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Low-Energy Chiral Lagrangian from the Gauged Nambu-Jona-Lasinio Model

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Low-Energy Chiral Lagrangian from the Gauged Nambu-Jona-Lasinio Model

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Abstract

The low-energy effective Lagrangian of the gauged Nambu-Jona-Lasinio model is derived as a series in powers of the derivatives of composite fields. Explicit expressions are obtained for the effective potential and the kinetic term (with two derivatives) along that part of critical line of the model for which the values of the gauge coupling satisfy $0 \leq \alpha < \alpha_c = \pi/3$. From the solution of linearized Dyson-Schwinger equation for the fermion mass function in planar quenched QED, formulas for the parameters of the effective Lagrangian (the pion decay constant F_π, the mass of scalar particle M_σ and Yukawa coupling g_Y) are given as functions of the suchgauge coupling. The role of the dynamical dimensions of the fields and the mechanism of scale symmetry breaking in this model are discussed.

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1 Introduction

The derivation of a low-energy effective action from some microscopic theory is an important task for different physical problems. Everybody knows that everything around us exists due to electromagnetic forces. However, nobody uses quantum electrodynamics (QED) to describe, for example, properties of solid and liquid states. Such a description is usually done in terms of an effective theory, which, in principle, should be derived from QED. Other famous examples are the Ginzburg-Landau effective action for the underlying BCS theory of superconductivity and the σ-model [1], which is a low energy version of the Nambu-Jona-Lasinio (NJL) model [2]. In the sixties the so-called chiral Lagrangians [3] were very popular, which describe the low-energy scattering processes of mesons and baryons rather well. Now we know the underlying microscopic theory: it is quantum chromodynamics (QCD) and the problem is how to derive such chiral Lagrangians from QCD.

Usually, following the path from a microscopic theory at high energies to the effective one at low energies we pass several such effective theories. For example, to describe properties of superconductors we have a chain of effective Hamiltonians: Fröhlich Hamiltonian $\rightarrow$ BCS Hamiltonian $\rightarrow$ Ginzburg-Landau Hamiltonian. In particle theory we would expect the following chain to be valid: string theories $\rightarrow$ unified theories (like $SU(5)$ or $O(10)$) $\rightarrow$ QCD + electroweak interactions $\rightarrow$ chiral Lagrangians $\rightarrow$ theory of nuclear forces. Each of these effective theories is characterized by its own energy scale. For QCD it is a Λ_{QCD}-parameter ($\sim$ 300 MeV). For energies much higher than Λ_{QCD} the description in terms of quarks and gluons using the QCD Lagrangian looks simpler, but for energies much lower than Λ_{QCD}, the description in terms of quarks bound states (π, ρ, K, N) with chiral Lagrangians becomes simpler. Thus the very important task is to find the dynamical mechanism for generating such a characteristic energy scale. In the Weinberg-Salam theory the fundamental scalar fields fulfil this role; the potential for the scalar fields has a nontrivial minimum which produces such a scale and this leads, in its turn, to mass generation for the W and Z bosons (Higgs mechanism). In QCD we have no fundamental scalar fields, instead we have the bound states of quarks and gluons, and we expect that the Higgs mechanism would operate at the level of the effective Lagrangian describing

the interaction of these bound states.

In these lectures I would like to consider the derivation of the low-energy effective action, taking the gauged NJL model (GNJL) as underlying microscopic theory. By the low-energy effective action we understand here the action derived after integration in the functional integral of the theory over some fields (for example, unobserved quarks and gluons in QCD, heavy particles in unified theories) or over high frequencies of the fields. After such an integration we are left with an effective action, which is known as a low-energy action to distinguish it from the one-particle irreducible (1PI) effective action generated by integrating over all fields. Usually it is represented by a very complicated nonlocal expression, which afterwards can be expanded in powers of derivatives of the observed fields, where each term is already a local expression. Such an understanding is closer to Wilson's approach to the effective action [4] and it is becoming now more and more popular among physicists (for a nice introduction to subject, see the recent review [5]). This leads to a new point of view about the meaning of renormalization and its intimate relation to the notions of compositeness, phase transitions, continuum limit.

The derivation of the low-energy action for pure NJL model has been considered in a number of papers [6]. The problem, however, is substantially more complicated if the microscopic Lagrangian includes gauge interactions, which is a typical situation in physics. The GNJL model is of particular interest, since nowadays it lies at the basis of various scenarios of dynamical electroweak symmetry breaking in standard Weinberg-Salam theory without a Higgs sector, in which the idea of a composite Higgs particle is realized as a bound state of $t\bar{t}$ quarks (for a review see Refs.[7-9]). It may also have a the promising application to more complex gauge theories with a slowly running coupling constant, such as walking extended technicolour models [10].

These lectures are based on the works published together with my co-authors (and good friends) Volodya Miransky and Volodya Kushnir [11-14], to whom I express my deep gratitude for fruitful cooperation.

2 Nambu-Jona-Lasinio Model and Avatars

The gauged NJL model is described by the Lagrangian

$$\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}i\gamma^\mu D_\mu\psi + \frac{G_0}{2}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2] \quad , \tag{2.1}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is a covariant derivative and the last term is a chirally invariant four-fermion interaction with G_0 the corresponding Fermi coupling constant. Another name for this model is QED with additional four-fermion interaction. In the absence of a fermion mass term that breaks the chiral symmetry explicitly, the Lagrangian (2.1) possesses a $U(1)$ gauge symmetry and a global $U_L(1) \times U_R(1)$ chiral symmetry. (It is not difficult to extend our results for $SU_L(N_f) \times SU_R(N_f)$ chiral symmetry with N_f fermion flavours.)

It is well known that in the planar quenched approximation the GNJL model (2.1) has a nontrivial phase structure [15] in the coupling constant plane (g, α), where $g \equiv G\Lambda^2/4\pi^2, \alpha \equiv e^2/4\pi^2$ as shown in Fig.1 (Λ is a cut-off). In fact there is a critical line

$$\begin{aligned} g_c(\alpha) = \tfrac{1}{4}(1+\omega)^2, \quad \omega = \sqrt{1-\tfrac{\alpha}{\alpha_c}}, \quad & \alpha \le \alpha_c = \tfrac{\pi}{3}, \\ g = \tfrac{1}{4}, \quad \alpha = \alpha_c, & \end{aligned} \tag{2.2}$$

above which the gap equation for the fermion self-energy $\Sigma(p)$ has a nontrivial solution. Thus the chiral symmetry is dynamically broken, which implies the existence of a nonzero vacuum condensate $\langle\bar{\psi}\psi\rangle$. One end point ($g = 1$, $\alpha = 0$) of the critical line (2.2) corresponds to the ordinary NJL model, while the other one ($g = 0$, $\alpha = \alpha_c = \pi/3$) corresponds to pure QED. Fine tuning the coupling g to g_c in such a way that $m_d/\Lambda \ll 1$, where $m_d \equiv \Sigma(0)$ is the dynamical mass of a fermion, and Λ the ultra-violet cut-off or the scale of

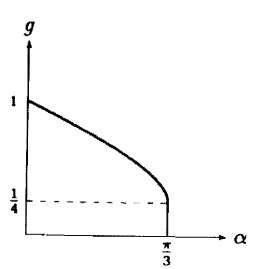


Fig. 1: Critical line in the (α, g) plane of the GNJL model separating spontaneously broken and unbroken phases of the chiral symmetry.

the "new physics", a nontrivial continuum limit ($m_d/\Lambda \to \infty$) can be reached just as in pure QED [16-19]. The spectrum of such a theory contains pseudoscalar (π) and scalar (σ) bound states which become dynamically active in the vicinity of the critical line. In these lectures it is our intention to derive the effective low-energy Lagrangian describing the interaction of such dynamical degrees of freedom as π and σ.

Let us introduce the chiral fields σ and π, rewriting the Lagrangian (2.1) in the form

$$\mathcal{L}_2 = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}i\gamma^\mu D_\mu\psi - \bar{\psi}(\sigma + i\gamma_5\pi)\psi - \frac{1}{2G_0}(\sigma^2 + \pi^2). \tag{2.3}$$

One can readily verify the equivalence of the Lagrangians (2.1) and (2.3) just making use of the Euler-Lagrange equations

$$\sigma = -G_0\bar{\psi}\psi\,, \qquad \pi = -G_0\bar{\psi}i\gamma_5\psi. \tag{2.4}$$

This can be expressed in another way using the language of path integrals. Consider the generating functional Z:

$$\begin{aligned} Z &= N\int d\psi d\bar{\psi} dA \exp\left[i\int d^4x \mathcal{L}_1(\psi,\bar{\psi},A)\right] \\ &= N'\int d\psi d\bar{\psi} dA d\sigma d\pi \exp\left[i\int d^4x \mathcal{L}_2(\psi,\bar{\psi},A,\sigma,\pi)\right], \end{aligned} \tag{2.5}$$

N, N' are some normalization constants, and it is understood that the integration measure of the gauge field A includes factors connected with the gauge fixing. The last equality in (2.5) is the so-called Hubbard-Stratonovich trick or the auxilary field method, reintroduced into quantum field theory by Gross and Neveu [20]. Note that $\mathcal{L}_1$ is quartic in ψ whereas $\mathcal{L}_2$ is only quadratic, and now we should integrate over fermion and gauge fields in the functional integral.

Suppose we have performed the integration over $\psi, \bar{\psi}$ and A (in QCD they are for example unobserved fields; yet we can "see" only composite particles), we then get

$$Z = \int d\sigma d\pi e^{i\Gamma(\sigma,\pi)} \quad , \tag{2.6}$$

where the effective action $\Gamma(\sigma,\pi)$ is defined by

$$\Gamma(\sigma,\pi) = -\frac{1}{2G_0}\int d^4x(\sigma^2+\pi^2) + \tilde{\Gamma}(\sigma,\pi) \quad , \tag{2.7}$$

and

$$e^{i\tilde{\Gamma}(\sigma,\pi)} = \int d\psi d\bar{\psi} dA \exp\{i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}[i\gamma^\mu D_\mu - (\sigma + i\gamma_5\pi)]\psi\right]\} \tag{2.8}$$

Precisely here a significant difference arises between the model under consideration and the standard NJL model. In the pure NJL model there are no gauge fields and we can easily perform the integration over fermion fields. Therefore, the problem reduces to the calculation of the fermion determinant in the external fields $\sigma(x)$ and $\pi(x)$

$$i\tilde{\Gamma}(\sigma,\pi) = \frac{1}{2}\ln\det \mathrm{H} = \frac{1}{2}\mathrm{Tr}\ln \mathrm{H}, \tag{2.9}$$

where

$$\mathrm{H} = [\mathrm{i}\hat{\partial} - (\sigma + \mathrm{i}\gamma_5\pi)]^2. \tag{2.10}$$

Further computation can be done using the Fock-Schwinger formalism of proper time or the generalized ζ-function method [21]. Both of them express $\mathrm{Tr}\ln \mathrm{H}$ through the trace of the heat kernel

$$K(x,x',t) = \langle x|e^{-t\mathrm{H}}|x'\rangle \tag{2.11}$$

(for convenience we are writing the operator H in Euclidean space where it is an elliptic operator). For example, we obtain

$$\mathrm{Tr}\ln \mathrm{H} = -\zeta'(0), \qquad \zeta(\mathrm{s}) = \frac{1}{\Gamma(\mathrm{s})}\int_0^\infty \mathrm{d}t\, t^{\mathrm{s}-1}\mathrm{Tr}e^{-t\mathrm{H}} \tag{2.12}$$

with the method of the generalized ζ-function, or

$$\mathrm{Tr}\ln \mathrm{H} = \int_{1/\Lambda^2}^\infty \frac{\mathrm{d}t}{t}\mathrm{Tr}\, e^{-t\mathrm{H}} \tag{2.13}$$

in the Fock-Schwinger formalism. (Note that the ζ-function works as analytical regularization and does not require an explicit cut-off, while we have to introduce such cut-off at the lower limit of integration in Eq. (2.13)).

A mathematical theorem due to Seeley [22] states that for a certain class of elliptic operators H, there exists at $t \to 0_+$ an expansion of the diagonal matrix elements of the heat kernel in the form

$$\mathrm{K}(x,x,t) \sim \sum_{m\geq 0} t^{\frac{m-n}{2r}}\mathrm{E_m}(\mathrm{x}|\mathrm{H}), \qquad \mathrm{t} \to 0_+ . \tag{2.14}$$

Here n is the dimension of the spacetime, and $2r$ is the order of the differential operator H. In the mathematical literature these coefficients $\mathrm{E_m}$ are called "Seeley-Gilkey" coefficients, whereas to physicists they are known as "Schwinger-DeWitt" coefficients. They are constructed from σ, π and their derivatives. For their explicit expressions see [23].

To derive the low-energy effective Lagrangians we need another type of expansion, the so-called expansion in powers of field derivatives, which is now widely utilized [24]. Since we are in the low-energy region, we may suppose the fields σ and π to be slowly varying. Then we can write the expansion of the effective action $\tilde{\Gamma}(\sigma,\pi)$ in the form

$$\tilde{\Gamma}(\sigma,\pi) = \int d^4x[\mathcal{L}_{kin}(\partial\sigma,\partial\pi,\sigma,\pi) - \mathrm{V}(\sigma,\pi)], \tag{2.15}$$

The potential part $\mathrm{V}(\sigma,\pi)$ does not contain derivatives of σ and π, whereas $\mathcal{L}_{kin}$ is an infinite series in the powers of such derivatives and $\mathcal{L}_{kin} = 0$ for constant fields.

It is possible to make a derivative expansion of the diagonal heat kernel elements as well [25] and then to obtain the corresponding expansion for the effective action. For concrete results in the pure NJL model we refer to the papers [6].

The problem becomes essentially more complicated for the gauged NJL model, due to the presence of quantized gauge fields. Now we cannot perform explicitly the functional integration in (2.8). Recently, the problem of evaluating the effective action in the GNJL model was considered by Bardeen and Love [26] (see also Ref.[27]). Here we follow an alternative method [12,13] which is based on the formalism of Green's functions for composite operators [11]. We believe that this approach is applicable to a broader class of models with dynamical symmetry breaking.

First of all we make use of the fact that $\tilde{\Gamma}(\sigma,\pi)$ coincides with a stationary point of the Cornwall-Jackiw-Tomboulis (CJT) [28] effective action Γ_{CJT},

calculated in QED with space dependent bare mass $m_0 = \sigma(x) + i\gamma_5\pi(x)$ (see(2.8)). In fact the CJT effective action Γ_{CJT} is a functional of the full propagator $G(x,y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle$ and of the external fields $\sigma(x)$ and $\pi(x)$ and is represented by the expression

$$\Gamma_{CJT}(G,\sigma,\pi) = -i\mathrm{Tr}[\ln G^{-1} + GS^{-1}(\sigma,\pi) - 1] + \sum_{n\geq 2}\Gamma_n(G). \qquad (2.16)$$

Here

$$iS^{-1}(x,y) = [i\hat{\partial}_x - (\sigma(x) + i\gamma_5\pi(x))]\delta(x-y)$$

is a free fermion propagator explicitly dependent on σ and π, $\Gamma_n(G)$ are n-loop two-particle irreducible vacuum diagrams with full propagators G for fermion lines and bare photon lines and vertices (see Fig. 2). Further, we take into account only the two first terms in the expansion (2.16)) (the so-called two-loop approximation). In this case

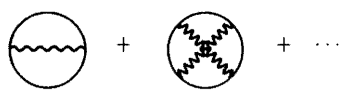


Fig. 2: Vacuum diagrams contributing to CJT effective action.

$$\Gamma_{CJT}(G,\sigma,\pi) = -i\mathrm{Tr}[\ln G^{-1} + GS^{-1} - 1]$$
$$+\tfrac{e^2}{2}\int d^4x d^4y\, \mathrm{Tr}[G(x,y)i\gamma_\mu G(y,x)i\gamma_\nu D^{\mu\nu}(x,y)]. \qquad (2.17)$$

The stationary condition

$$\frac{\delta\Gamma_{CJT}(G,\sigma,\pi)}{\delta G} = 0 \qquad (2.18)$$

gives the Dyson-Schwinger (DS) equation for the fermion propagator in the ladder approximation,

$$G^{-1}(x,y) = S^{-1}(x,y) - e^2 i\gamma^\mu G(x,y) i\gamma^\nu D_{\mu\nu}(x,y), \qquad (2.19)$$

which is depicted in Fig. 3. The solution of the (2.19)), $G(\sigma,\pi)$, is a function of the external fields σ and π, and at the stationary point of Γ_{CJT} we have

$$\tilde{\Gamma}(\sigma,\pi) = \Gamma_{CJT}(G(\sigma,\pi),\sigma,\pi). \qquad (2.20)$$

Keeping other terms in the expansion (2.16) leads to the quenched approximation for the fermion propagator. It is known [29] that qualitatively the quenched approximation does not change the main results of the ladder approximation for the fermion mass function (see below).

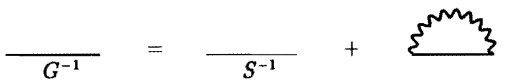


Fig. 3: Dyson-Schwinger equation for the fermion propagator in the ladder approximation.

3 Derivation of the Effective Potential

We start with the derivation of the effective potential. Although the expression for the potential for the part of the critical line $\alpha < \alpha_c = \pi/3$ can be obtained by different methods [26], we prefer to follow the above outlined method and discuss in more detail the local limit, which turns out to be relevant to the explanation of the scale-symmetry breaking mechanism in the model under consideration. We also obtain the potential in the case of supercritical dynamics, with $\alpha \geq \alpha_c$.

For evaluating the effective potential we consider constant field configurations $\sigma = \pi = \text{const}$ for which $\mathcal{L}_{kin} = 0$. Moreover, due to chiral symmetry it is sufficient to consider the configuration $\sigma = \text{const}$, $\pi = 0$ (to restore π we will replace σ by the chiral invariant $(\sigma^2 + \pi^2)^{1/2}$ at the end of computation). Thus, we need to calculate CJT effective potential in QED with bare mass $m_0 = \sigma = \text{const}$. In the momentum representation ($G^{-1}(p) = \hat{p}A(p^2) - \Sigma(p^2)$), after the integration over directions in Euclidean

space, we get the following expression for it [30]:

$$V_{CJT}(\Sigma) = -\frac{1}{8\pi^2}\Big\{\int_0^{\Lambda^2} dp^2 p^2 \Big[\ln\Big(1+\frac{\Sigma^2(p^2)}{p^2}\Big) - \frac{2\Sigma^2(p^2)}{p^2+\Sigma^2(p^2)}\Big] +$$
$$+\lambda\int_0^{\Lambda^2} dp^2 \frac{p^2\Sigma(p^2)}{p^2+\Sigma^2(p^2)}\int_0^{\Lambda^2} dk^2 \frac{k^2\Sigma(k^2)}{k^2+\Sigma^2(k^2)}\Big(\frac{\theta(p^2-k^2)}{p^2}+\frac{\theta(k^2-p^2)}{k^2}\Big)\Big\}$$
$$-\frac{\sigma}{4\pi^2}\int_0^{\Lambda^2} dp^2 \frac{p^2\Sigma(p^2)}{p^2+\Sigma^2(p^2)} \equiv \mathrm{V}_1 + \mathrm{V}_2. \tag{3.1}$$

Here $\lambda = 3\alpha/4\pi$ and we used the Landau gauge in which the function $A(p^2) = 1$ in the approximation under consideration. The stationary condition $\delta\mathrm{V}/\delta\Sigma = 0$ leads to the famous nonlinear integral equation for the mass function Σ:

$$\Sigma(p^2) = \sigma + \lambda\int_0^{\Lambda} dk^2 \frac{k^2\Sigma(k^2)}{k^2+\Sigma^2(k^2)}\left[\frac{\theta(p^2-k^2)}{p^2}+\frac{\theta(k^2-p^2)}{k^2}\right]. \tag{3.2}$$

It is easy to show that Eq.(3.2)) can be reduced to the following nonlinear differential equation of the second order,

$$\frac{d}{dp^2}\Big(p^4\frac{d\Sigma}{dp^2}\Big) + \lambda\frac{p^2\Sigma(p^2)}{p^2+\Sigma^2(p^2)} = 0, \tag{3.3}$$

with infra-red and ultra-violet boundary conditions

$$p^4\frac{d\Sigma}{dp^2}\Big|_{p^2=0} = 0, \qquad \text{IRBC}, \tag{3.4}$$

$$\frac{d}{dp^2}[p^2\Sigma(p^2)]\Big|_{p^2=\Lambda^2} = \sigma, \qquad \text{UVBC}. \tag{3.5}$$

It is remarkable that IRBC (3.4) determines uniquely the solution of the Eq.(3.3), leaving the value $\Sigma(0)$ as a free parameter. Then UVBC (3.5) gives the relation between $\Sigma(0)$, σ, and the ultra-violet cut-off, Λ, which might be satisfied or not. Thus we have a nonlinear eigenvalue problem, the natural eigenvalue parameter being $\Sigma(0)$. It is known that for the values $\lambda < 1/4$ ($\alpha < \alpha_c = \pi/3$) it has a unique solution for finite cut-off [31].

The asymptotic behaviour of the mass function at large momentum has the form

$$\Sigma(p) \approx A(\alpha)\frac{\Sigma_0}{p\omega}\sinh\Big[\omega\Big(\ln\frac{p}{\Sigma_0}+\delta(\alpha)\Big)\Big], \quad \alpha < \alpha_c, \tag{3.6}$$

$$\Sigma(p) \approx A(\alpha)\frac{\Sigma_0}{p\nu}\sin\Big[\nu\Big(\ln\frac{p}{\Sigma_0}+\delta(\alpha)\Big)\Big], \quad \alpha > \alpha_c, \tag{3.7}$$

where $\omega = \sqrt{1-\alpha/\alpha_c}$ and $\nu = \sqrt{\alpha/\alpha_c - 1}$. Here $A(\alpha)$ and $\delta(\alpha)$ are parameters (the amplitude and the phase) of the solution, while $\Sigma_0 \equiv \Sigma(0)$ is the fermion mass-scale, which depends in our case on the external field σ.

To get the functions $A(\alpha)$ and $\delta(\alpha)$ we need formulas relating the behaviour of $\Sigma(p)$ at the point $p = 0$ to that at $p = \infty$. It is a connection problem for nonlinear ordinary differential equations and it is being solved only now for the famous Painlevé transcendents [32]. However, Painlevé equations belong to the unique class of nonlinear equations which only have poles as moving singularities. As was shown in [33] the Eq.[20] has moving branch-point singularities in the complex p^2–plane, and so the problem of calculating the functions $A(\alpha)$ and $\delta(\alpha)$ becomes even more complicated than in the case of the Painlevé equations. It is a challenge indeed for mathematicians to solve the connection problem for Eq.(3.3)!

For the linearized version of Eq.(3.3) (when we replace the term $\Sigma^2(p^2)$ in the denominator by the constant $\Sigma^2(0)$), which approximates the nonlinear equation well for the entire range of momenta (see,however, Sect.4), the solution is expressed through the Gauss hypergeometric function [18],

$$\Sigma(p^2) = \Sigma_0 F\left(\frac{1+\omega}{2}, \frac{1-\omega}{2}, 2; -\frac{p^2}{\Sigma_o^2}\right). \tag{3.8}$$

In this case the functions $A(\alpha)$ and $\delta(\alpha)$ reduce to

$$A(\alpha) = 2\left[\frac{\Gamma(1+\omega)\Gamma(1-\omega)}{\Gamma(\frac{3+\omega}{2})\Gamma(\frac{3-\omega}{2})\Gamma(\frac{1+\omega}{2})\Gamma(\frac{1-\omega}{2})}\right]^{\frac{1}{2}}, \tag{3.9}$$

$$\delta(\alpha) = \frac{1}{2\omega}\ln\left[\frac{\Gamma(1+\omega)\Gamma(\frac{3-\omega}{2})\Gamma(\frac{1-\omega}{2})}{\Gamma(1-\omega)\Gamma(\frac{3+\omega}{2})\Gamma(\frac{1+\omega}{2})}\right]. \tag{3.10}$$

In the region $\alpha > \alpha_c$, the solution is obtained from (3.8)) replacing ω by $i\nu$.

To compute the value of the effective potential (3.2) for the solution of the DS equation it is useful to employ a kind of virial relation for $\mathrm{V}(\Sigma,\sigma)$:

$$\mathrm{V}_{\mathrm{CJT}}(\Sigma,\sigma) = \frac{1}{2}\frac{\partial \mathrm{V}_{\mathrm{CJT}}}{\partial \ln\Lambda^2} + \frac{1}{4}\mathrm{V}_{2\,\mathrm{CJT}}, \tag{3.11}$$

which is a consequence of the scale transformation property of V_{CJT}. Indeed, considering variations of V_{CJT} with respect to the scale transformation

$$\Sigma(p^2) \to \Sigma^{(s)}(p^2) = s\Sigma\left(\frac{p^2}{s^2}\right),$$
$$V_{CJT}(\Sigma^{(s)}, \Lambda^2, \sigma) = s^4 V_{CJT}\left(\Sigma, \frac{\Lambda^2}{s^2}, \frac{\sigma}{s}\right), \tag{3.12}$$

where s is a parameter, and taking into account that

$$\left.\frac{dV_{CJT}(\Sigma^{(s)}, \Lambda^2, \sigma)}{ds}\right|_{s=1} = \int \left[\frac{\delta V_{CJT}}{\delta\Sigma^{(s)}(p^2)}\right]_{\Sigma^{(s)}=\Sigma} \left[\frac{d\Sigma^{(s)}(p^2)}{ds}\right]_{s=1} d^4p = 0, \tag{3.13}$$

we find the relation (3.11).

From (3.11) and DS Eq.(3.2), it follows that

$$\tilde{V}(\Sigma, \sigma) = -\frac{1}{16\pi^2}\left[\Lambda^4 \ln\left(1 + \frac{\Sigma^2(\Lambda^2)}{\Lambda^2}\right) + \sigma \int_0^{\Lambda^2} \frac{dp^2 p^2 \Sigma(p^2)}{p^2 + \Sigma^2(p^2)}\right]. \tag{3.14}$$

Hence, and from Eq.(3.2) at $p^2 = \Lambda^2$, we have

$$\tilde{V}(\Sigma_0, \sigma) = -\frac{1}{16\pi^2}\left[\Lambda^4 \ln\left(1 + \frac{\Sigma^2(\Lambda^2)}{\Lambda^2}\right) + \frac{\sigma}{\lambda}\Lambda^2\left(\Sigma(\Lambda^2) - \sigma\right)\right]. \tag{3.15}$$

One can see that $\tilde{V}$ depends on Σ_0 and σ, and the field σ is related to Σ_0 by the expression (see (3.12))

$$\sigma = \frac{d}{dp^2}\left(p^2\Sigma(p^2)\right)\Big|_{p^2=\Lambda^2} \approx A(\alpha)\frac{\Sigma_0^2}{2\Lambda}\left(\frac{\sinh\theta}{\omega} + \cosh\theta\right), \Lambda \to \infty$$
$$\theta = \omega \ln \frac{\Lambda e^{\delta}}{\Sigma_0}. \tag{3.16}$$

Eq.(3.5) (or (3.16)) allows $\tilde{V}$ to be written as

$$\tilde{V}(\Sigma_0(\sigma)) = -\frac{\Lambda^4}{16\pi^2}\left[\ln\left(1 + \frac{\Sigma^2(\Lambda^2)}{\Lambda^2}\right) - \frac{4\Sigma'(\Lambda^2)}{1-\omega^2}\left(\Sigma(\Lambda^2) + \Lambda^2\Sigma'(\Lambda^2)\right)\right]. \tag{3.17}$$

Note that Eq.(3.17) is an exact expression for the effective potential; it does not contain integrals and is expressed through the values of the mass function and its derivatives at the cut-off Λ. At large Λ, (3.17) can be simplified by using asymptotic forms (3.6) and (3.16), and we obtain

$$\tilde{V}(\Sigma_0(\sigma)) = \frac{A^2(\alpha)}{16\pi^2\omega^2}\Sigma_0^4\left[\frac{1}{1-\omega^2} - \cosh 2\theta\right]. \tag{3.18}$$

As is well known, [16,18,31,34] there is no solution corresponding to spontaneous chiral symmetry breaking in ladder QED at $\alpha < \alpha_c$. This is seen also from the QED effective potential (3.18), because there is only a trivial solution $\sigma = 0$ of the stationary equation $dV/d\sigma = 0$. However, the situation changes if we take the four-fermion interaction into account. In that case the potential takes the form (see (2.7))

$$\begin{aligned} V(\Sigma_0(\sigma)) &= \tilde{V}(\Sigma_0(\sigma)) + \frac{\sigma^2}{2G_0} \\ &= \frac{1}{16\pi^2}\Big\{\frac{8\pi^2}{G_0}\left[\Sigma(\Lambda^2) + \Lambda^2\Sigma'(\Lambda^2)\right]^2 - \Lambda^4 \ln\left(1 + \frac{\Sigma^2(\Lambda^2)}{\Lambda^2}\right) \\ &\quad + \frac{4\Lambda^4\Sigma'(\Lambda^2)}{1-\omega^2}\left(\Sigma(\Lambda^2) + \Lambda^2\Sigma'(\Lambda^2)\right)\Big\}, \end{aligned} \tag{3.19}$$

or at large Λ

$$V(\Sigma_0) \approx \frac{A^2(\alpha)}{16\pi^2\omega^2}\Sigma_0^4\Big\{\frac{\pi^2}{G_0\Lambda^2}\left[(1+\omega^2)\cosh 2\theta + 2\omega \sinh 2\theta + \omega^2 - 1\right] + \frac{1}{1-\omega^2} - \cosh 2\theta\Big\}, \quad \alpha < \alpha_c, \tag{3.20}$$

$$V(\Sigma_0) \approx \frac{A^2(\alpha_c)}{8\pi^2}\Sigma_0^4\Big\{\frac{1}{2} + \frac{\pi^2}{G_0\Lambda^2}(L+1)^2 - L^2\Big\}, \quad \alpha = \alpha_c \tag{3.21}$$

(here $L = \ln \Lambda e^{\delta}/\Sigma_0$). From the stationary equation $dV/d\Sigma_0 = 0$ we can determine now the value $\bar{\Sigma}_0 \equiv m_d$ corresponding to the minimum of the potential V:

$$\frac{1}{4g} \equiv \frac{\pi^2}{G_0\Lambda^2} = \frac{2\cosh 2\bar{\theta} - \omega \sinh 2\bar{\theta} + 2(\omega^2-1)^{-1}}{2\cosh 2\bar{\theta} + (3\omega - \omega^3)\sinh 2\bar{\theta} + 2(\omega^2 - 1)}, \tag{3.22}$$

$$\frac{1}{4g} = \frac{\bar{L}^2 - \frac{1}{2}\bar{L} - \frac{1}{2}}{\bar{L}^2 + \frac{3}{2}\bar{L} + \frac{1}{2}} \tag{3.23}$$

for values $\alpha < \alpha_c$ and $\alpha = \alpha_c$, respectively ($\bar{\theta} = \omega \ln(\frac{\Lambda e^{\delta}}{\bar{\Sigma}_0})$, $\bar{L} = \ln(\frac{\Lambda e^{\delta}}{\bar{\Sigma}_0})$).

In the local limit $\bar{\Sigma}_0/\Lambda \to 0$ we immediately see that Eqs.(3.22), (3.23) determine the critical line (2.2) separating the phases with broken and unbroken chiral symmetry. For values $\alpha < \alpha_c$ spontaneuos symmetry breaking takes place for all $g > g_c(\alpha)$, and the effective potential $V(\Sigma_0)$ (3.20),(3.21) describes the fluctuations near the nonsymmetric vacuum $\bar{\sigma} \neq 0\,(\bar{\Sigma}_0 \neq 0)$.The situation is different for $\alpha > \alpha_c$. While for $\alpha < \alpha_c$ it is mainly the four-fermion interaction that is responsable for the spontaneous breaking of chiral symmetry, for $\alpha > \alpha_c$ the main role is played by the electromagnetic interaction, and the critical line has the form

$$\alpha = \alpha_c, \quad g \leq \frac{1}{4}. \tag{3.24}$$

In that case spontaneous chiral symmetry breaking takes place for all $\alpha > \alpha_c$ and, in particular, for $\alpha > \alpha_c$, $g = 0$ (pure QED) [16-19,31,34]. We shall discuss this in more detail at the end of the section.

Let us note that the point $\alpha = 0$, $g = 1$ (Nambu-Jona-Lasinio point) requires more careful treatment. In fact, using formulas

$$\frac{\partial\Sigma(p^2)}{\partial\Sigma_0} = \frac{\Sigma(p^2) - 2p^2\Sigma'(p^2)}{\Sigma_0}, \quad \frac{\partial\Sigma'(p^2)}{\partial\Sigma_0} = -\frac{\Sigma'(p^2) + 2p^2\Sigma''(p^2)}{\Sigma_0} \tag{3.25}$$

and the differential equation (3.3), we can write the exact equation for the critical line $\partial V/\partial\Sigma_0 = 0$:

$$\frac{1}{g} = -\frac{4}{1-\omega^2}\frac{\Lambda^2\Sigma'(\Lambda^2)}{\Sigma + \Lambda^2\Sigma'}. \tag{3.26}$$

In the case that Σ is given by the hypergeometric function (3.8), Eq.(3.26) reduces to

$$\frac{1}{g} = \frac{\frac{\Lambda^2}{\Sigma_0^2} F\left(\frac{3+\omega}{2}, \frac{3-\omega}{2}, 3; -\frac{\Lambda^2}{\Sigma_0^2}\right)}{2F\left(\frac{1+\omega}{2}, \frac{1-\omega}{2}, 2; -\frac{\Lambda^2}{\Sigma_0^2}\right) - \frac{1-\omega^2}{4}\frac{\Lambda^2}{\Sigma_0^2} F\left(\frac{3+\omega}{2}, \frac{3-\omega}{3}, 3; -\frac{\Lambda^2}{\Sigma_0^2}\right)} \tag{3.27}$$

and for $\alpha = 0$ ($\omega = 1$) it yields the well-known "gap-equation" of the NJL model[2]

$$\frac{1}{g} = \frac{\Lambda^2}{2\bar{\Sigma}_0^2} F(2,1,3; -\frac{\Lambda^2}{\bar{\Sigma}_0^2}) = 1 - \frac{\bar{\Sigma}_0^2}{\Lambda^2}\ln(\frac{\Lambda^2}{\bar{\Sigma}_0^2} + 1). \tag{3.28}$$

Thus the relation (3.26) can be considered as a generalization of the gap equation of the NJL model. It is clear that for the dynamical mass $\bar{\Sigma}_0$ we obtain from (3.26) the generic form, $\bar{\Sigma}_0 = \Lambda f(\alpha, g)$, that is nothing but the generalization of the scaling law in pure QED [16-19]. It should be noted that for $\alpha < \alpha_c$ this scaling is not of the exponential type, as in pure QED, but of the power type [15]. The renormalization in the spontaneously broken phase is performed, in accordance with this scaling relation, by choosing the couplings α and g dependent on the cut-off Λ, in such a way that the minimum of the effective potential V remains fixed at a finite value, $\Sigma_0 = \bar{\Sigma}_0$, as Λ goes to infinity. As in pure QED, we may expect that such a "fine-tuning" of the couplings will define a nontrivial theory as we approach the fixed points on the critical line (2.2), when $\Lambda \to \infty$. Inserting the expressions for g (3.22),(3.23) into (3.20),(3.21) and taking the limit $\Lambda \to \infty$, we find

$$V(\Sigma_0) = \frac{A^2(\alpha)}{16\pi^2\omega(1-\omega^2)}\Sigma_0^4\left[2 - \omega - 2\left(\frac{\bar{\Sigma}_0}{\Sigma_0}\right)^{2\omega}\right], \quad \alpha < \alpha_c, \tag{3.29}$$

$$V(\Sigma_0) = \frac{A^2(\alpha_c)}{4\pi^2}\Sigma_0^4\left[\ln\frac{\Sigma_0}{\bar{\Sigma}_0} - \frac{1}{4}\right], \quad \alpha = \alpha_c. \tag{3.30}$$

The next task is to re-express the effective potential in terms of the field σ. Let us introduce the renormalized fields, $\sigma_r = Z_m^{-1}\sigma, \pi_r = Z_m^{-1}\pi$. The normalization can be chosen in such a way that the value of the renormalized field $\bar{\sigma}_r$ at the minimum of potential coincides with $\bar{\Sigma}_0$. Thus, from (3.16) we obtain for the constant Z_m:

$$Z_m \approx A(\alpha)\frac{\bar{\Sigma}_0}{2\Lambda}\left(\frac{\sinh\bar{\theta}}{\omega} + \cosh\bar{\theta}\right), \quad \alpha < \alpha_c, \tag{3.31}$$

$$Z_m \approx A(\alpha_c)\frac{\bar{\Sigma}_0}{2\Lambda}\left(\bar{L} + 1\right), \quad \alpha = \alpha_c. \tag{3.32}$$

Then in the local limit ($\Lambda \to \infty$) we have the relation

$$\sigma_r = \bar{\Sigma}_0\left(\frac{\Sigma_0}{\bar{\Sigma}_0}\right)^{2-\omega}, \tag{3.33}$$

which allows us to rewrite (3.29) and (3.30) in the following form

$$V(\rho_r) = \frac{A^2(\alpha)}{16\pi^2\omega(1-\omega^2)}\bar{\Sigma}_0^4\left[(2-\omega)\left(\frac{\rho_r^2}{\bar{\Sigma}_0^2}\right)^{\frac{2}{2-\omega}} - 2\frac{\rho_r^2}{\bar{\Sigma}_0^2}\right] \tag{3.34}$$

for $\alpha < \alpha_c$ and $g > 1/4$,

$$V(\rho_r) = \frac{A^2(\alpha_c)}{16\pi^2}\bar{\Sigma}_0^2\rho_r^2\left[\ln\frac{\rho_r^2}{\bar{\Sigma}_0^2} - 1\right] \tag{3.35}$$

for $\alpha = \alpha_c, g = 1/4$. Here $\rho_r^2 = \sigma_r^2 + \pi_r^2$, and we used the fact that, as a consequence of the $U(1)\times U(1)$ symmetry, $V(\rho_r)$ can be obtained from $V(\sigma_r)$ by the replacement of σ_r by the chiral invariant ρ_r.

We note that the fermion dynamical mass $\bar{\Sigma}_0$ appeared in the potential V as a result of the dimensional transmutation phenomenon: from the requirement that the value $V(\Sigma_0 = \bar{\Sigma}_0)$ be a minimum the constant g was determined as a function of $\bar{\Sigma}_0, \Lambda$ and α. As a result, in the local limit $\Lambda \to \infty$, the dimensionless coupling constant g was replaced by the dimensional parameter $\bar{\Sigma}_0$.

The important point is the appearance in the potential (3.34) of the term proportional to $(\rho_r^2/\bar{\Sigma}_0^2)^{2/(2-\omega)}$. It can be shown that the contribution of this term to the effective action is scale-invariant. Indeed, under scale transformations Σ_0 transforms as $\Sigma_0 \to s\Sigma_0$ and then it follows from (3.33) that the dynamical dimension of the composite field σ equals $d_\sigma = 2-\omega$. This in turn means that the dynamical dimension of the term $(\rho_r^2/\bar{\Sigma}_0^2)^{2/(2-\omega)}$ equals four. Consequently, in the potential (3.34) only the second term (the mass term) violates scale symmetry, and the corresponding violation is soft (the dynamical dimension of this mass term obeys $2(2-\omega) < 4$). We also note the negative sign of the mass term, which is the reason for the appearance of a nontrivial vacuum expectation value for the field ρ_r and, consequently, for spontaneous breaking of chiral symmetry in the manner analogous to the standard Goldstone mechanism.

For $\alpha = \alpha_c$, scale invariance is violated in the potential (3.35) by the logarithmic term. The form of the potential (3.35) is reminiscent of the one-loop effective potential in the Coleman-Weinberg (CW) model [35]. However,

there are also substantial differences between them: while in the CW potential the power of the scalar field is equal to four, in our case this power is equal to two. This, of course, reflects the fact that for $\alpha = \alpha_c$ the dynamical dimension of the composite field σ equals $d_\sigma = (2-\omega)|_{\alpha=\alpha_c} = 2$ and the mass term is scale invariant.

As we shall see later, the kinetic terms in the GNJL-model are scale invariant in the continuum limit. Therefore in this model the dynamics of scale symmetry breakdown is completely governed by the effective potential. This implies that the hypothesis of partially conserved dilatation current (PCDC) is realized in the tree approximation in this model, i.e. the σ scalar can be considered as a massive dilaton [30,36]. The main point of the PCDC approach is precisely the inclusion of the noncanonical dynamical dimensions of the spinless fields. The derived effective potential (3.34) depends on dynamical dimensions in an explicit way, reflecting the dynamics of formation of these tight bound states in the GNJL-model (see below). Therefore the dynamical dimensions of composite fields should be considered as important phenomenological parameters in the construction of the low-energy Lagrangians. Consequences of the PCDC hypothesis for describing the dynamics of the σ scalar are discussed in more detail in Refs.[30,36].

Up to now we discussed the dynamics of the model corresponding to a part of the critical line $\alpha \le \alpha_c$ and $g \ge 1/4$. The picture of scale-symmetry breaking for $\alpha > \alpha_c$ and $g < 1/4$, where the breaking of chiral symmetry is due mainly to electromagnetic interaction, is somewhat different from that discussed above. Using the expression (3.7) for the asymptotic behaviour of the mass function in this region, we can perform all calculations similarly to the case $\alpha < \alpha_c$. As a result we obtain

$$V(\Sigma_0(\rho)) = \frac{A^2(\alpha)\Sigma_0^4}{16\pi^2\nu^2}\left[(1+\frac{\nu^2-1}{4g})\cos 2\theta + \frac{\nu}{2g}\sin 2\theta + \frac{1+\nu^2}{4g} - \frac{1}{1+\nu^2}\right], \tag{3.36}$$

where $\theta = \nu\ln(\Lambda e^{\delta}/\Sigma_0)$, while the function $\Sigma_0(\rho)$ is determined from the equation (compare with (3.16))

$$\sigma = \frac{A(\alpha)\Sigma_0^2}{2\Lambda}\left(\frac{\sin\theta}{\nu} + \cos\theta\right) \tag{3.37}$$

with the replacement $\sigma \to \rho$. The equation $dV/d\Sigma_0 = 0$ now takes the form

$$\frac{1}{4g} = \frac{2\cos 2\bar{\theta} + \nu \sin 2\bar{\theta} - 2(1+\nu^2)^{-1}}{2\cos 2\bar{\theta} + (3\nu + \nu^3)\sin 2\bar{\theta} - 2(1+\nu^2)}, \tag{3.38}$$

where $\bar{\theta} = \nu \ln(\Lambda e^{\delta}/\bar{\Sigma}_0)$. From (3.38) we find the solution for $\bar{\theta}$ near $\alpha = \alpha_c$:

$$\bar{\theta} = \pi - \arctan\left(\nu \frac{1+4g}{1-4g}\right) \approx \pi - \nu \frac{1+4g}{1-4g}, \quad \nu \to 0. \tag{3.39}$$

Writing θ as $\theta = \bar{\theta} - \nu \ln(\Sigma_0/\bar{\Sigma}_0)$ we obtain in the local limit the following expression for the potential (3.36) on the critical line $\alpha = \alpha_c, g < 1/4\,(\nu = 0)$:

$$V(\Sigma_0(\rho)) = \frac{A^2(\alpha_c)\Sigma_0^4}{16\pi^2}\Big[\frac{2}{g}(\frac{1}{4} - g)\ln^2\frac{\Sigma_0}{\bar{\Sigma}_0} + 4\ln\frac{\Sigma_0}{\bar{\Sigma}_0} - 1\Big], \quad g < \frac{1}{4}. \tag{3.40}$$

Equation (3.37) also simplifies in that limit:

$$\sigma = \frac{A(\alpha_c)\Sigma_0^2}{2\Lambda}\left(\ln\frac{\Sigma_0}{\bar{\Sigma}_0} + \frac{8g}{1-4g}\right). \tag{3.41}$$

Choosing as before the renormalization constant Z_m so that $\bar{\sigma}_r = Z_m^{-1}\bar{\sigma} \equiv \bar{\Sigma}_0$, we find

$$Z_m = \frac{4gA(\alpha_c)}{1-4g}\frac{\bar{\Sigma}_0}{\Lambda} \tag{3.42}$$

$$\sigma_r = \frac{\Sigma_0^2}{\bar{\Sigma}_0}\Big[1 + \frac{1-4g}{8g}\ln\frac{\Sigma_0}{\bar{\Sigma}_0}\Big] \tag{3.43}$$

The presence of the singularity at $g = 0$ (pure QED) in the relations (3.40), (3.43) is connected with the fact that the fields σ and π do not correspond any more to real bound states if $\alpha > \alpha_c$. In fact the Euler- Lagrange equation (2.4) gives in that case $\sigma = \pi = 0$ for $g = 0$. Also from Eq.(3.43) we find $\Sigma_0 = \bar{\Sigma}_0 = \text{const}$ when $g = 0$, and Eq.(3.40) reduces to $V = -\frac{A^2(\alpha_c)\Sigma_0^4}{16\pi^2}$. It means that in the region $\alpha > \alpha_c, g < 1/4$, where the QED interaction is primarily responsible for the dynamical chiral symmetry breaking and for the formation of bound states, the fields σ and π corresponding to these bound states should be introduced in another way. The dynamics of pure QED is discussed in more detail in Ref.[37].

4 Derivation of the Kinetic Terms in the Effective Action

The kinetic part $\mathcal{L}_{kin}(\sigma,\pi)$ of the effective action (2.15) can be represented in general as a power series of derivatives of the composite fields σ and π. The computation of $\mathcal{L}_{kin}$ is reduced to the calculation of the Green's functions $G^{(n)}(q_1,\cdots,q_{n-1})$ of these fields near the point $q_1 = q_2 = \cdots = q_{n-1}$ in QED with bare mass $m_0 = \sigma + i\gamma_5\pi$ [12,13]. The general method of calculation of Green's functions for composite operators was developed in Ref.[11] and in what follows we use the technique described in that work.

The $U(1)\times U(1)$ symmetry implies the following general form for the kinetic terms with the two derivatives in the local ($\Lambda \to \infty$) limit:

$$\mathcal{L}_{kin} = \frac{F_1}{2}(\partial_\mu\sigma_r\partial^\mu\sigma_r + \partial_\mu\pi_r\partial^\mu\pi_r) + \frac{F_2}{\rho_r^2}(\sigma_r\partial^\mu\sigma_r + \pi_r\partial^\mu\pi_r)(\sigma_r\partial_\mu\sigma_r + \pi_r\partial_\mu\pi_r) \quad , \tag{4.1}$$

where F_1, F_2 are the functions of α , $\rho_r = (\sigma_r^2 + \pi_r^2)^{\frac{1}{2}}$ and $\bar{\rho}_r$ (the value $\bar{\rho}_r$ minimizes the potential). For the evaluation of F_1 and F_2 it is sufficient to consider only constant field configurations as these functions do not depend on the derivatives of the fields σ and π. To find F_1 and F_2 let us consider the functional derivatives

$$\frac{\delta^2\tilde{\Gamma}_{kin}(\sigma,\pi)}{\delta\sigma_r(x)\delta\sigma_r(0)}\Big|_{\substack{\sigma_r=const\\ \pi_r=0}} = -(F_1 + 2F_2)\Box\delta(x) \quad , \tag{4.2}$$

$$\frac{\delta^2\tilde{\Gamma}_{kin}(\sigma,\pi)}{\delta\pi_r(x)\delta\pi_r(0)}\Big|_{\substack{\sigma_r=const\\ \pi_r=0}} = -F_1\Box\delta(x) \quad . \tag{4.3}$$

From this it follows that

$$F_1 = -\frac{1}{8}\int d^4x x^2 \frac{\delta^2\tilde{\Gamma}(\sigma,\pi)}{\delta\pi_r(x)\delta\pi_r(0)}\Big|_{\substack{\sigma_r=const\\ \pi_r=0}} \quad , \tag{4.4}$$

$$F_2 = -\frac{1}{16}\int d^4x x^2\Big[\frac{\delta^2\tilde{\Gamma}(\sigma,\pi)}{\delta\sigma_r(x)\delta\sigma_r(0)} - \frac{\delta^2\tilde{\Gamma}(\sigma,\pi)}{\delta\pi_r(x)\delta\pi_r(0)}\Big]\Big|_{\substack{\sigma_r=const\\ \pi_r=0}} \quad . \tag{4.5}$$

In (4.4), (4.5) we have omitted the subscript kin and used $\tilde{\Gamma}$ instead of $\tilde{\Gamma}_{kin}$ because the contribution from the effective potential is proportional to $\delta(x)$

$$
\begin{aligned}
xf &= \frac{\lambda}{2}\int_0^{\Lambda^2}\frac{dyy}{(y+\Sigma^2)^2}[(y+\Sigma^2)f+\Sigma(1+2\Sigma\Sigma')]* \\
&\quad [\frac{y}{x}\theta(x-y)+\frac{x}{y}\theta(y-x)], \qquad (4.26)\\
3g_1 &= \frac{\lambda}{12}\int_0^{\Lambda^2}\frac{dyy}{(y+\Sigma^2)^2}\Big[-\Sigma^2(y)K_{10}(x,y)+[(y-\Sigma^2)g_1-2y\Sigma h]K_{11}(x,y) \\
&\quad +[-(y+\Sigma^2)g_2+2y\Sigma\Sigma']K_{12}(x,y)\Big], \qquad (4.27)\\
g_2 &= \frac{\lambda}{12}\int_0^{\Lambda^2}\frac{dyy}{(y+\Sigma^2)^2}\Big[-\Sigma^2(y)K_{20}(x,y)+[(y-\Sigma^2)g_1-2y\Sigma h]K_{21}(x,y) \\
&\quad +[-(y+\Sigma^2)g_2+2y\Sigma\Sigma']K_{22}(x,y)\Big], \qquad (4.28)\\
xh &= \frac{\lambda}{6}\int_0^{\Lambda^2}\frac{dyy}{(y+\Sigma^2)^2}[(y-\Sigma^2)h+2\Sigma g_1+\Sigma]* \\
&\quad [\frac{y}{x}\theta(x-y)+\frac{x}{y}\theta(y-x)]. \qquad (4.29)
\end{aligned}
$$

The kernels in (4.27),(4.28) are

$$
\begin{aligned}
K_{10}(x,y) &= \theta(x-y)(\frac{12}{x}+\frac{6y}{x^2})+\theta(y-x)\cdot\frac{18}{y},\\
K_{11}(x,y) &= \theta(x-y)(\frac{9}{x}+\frac{5y}{x^2})+\theta(y-x)(\frac{9}{y}+\frac{5x}{y^2}),\\
K_{12}(x,y) &= \theta(x-y)(\frac{3}{x}+\frac{y}{x^2})+\theta(y-x)(\frac{9}{y}-\frac{5x}{y^2}),\\
K_{20}(x,y) &= \theta(x-y)(\frac{12}{x}-\frac{6y}{x^2})+\theta(y-x)\frac{6}{y},\\
K_{21}(x,y) &= \theta(x-y)(\frac{9}{x}-\frac{5y}{x^2})+\theta(y-x)(\frac{3}{y}+\frac{x}{y^2}),\\
K_{22}(x,y) &= \theta(x-y)(\frac{3}{y}-\frac{y}{x^2})+\theta(y-x)(\frac{3}{y}-\frac{x}{y^2}), \qquad (4.30)
\end{aligned}
$$

with the relations $K_{i1}+K_{i2}=K_{i0}$. We note that Eq.(4.27)-(4.29) have appeared in another approach in Ref.[39]. These equations have been analyzed in Ref.[14] in particular, the exact solution for the function f,

$$
f=-\Sigma', \qquad (4.31)
$$

was obtained at $\Lambda=\infty$. Combining all formulas, we get the following expression for the function F_1 in the local limit:

$$
F_1=(\frac{\rho_r^2}{\bar{\Sigma}_0^2})^{\frac{\omega-1}{2-\omega}}f_1(\alpha), \qquad (4.32)
$$

where the function $f_1(\alpha)$ is given by

$$
f_1(\alpha)=\frac{1}{8\pi^2\Sigma_0^2}\int_0^\infty\frac{dxx\Sigma}{(x+\Sigma^2)^2}[\Sigma-\frac{1}{2}x\Sigma'+\frac{3}{2}\Sigma\cdot g_1+\frac{1}{2}(\Sigma-2x\Sigma')g_2+\frac{3}{2}xh], \qquad (4.33)
$$

(the function $f_1(\alpha)$ does not depend on Σ_0, as can be seen after rescaling x and $\Sigma(x)$). In deriving (4.33) we have performed an integration by parts of the term with function $f=-\Sigma'$ in (4.25). An analogous expression can be obtained for the function F_2 too. At last we come to the final expression for the effective Lagrangian

$$
\begin{aligned}
\mathcal{L}_{eff}(\sigma,\pi)=&(\frac{\rho^2}{\bar{\rho}^2})^{\frac{\omega-1}{2-\omega}}\Big\{\frac{f_1(\alpha)}{2}(\partial^\mu\sigma\partial_\mu\sigma+\partial^\mu\pi\partial_\mu\pi)+\frac{f_2(\alpha)}{\rho^2}(\sigma\partial^\mu\sigma+\pi\partial^\mu\pi)(\sigma\partial_\mu\sigma+\pi\partial_\mu\pi)\Big\}\\
&+\frac{A^2(\alpha)}{16\pi^2\omega(1-\omega^2)}\bar{\rho}^4\Big[(2-\omega)(\frac{\rho^2}{\bar{\rho}^2})^{\frac{2}{2-\omega}}-2\frac{\rho^2}{\bar{\rho}^2}\Big], \qquad (4.34)
\end{aligned}
$$

where we denoted $\bar{\rho}\equiv\bar{\Sigma}_0$ and omitted the subscript r. Since the dynamical dimension of the fields σ and π equals $d_\sigma=2-\omega$ (see (3.33)), it follows from (4.34) that the dynamical dimension of the kinetic term $\mathcal{L}_{kin}$ equals four, and its contribution to the effective action is scale-invariant.

It can be proved that all the terms with higher derivatives in the effective action are also scale invariant (for the details of this proof see [13,37]). We emphasize only that the key points in that proof are:

i) multiplicative renormalizability of the correlators with the number of composite fields $n>2$;

ii) existence of but one dimensionful parameter Σ_0 in ladder QED in the local limit;

iii) absence of renormalization of the coupling constant α in ladder QED, which gives rise to power dependence of the renormalization constant $Z_m(\Sigma_0/\Lambda)$.

The correlators $\Delta_p(q)$ and $\Delta_s(q)$ are not multiplicatively renormalizable. However, the evaluation of the terms with two derivatives uses the quantity $\partial^2\Delta_{p,s}(q)/\partial q^\mu \partial q_\mu$, the multiplicative renormalizability of which can be easily established by looking at Eq.(4.21) and taking into account that multiplicative renormalizability is true for the vertex $\Gamma_p(q,k)$, containing only one composite operator.

The proof is not valid for the effective potential $\tilde{V}$, which can be expressed in terms of the values of Green's functions $\Delta_{p,s}(q)$ at $q=0$. The reason for this is that to get finite expressions for $\Delta_{p,s}(q)$ one needs (besides a multiplicative renormalization) an additional subtraction of a quadratic divergence present in $\Delta_{p,s}(q)$. In fact, it is just this divergence that cancels the one in the term $\frac{1}{2G_0}(\sigma^2+\pi^2)$ in the potential (3.19) near the critical line. As a result, the only terms which break scale symmetry are the mass term and the logarithmic term in the potentials (3.34) and (3.35) at
$\alpha < \alpha_c$, $g > 1/4$ and $\alpha = \alpha_c$, $g = 1/4$, respectively.

We note that the effective action with a potential of the form (3.35) and a scale-invariant kinetic term has been postulated before in Ref.[30]. It can be obtained from (4.34), taking the limit $\alpha = \alpha_c$.

Let us discuss in more detail the effective Lagrangian (4.34). First of all, we redefine fields σ and π: $\sigma \to f_1^{-1/2}(\alpha)\sigma$, $\pi \to f_1^{-1/2}(\alpha)\pi$ and rewrite (4.34) in the form

$$\begin{aligned}\mathcal{L}_{eff}(\sigma,\pi) &= \left(\frac{\rho^2}{\bar{\rho}^2}\right)^{\frac{\omega-1}{2-\omega}}\Big\{\frac{1}{2}(\partial^\mu\sigma\partial_\mu\sigma+\partial^\mu\pi\partial_\mu\pi)+\frac{f_2(\alpha)}{\rho^2}(\sigma\partial^\mu\sigma+\pi\partial^\mu\pi)(\sigma\partial_\mu\sigma+\pi\partial_\mu\pi)\Big\}\\ &\quad -\frac{A^2(\alpha)\bar{\rho}^4}{16\pi^2\omega(1-\omega^2)f_1^2(\alpha)}\Big[(2-\omega)\Big(\frac{\rho^2}{\bar{\rho}^2}\Big)^{\frac{2}{2-\omega}}-2\frac{\rho^2}{\bar{\rho}^2}\Big]. \qquad (4.35)\end{aligned}$$

Now the minimum of the effective potential is determined from the equation $\bar{\sigma}^2+\bar{\pi}^2 = f_1(\alpha)\bar{\Sigma}_0^2$. Choosing the vacuum configuration to be
$\bar{\sigma} = f_1^{1/2}(\alpha)\bar{\Sigma}_0$, $\bar{\pi}=0$, we find that the excitation spectrum contains a massless pseudoscalar Goldstone boson, as is necessary for spontaneous breaking of chiral symmetry. The mass of the scalar particle σ is obtained from the

second derivative of the potential

$$M_\sigma^2 = \left.\frac{\partial^2 V}{\partial\sigma^2}\right|_{\substack{\sigma=\bar{\sigma}\\ \pi=0}} = \frac{A^2(\alpha)}{2\pi^2(1-\omega^2)(2-\omega)f_1(\alpha)}\bar{\Sigma}_0^2. \qquad (4.36)$$

In fact the function $f_1(\alpha)$ is related to the pion decay constant F_π. Actually, since the composite fields $\sigma \sim \bar{\psi}\psi$ and $\pi \sim \bar{\psi}i\gamma_5\psi$ transform under chiral rotations $\psi \to \exp(i\gamma_5\varphi)\psi$ as

$$\begin{aligned}\sigma &\to \sigma\cos 2\varphi + \pi\sin 2\varphi,\\ \pi &\to -\sigma\sin 2\varphi + \pi\sin 2\varphi,\end{aligned} \qquad (4.37)$$

the axial current is determined by the expression

$$\begin{aligned}j_{\mu 5} &= 2\left(\sigma\frac{\partial\mathcal{L}_{eff}}{\partial\partial_\mu\pi} - \pi\frac{\partial\mathcal{L}_{eff}}{\partial\partial_\mu\sigma}\right)\\ &= 2\Big(\frac{\rho^2}{\bar{\rho}^2}\Big)^{\frac{\omega-1}{2-\omega}}(\sigma\partial_\mu\pi - \pi\partial_\mu\sigma).\end{aligned} \qquad (4.38)$$

From the definition of the decay constant F_π, we have

$$\langle 0|j_{\mu 5}(0)|\pi\rangle = iP_\mu F_\pi, \qquad (4.39)$$

and in the tree approximation we find

$$2\bar{\sigma} = -F_\pi. \qquad (4.40)$$

Hence

$$\begin{aligned}F_\pi^2 &= 4f_1(\alpha)\bar{\Sigma}_0^2 \qquad (4.41)\\ &= \frac{1}{2\pi^2}\int_0^\infty \frac{dx\,x\Sigma}{(x+\Sigma^2)^2}\Big\{\Sigma - \frac{1}{2}x\Sigma' + \frac{3}{2}\Sigma g_1 + \frac{1}{2}(\Sigma - 2x\Sigma')g_2 + \frac{3}{2}xh\Big\}.\end{aligned}$$

This is the exact (ladder approximation!) expression for F_π^2 (in the case of $SU_L(N_f)\times SU_R(N_f)$ chiral symmetry group, Eq.(4.42) should be multiplied by a factor 1/2 due to the normalization of the Gell-Mann matrices
$\mathrm{Tr}\frac{\lambda^a}{2}\frac{\lambda^b}{2} = \frac{1}{2}\delta^{ab}$). One can show [14] that F_π^2, as given by (4.42), is connected

and vanishes after integration over x. On the other hand the functional derivatives are expressed through the correlation functions of the composite operators $\bar{\psi}\psi$, $\bar{\psi}i\gamma_5\psi$ in QED with bare mass σ.

$$\frac{\delta^2\tilde{\Gamma}(\sigma,\pi)}{\delta\sigma(x)\delta\sigma(0)}\Big|_{\substack{\sigma=const\\ \pi=0}} = i < 0|T\bar{\psi}\psi(x)\bar{\psi}\psi(0)|0 >\Big|_{\substack{\sigma=const\\ \pi=0}} , \tag{4.6}$$

$$\frac{\delta^2\tilde{\Gamma}(\sigma,\pi)}{\delta\pi(x)\delta\pi(0)}\Big|_{\substack{\sigma=const\\ \pi=0}} = i < 0|T\bar{\psi}i\gamma_5\psi(x)\bar{\psi}i\gamma_5\psi(0)|0 >\Big|_{\substack{\sigma=const\\ \pi=0}} . \tag{4.7}$$

Then defining the Fourier transforms of these correlators

$$\Delta_s(q) = i\int d^4x e^{iqx} < 0|T\bar{\psi}\psi(x)\bar{\psi}\psi(0)|0 >\Big|_{\substack{\sigma=const\\ \pi=0}} , \tag{4.8}$$

$$\Delta_p(q) = i\int d^4x e^{iqx} < 0|T\bar{\psi}i\gamma_5\psi(x)\bar{\psi}i\gamma_5\psi(0)|0 >\Big|_{\substack{\sigma=const\\ \pi=0}} , \tag{4.9}$$

and taking into account the renormalization relations $\sigma_r = Z_m^{-1}(\bar{\Sigma}_0/\Lambda)\sigma$, $\pi_r = Z_m^{-1}(\bar{\Sigma}_0/\Lambda)\pi$, we obtain

$$F_1 = \frac{1}{8}Z_m^2(\frac{\bar{\Sigma}_0}{\Lambda})\frac{\partial^2\Delta_p(q)}{\partial q^\mu\partial q_\mu}\Big|_{q=0} , \tag{4.10}$$

$$F_2 = \frac{1}{16}Z_m^2(\frac{\bar{\Sigma}_0}{\Lambda})\Big[\frac{\partial^2\Delta_s(q)}{\partial q^\mu\partial q_\mu} - \frac{\partial^2\Delta_p(q)}{\partial q^\mu\partial q_\mu}\Big]\Big|_{q=0} . \tag{4.11}$$

Therefore the functions F_1 and F_2 are given by the values of the second derivatives of the correlators $\Delta_p(q)$ and $\Delta_s(q)$ at zero momemtum. In the following we discuss only the evaluation of $\partial^2\Delta_p/\partial q^\mu\partial q_\mu|_{q=0}$ (the evaluation of $\partial^2\Delta_s/\partial q^\mu\partial q_\mu|_{q=0}$ is similar; some specific features of that calculation will be mentioned below).

To compute the expression for $\Delta_p(q)$ we differentiate Eq.(2.20) twice with respect to π, taking into account that $G(\sigma,\pi)$ is the solution of $\delta\tilde{\Gamma}/G = 0$ (or Eq.(2.19). We find

$$\frac{\delta\tilde{\Gamma}}{\delta\pi(x)} = \mathrm{Tr}\, G(x,x)i\gamma_5 , \tag{4.12}$$

$$\frac{\delta^2\tilde{\Gamma}}{\delta\pi(x)\delta\pi(z)} = \mathrm{Tr}\frac{\delta G(x,x)}{\delta\pi(z)}i\gamma_5 . \tag{4.13}$$

Defining now the amputated proper vertex $\Gamma_p(k,p)$ through the relation

$$\frac{\delta G(u,v)}{\delta\pi(z)}\Big|_{\substack{\sigma=const\\ \pi=0}} = -i< 0|T\psi(u)\bar{\psi}(v)\bar{\psi}i\gamma_5\psi(z)|0 >\Big|_{\substack{\sigma=const\\ \pi=0}} = \tag{4.14}$$
$$= -i\int\frac{d^4kd^4p}{(2\pi)^8}e^{-ik(u-z)+ip(v-z)}G(k)\Gamma_p(k,p)G(p)$$

and taking the Fourier transform of Eq.(4.13), we come to the following expression for the correlator $\Delta_p(q)$: [11]

$$\Delta_p(q) = -i\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[G(k)\Gamma_p(k,k+q)G(k+q)i\gamma_5] . \tag{4.15}$$

Graphically Eq.(4.15) is pictured in Fig. 4. In the ladder approximation the fermion propagator $G(x) = i[\hat{k} - \Sigma(k^2)]^{-1}$ (in Landau gauge) and the equation for $\Gamma_p(k,k+q)$ can be obtained by functionally differentiating Eq.(2.19) with respect to $\pi(z)$

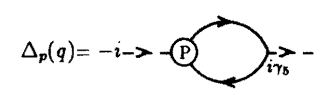


Fig. 4: Diagram for the correlator $\Delta_p(q)$.

$$-\int d^4u\, d^4v\, G^{-1}(x,u)\frac{\delta G(u,v)}{\delta\pi(z)}G(v,y) = -\gamma_5 - e^2 i\gamma^\mu\frac{\delta G(x,y)}{\delta\pi(z)}i\gamma^\nu D_{\mu\nu}(x,y) \tag{4.16}$$

and taking the Fourier transform. Thus the vertex $\Gamma_p(k,k+q)$ satisfies the equation [11] (see Fig. 5)

$$\Gamma_p(k,k+q) = i\gamma_5 + e^2\int\frac{d^4r}{(2\pi)^4}i\gamma^\mu G(r)\Gamma_p(r,r+q)G(r+q)i\gamma^\nu D_{\mu\nu}(k-r) . \tag{4.17}$$

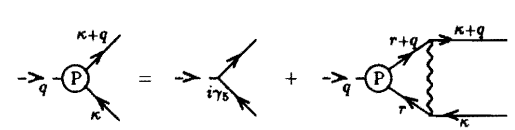
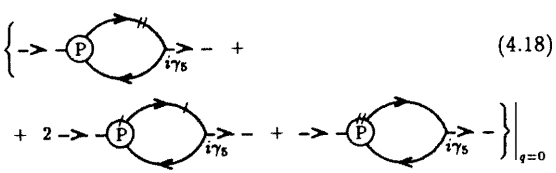


Fig. 5: Equation for the proper vertex $\Gamma_p(k,k+q)$ in the ladder approximation.

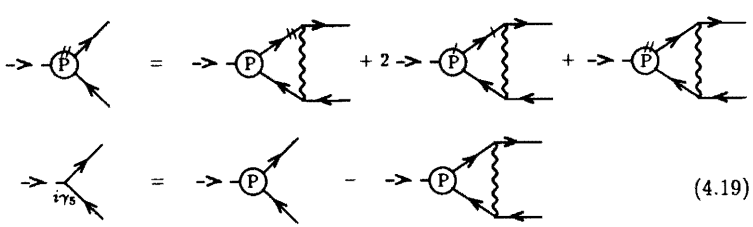
From Eq.(4.15) we obtain the expression for $\left.\frac{\partial^2\Delta_p(q)}{\partial q^\mu \partial q_\mu}\right|_{q=0}$:

$$\left.\frac{\partial^2\Delta_p(q)}{\partial q^\mu \partial q_\mu}\right|_{q=0} = -i\Big\{$$



$$\Big\}\Big|_{q=0} \tag{4.18}$$

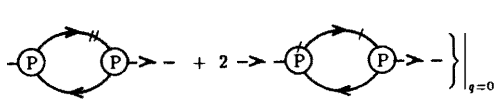
where the prime denotes a derivative with respect to q. The last expression can be simplified if use is made of the following equations, which result from (4.17):



(4.19)

One gets

$$\left.\frac{\partial^2\Delta_p(q)}{\partial q^\mu \partial q_\mu}\right|_{q=0} = -i\Big\{$$



$$\Big\}\Big|_{q=0} \tag{4.20}$$

or in analytical form

$$\left.\frac{\partial^2\Delta_p(q)}{\partial q^\mu \partial q_\mu}\right|_{q=0} = -i\Big\{\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}[G(k)\Gamma_p(k,k)G^\mu_{,\mu}(k)\Gamma_p(k,k)] + 2\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}[G(k)\Gamma_{p,\mu}(k,k)G^\mu_{,}(k)\Gamma_p(k,k)]\Big\}. \tag{4.21}$$

For the vertex function at zero transfer-momentum, $\Gamma_p(k,k)$, the exact expression

$$\Gamma_p(k,k) = i\gamma_5\frac{\Sigma(k^2)}{\sigma} \tag{4.22}$$

was obtained in Ref.[11,38] (in fact it is a consequence of the axial-vector Ward identity with nonzero fermion bare mass). We note that for the scalar vertex $\Gamma_s(k,k)$, connecting the composite operator $\bar\psi\psi$ with $\bar\psi$ and ψ (and entering into the expression for $\partial^2\Delta_s(q)/\partial q^\mu\partial q_\mu|_{q=0}$) the analogous relation has the form [11,38]

$$\Gamma_s(k,k) = i\frac{\partial\Sigma(k^2)}{\partial\sigma} \quad . \tag{4.23}$$

For the derivative $\Gamma_{p,\mu}(k,k)$ we can write a general expansion containing four scalar functions $f(k^2), g_1(k^2), g_2(k^2), h(k^2)$:

$$\Gamma_{p,\mu}(k,k) \equiv \left.\frac{\partial\Gamma_p(k,k+q)}{\partial q^\mu}\right|_{q=0} = i\frac{\gamma_5}{\sigma}[k_\mu f(k^2) + (\gamma_\mu - \frac{k_\mu\hat k}{k^2})g_1(k^2) + \tag{4.24}$$
$$+\frac{k_\mu\hat k}{k^2}g_2(k^2) + i\sigma_{\mu\nu}k^\nu h(k^2)] \quad , \quad \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu].$$

Thus for $\Delta''_p(0)$ we have

$$\left.\frac{\partial^2\Delta_p(q)}{\partial q^\mu \partial q_\mu}\right|_{q=0} = \frac{1}{\pi^2\sigma^2}\Big\{\int_0^{\Lambda^2}\frac{dx\,x\Sigma^2(x)}{(x+\Sigma^2(x))^2}[1+2\Sigma\Sigma' + x\Sigma\Sigma'' + \frac{\Sigma^2 - 4x\Sigma\Sigma' + 2x(x-\Sigma^2)(\Sigma')^2}{x+\Sigma^2}] \tag{4.25}$$
$$+\frac{1}{2}\int_0^{\Lambda^2}\frac{dx\,x\Sigma}{(x+\Sigma^2)^2}[-x(1+2\Sigma\Sigma')f + 3\Sigma\cdot g_1 + (\Sigma - 2x\Sigma')g_2 + 3xh]\Big\}.$$

The functions f, g_1, g_2, h satisfy the system of integral equations which follow from (4.25) and from the equation for $\Gamma_{p,\mu}(k,k)$ (for deriving it we should differentiate Eq.(4.17) once at $q=0$):

in a very simple way with the Langacker-Pagels representation for the decay constant [40]; however, our formula (4.42) is more useful for the numerical calculations because of its better convergence at large momenta.

To calculate F_π^2 we need information about the functions g_1, g_2, h. The analysis shows [14] that these functions decrease at $x \to \infty$ and, as they are proportional to α, we may expect their contribution to be not large. If we neglect these functions we recover the famous Pagels-Stokar (PS) formula [41]:

$$F_\pi^2 = \frac{1}{2\pi^2}\int_0^\infty \frac{dx x\Sigma(\Sigma - \frac{1}{2}x\Sigma')}{(x+\Sigma^2)^2}. \tag{4.42}$$

In the conventional approximation, when we substitute $\Sigma^2(x)$ for $\bar{\Sigma}^2(0)$ in the denominator and use the solution (3.8), we have for F_π^2 the integral

$$\begin{aligned} F_\pi^2 = \; & \frac{\bar{\Sigma}_0^2}{2\pi^2}\int_0^\infty \frac{dx x}{(x+1)^2} F\left(\frac{1+\omega}{2}, \frac{1-\omega}{2}, 2; -x\right) * \\ & \left[F\left(\frac{1+\omega}{2}, \frac{1-\omega}{2}, 2; -x\right) + \frac{1-\omega^2}{16} x F\left(\frac{3+\omega}{2}, \frac{3-\omega}{3}, 3; -x\right)\right]. \end{aligned} \tag{4.43}$$

The expression (4.44) is computed with the help of formulas for integrating the product of hypergeometric functions (for details see [14]); one finds

$$\begin{aligned} F_\pi^2 \; = \; & \frac{\bar{\Sigma}_0^2}{2\pi^2}\frac{\pi\omega}{\sin\pi\omega\,\Gamma^2(\frac{3+\omega}{2})\Gamma^2(\frac{3-\omega}{2})}\Big\{1 + \frac{1}{8}(1-\omega^2) \\ & + \frac{1}{32}(1-\omega^2)^2\frac{\psi'(\frac{3+\omega}{2}) - \psi'(\frac{3-\omega}{2})}{\omega}\Big\}, \quad \psi(x) = \frac{d\ln\Gamma(x)}{dx}. \end{aligned} \tag{4.44}$$

Now we have the most important parameters of the effective Lagrangian (the decay constant F_π, the mass of the scalar particle M_σ, and also the renormalized Yukawa coupling defined through $g_Y = 2\bar{\Sigma}_0/F_{\pi^2}$) as explicit functions of the gauge coupling on the critical line.

Let us consider the very weak gauge coupling limit $\alpha \to 0$. From Eq.(4.45)

it follows that

$$F_\pi^2 \approx \frac{\bar{\Sigma}_0^2}{3\pi\alpha}, \quad \alpha \to 0, \tag{4.45}$$

and for M_σ^2:

$$M_\sigma^2 = \frac{2A^2(\alpha)}{\pi^2(1-\omega^2)(2-\omega)}\frac{\bar{\Sigma}_0^4}{F_\pi^2} \approx 4\bar{\Sigma}_0^2, \quad \alpha \to 0, \tag{4.46}$$

where we have used for $A(\alpha)$ the expression (3.9) which gives $A(0) = \sqrt{2}$. For Yukawa and scalar quartic self-couplings we find, respectively

$$g_Y = (12\pi\alpha)^{1/2}, \quad \lambda = 6\pi\alpha. \tag{4.47}$$

Thus, our effective Lagrangian (4.35) takes a very simple form when $\alpha \to 0$:

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial^\mu\sigma\partial_\mu\sigma + \partial^\mu\pi\partial_\mu\pi) + \bar{\Sigma}_0^2(\sigma^2+\pi^2) - \lambda(\sigma^2+\pi^2)^2, \tag{4.48}$$

and describes the dynamics of spontaneously broken chiral symmetry with weakly interacting spinless bosons. The couplings λ and g_Y vanish as $\alpha \to 0$,which corresponds to the familiar fact that in the local limit the NJL model represents a free theory. One remark should be made here. While computing the mass of scalar particle (4.46) we have used the expression for the amplitude $A(\alpha)$ given by the solution of linearized eqution for the mass function. However, it was observed, firstly by numerical calculations [36] and then analytically [26,43], that the real amplitude recovered from the nonlinear equation differs from (3.9). In fact we have $A(0) = 1$, instead of $A(0) = \sqrt{2}$, and this leads to the mass relation $M_\sigma^2 = 2\bar{\Sigma}_0^2$ (nevertheless, for any fixed cut-off $\Lambda >> \bar{\Sigma}_0$ and $\alpha \to 0$, we obtain the usual NJL relation $M_\sigma^2 = 4\bar{\Sigma}_0^2$). This underlines once more the importance of getting the functions $A(\alpha)$ and $\delta(\alpha)$ from the nonlinear equation.[2] At another limit $\alpha \to \alpha_c = \pi/3, g \to 1/4$ (Bardeen, Leung, Love point on the critical line) we find

$$F_\pi^2 = \frac{8}{\pi^4}\Big\{1 + [\frac{1}{8} + \frac{1}{32}\psi''(\frac{3}{2})]\Big\} = \frac{8}{\pi^4}\Big\{1 + [\frac{5}{8} - \frac{7}{16}\zeta(3)]\Big\} \approx \frac{8}{\pi^4} \times 1.1, \tag{4.49}$$

[2]In this connection we would like to mention the recent paper by A.T.Filippov [43] in which an approximate analytic solution of nonlinear DS eqution (3.3) was derived.

where $\zeta(z)$ is the Riemann zeta-function [$\zeta(3) = 1.202$]. It is interesting to compare the contributions from the two terms Σ^2 and $-\frac{1}{2}x\Sigma\Sigma'$ to the PS formula. It is seen that the amount coming from the second of these two terms is rather small. Even in the strong coupling regime ($\alpha \to \alpha_c = \pi/3$), it is amounts to only 10%.This may be also underlying reason for neglecting the functions g_1, g_2, h in the expression for F_π^2.

For the mass of the scalar particle we get ($\alpha = \alpha_c$, $g = 1/4$)

$$M_\sigma = 1.35\bar{\Sigma}_0, \tag{4.50}$$

while combined analytical and numerical calculations of the nonlinear equation give $M_\sigma = 1.08\bar{\Sigma}_0$ [42].

We note that the value of M_σ for $\alpha = \alpha_c$ was recently computed by Bardeen and Love, based on a dispersion relation [26]. They have obtained the result $M_\sigma = 2\sqrt{1 - \exp(-1/(1-c))}\bar{\Sigma}_0$ with c an undetermined constant, which is expected to be of order one.

We should also remark that the derived effective action corresponds to the tree approximation of the effective theory. In particular, the exact mass relation for M_σ may be different from that obtained in the present approximation, if we take into account the quantum fluctuations of the fields σ and π. Ananother reason for such a deviation could be that in the above analysis we disregarded the effects of Coulomb bound states in the scalar channel. For small values of α, when the dominant binding arises from the four-fermion interaction, the effects of these Coulomb bound states should not be very significant. However, on the part of critical line $\alpha = \alpha_c$, $g < 1/4$ the binding due to electromagnetic interactions becomes important and the Coulomb bound states should play a more prominent role (see, also, discussion at the end of Sect.3 and in Ref.[37]).

5 Conclusions

The low-energy effective action in the GNJL model corresponds to the σ model with noncanonical dynamical dimension, $d_\sigma = 2 - \sqrt{1 - \alpha/\alpha_c}$, for the

composite fields $\sigma \sim \bar{\psi}\psi$ and $\pi \sim \bar{\psi}i\gamma_5\psi$. From the physical piont of view the dynamical dimensions characterize the behaviour of composite operators at small distances, which is reflected in the wave-function behaviour of the corresponding bound states. Let us consider the amputated Bethe-Salpeter (BS) wave function of a spinless fermion-antifermion bound state $\chi(q)$, where q is the relative momentum of fermions. It has the UV asymptotics of the form (up to logarithmic factors)

$$\chi(q) \sim q^{\gamma_m - 2} \tag{5.1}$$

where γ_m is the anomalous dimension of the composite operators $\bar{\psi}\psi$ or $\bar{\psi}i\gamma_5\psi$ and $d_\sigma = 3 - \gamma_m$ (thus $\gamma_m = 1 + \sqrt{1 - \alpha/\alpha_c}$). Since in the model under consideration $d_\sigma \leq 2(\gamma_m \geq 1)$, the wave function $\chi(q)$ (which determines the form-factors of the bound states) decreases for large momenta much more slowly than in theories like QCD, where $d_\sigma = 3$ (canonical dimension of the $\bar{\psi}\psi$ operator). The QCD bound states are characterized by the scale (the size of the bound state) $r \sim \Lambda_{QCD}^{-1}$ and for $q \gg \Lambda_{QCD}$ their form-factor rapidly decreases and they are soft bound states. On the contrary, the bound states of GNJL model may be called "tight" bound sytates (for the pure NJL model $d_\sigma = 1\,(\gamma_m = 2)$ and we deal with structureless or pointlike particles). A characteristic feature of such bound states is that, in contrast to QCD, they are relevant degrees of freedom for the dynamics at high energies. This property might turn out to be more important for the description of the dynamics of the electroweak symmetry breaking in scenarios based on the GNJL model. In that case the bound states play the role of the Higgs bosons. The properties of such states are discussed in detail in Ref.[44].

Another important application of the GNJL model would be its use in the analysis of data of a computer calculation on the lattice for asymptotically nonfree theories of the type of QED [45]. From the effective potential one can determine various critical exponents [27,46] describing the critical behaviour of the theory near the line of phase transition and then compare them with those obtained in lattice calculations. As was noted in Refs.[27,46], the results of lattice computations for noncompact QED with sufficiently large bare coupling constant [45] can be understood if it is supposed that in the scaling region an essential role is played by the induced four-fermion interaction. The derivation of the effective action in the GNJL model shows clearly how in the scaling region, $\bar{\Sigma}_0/\Lambda \ll 1$, this formally inessential interaction is trans-

formed into essential degrees of freedom, like the fields σ and π that become dynamically active and relevant for an accurate description of the physics. Further investigations are welcome in order to take into account the effects of Coulomb bound states manifested at large gauge coupling.

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