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# EINSTEIN'S NOTION OF WEAK GRAVITY, THE EQUIVALENCE PRINCIPLE AND CRITERIA FOR GRAVITATIONAL PLANE WAVES

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Abstract

Einstein's notion of weak gravity implies Newton's static theory as a first approximation, and is the basic assumption of Einstein's radiation formula. But, Einstein suggested in 1916 that any Gaussian curvilinear coordinate system is valid as a space-time coordinate system. Therefore, boundedness of a space-time metric and thus the notion of weak gravity would be meaningless. To resolve this paradox, one must note, as Einstein pointed out in 1921, that, to serve as a space-time coordinate system, a Gaussian system must be additional compatible with the equivalence principle. Also, when coordinate variables do not have length as the unit, to consider boundedness, the compensative factors must be removed to obtain a *normalized* metric. Then, it is proven, in agreement with Einstein's notion, that a normalized metric element must be bounded for weak gravity. Thus, a solution of the space-time metric is valid in physics only if it is compatible with Einstein's notion of weak gravity. Since the gravitationl plane-waves must be justified as a (spatial) local idealization of a weak wave from an isolated finite source at a far distance, the unbounded "plane-waves" of Bondi, Pirani, and Robinson are invalid as a physical idealization. This supports the correctness of Einstein's 1936 discovery and that, for dynamic problems, the Einstein equation of 1915 is incompatible with weak gravity.

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"According to the principle of equivalence, the metrical relations of the Euclidean geometry are valid relatively to a Cartesian system of reference of infinitely small dimensions, and in a suitable state of motion (free falling, and without rotation)." -- A. Einstein (The Meaning of Relativity, p. 90).

## 1. Introduction.

A major problem in general relatvity is that any Riemannian geometry metric with the proper metric signature would be accepted as a valid solution of Einstein's equation of 1915 (Kramer, Stephani, Herlt & MacCallum 1980). This is, in part, due to the fact that the nature of the source term has been obscure since the beginning (Einstein 1954). When a source term is given, the adequacy of this term for a physical situation is often not clear. For instance, the electromagnetic energy-stress tensor may not be adequate for a dynamic problem (Lo 1997b) although it has been adequate for a static problem (Wald 1964). To determine whether a solution is valid and whether a given source term is adequate, it is desirable to develope general physical requirements which are not specific for just a particular problem.

Einstein's notion of weak gravity requires that the elements of a space-time metric deviate from the flat metric should be very small in comparison with one if the strength of the gravitational source is not large. Since the gravitational coupling constant is very small, the boundedness of a solution is often sufficient. This notion is based on the principle of causality (i.e., the causes of phenomena are identifiable [Lo 1994, 1997b]), and is also consistent with the correspondence principle since this notion leads to a static limit which agrees with Newton's theory. Thus, the notion of weak gravity is a physical requirement with which a solution of the field equation must be compatible (see §2). This physical requirement, due to its simplicity, is useful in determining whether the source term of an Einstein equation is adequate; and can be used even before a solution is obtained (Lo 1994, 1997b).

Moreover, boundedness is a natural requirement. In physics, every physical quantity is finite; and, for an isolated system, the magnitude of any physical quantity is bounded. In general relativity, Einstein's weak gravity assumption suggests that the space-time metric $g_{ab}$, being a physical quantity, is usually not an exception. In other words, for a finite isolated system, one has

$$|g_{ab}(x,y,z,t)| < \text{constant} \tag{1}$$

over the entirety of space-time. In fact, eq. (1) is also a necessary implicit assumption of Einstein's radiation formula (Weinberg 1972). However, there are complications since some coordinate variables may not have the unit of length (§ 2). Also, in part due to Einstein's (1916) early error (§ 3), many theorists failed to recognize the crucial differences between an arbitrary mathematical coordinate system and a valid physical space-time coordinate system (Lo 1994, 1998); and thus they believed that eq. (1) had no meaning in physics. This illustrates that "Too great an emphasis on geometry can only obscure the deep connection between gravitation and the rest of physics (Weinberg 1972)."

As a result of these confusions (§§ 3-6), the unbounded "plane-wave" solutions presented by Bondi, Pirani and Robinson (1959) were accepted as a valid local idealization. Moreover, there is a book on colliding "plane waves" (Griffiths 1991). Note also that Damour and Schmidt (1990) claimed the existence of dynamic solutions, although they have not proven the crucial characteristic, their compatibility with weak gravity (Newton 1997). Thus, from either the viewpoint of theoretical consistency or the viewpoint of applications of the theory, the question of compatibility with the notion of weak gravity must be clarified.

## 2. Normalization of a Space-Time Metric, and Compatibility with Weak Gravity.

Some theorists disregard eq. (1) because its validity would depend on the choice of variables. For instance, for a spherical coordinate system, the Minkowski metric is

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2), \tag{2}$$

instead of the metric, $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ in a Cartesian coordinate system. The metric elements such as $r^2$ and $\sin\theta^2 r^2$ are not bounded. But, they also are ***compensative factors*** which arise from the fact that both the coordinate variables $\theta$ and $\phi$ have a dimension other than length. To consider the physics of boundedness, these factors should be eliminated to have a normalized metric.

Definition. A metric $\hat{g}_{\mu\nu}$ is said to be the ***normalization*** of a space-time metric $g_{ab}$ if $F_{\mu\nu} = \eta_{\mu\nu} f_\mu f_\nu$

($\eta_{\mu\nu}$ is the flat metric) is the flat metric representation in that coordinate system, and $\hat{g}_{\mu\nu} = g_{\mu\nu}/f_\mu f_\nu$ .

For instance, consider the Schwarzschild metric (Wald 1984) solution of the Einstein equation of 1915,

$$ds^2 = (1 - \frac{C}{r})dt^2 - (1 - \frac{C}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2), \tag{3a}$$

where $C = KM/4\pi$, K is the coupling constant, and M is the total mass. Metric (3a) is an exterior solution for a star. (According to the perfect fluid model, an interior solution requires the radius of a star, $R > 9C/8$ [Wald 1984].) For the Schwarzschild metric (3a), the ***normalized*** metric elements $\hat{g}_{ab}$ are:

$$\hat{g}_{tt} = (1 - \frac{C}{r}), \quad \hat{g}_{rr} = -(1 - \frac{C}{r})^{-1}, \quad \text{and} \quad \hat{g}_{\theta\theta} = \hat{g}_{\phi\phi} = -1 \tag{3b}$$

Therefore, for a star (of radius $R > 1.125C$) the normalized metric elements of the exterior solution are bounded, and similarly that of the interior solution. Thus, the normalized metric elements are bounded.

The equivalence principle is readily related to the normalized metric since a light speed is related to the normalized metric elements (Einstein 1954) Then, based on $ds^2 = 0$, the light speed in the $\theta$-direction is

$$\frac{rd\theta}{dt} = \pm\{-\hat{g}_{tt}/\hat{g}_{\theta\theta}\}^{1/2} = \pm (1 - \frac{C}{r})^{1/2}, \quad \frac{1}{3} < \left|\frac{rd\theta}{dt}\right| \leq 1, \tag{4a}$$

and

$$\frac{dr}{dt} = \pm\{-\hat{g}_{tt}/\hat{g}_{rr}\}^{1/2} = \pm (1 - \frac{C}{r}), \quad \frac{1}{9} < \left|\frac{dr}{dt}\right| \leq 1 \tag{4b}$$

for the r-direction. On the other hand, the metric element $g_{\theta\theta}$ is related to angular velocity, i.e.,

$$\frac{d\theta}{dt} = \pm\frac{1}{r}\{-g_{tt}/g_{\theta\theta}\}^{1/2} = \pm \frac{1}{r}(1 - \frac{C}{r})^{1/2}, \quad \text{and} \quad \text{limit} \left|\frac{d\theta}{dt}\right| = 0, \text{ as } r \to \infty \tag{4c}$$

Thus, that the metric element $g_{\theta\theta}$ is unbounded, reflects the fact that the angular velocity of light must go to zero as r increases, since the light speeds have a maximum. But, $\hat{g}_{\theta\theta}$ is bounded.

Boundedness of a normalized metric is independent of the specific space-time coordinate system used.

For example, in a Cartesian coordinate system, Schwarzschild metric (3a) is transformed to (Pauli 1958)

$$ds^2 = (1 - \frac{C}{r})dt^2 - (dx^2 + dy^2 + dz^2) - \frac{C}{r}(1 - \frac{C}{r})^{-1}r^{-2}(xdx + ydy + zdz)^2. \qquad (3c)$$

It is clear that the elements of metric (3c) are bounded for $r > 1.125\ C$. It will be shown in § 4 that this is guaranteed by general physical considerations such as the equivalence principle (Einstein 1954; see also § 3).

It should be noted that, for $r < C$, eq. (4) is no longer valid since both $g_{tt}$ and $g_{rr}$ change sign; t becomes space-like and r becomes time-like. Thus, metric (3) is neither static nor spherically symmetric. Moreover, Bonnor (1992) pointed out, "This is the region of final approach to the black hole which is supposed to exist at $r = 0$; notice that this (physical) singularity is space-like." Thus, the Schwarzschild solution actually supports Einstein's (1954) remark that his equation may not be valid for very compact objects However, if only compatibility with weak gravity is interested, one does not have to consider a very strong gravity.

Definition. For an isolated physical system, a space-time metric $g_{ab}(x)$ in a Cartesian coordinate system is said to be ***compatible with the notion of weak gravity*** if i) there exists a set of n-parameters ($\lambda_1$, $\lambda_2$, ... , $\lambda_n$) which represent the strength of the sources for gravity such that $g_{ab}(x) = g_{ab}(x;\lambda_1,...,\lambda_n)$ and $g_{ab}(x;0,...,0) = \eta_{ab}$ , the flat metric; ii) there exists a parameter set value $(\bar{\lambda}_1,...,\bar{\lambda}_n)$ such that the space-time metric $g_{ab}(x;\bar{\lambda}_1,...,\bar{\lambda}_n)$ satisfies the weak gravity condition, $|g_{ab}(x;\bar{\lambda}_1,...,\bar{\lambda}_n) - \eta_{ab}| \ll 1$ [2] (to be more precise, $\forall\ \epsilon,\ \exists\ \delta,\ \ni\ |g_{ab}(x;\bar{\lambda}_1,...,\bar{\lambda}_n) - \eta_{ab}| < \epsilon$, if $|\bar{\lambda}_i| < \delta$ for $i = 1,...,n$).

For example, if a star is so compact that its radius $R < C$ ($= KM/4\pi$), the Schwarzschild metric (3a) is not bounded. Let us consider a very small $\bar{\bar{C}}$ ($\ll R$). Then, metric (3c) becomes regular (i.e., no singularity) and bounded. Thus, metric (3a) is compatible with weak gravity. Note that the compatibility with weak gravity is based on the principle of causality, and is satisfied by linear field equation (14) in § 5.

Nevertheless, many solutions of Einstein's equation of 1915 fail this physical requirement of compatibility with the notion of weak gravity, and therefore can be intrinsically unphysical. Let us consider, for instance,

$$ds^2 = du\ dv + h_{ij}(u)x^i x^j\ du^2 - dx^i\ dx^j\ , \qquad (5)$$

where $u = t - z$, $v = t + z$, $h_{ii}(u) \geq 0$, and $h_{ij} = h_{ji}$. Its physical cause can be an electromagnetic plane wave (Penrose 1965). This metric is incompatible with weak gravity since $h_{ij}(u)x^i x^j$ can be arbitrarily large. Moreover, the gravitational force (related to $\Gamma^z{}_{tt} = \frac{1}{2}\partial(h_{ij}x^i x^j)/\partial t$ has arbitrary parameters (the coordinate origin). This arbitrariness in the metric violates the principle of causality (Lo 1994). Metric (5a) also does not satisfy coordinate relativistic causality and therefore the equivalence principle because the requirement, $1 > (1 + H)/(1 - H)$ (where $H = h_{ij}x^i x^j$) may not be satisfied. The cause of these problems has been identified as due to an inadequacy of the source term of the Einstein equation of 1915 (Lo 1997b).

In general, the metric can be written, instead of

$$ds^2 = g_{ab}\, dx^a\, dx^b\ , \tag{6a}$$

but as

$$ds^2 = \hat{g}_{\mu\nu}\, d\hat{x}^\mu\, d\hat{x}^\nu\ , \quad \text{where} \quad d\hat{x}^\mu = f_\mu dx^\mu \tag{6b}$$

are compensated differentials. For example, $d\hat{\theta} = rd\theta$. Then, eq. (1) is modified to be generally

$$|\hat{g}_{\mu\nu}| < \text{constant}. \tag{1$'$}$$

It will be shown that eq. (1)' is always valid for weak gravity due to the equivalence principle. However, to prove this, one must first clarify the misunderstandings related to the equivalence principle.

## 3. Covariance, the Equivalence Principle, and the Light Cone Condition.

Einstein (1916) argued that a Gaussian curvilinear coordinate system can serve as a space-time coordinate system in physics. This is incorrect because a space-time coordinate system must satisfy the equivalence principle and other physical requirements, as Einstein (1954) clarified in 1921. Einstein's earlier desire for the validity of a Gaussian system as a space-time coordinate system came from his over-stated principle of general covariance. In practice, there are two features in the general covariance principle: 1) the mathematical formulation in terms of the Riemannian geometry; and 2) the general validity of any Gaussian

coordinate system as a space–time coordinate system in physics.

To establish the need of Riemannian geometry, Einstein (1916) argued "The law of physics must be of such a nature that they apply to systems of reference in any kind of motion." Then, he successfully demonstrated that Riemannian geometry is needed for a rotating coordinate system. However, he has not been able to establish feature 2). His arguments which are absent from his later book (Einstein 1954) are as follows:

> "That this requirement of general covariance, which takes away from space and time the last remnant of physical objectivity, is a natural one, will be seen from the following reflexion. All our space–time verifications invariably amount to a determination of space–time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point–events happening at the same place at the same time. The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences."

However, while all verifications indeed amount to a determination of space–time coincidences, in order to ***predict*** such coincidences theoretically, one must able to relate events of different locations in a definite manner. Eddington (1975) commented "space is not a lot of points close together; it is a lot of distances interlocked." Thus, a coordinate system must be related to objective physical measurements.

For instance, the equivalence principle actually restores the physical objectivity of space and time. As a result, we can calculate the gravitational red shifts which relate events of different locations. Such a prediction is possible because the equivalence principle does relate events of different locations. Thus, a space–time coordinate system is not just a Gaussian system. On the contrary, using a Gaussian system can be ***incompatible with experiments*** if the equivalence principle is ***not*** satisfied (see metrices [8a] & [10]). Thus, the physics of the covariance principle lies not only in the validity of feature 1) but also the restriction on feature 2), the conditional validity of a Gaussian system in physics.

If, at the earlier stage, Einstein's arguments are not so perfect, he seldom allowed such defects be used in his calculations. In other words, unlike his earlier arguments, his calculations are self–consistent. This is

evident in his book, 'The Meaning of Relativity' which he edited in 1954. Thus, we can actually obtain his final viewpoints from his calculations and this book. Einstein's viewpoints on space-time coordinates are:

1) A physical (space-time) coordinate system must be physically realizable (see also 2) & 3) below).

   Einstein (1982a) made clear in 'What is the Theory of Relativity?' that "In physics, the body to which events are spatially referred is called the coordinate system." Furthermore, Einstein wrote "If it is necessary for the purpose of describing nature, to make use of a coordinate system arbitrarily introduced by us, then the choice of its state of motion ought to be subject to no restriction; the laws ought to be entirely independent of this choice (general principle of relativity). Thus, Einstein's coordinate system has a state of motion and is usually referred to a physical body. Also, his calculations would not be self-consistent if an arbitrary Gaussian system were used (see metrices [8a] & [10]).

2) A physical coordinate system is a Gaussian system such that the equivalence principle is satisfied.

   In some text books of general relativity (Misner, Thorne, & Wheeler 1973; Wald 1984; Ohanian & Ruffini 1994) any Gaussian system is accepted as a space-time coordinate system. One might attempt to justify this viewpoint by pointing out that Einstein (1954) also wrote in his book that "In an analogous way (to Gaussian curvilinear coordinates) we shall introduce in the general theory of relativity arbitrary co-ordinates, $x_1$, $x_2$, $x_3$, $x_4$, which shall number uniquely the space-time points, so that neighbouring events are associated with neighbouring values of the coordinates; otherwise, the choice of co-ordinate is arbitrary." But, Einstein (1954) qualified this with a physical statement that "In the immediate neighbour of an observer, falling freely in a gravitational field, there exists no gravitational field." Later, the usage of the equivalence principle is demonstrated.

3) The equivalence principle requires not only, at each point, the existence of a local Minkowski space

$$ds^2 = c^2dT^2 - dX^2 - dY^2 - dZ^2, \tag{7}$$

*but such a local Minkowskian space must be obtainable by free falling.*

   Einstein (1954) wrote, "According to the principle of equivalence, the metrical relation of the Euclidean geometry are valid relative to a Cartesian system of reference of infinitely small dimensions,

and in a suitable state of motion (free falling, and without rotation)." Thus, in addition to having the proper metric signature, a solution of the Einstein equation (which is approximately valid for the case considered [Lo 1995]) must satisfy the equivalence principle as an independent physical requirement. Note that the existence of local Minkowski spaces obtainable by the "free falling" is a physical requirement. In other words, for a manifold whose metric has the proper signature (+,−,−,−), its local Minkowski spaces may not be obtainable by "free falling". Such a possibility was over-looked by Einstein's paper of 1916.

In short, a Gaussian coordinate system with a proper metric signature, is not necessarily a space-time coordinate system whose local Minkowski spaces are obtainable by free falling. Although Einstein's equivalence principle is eloquently presented in many modern texts books (see for instance, § 3 of chapter one in the excellent book written by Weinberg [1972]), they have neglected to discuss this characteristic which makes a space-time coordinate system distinct from an arbirary Gaussian coordinate system.

If the covariance principle should be restricted because of the equivalence principle, the current notion of gauge is inadequate. This notion of gauge can be based on only the mathematical diffeomorphism (Wald 1984) which is a one-one, onto, $C^\infty$ (infinitely differentiable) map between manifolds and its inverse map is $C^\infty$. It was also claimed that two differeomorphic manifolds have physically identical properties. But, Bonnor et al. (1994) found that such a coordinate freedom leads to difficulties in physical interpretations. Moreover, according to this notion, coordinates cannot have any physical meaning other than identifying space-time points so that neighboring events are associated with neighbouring values of the coordinates. (The inadequacy of such a coordinate system has already criticized by Eddington [1975].) Thus, a space-time coordinate system is incorrectly reduced to just a mathematical Gaussian system. Note that the equivalence principle restricts the valid space-time coordinate systems for covariance and therefore the valid gauge transformations

If a coordinate system for which the equivalence principle is not satisfied, is mistaken as a space-time coodinate system, inconsistence would happen. To illustrate this, let us consider the constant metric,

$$ds^2 = [(c - v)dt' + dz'][(c + v)dt' - dz'] - dx'^2 - dy'^2, \tag{8a}$$

which satisfies the Einstein equation $G_{\mu\nu} = 0$. Then, if the coordinate system were a space-time coordinate

system, for light rays in the z'-direction, $ds^2 = 0$ would imply the light speeds were

$$\frac{dz'}{dt'} = c + v, \quad \text{or} \quad \frac{dz'}{dt'} = -c + v. \tag{8b}$$

Eq. (8b) gives a light speed larger than c. On the other hand, metric (8a) implies no gravity. When gravity is absent, the Michelson-Morley experiment supports that light speed is c. Thus, eq. (8b) disagrees with experiment and violates ***coordinate relativistic causality*** (i.e. no cause event can propagate faster than c). Thus, if metric (8a) were valid in physics, Einstein's method would be wrong.

Moreover, metric (8a) is obtained from the flat metric ($ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$) by

$$x' = x, \quad y' = y, \quad z' = z + vt, \quad \text{and} \quad t' = t, \tag{9}$$

where t is the time coordinate and v is a constant. The Galilean transformation (9) is clearly a diffeomorphism. But, one cannot justify transformation (9) as physical since metric (8a) is ***not*** physically realizable as verified by the Michelson-Morley experiment. In other words, the prime system (8a) is merely a manifold in mathematics whose coordinate system has no physical meaning (please see also Appendix A). In conclusion, ***the Galilean transformation (9) is also not valid in general relativity.***

In metric (8a), there is a non-zero cross term 2vdt'dz' although this is not the problem (see Appendix A). To see the need of satisfying the equivalence principle further, we consider another constant metric,

$$ds^2 = \alpha^2dt^2 - dx^2 - dy^2 - dz^2, \tag{10}$$

where $\alpha$ ($\geq 2c$) is a constant. Metric (10) is a solution of the Einstein equation $G_{\mu\nu} = 0$. Then, $ds^2 = 0$ would imply that the velocity of light is $\alpha$. One might argue that metrice (10) can be transformed to

$$ds^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2, \tag{11}$$

by the following diffeomorphism,

$$x' = x, \quad y' = y, \; z' = z, \quad \text{and} \quad t' = t\alpha/c. \tag{12}$$

Eq. (12) implies, however, that the units of t and t' are distinct and the light speed remains $\alpha$ but ***not*** c.

If one examines the rate of a unit clock, which is arranged to be at rest in the gravitational field (10), then according to Einstein (1916, 1954) for a clock period ds = 1; dx = dy = dz = 0, one has $1 = g_{tt}\, dt^2$. Hence, $dt = 1/\alpha$. It follows that the light speed is $\alpha$ since there is no contraction for the space coordinates.

Eq. (12) is not a rescaling since all the physical units remain the same. In a rescaling only the physical units, but not the physics, are changed. For example, the light speed can be expressed as 1 lightyear per year or $3 \times 10^{10}$ cm/sec. However, if $\alpha = 2c$, metric (10) implies that the light speed would be 2c, i.e., $6 \times 10^{10}$ cm/sec; and metric (11) implies that the light speed is $3 \times 10^{10}$ cm/half-sec. Thus, if metric (11) were considered as Minkowski, the diffeomorphism (12) would amount to redefining the space.

To resolve the difficulties, let us check whether the equivalence principle is satisfied. Since (8a) and (10) are constant metrices, all their Christoffel symbols $\Gamma^{\mu}{}_{\alpha\beta}$ are zero. Therefore, the manifolds are flat and, according to the geodesic equation, there is no gravitational acceleration. Then, in a non-rotating free falling, the velocity of an observer is a constant. According to special relativity, this observer carries with himself a new coordinate system which must be obtained by a Lorentz transformation. But, a Lorentz transformation cannot transform metrices (8a) nor (10) to a local Minkowski space. Thus, the equivalence principle is not satisfied. In conclusion, ***the cause of an incorrect light speed and other problems is identified as due to the failure of satisfying the equivalence principle*** (see also Appendix B).

Due to the requirement of the equivalence principle, a diffeomorphism may not be adequate in physics although such a diffeomorphism could be useful for the purpose of calculation in mathematics. But, whatever mathematical coordinate systems one may use in calculations, one must first verify that the equivalence principle is valid in a space-time coordinate system on which physical interpretations can be based.

The above analysis also raises an important question, would a manifold whose metric has the proper signature (+,−,−,−) be alway diffeomorphic to a physical space? Unfortunately, the answer is negative

because such a metric may fail other physical requirements (Lo 1994, 1998) such as the principle of causality and etc. For instance, the current solution for an electromagnetic plane-wave (Lo 1994, 1997b) and the "plane-waves" of Bondi et al. (1959) (as will be shown in § 7) are incompatible with Einstein's notion of weak gravity. Thus, "defining" the light speed in the local Minkowskian space, though used in text books, is only an illusion because such a definition is not mathematically well-defined and therefore physically misleading. This is one of the manner that currently the equivalence principle is practically ignored.

Historically, as shown by Hilbert (1915), the Einstein equation is insufficient to obtain a solution, and four more conditions are needed. These conditions, which result in a choice of coordinates, is called the gauge. But, because a physical space must satisfy the equivalence principle, the choice is actually restricted. This may explain why this gauge notion was not accepted by Eddington (1975). Moreover, the full extend of such a freedom of choice, as pointed out by Bonnor et al. (1994), is a problem for a consistent physical interpretation. Note that Einstein's viewpoints on the space-time coordinates have changed over years. A clear example to support the belief of Einstein's change is his view on the equivalence principle. In 'Relativity and Problem of Space (1954)', Einstein (1982b) added the crucial phrase, "at least to a first approximatioin" on the indistinguishability between gravity and acceleration. In view of these facts, one should take from the viewpoints of his later years as Einstein's viewpoint on a subject.

In short, theoretical self-consistency needs a restriction of the mathematical covariance to ***physical covariance*** among valid space-time coordinate systems (Lo 1994, 1998). In particular, the light-cone condition gives the valid light speed only if the equivalence principle is satisfied. Nevertheless, since Einstein has made little efforts beyond his book (Einstein 1954) to rectify this mistake, the confusion persists. Moreover, many relativists are really mathematician at heart, and are interested in solutions instead of their physical meaning (Kramer et al. 1980). Consequently, they ignore the need of satisfying the equivalence principle as a physical requirement and are unwilling to give up the convenience of having the unrestrict gauge. Among Einstein's peers, Eddington (1975) is probably the only one who realized this mistake.

## 4. The Equivalence Principle and Boundedness of the Space-Time Metric

To show the need of restrictions by physical principles, let us consider the gravity of an isolated system.

If any Gaussian system could be a space-time coordinate system, then it can be shown that the "space-time" metric were not bounded. For instance, consider the transformation (without changing the time unit),

$$t = C\{\exp(T/C) - \exp(-T/C)\}/2. \tag{13a}$$

Then

$$ds^2 = \frac{1}{4}\{\exp(T/C) + \exp(-T/C)\}^2 dT^2 - dx^2 - dy^2 - dz^2 \tag{13b}$$

repesents the Minkowski metric. Metric (13b) does not involve rescaling since (13a) implies, when $T = 0$, $t = 0$ and the coefficient of $dT^2$ is 1. But, the metric element $\{\exp(T/C) + \exp(-T/C)\}^2/4$ is not bounded, and $(T,x,y,z)$ is not a space-time coordinate system. In other words, metric (13b) is not realizable in physics. If it were, the light speed in the x-direction, according to $ds^2 = 0$ (Einstein 1954), would be

$$\frac{dx}{dT} = \pm\frac{1}{2}\{\exp(T/C) + \exp(-T/C)\}, \tag{13c}$$

and

$$\lim\left|\frac{dx}{dT}\right| = \infty, \quad \text{as} \quad T \to \infty .$$

Thus, in violation of coordinate relativistic causality, the light speed could be infinite. This illustrates that the boundedness of a normalized space-time metric has its root on the equivalence principle (Lo 1998). It can be shown directly that metric (13b) violates the equivalence principle (see Appendix B).

Now, consider another diffeomorphic transformation (without changing the distant unit),

$$x = C\{\exp(X/C) - \exp(-X/C)\}/2. \tag{14a}$$

Then

$$ds^2 = dt^2 - \frac{1}{4}\{\exp(X/C) + \exp(-X/C)\}^2 dX^2 - dy^2 - dz^2 \tag{14b}$$

repesents the Minkowski metric. Therefore, metric (14b) satisfies the equation $G_{ab} = 0$. But, the metric element $\{\exp(X/C) + \exp(-X/C)\}^2/4$ is not bounded. Consequently, $(t,X,y,z)$ is not a valid space-time

coordinate system in physics. If it were, the light speed in the X-direction, according to the light cone condition $ds^2 = 0$ (Einstein 1954), would be

$$\frac{dX}{dt} = \pm 2\{\exp(X/C) + \exp(-X/C)\}^{-1}, \tag{14c}$$

and

$$\lim \left| \frac{dX}{dt} \right| = 0, \quad \text{as} \quad X \to \infty .$$

Thus, the light speed would approach zero (but the light speed is 1 in y and z-directions) without an identifiable physical cause since $G_{ab} = 0$. In general relativity, a very slow speed of light is due to strong gravity.

Moreover, metrices (13) and (14) are incompatible with Einstein's notion of weak gravity since it cannot be achieved with adjusted parameters. Thus, the need of physical principles is illustrated.

Now, let us consider whether a normalized metric for an isolated system is bounded. First, according to Einstein (1954), the gravitational time dilation implies that $\hat{g}_{tt} \leq 1$; and we observe gravitational red shifts but not blue shifts. In general, for an orthogonal coordinate system, the light speed L(s) in the s-direction is

$$|L(s)| = \{-\hat{g}_{tt}/\hat{g}_{ss}\}^{1/2} \leq 1. \tag{15a}$$

However, an upper light speed limit leads to only $\hat{g}_{tt} \leq -\hat{g}_{ss}$ , and would still allow $|\hat{g}_{ss}|$ to be unbounded. As shown by metric (3), if a star could be so compact that r = C belonged to the exterior of the star, one would have $\hat{g}_{tt} = 0$ and $\hat{g}_{rr} = \infty$ and the light speed is zero. However, if one limits the region to r > 1.125 C, then all the normalized metric elements are bounded, and the light speed has a lower limit.

Now, if one assumes that the light speed has a lower limit $L_m$ ($\leq 1$), then one has

$$|\hat{g}_{tt}| \leq |\hat{g}_{ss}| \leq |\hat{g}_{tt}|(L_m)^{-2} \leq (L_m)^{-2}. \tag{15b}$$

Thus, the space metric elements are bounded above. They would also be bounded below, if $|\hat{g}_{tt}|$ also has a lower bound $g_L$ . Since the light speeds has a lower bound, the existence of $g_L$ is likely.

For a non-orthogonal space-time coordinate system, it can also be shown (please see Appendix C) that all normalized elements of a space-time metric are bounded above if the light has a lower speed limit. Thus, eq. (1)' is valid if the small regions, which are associated with singularities, are excluded. This means that the normalized space-time metric is bounded for weak gravity since, as pointed out in § 2, singularities would disappear for weak gravity. Thus, ***Einstein's notion of weak gravity is consistent with the equivalence principle.*** Consequently, ***compatibility with the weak gravity notion is valid in general relativity and should be a physical requirement for the field equation.***

## 5. The Notion of Weak Gravity and The Existence of Gravitational Waves

Relativity requires the existence of gravitational waves because physical influence must be propagated with a finite speed (Lorentz 1900). Thus, the existence of the gravitational wave is also a physical requirement for a valid field equation. The Einstein equation of 1915 seemed to satisfy this requirement since its linearized "approximation" (see eq. [14]) clearly gives a wave solution. The Einstein equation of 1915 is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -K\, T(m)_{\mu\nu} , \tag{16}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, $T(m)_{\mu\nu}$ is the energy-stress tensors for massive matter, and K (= $8\pi\kappa c^{-2}$, where $\kappa$ is the Newtonian coupling constant) is the coupling constant.

However, Einstein discovered in 1936 (Born 1968) that his equation does not admit a propagating wave solution and therefore linearized gravity is not reliable. Recently, it has been shown that this linearization procedure is not generally valid in mathematics (Lo 1995, 1996). Thus, it would be necessary to justify the existence of a propagating wave solution independent of the Einstein equation of 1915 (see also § 7).

Fortunately, for a massive source, the linear field equation as the first order approximation, can be justified (Lo in preparation) on the basis of the equivalence principle which requires modifying the Newtonian gravity, and earlier experiments such as the bending of light. The linear field equation for weak gravity is

$$\frac{1}{2} \partial_c \partial^c \bar{\gamma}_{ab} = -KT(m)_{ab} , \tag{17a}$$

where $\bar{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\eta_{ab}\,\gamma$, $\gamma_{ab} = g_{ab} - \eta_{ab}$ , $\gamma = \eta^{cd}\gamma_{cd}$ , and $\eta_{ab}$ is the flat metric. Eq. (17a) has a mathematical structure similar to that of Maxwell's equations. An inhomogeneous solution of eq. (17a) is

$$\bar{\gamma}_{ab}\,(x^i,t) = -\frac{K}{2\pi}\int \frac{1}{R} T_{ab}[y^i,\ (t-R)]\,d^3y, \quad \text{where } R^2 = \sum_{i=1}^{3}(x^i - y^i)^2 . \tag{17b}$$

Note that the Schwarzschild solution, after a gauge transformation, can also be approximated by (17b). Solution (17b) would represent a wave if $T_{ab}$ has a dynamical dependency on time t' (= t − R). Thus, the theoretical existence of gravitational waves is assured as a certainty as believed (Pauli 1958; Weinberg 1972; Misner et al. 1973; Wald 1984; Ohanian & Ruffini 1994; Lo 1995).

Moreover, from linear field eq. (17), some characteristics of an exact solution for the weak gravity due to dynamic massive matter can be obtained. They are as follows:

i) Solution (17b) manifests that the first order approximation of the space-time metric includes a propagating wave, and is an almost periodic function of time for a source in an almost periodic motion. Therefore, the principle of causality implies that the exact solution is also an almost periodic function.

ii) Moreover, if eq. (17a) gives a first order approximation, $\partial^a\,\bar{\gamma}_{ab}$ should be of second order (Eddington 1975) due to the conservation law, $\nabla^a\,T_{ab} = 0$ (which is independent of the notion of gauge).

iii) By definition, an exact space-time metric element (in a Cartesian coordinate system) for weak gravity is a small deviation from the flat metric (Einstein 1954) and therefore must be bounded (i.e. $|g_{ab}| <$ constant). On the other hand, according solution (17b), the first order approximation of a $\gamma_{ab}$ is also bounded. Thus, eq. (17) is consistent with the requirement of weak gravity. Moreover, for the case of including singular mass distributions, in the region too close to the singular source, $\gamma_{ab}$ is no longer small and therefore eq. (17a) is not valid for this problem. Thus, one must remodel $T_{ab}$ such that weak gravity can be applied without significantly changing the gravity of the other regions.

These characteristics assure the validity of weak gravity, and the existence of gravitational waves. Note that conditions i) and iii) are also satisfied by the electromagnetic wave and conditon ii) is similar to the Lorentz gauge. Thus, there should be a fruitful analogy with electromagnetism (Weinberg 1972) (see also § 6).

From field eq. (17), the first order approximation of a space-time metric is bounded. As shown in §4,

for weak gravity, the exact space-time metric must be bounded. Due to the coupling constant K is very small, the first order approximation is dominant in the exact space-time metric. This dominance is used in the derivation of Einstein's radiation formula which is supported by binary pulsar experiments. Einstein's radiation formula is, in turn, the base to show (Lo 1995) eq. (17) to be incompatible with the Einstein eq. (16).

## 6. The Plane-Wave as a Spatial Local Idealization

In general relativity, the equation for weak gravity is a Maxwell-type equation, and thus one may expect a fruitful analogy between the electromagnetic wave and the gravitational wave. In classical electrodynamcis, as well as general relativity, as stated by Bondi, Pirani and Robinson (1959), the plane-wave is a spatial local idealization of a wave from a distant source. However, a plane-wave, though has a finite energy density in electrodynamics, is not a valid global idealization since its total energy is infinite.

Nevertheless, it is the plane-wave solutions of Maxwell's equations that lead most naturally to an interpretation in terms of a particle, the photon. Many physicists believe also that similarly it is the radiative solutions of Einstein's equations that will lead to the concept of a particle of gravitational radiationis, the graviton (Weinberg 1972; Misner et al. 1973). Thus, gravitational plane-wave solutions are important.

To show a plane-wave as a local idealization, let us consider a field equation of d'Alembertian type,

$$\partial_c \partial^c W(x^i,t;\lambda_\alpha) = -2KS(x^i,t;\lambda_\alpha) \ . \tag{18}$$

which governs the wave W with a finite source term S where $\lambda_\alpha$ are the parameters for the source strength. When $\lambda_\alpha = 0$, $S = 0$; and $S(x^i,t;\lambda_\alpha)$ is a uniformly continuous function of $\lambda_\alpha$. A general solution of (18) is

$$W(x^i,t;\lambda_\alpha) = \frac{-K}{2\pi}\int \frac{1}{R} S\left[y^i,\ (t-R);\lambda_\alpha\right] d^3y, \quad \text{where} \quad R^2 = \sum_{i=1}^{3}(x^i - y^i)^2 \ , \tag{19}$$

since K is a constant. Then, clearly solution (19) is compatible with the notion of a weak field. When the source S is a periodic function of t, then W is a wave propagating outward.

When $x^i$ is far away from the region of the source $S(y^i,t';\lambda_\alpha)$, $W(x^i,t;\lambda_\alpha)$ can be approximated as

$$W(x^i,t;\lambda_\alpha) \simeq \frac{-K}{2\pi r}\int S[y^i,\ (t-r);\lambda_\alpha]\,d^3y = \frac{-K}{2\pi r}\,F(t-r;\lambda_\alpha), \quad \text{where} \quad r^2 = \sum_{i=1}^{3}(x^i)^2. \tag{20}$$

Consider

$$\frac{\partial}{\partial x^a}\left[\frac{1}{r}\,F(t-r;\lambda_\alpha)\right] = \frac{-x^a}{r^3}\,F(t-r;\lambda_\alpha) + \frac{-x^a}{r^2}\,F'(t-r;\lambda_\alpha). \tag{21}$$

When $S(x^i,t;\lambda_\alpha)$ is a periodic function of t, F and F' are of the same order. Then, if r is very large, the first term in right hand side of (21) is negligible because of the extra factor 1/r.

Thus, for a large $r_0$ , $W(x^i,t;\lambda_\alpha)$ can be treated ***as if the denominator*** $r_0$ ***were a constant***, i.e.

$$W(x^i,t;\lambda_\alpha) \simeq \beta\,F(t-r;\lambda_\alpha), \quad \text{where} \quad \beta = \frac{-K}{2\pi r_0} \tag{22}$$

can be thought of as if a constant. Note that both W and F are propagating to the same r-direction; and that, at radius $r_0$ (i.e. spatially local), F satisfies approximately the same equation which W satisfies exactly;

$$\partial_c\partial^c\,W(x^i,t;\lambda_\alpha) = 0 \quad \text{and} \quad \partial_c\partial^c\,F(x^i,t;\lambda_\alpha) \approx 0 \tag{23}$$

if the terms with a factor of higher order $1/r^n$ are neglected.

When considering a narrow ray in the z-direction, according to (21) its derivatives with respect of x, and y are very close to zero. Thus, within the narrow ray in the z-direction, one may consider that in effect

$$F(x^i,t;\lambda_\alpha) \approx F(t-z;\lambda_\alpha)\ . \tag{24}$$

$\beta F(t-z;\lambda_\alpha)$ is called a plane-wave and it satisfies eq. (23), the wave equation in vacuum. The above proceedure is idealizing a wave with a distant source as a plane wave. However, this is a spatial local idealization as an approximation of W in the z-direction at $r_0$. Although $\beta F(t-z;\lambda_\alpha)$ ***does not diminish at infinite, it is compatible with the notion of weak field*** if S is nonzero only in a finite region. As such, it is useful for the investigation of spatial local characteristics although the total energy would be infinite.

Because the source $S(x^i,t;\lambda_\alpha)$ is finite and periodic in t, the function $\beta F(u;\lambda_\alpha)$ $(u = t-z)$ is a bounded

periodic function of u, and small for a weak source. But, a plane solution of eq. (23), f(u) may not be a plane-wave. For example, since $f(u) = u^2$ is neither bounded nor periodical, f(u) is not a wave. Similarly, $f(u) = \exp(u^2)$ or $\exp(-u^2)$ is also not a wave. These example illustrate the mistake of superficial mathematics which forgets the crucial physics other than the field equations. Note that both Weinberg (1972) and Misner, Thorne, and Wheeler (1973) have concluded correctly that a gravitational plane-wave is bounded.

If the cause of the field, and thus the source $S(x^i,t;\lambda_\alpha)$ is a periodic function of t, then due to the principle of causality, the exact solution of gravity should also be a periodic function of t. Also, for a very large $r_0$, $1/r_0$ can even be much smaller than $K^n$ the nth order. If the first order approximation (17b) for weak gravity is dominating due to the smallness of the coupling constant K, it may be possible to have the plane-wave idealization for the exact equation. In short, if eq. (17) is a valid approximation, a gravitational exact plane-wave should also be a periodic function of u and compatible with weak gravity although such a wave does not diminish at infinite. It remains to check whether $G_{ab} = 0$ is compatible with this *requirement.*

## 7. Compatibility with Weak Gravity and Invalidity of Existing "Plane-Wave" Solutions

From the above analysis, a crucial characteristic of the plane-wave is its boundedness. As pointed out by Bondi, Pirani and Bobinson (1959), the gravitational plane wave is an idealization of a weak wave from a distant soure. They wrote "Our interest in plane waves derives not, of course, from the expectation that such waves might exist in nature, but from the presumption that at a great distances from a finite source of gravitational waves, these waves must appear to be approximately plane."

However, their exact "plane wave" is

$$ds^2 = \exp(2\phi)(d\tau^2-d\xi^2)-u^2[\mathrm{ch}2\beta(d\eta^2+d\zeta^2)+\mathrm{sh}2\beta\cos 2\theta(d\eta^2-d\zeta^2)-2\mathrm{sh}2\beta\sin 2\theta d\eta d\zeta], \quad (25a)$$

where $\phi$, $\beta$, and $\theta$ are functions of u ($= \tau-\xi$). It satisfies the differential equation (i.e. their eq. [2.8]),

$$2\phi' = u(\beta'^2 + \theta'^2\,\mathrm{sh}^2\,2\beta). \quad (25b)$$

Clearly, the variables have the unit of length. Although $\exp(2\phi)\leq 1$ is possible, metric (25a) is not bounded because this would require the impossibility of $|u^2| <$ constant. A wave whose amplitude grows with time due to the factor function $u^2$, can hardly be regarded as due to an isolated finite source which radiates energy over time. Moreover, according to metric (25a), the velocity of light has a very strange directional behavour. The light speed in $\eta$- or $\zeta$-direction has the time limit zero although the light speed is 1 in the $\xi$-direction.

A plane-wave is an idealization for (spatial) local characteristics (see § 6). Accordingly, a plane-wave must satisfy the three physical requirements (which is not too severe as Rosen's [1937]) in § 5. However, metric (25) fails them and thus ***is invalid in physics as a spatial local idealization***. Metric (25) also illustrates that the mathematical characteristic of planeness is insufficient for plane-waves.

Note that because of the factor function $u^2$ in metric (25), there is no parameter whose change would lead to a weak gravity. Thus, metric (25) is incompatible with Einstein's notion of weak gravity. It follows that ***metric (25) for $G_{ab} = 0$ is incompatible with linear field eq. (17).***

Moreover, with metric (25), one may illustrates that an unbounded one-parameter family of solutions $g_{ab}(x^i,\lambda)$ for $G_{ab} = 0$ may have no meaning for a physical perturbation. Although metric (25a) is not bounded, a one-parameter family $g_{ab}(x^i,\lambda)$ can be obtained if one substitutes $\phi$ and $\theta$ in metric (25a) respectively with $\phi+\lambda$ and $\theta+\lambda$ since they satisfy the same differential equation (25b). However, not only $g_{ab}(x^i,0)$, but the expansion of $\lambda$ order terms are not bounded. They are as folllows:

$$h_{(1)\eta\eta} = -h_{(1)\zeta\zeta} = 2u^2 \mathrm{sh}2\beta \sin 2\theta,$$

and

$$h_{(1)\eta\zeta} = -2u^2 \mathrm{sh}2\beta \cos 2\theta. \tag{26}$$

Thus, no matter how small $\lambda$ can be chosen, one cannot relate the family $g_{ab}(x^i,\lambda)$ to a physical perturbation. The existence of a smooth one-parameter family $g_{ab}(x^i,\lambda)$ for $G_{ab} = 0$ has meaning to the reliability of perturbation theory only if 1) globally $g_{ab}(x^i,0) \approx \eta_{ab}$, the flat metric, and 2) the family is bounded, i.e. $|g_{ab}(x^i,\lambda)| \leq$ constant. This means that $g_{ab}(x^i,\lambda)$ should be compatible with the notion of weak gravity.

Since Misner et al. (1973) conclude that a plane-wave is bounded, let us examine their plane-wave,

$$- ds^2 = L^2(e^{2\beta}\, dx^2 + e^{-2\beta}\, dy^2) + dz^2 - dt^2 \,, \tag{27}$$

where both L (background factor) and $\beta$ (wave factor) are functions of $u(\equiv t - z)$. First, form (27) would be incompatible with coordinate relativistic causality which requires:

$$L^{-2}e^{2\beta} \leq 1; \qquad \text{and} \qquad L^{-2}e^{-2\beta} \leq 1. \tag{28}$$

However, relation (28) may not be possible if $L = 1$ and $\beta \neq 0$ is a weak wave limit as stated by Misner et al. (1973). Note that form (27) cannot accommodate a circularly polarized wave (1996). Wald (1984) also pointed out that there is insufficient evidence to support the decomposition in form (27) as natural.

Although form (27) does not have an appearance of obvious unboundedness, the problem remains. Plane-wave form (27) reduces the Ricci curvature to zero except

$$R_{uu} = 2[L'' + (\beta')^2 L]/L \,. \tag{29}$$

However, no exact wave solution has been obtained for $R_{uu} = 0$ (Kramer et al. 1980) because this is impossible (Lo 1997b). Thus, the theoretical approach of Misner et al. (1973) on the plane-waves, is actually not yet self-consistent.

In both cases the transverse metric elements cannot be bounded. This demonstrates also that, for problems related to waves, eq. (16) and eq. (17) are not compatible. Physically, this would mean that eq. (16) is not valid for a dynamic problem since gravitational waves will be created in a dynamic interaction (Lorentz 1900; Wheeler 1990). It should be noted that although compatibility with the notion of weak gravity was not discussed in the literature (Kramer et al. 1980; Blanchet & Damour 1986) of gravitational radiation, this compatibility is crucial for the problem of radiation.

It has been known for some time that any attempt to extend the linear appproximation to higher approximations in a simple way leads to divergent terms. However, the validity of eq. (16) was not questioned because the arguments before 1995 were not clean-cut due to the long history of the solution of linear app-

roximation must be involved. The incompatibility between eqs. (16) and (17) are clearly established in 1995 because symmetry considerations do not need details of the long history of a linear approximation (Lo 1995). Nevertheless, this incompatibility could be interpreted as that contribution of the complicated non-linear terms in the Einstein equations cannot be dealt with satisfactorily following this method and that some other approach is needed. But, the binary pulsar experiments support Einstein's radiation formula, and therefore the linear approximation is a valid approximation. Also, a theoretical proof for the validity of eq. (17) as a valid approximation (Lo in preparation) leads to the conclusion that the need of modifying eq. (16) is beyond doubt.

## 8. Conclusions and Discussions

To see the invalidity of $G_{ab} = 0$ for waves, one can consider the Einstein equation of 1995 (Lo 1995),

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = -K\left[T(m)_{\mu\nu} - t(g)_{\mu\nu}\right], \tag{30a}$$

and

$$\nabla^{\mu}T(m)_{\mu\nu} = 0, \qquad \text{and} \qquad \nabla^{\mu}t(g)_{\mu\nu} = 0, \tag{30b}$$

where $t(g)_{\mu\nu}$ is the energy-stress tensor for gravity. Eq. (30b) implies that the equivalence principle is satisfied (Lo 1995). Since eq. (17) is an approximation of (30a), this equation is compatible with Einstein's notion of weak gravity and has bounded wave solutions.

Eq. (30) supports Einstein's discovery (Born 1968) and Hogarth's (1953) conjecture that there is no physical dynamic solution unless the gravitational energy tensor is added to eq. (13) although some of its static solutions are supported by experiments (Ohanian & Ruffini 1994). The anti-gravity coupling (Lo 1997c) is necessary due to the existence of gravitational waves (Lo 1997b, 1995). It should be noted that such a coupling implies the energy assumptions of the singualrity theorems (Wald 1984) invalid. Thus, space-time singularities may not necessarily exist even in theory.

*Since compatibility with weak gravity is established as a physical requirement, the question of boundedness is crucial for a dynamic solution of eq. (16)*, the Einstein equation of 1915. From eq. (30), it is clear that eq. (16) does not have the physical feature that allows a gravitational wave,

like an electromagnetic wave, to carry energy–momentum. Thus, without this physical feature, there is no possibility that eq. (16) would admit a physical gravitational wave which is necessarily bounded.

In other words, the unboundedness of metrices (25) and (27) is not accidential, but is due to the deficiency of eq. (16). From the linear field eq. (17) for weak gravity, boundedness is a necessary condition for the plane wave as well as Einstein's radiation formula (Weinberg 1972; Misner et al. 1973). It is based on the validity of Einstein's radiation formula (on the time average) that eq. (30) is derived (Lo 1995) such that a bounded dynamic solution would become possible. Thus, due to these physical considerations, if general relativity is essentially correct, ***there is no bounded gravitational plane wave which satisfies*** $G_{ab} = 0$. Nevertheless, based on the notion of weak gravity, an unverified implicit assumption for the Einstein equation of 1915 was that its exact dynamic solutions are bounded (Blanchet & Damour 1986; Damour 1987). Apparently, many theorists were not aware of that ***a physical requirement may not necessarily be satisfied by a nonlinear field equation*** (Lo 1996).

Moreover, Einstein's radiation formula, which implies the existence of radiation, directly leads to the need of modifying eq. (16). Although Einstein's notion of gravitational energy–stress (Pauli 1958) is not localizable, physically it is impossible to transform a radiation away by changing the coordinate system. Thus, Einstein's notion can only be an approximation for some coordinate systems. It follows that eq. (16), which implies the exact validity of Einstein's conservation law (Weinberg 1972), must be modified.

The theoretical framework of general relativity has been firmly established by the accurate agreements between its predictions and observations. However, such confirmations also lead some theorists to develop, without adeqaute analysis, a strong faith on its equation (Weisskopf 1988). Moreover, there are implicit assumptions which may or may not be valid (Lo 1996, 1997a, 1997c). Ideally, our confidence on general relativity should be based on both expereimental agreements as well as thorough theoretical analysis. (By the way, it is thorough theoretical analysis that leads to the creation of general relativity.) Unfortunately, except Einstein's peers such as Eddington (1975), Lorentz, Pauli (1958) and etc., few identified the implicit assumptions and analyzed the theoretical foundation of general relativity. Even to–day, to many relativists, the implications of the equivalence principle remains not well understood, and the covariance remains overstated. As illustrated by metric (13), once the spatial coordinates are chosen, the time–coordinate is fixed.

Consequently, contradictory statements from theorists in the field of general relativity are not uncommon. For instance, as shown in § 3, the Galilean transformation, is incorrectly considered as "valid" although such a transformation disagrees with experiments. The exchange of the time coordinate and a space coordinate is a diffeomorphism. According to this, Hawking (1988) claims in his popular book that "In relativity, there is no real distinction between the space and time coordinates just as there is no real difference between any two space coordinates." Nevertheless, from physics he also realizes and writes in the same book that "something that distinguishes the past from the future, giving a direction to time". Apparently, based on general mathematical covariance, many theorists believe that it is meaningless to consider the question of boundedness of a space-time metric. But, this does not prevent them from accepting Einstein's notion of weak gravity. Note also that, assuming the general mathematical covariance, leads to the absurd "conclusion" that, depending on the choice of coordinates, the total energy flow due to radiation could be negative, positive or zero (Denisov Logunov 1982; Vlasov & Denisov 1982).

Unfortunately, Einstein's (1954) book which provides the corrected view (see § 3), was seldom referred to in the literature. Most authors of text books took the so-called "orthodox view" of 1916 (Einstein 1916), believed the validity of an arbitrary Gaussian system as a space-time coordinate system, and thus accepted unphysical solutions (Kramer et al. 1980; Bonnor 1992). I believe that they would be greatly benefited by Einstein's book. It is hope that this paper would give an added impetus for studying the fundamental problems in connection with applications.

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## Appendix A: On Galilean Transformations and Space–Time Coordinate systems

To illustrate the meaning of a space time coordinate system, let us start with the flat metric,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{A1}$$

for a Minkowski space K. It is clear that the light condition $ds^2 = 0$ implies that the light speed is c for any direction. For a system K' related to K with a velocity v in the z–direction, the Galilean transformation is

$$x' = x, \quad y' = y, \quad z' = z + vt, \quad \text{and} \quad t' = t. \tag{A2}$$

It follows that $dz'/dt' = dz/dt + v$. Since light speed $dz/dt = \pm c$, one would have light speed in z'–direction

$$L_{z'} = v \pm c. \tag{A3a}$$

For a light velocity in the x'–direction, the light speed is no longer in the x–direction alone; and

$$L_{x'} = \pm(c^2 - v^2)^{1/2}. \tag{A3b}$$

But, experiment such as the Michelson–Morley experiment, supports the constancy of the light speed. Thus, the Lorentz transofrmation, but not the Galilean transformation is valid in physics. It is true that the Galilean transformation does not preserve the form of the Maxwell equation; but what leads to the rejection of such a transformation is the fact that such a transformation is not physically realizable.

However, it should be noted that, as a mathematical transformation, the Galilean transformation (A2) is a valid diffeomorphism, and the coordinate system K' is a valid mathematical coordinate system although system K' is not physically realizable. For a mathematical system to be able to serve as a space–time coordinate, Einstein proposed an additional physical condition that the Equivalence Principle must be satisfied.

To illustrate that the equivalence principle restricts the valid space–time coordinate systems for co–

variance, let us use the Galilean transformation (A2) to transform metric (A1) to the constant metric,

$$ds^2 = [(c - v)dt' + dz'][(c + v)dt' - dz'] - dx'^2 - dy'^2. \tag{8a}$$

If K' were a space-time coordinate system, for light rays in the z'-direction, $ds^2 = 0$ would imply

$$[(c - v)dt' + dz'][(c + v)dt' - dz'] = 0. \tag{A4}$$

Thus, the light speeds in the z-direction would, in agreement with (A3), be

$$L_{z'} = \frac{dz'}{dt'} = c + v, \quad \text{or} \quad \frac{dz'}{dt'} = -c + v. \tag{8b}$$

For light speeds in the x'-direction, according to metric (8a), $ds^2 = 0$ would imply

$$(c - v)dt'\ (c + v)dt' - dx'^2 = 0. \tag{A5}$$

Thus, the light speed in the x'-direction would, in agreement with (A3), be

$$L_{x'} = \frac{dx'}{dt'} = \pm\ (c^2 - v^2)^{1/2}. \tag{A6}$$

On the other hand, constant metric (8a) implies that there is no gravity. Thus, eqs. (8b) and (A6) which reproduce (A3), disagree with experiment. The above calculations show that the light-cone condition needs not imply the valid light speeds unless the metric is compatible with the equivalence principle.

However, ***mathematically*** a local Minkowki metric can be obtained by choosing first the path of a particle to be the time coordinate and then the other three space coordinates by orthogonality. This illustrates that having a proper metric signature, which ensures the existence of a local Minkowski space, may not be sufficient for satisfying the equivalent principle (see also Appendix B).

Nevertheless, some attribute the above problem as due to the fact that metric (8a) has a cross term $g_{tz}$ . This attribution is groundless. To illustrate this, let us consider the Kerr metric (Wald 1984) as follows:

$$ds^2 = g_{tt}\, dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + 2g_{\phi t}\, dtd\phi + g_{\phi\phi}\, d\phi^2 \tag{A7}$$

where

$$g_{tt} = \frac{\Delta - a^2\sin^2\theta}{\Sigma} \quad (0 < g_{tt} \leq 1), \qquad g_{rr} = -\frac{\Sigma}{\Delta}, \qquad g_{\theta\theta} = -\Sigma$$

$$g_{\phi t} = \frac{a\, \sin^2\theta\, [r^2 + a^2 - \Delta]}{\Sigma}, \quad \text{and} \quad g_{\phi\phi} = -\{\frac{[r^2 + a^2]^2 - \Delta a^2\sin^2\theta}{\Sigma}\}\sin^2\theta$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2Cr, \text{ and } C = KM/4\pi.$$

For the light speed in the $\phi$-direction, according to Einstein (1954), one has

$$L_\phi = \frac{r\, \sin\theta\, d\phi}{dt} = [2Cr\, a\, \sin\theta \pm \Delta^{1/2}\Sigma]\, r/[(r^2 + a^2)^2 - \Delta\, a^2\, \sin^2\theta] \tag{A8}$$

But, in spite of the cross term $g_{\phi t}$ , calculation shows that

$$\left|\frac{r\, \sin\theta\, d\phi}{dt}\right| < 1. \tag{A9}$$

because (A9) is equivalent to

$$\Sigma^2\, (a^2+r^2)a^2 + 2Cr\Sigma[(r^2+a^2)(r - a\, \sin\theta)^2 + a^4\, \sin^2\theta] + [2Cra\, \sin\theta(r - a\, \sin\theta)]^2 > 0.$$

Thus, a cross term is not the problem, but the failure to satisfy the equivalence principle is.

In short, the equivalence principle makes that space-time coordinates have the necessary distinct characteristics. Note that Hawking (1988) observed in his popular book that "something that distinguishing the past from the future, giving a direction to time."

**Appendix B: Another Example of Non-physical Spaces and its Violation of the Equivalence Principle.**

Consider the transformation, which is a diffeomorphism,

$$t = C\{\exp(T/C) - \exp(-T/C)\}/2. \tag{13a}$$

Then

$$ds^2 = \frac{1}{4}\{\exp(T/C) + \exp(-T/C)\}^2 dT^2 - dx^2 - dy^2 - dz^2 \tag{13b}$$

repesents the Minkowski metric after the transformation. If metric (13b) is realizable, according to $ds^2 = 0$, the measured light speed would be $\{\exp(T/C) + \exp(-T/C)\}/2$.

From (13b), the Christoffel symbols $\Gamma_{\nu,\alpha\beta}$ $(= [\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}]/2)$, are zeros except

$$\Gamma_{t,tt} = \partial_t g_{tt}/2 \tag{B1}$$

Then, according to the geodesic equation, the equation of motion for a particle is

$$\frac{d^2T}{ds^2} + \Gamma^t{}_{tt}\frac{dT}{ds}\frac{dT}{ds} = 0, \quad \text{and} \quad \frac{d^2x^\mu}{ds^2} = 0 \tag{B2}$$

where

$$\Gamma^t{}_{tt} = \frac{d}{dT}(\ln\{\exp(T/C) + \exp(-T/C)\})$$

It follows eq. (B2) one obtains, for some consstant k

$$\frac{dT}{ds} = k\{\exp(T/C) + \exp(-T/C)\}^{-1} \quad \text{and} \quad \frac{dx^\mu}{ds} = \text{constant} \tag{B3}$$

Now, consider the case $dx/dT = dy/dT = dz/dT = 0$; and therefore $dx/ds = dy/ds = dz/ds = 0$. Thus, in such a free falling, there is no change in the spatial position. Physically, this means any measurement made by such an observer would not be changed by this "free falling". On the other hand, the equivalence principle would require the measured light speed is 1. Thus, the equivalence principle is not satisfied.

## Appendix C: Boundedness of a Normalized Space–Time Metric

In this appendix, it will be shown that the elements of a space–time metric will be bounded in absolute value if the light speed has has a maxmum $L_M$ (= 1) and a minimum $L_m$. Consider a normalized metric $ds^2 = \hat{g}_{\mu\nu}\, d\hat{x}^{\mu}\, d\hat{x}^{\nu}$. For a time–like differential, $ds^2 > 0$; and a space–like differential, $ds^2 < 0$. It follows that

$$\hat{g}_{tt} > 0, \quad \hat{g}_{ss} < 0, \quad \text{and} \quad (\hat{g}_{ss'})^2 < \hat{g}_{ss}\, \hat{g}_{s's'} \tag{C1}$$

for the time–time component $\hat{g}_{tt}$ , space–space components $\hat{g}_{ss}$ , $\hat{g}_{s's'}$ , spce–space cross elements $\hat{g}_{ss'}$ . Thus, it remains to show that $\hat{g}_{tt}$ , $\hat{g}_{ss}$ and $\hat{g}_{st}$ are bounded. If the equivalence principle is satisfied, according to Einstein (1954) based on the light–condition $ds^2 = 0$, the light speed $L_s$ in the s–direction is

$$L_s = \{-\hat{g}_{ts} \pm [(\hat{g}_{st})^2 - \hat{g}_{ss}\, \hat{g}_{tt}]^{1/2}\}/\hat{g}_{ss}\,, \quad \text{and} \quad L_m \le |L_s| \le 1\,. \tag{C2}$$

It follows from eqs. (C1) and (C2) that

$$2\,|\hat{g}_{ts}| \le (1 - L_m)\,|\hat{g}_{ss}|\,, \tag{C3a}$$

and

$$L_m\,|\hat{g}_{ss}| + |\hat{g}_{ts}| \le [(\hat{g}_{st})^2 - \hat{g}_{ss}\, \hat{g}_{tt}]^{1/2} \le |\hat{g}_{ss}| - |\hat{g}_{ts}|\,. \tag{C3b}$$

Relation (C3) implies

$$|\hat{g}_{tt}| \le |\hat{g}_{ss}| \le |\hat{g}_{tt}|\,|L_m|^{-2}\,, \qquad |\hat{g}_{ss'}| < |\hat{g}_{tt}|\,|L_m|^{-2}, \tag{C4a}$$

and

$$|\hat{g}_{ts}| \le \frac{1}{2}(1 - L_m)\,|\hat{g}_{tt}|\,|L_m|^{-2}. \tag{C4b}$$

Note that relation (C4a) for non–crossing terms is the same as relation (12b). Thus, according to (C4), all the metric elements are bounded since $g_{tt}$ ($\le 1$) is bounded due to time dilation.

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