathrm{d}t \end{aligned} \tag{1}$$

in natural units, *i.e.* $\hbar = c = G = 1$ as well as $k = 1$, where

$$\Delta = r^2-2\tilde{m}r+\alpha^2+\tilde{q}^2\ ;\ \rho^2 = r^2+\alpha^2\cos^2\theta\ . \tag{2}$$

Here [9, 10]

$$\tilde{m} = m\left(1+\frac{a^2q^2}{6m^2}\right)\ ;\ \tilde{q} = q\left(1+\frac{a^2q^2}{3m^2}\right)\ ;\ \tilde{j} = \alpha\tilde{m} \tag{3}$$

up to order $(q/m)^2$, where a is the dilaton coupling constant. It may be of interest to note that the newfashioned KN dilatonic solution induces magnetic dipole moment $\tilde{\mu}$ to a distant obserber as follows [9, 10, 16]:

$$\tilde{\mu} \equiv \tilde{g}\frac{\tilde{q}\tilde{j}}{2\tilde{m}} = \alpha\tilde{q}\left(1-\frac{a^2q^2}{6m^2}\right)\ ;\ \tilde{g} = 2\left(1-\frac{a^2q^2}{6m^2}\right) \tag{4}$$

in proper reference to $\mu \equiv g/2\cdot qj/m = \alpha q$; $g = 2$ [16 - 18] for the original KN solution, where $^{[}\tilde{g}^{]}$ reads gyromagnetic ratio. As can transparently be seen from the matric (1), the causal structure of the original KN black hole solution is not affected at order $(q/m)^2$ by the presence of a nonvanishing dilaton field. Thus we obtain both outer and inner horizons $r_\pm$ which read

$$r_\pm = \tilde{m} \pm \sqrt{\tilde{m}^2-\alpha^2-\tilde{q}^2}\ . \tag{5}$$

The extremal black hole has the horizon at $r = r_{\mathrm{E}} = \tilde{m}_{\mathrm{E}} = \sqrt{\alpha^2+\tilde{q}^2}$. The surface area A_{KN} of the event horizon r_+ evidently reads

$$A_{\mathrm{KN}}(\tilde{m},\ \alpha,\ \tilde{q}) = 4\pi\,({r_+}^2+\alpha^2)\ . \tag{6}$$

The inverse Hawking temperature $\beta_{\mathrm{H}} = {T_{\mathrm{H}}}^{-1}$ on the horizon r_+ is determined through the postulate of Eucleadian regularity as

$$\beta_{\mathrm{H}} = 2\pi\,\frac{{r_+}^2+\alpha^2}{\sqrt{\tilde{m}^2-\alpha^2-\tilde{q}^2}}\ . \tag{7}$$

The Bekenstein-Hawking entropy S_{KN} is associated with β_{H} through

$$\beta_{\mathrm{H}} = \left(\frac{\partial S_{\mathrm{KN}}}{\partial\tilde{m}}\right)_{\tilde{j},\,\tilde{q}} \tag{8}$$

which eventually yields $S_{\mathrm{KN}} = \pi({r_+}^2+\alpha^2) = 1/4\cdot A_{\mathrm{KN}}$. The so-called area theorem of the horizon is then transparently confirmed for the KN dilatonic black hole, irrespective of the value of a . It is parenthetically mentioned that the extremal black hole has finite entropy $S = \pi({\tilde{m}_{\mathrm{E}}}^2+\alpha^2)$ and vanishing temperature $T_{\mathrm{H}} = 0$, i.e. the demise of the Hawking radiation is brought to realization at extremality. The angular velocity Ω of the horizon is prescribed through the Christodoulou-Bekenstein formula as

$$\Omega = -\,T_{\mathrm{H}}\left(\frac{\partial S_{\mathrm{KN}}}{\partial\tilde{j}}\right)_{\tilde{m},\,\tilde{q}} = \frac{\alpha}{{r_+}^2+\alpha^2}\ . \tag{9}$$

Similarly the electromagnetic potential Φ measured at infinity with respect to the horizon is described as

$$\Phi = -\,T_{\mathrm{H}}\left(\frac{\partial S_{\mathrm{KN}}}{\partial\tilde{q}}\right)_{\tilde{m},\,\tilde{j}} = \frac{r_+\tilde{q}}{{r_+}^2+\alpha^2}\ . \tag{10}$$

As a consequence, we are self-consistently led to the first law of thermodynamics

$$\mathrm{d}\tilde{m} = T_{\mathrm{H}}\,\mathrm{d}S_{\mathrm{KN}}+\Omega\,\mathrm{d}\tilde{j}+\Phi\,\mathrm{d}\tilde{q} \tag{11}$$

for the KN dilatonic black hole system. The specific heat $C_{\tilde{j},\,\tilde{q}}$ of the black hole is represented as

$$C_{\tilde{j},\,\tilde{q}} = -\,{\beta_{\mathrm{H}}}^2\left(\frac{\partial\tilde{m}}{\partial\beta}\right)_{\tilde{j},\,\tilde{q}} = -\,\frac{\tilde{m}\beta_{\mathrm{H}}}{1+\beta_{\mathrm{H}}(8\pi\tilde{m}-\beta_{\mathrm{H}})\,/\,4\pi S_{\mathrm{KN}}}\ . \tag{12}$$

It is to be remembered that the specific heat (12) vanishes at expremality, turns out to be negative so long as $0 < \beta_{\mathrm{H}} < 4\pi\tilde{m}[1+\sqrt{1+({r_+}^2+\alpha^2)/4\tilde{m}^2}]$ and is then reduced to $C_{\tilde{j},\,\tilde{q}} \simeq -\tilde{m}\beta_{\mathrm{H}} \sim -2\pi{r_+}^2 \sim -8\pi m^2$ at the asymptopia, i.e., $r_+ \sim 2m \gg \alpha, q$.

The quantum degeneracy $\rho_{\rm KN}$ of the KN dilatonic black hole is described as

$$\rho_{\rm KN}(\tilde{m},\ \tilde{j},\ \tilde{q}\ ;\ a)\ \sim\ \bar{c}\ \exp[S_{\rm KN}(\tilde{m},\ \tilde{j},\ \tilde{q}\ ;\ a)] \tag{13}$$

in the small chage-to-mass approximation, where $S_{\rm KN}$ reads the Bekenstein-Hawking entropy and $\bar{c}$ remains to be self-consistently determined. Since the degeneracy (13) dominates over the Boltzmann factor $\exp[-\beta m]$ at $0 \leqq \beta < 4\pi m$, the (grand) canonical partition function of the black hole system diverges at any temperature higher than $(4\pi m)^{-1}$. Statistical mechanics of such system is then formulated in the microcanonical ensemble approach. Let us now consider a dilute gas of KN dilatonic black holes in the small charge-to-mass approximation. The microcanonical density $\Omega_{\rm KN}$ of states is then expressed as

$$\Omega_{\rm KN}(E,\ J,\ Q\ ;\ a)\ =\ \sum_{n=1}^{\infty}\ \Omega_n\ (E,\ J,\ Q\ ;\ a)\quad , \tag{14}$$

where the density of the n black hole configuration is asymptotically given by [1 - 4, 7, 8]

$$\begin{aligned}\Omega_n(M,J,Q\ ;\ a)\ &=\ \frac{1}{n!}\prod_{i=1}^{n}\int_{\tilde{m}_0}^{\infty}{\rm d}\tilde{m}_i\ \int_{-\tilde{m}_i{}^2}^{\tilde{m}_i{}^2}{\rm d}\tilde{j}_i\int_{-\sqrt{\tilde{m}_i{}^2-\alpha_i{}^2}}^{\sqrt{\tilde{m}_i{}^2-\alpha_i{}^2}}{\rm d}\tilde{q}_i\ \rho_{\rm KN}(\tilde{m}_i,\tilde{j}_i,\tilde{q}_i,;a)\\ &\times\ \delta(M-\sum_{i=1}^{n}\tilde{m}_i)\,\delta(J-\sum_{i=1}^{n}\tilde{j}_i)\,\delta(Q-\sum_{i=1}^{n}\tilde{q}_i)\end{aligned} \tag{15}$$

at high energy. Here we have set up $E = M$ and $\tilde{m}_0$ reads the minimum allowed mass for each black hole. The Bekenstein-Hawking entropy S_n for such a gas of n black holes is written in the form $S_n\ (M,\ J,\ Q\ ;\ a)\ =\ \sum_{i=1}^{n}\ S_{\rm KN}(\tilde{m}_i,\ \tilde{j}_i,\ \tilde{q}_i\ ;\ a)$, where $\sum_{i=1}^{n}\tilde{m}_i = M,\ \sum_{i=1}^{n}\tilde{j}_i = J,\ \sum_{i=1}^{n}\tilde{q}_i = Q,\ \tilde{m}_i{}^2 \geqq \alpha_i{}^2 + \tilde{q}_i{}^2\ ;\ \alpha_i \equiv \tilde{j}_i/\tilde{m}_i\ ;\ i = 1,2,\ldots,n$. The extrema of the entropy S_n are obtained by solving

$$\begin{pmatrix}\beta_{\rm H}(\tilde{m}_i,\ \tilde{j}_i,\ \tilde{q}_i\ ;\ a)\\ \beta_{\rm H}(\tilde{m}_i,\ \tilde{j}_i,\ \tilde{q}_i\ ;\ a)\Omega(\tilde{m}_i,\ \tilde{j}_i,\ \tilde{q}_i\ ;\ a)\\ \beta_{\rm H}(\tilde{m}_i,\ \tilde{j}_i,\ \tilde{q}_i\ ;\ a)\Phi(\tilde{m}_i,\ \tilde{j}_i,\ \tilde{q}_i\ ;\ a)\end{pmatrix} = \begin{pmatrix}\lambda_M\\ -\lambda_J\\ -\lambda_Q\end{pmatrix};i = 1,\ 2,\ \ldots,\ n\quad , \tag{16}$$

where the Lagrange multipliers $\lambda_M,\ \lambda_J$ and λ_Q are associated with the mass, angular momentum and charge constraints, respectively. There exists the equipartition configuration $S_n^{\rm E}$ of n black holes in which $\tilde{m}_i = M/n,\ \tilde{j}_i = J/n$ and $\tilde{q}_i = Q/n; i = 1,2,\ldots,n$

for $n \lesssim M^2/J$. We then obtain $S_n^{\rm E} \sim 1/n \cdot S_{\rm KN}(M,\ J,\ Q\ ;\ a)$ for sufficiently large values of mass M, where $S_{\rm KN}(M,\ J,\ Q\ ;\ a)$ in turn is the Bekenstein-Hawking entropy for the hypothetical configuration in which the only one KN dilatonic black hole acquires all the mass, all the angular momentum and all the charge of the whole system. Thus the most probable configuration in association with the maximum of the entropy S_n is very far from thermal equilibrium. We are rather led *à la* refs. [3] (1992), [4], [7] and [8] (2001) to the most probable configuration in which a single massive KN dilatonic black hole carries most of the mass $M - (n-1)\tilde{q}$, most of the charge $Q - (n-1)\tilde{q}$ and almost all the angular momentum J, and consequently is surrounded by the resultant $(n-1)$ highly light extremal RN black holes of charge $\tilde{q}$. Equation (15) will then be reduced to

$$\begin{aligned}\Omega_n(M,\ J,\ Q\ ;\ a)\ &\simeq\ \frac{\bar{c}^n}{n!}\exp[(n-1)\,S_{\rm KN}(\tilde{q},\ 0,\ \tilde{q}\ ;\ a)]\\ &\times\ \exp[S_{\rm KN}(M-(n-1)\tilde{q},\ J,\ Q-(n-1)\tilde{q}\ ;\ a)]\end{aligned} \tag{17}$$

at sufficiently high energy. Let us recall to remembrance that $S_{\rm KN}(q,\ 0,\ q,\ ;\ a) = \pi q^2$ is identical with the Bekenstein-Hawking entropy of the extremal RN black hole which has vanishing Hawking temperature $T_{\rm H} = 0$. The most probable configuration $\Omega_{\bar{N}}$ of black holes is obtained from the extremum condition

$$\left.\frac{{\rm d}\Omega_n(M,\ J,\ Q\ ;\ a)}{{\rm d}n}\right|_{n=\bar{N}(M,\ J,\ Q\ ;\ a)} = 0\quad . \tag{18}$$

Let us now suppose $M \gg Q$ and $M \gg (\bar{N}-1)\tilde{q}$. There will then appear the maximum energy M_c beyond which no microcanonical equilibrium configuration is materialized. Thus we obtain

$$M_c\ \sim\ \frac{1}{8\pi q}\ \{\ln\ \bar{c}\ -\psi(2)\}\ ;\ \ln\ \bar{c}\ >\ \psi(2) \tag{19}$$

which in turn plays the role of the self-consistency condition on the microcanonical density $\Omega_{\rm KN}$ at the asymptopia $M \simeq M_c$. Here $\psi(x) = {\rm d}\ln\Gamma(x)/{\rm d}x$ reads the psi function. The inverse microcanonical critical temperature β_c literally reads

$$\beta_c\ \simeq\ \left(\frac{\partial\ln\Omega_{\bar{N}=1}(M_c,\ J,\ Q\ ;\ a)}{\partial M_c}\right)_{J,\,Q}\ =\ \beta_{\rm H}(M_c,\ J,\ Q\ ;\ a)\ \sim\ 8\pi\ M_c \tag{20}$$

in full consonance with the Bekenstein-Hawking entropy $S_{\rm KN}(M_c,\ J,\ Q\ ;\ a) \sim 4\pi {M_c}^2$. The present observation is simply and naturally paraphrased as follows: The gas is composed of the only one highly massive KN dilatonic black hole at the asymptopia $M \simeq M_c$ whose mass, angular momentum and charge are M_c, J and Q, respectively, and spatial extension is of the order of the Planck volume. It still remains to be clarified whether or not the maximum energy M_c really plays the role of the disintegration point for the "enigmatic" phase transition of the black hole system. We do not enter into the deeper details but merely refer to refs.[7] and [8] (2001).

Statistical mechanics of a dilute gas of KN dilatonic black holes has been studied in the microcanonical ensemble paradigm within the framework of the small charge-to-mass approximation *à la* refs. [9] and [10]. As anticipated, the present argument is resemblant in every detail to the previous one [7] of ourselves for a dilute system of standard KN black holes. The principal observation is as follows : Irrespective of the value of a, the most probable microcanonical distribution is far from thermal equilibrium and rather turns out to be the configuration in which most of the mass, most of the charge and almost all of the angular momentum of the whole gas are asymptotically associated with a single energetic KN dilatonic black hole. The fruitful thermodynamical investigation of string excitations will be prerequisite for the real solid materialization of the physical significance of the maximum mass in the microcanonical black hole ensemble.

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