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RELATIVISTIC CAUSALITY, PRINCIPLE OF EQUIVALENCE AND THE QUESTION OF GAUGE IN GENERAL RELATIVITY

by

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Abstract

Relativistic causality and the principle of equivalence are physical requirements which are mathematically independent of Einstein's field equation. Therefore, they may not be consistent with a gauge which is based only on diffeomorphism. It is shown that a diffeomorphism may not necessarily be compatible with the principle of equivalence, and that the light cone condition is inadequate to ensure relativistic causality. Thus, the current notion of gauge is inadequate. For example, a Galilean transformation, though a diffeomorphism, not only leads to different light speeds for different directions in a vacuum, but also is incompatible with the principle of equivalence. The Michelson–Morley experiment manifests that a Galilean transformation leads to a coordinate system which is not physically realizable. Moreover, this analysis confirms that, in current theory, there are no valid physical gravity solutions for electromagnetic plane waves.

Relativistic Causality, Principle of Equivalence

and

the Question of Gauge in General Relativity

1. Introduction

In general relativity, gauge is related to a coordinate transformation instead of a gauge function in a fixed coordinate system as in electrodynamics [1]. Thus, a question arises as to whether the coordinates for a gauge are physically realizable. (Einstein [2] pointed out that "In physics, the body to which events are spatially referred is called the coordinate system." and that "Natural laws are to be formulated in such a way that their form is identical for coordinate systems of any kind of state of motion.") A current viewpoint is that all mathematical coordinate systems are equivalently valid in *physics* [3]. Since one must use a physical coordinate system to compare experiments with calculations, such a viewpoint may not be valid unless every mathematical coordinate system is physically realizable. It will be shown that both relativity and experiments imply that the realizability of such a coordinate system is only conditional (see §2 and § 3).

In current theory, the concept of gauge is based on the mathematical concept, *diffeomorphism*, which is a one-one onto C^∞ (infinitely differentiable) map between two manifolds and its inverse map is C^∞ [3]. Since two diffeomorphic manifolds have identical structure, it is claimed that such manifolds have physically identical properties. Then, the diffeomorphism comprises the gauge freedom of any theory formulated in terms of tensor fields on a spacetime manifold. Note that this notion of gauge hinges on an implicit assumption that every mathematical coordinate system is physically realizable. However, a pure mathematical condition may not always be compatible with physics. In fact, the choice of physical coordinates is restricted (see §2 & §3) by relativistic causality, i.e. no cause event can propagate faster than the velocity of light in a vacuum.

The satisfaction of relativistic causality is a primary motive of general relativity. In fact, relativistic causality is guaranteed by Einstein's principle of equivalence since in a free falling coordinate system, the space is locally flat. However, both the principle of equivalence and relativistic causality are physical requirements. Since Einstein's field equation may allow solutions, which are clearly *incompatible* with relativistic causality (see §2), its satisfaction for any solution must be proven.

Nevertheless, a current belief is that any solution of Einstein equation satisfies relativistic causality by virture of the principle of equivalence. Consequently, many solutions in the literature are ***not*** valid in physics [4]. Some of such source tensors may actually not even have any physical solution [5,6]. To show that the above belief is incorrect, for clarity, it is necessary to consider physical situations in which the physical picture is very clear. Since the assumptions used in general relativity should be applicable to any physical situations, the simplest situation is that there is no gravitational force.

In this paper, it will be shown, through examples, that the principle of equivalence may not be applicable to a solution, and that a diffeomorphism induced manifold may not be a physical space (see §2 and also §4). Thus, based on diffeomorphism alone, the current notion of gauge is inadequate. To illustrate an unphysical metric further, an example which does not satisfy relativistic causality, is analyzed in §3.

2. Diffeomorphism, the Principle of Equivalence, and Physical Space

Two manifolds with an identical mathematical structure do not necessarily have identical properties in physics since a diffeomorphism ignores the physical nature of the space (see also §4). For instance, in the lauguage of manifold, there is no distinction between the time coordinate and a space coordinate. Therefore, the exchange of the time coordinate and a space coordinate is a diffeomorphism which is clearly not valid in physics. Thus, two diffeomorphic manifolds may not necessarily also be equivalence in physics. In general, a diffeomorphism may lead to a mathematical coordinate system whose physical implication is incompatible with experiments. Therefore, such a space is only a pure mathematical object, but is ***not*** a physical space.

To distinguish a physical space from merely a manifold, for clarity, constant metrices and linear transformations shall be considered here. Note that a space with a constant metric is diffeomorphic to a Minkowski space. Also, it is illustrated that a light speed is ***not*** generally invariant.

It is well known that the Galilean transformation leads to an unphysically coordinate system which is not compatible with the Michelson–Morley experiment. Let us consider a coordinate system $(x, y, z, t,)$ where t is the time coordinate, and a Galilean transformation,

$$x' = x, \quad y' = y, \quad z' = z + vt, \quad \text{and} \quad t' = t, \tag{1}$$

where v is a constant. Transformation (1) is clearly a diffeomorphism for the flat space–time in which the light speed $dz/dt = \pm c$. Then, in the prime coordinate system, the forward light speed in the z–direction is

$$\frac{dz'}{dt'} = \frac{dz}{dt} + v = c + v; \quad \text{and} \quad \frac{dz'}{dt'} = -c + v \tag{2}$$

is the light speed in the backward z–direction. Eq. (2) clearly violates relativistic causality, the light speed in a vacuum is the maximum speed possible. Nevertheless, eq. (1) transforms the flat metric,

$$ds^2 = c^2dt^2 - dz^2 - dx^2 - dy^2, \tag{3}$$

to

$$ds^2 = [dz' + (c - v)dt'][-dz' + (c + v)dt'] - dx'^2 - dy'^2. \tag{4}$$

Note that metric (4) and the light cone condition ($ds^2 = 0$) for the light in z–direction would produce eq. (2). Thus, it has been shown that the light cone condition is inadequate to ensure relativistic causality.

On the other hand, general relativity cannot justify transformation (1) as physical. According to the relation, $z' = z + vt$, the prime coordinate system would be associated with an observer on an object moving with a uniform velocity in the z–direction. However, if this prime observer measures light speeds from this object, he finds that a light velocity in the z–direction is again $\pm c$, but not according to eq. (2). Thus, transformation (1) and therefore metric (4) is ***not*** physically realizable. And calculations must be interpreted in terms of the old coordinate system. In other words, the prime system is ***not*** a physical space. Note that, for a Lorentz transformation, the physics of calculations can also be interpreted with the new coordinate system.

If the prime observer and an observer in the original coordinate system communicate their observed results to each other, they will conclude that either transformation (1) is not valid or the other observer performs his experiments incorrectly. If they finally agree that both of them conducted their experiments correctly, to reconcile the experimental facts, they will get the theory of special relativity. Thus, special relativity is based on experiments that physicists have to conclude that ***Galilean transformations and therefore some mathematical coordinate systems are not physically realizable.***

Nevertheless, metric (4) satisfies Einstein's equation, $G_{ab} = 0$. This shows that ***relativistic causality is independent of Einstein's equation.*** Moreover, Einstein's equation does not even imply the validity of the light cone condition ($ds^2 = 0$) for a light ray. For example, the positive definite metric,

$$ds^2 = dt^2 + dz^2 + dx^2 + dy^2, \tag{5}$$

is obviously also a solution of $G_{ab} = 0$. For a constant metric, there is no physical free falling which relates to a local Minkowski space. Metrices (4) and (5) illustrate that ***the principle of equivalence is*** inapplicable to a non-physical space, and is also ***incompatible with Galilean transformations.***

To illustrate the inadequacy of diffeomorphism further, let us consider one more simple example. Now, consider a coordinate transformation,

$$\alpha x' = x, \quad y' = y, \quad z' = z, \quad \text{and} \quad t' = t, \tag{6}$$

where α (> 0) is a unitless constant. Diffeomorphism (6) is not a rescaling. (Note that the flat metric (3) requires that every coordinate uses the same unit. A rescaling will not change the flat metric η_{ab} since the light speed makes the compensational change.) Then, the flat metric (3) is transformed to

$$ds^2 = c^2dt'^2 - dz'^2 - dy'^2 - \alpha^2dx'^2. \tag{7}$$

The coordinate system (x', y', z', t') is not realizable since the principle of equivalence implies that the flat metric is the ***only*** constant metric. Otherwise, this would mean that, for the x-direction, the light speed in a vacuum had changed to $\pm c/\alpha$. (If $\alpha > 1$, relativistic causality would appear to have been satisfied. Thus, relativistic causality may be only a necessary condition for a physical space.) Therefore, metric (7) would imply that light speeds in a vacuum were no longer homogeneous, but depended on the direction.

For two diffeomorphic manifolds, any relation among different tensors is identical. But, a diffeomorphism does not ensure that the numerical value of a tensor component is valid. In other words, there is no as-

surance to the physical reality. From the above considerations, calculated results must be understood in terms of a physical coordinate system since a non-physical coordinate system can lead to completely wrong conclusions. In view of the fact that a current source tensor may not necessarily be appropriate as pointed out by Einstein [7], the failure of finding a physical coordinate system may mean that the source is inadequately considered [5] or even invalid [8]. Thus, whether a coordinate system is physical, is important.

3. Velocity of Light, Relativistic Causality, and Gauge

From the viewpoint of relativity, a necessary condition for a physical space is that relativistic causality must be satisfied. Then, for a constant metric, the flat metric (3) is the only solution. A light speed is determined by components of the space-time metric, and therefore the requirement that a speed of light is smaller than or equal to c (the light speed in a vacuum) may not be covariant for all diffeomorphic manifolds *(see §2). Nevertheless, the principle of covariance is compatible with relativistic causality since its violation means that the choice of coordinates is not valid in physics.*

Also, it is not difficult to see that relativistic causality is satisfied by the Schwarzschild solution [3],

$$ds^2 = (1 - \frac{C}{r})dt^2 - (1 - \frac{C}{r})^{-1}dr^2 - r^2 d\Omega^2, \tag{8}$$

where C is a positive constant, $d\Omega^2 = (d\theta^2 + \sin^2\theta\, d\phi^2)$, and (r,θ,ϕ) are spherical coordinates. The light speeds in the r-direction and θ-direction are respectively,

$$\frac{dr}{dt} = \pm\,(1 - \frac{C}{r}), \quad \text{and} \quad \frac{rd\theta}{dt} = \pm\,(1 - \frac{C}{r})^{1/2} \tag{9}$$

Thus, due to gravity, light speeds are slower. A light ray bends as if being attracted because light speeds are directionally dependent in a special way. Then, it follows the principle of equivalence that light speeds in eq. (9) support Einstein's viewpoint [9] that "The principle of inertia and the principle of the constancy of the velocity of light are valid only with respect to an inertial system." On the other hand, the local light speed is always 1 in a free falling coordinate system. This difference is due to that a light speed is not invariant.

One might argue on the ground that, based on the simultaneous distance in the r-direction and the local time, light speeds in the r-direction would remain to be $[(1 - C/r)^{-1}dr^2/(1 - C/r)dt^2]^{1/2} = \pm 1$. But, this is not valid in physics because a velocity is that a difference of distance at different time divided by the difference of time at different places. Moreover, from metric (4), for a light ray in the z'-direction, the simultaneous distance is determimed by $ds^2 = -dz'^2$, and the local time is determined by $ds^2 = (c^2 - v^2)dt^2$. If such a z'-directional light speed were $\pm$ c, then one obtains $ds^2 = 2vdt'dz' \neq 0$. In other words, the light cone condition would be violated. For a light ray in the x-direction, althought light cone condition is not violated, one would still obtain the unphysical relation that $dx^2/\{(1 - v^2/c^2)dt^2\} = c^2$.

All the above metrices satisfy the harmonic gauge. In the literature [4], there are non-trivial metric solutions which do not satisfy relativistic causality. For example, an accepted metric [10] is as follows,

$$ds^2 = du\, dv + h_{ij}(u)x_i x_j\, du^2 - dx_i\, dx_i\ , \tag{10}$$

where $u = t-z$, $v = t+z$ (the light speed in a vacuum is denoted as 1) , $h_{ii}(u) \geq 0$, and $h_{ij} = h_{ji}$. This metric also satisfies the harmonic gauge. The cause of metric (10) can be an electromagnetic plane wave. Metric (10) is related to a z-directional plane wave metric by a coordinate transformation. For a plane wave, the origin of the x-y plane is arbitrary. Metric (10) depends on $x(= x_1)$ and $y(= x_2)$; and is not bounded nor weak. Thus, the transformation is likely to be ***not*** physical since weak gravity is a physical condition which can be related to a plane wave. Metric (10), in terms of the deviation γ_{ab}, satisfies

$$\eta^{ab}\partial_a\partial_b\, \gamma_{ll} = -2\,[h_{xx}(u) + h_{yy}(u)]\,, \qquad \text{and} \qquad \gamma = \eta^{cd}\gamma_{cd} = 0. \tag{11}$$

Clearly, metric (10) is also rather arbitrary as the similar case derived by Peres and Bonnor [11]. It will be shown that metric (10) is ***not*** compatible with relativistic causality.

Metric (10) and similar metrics [11] have a general form as follows:

$$ds^2 = du\, dv + H du^2 - dx_i\, dx_i\ . \tag{12}$$

A light trajectory satisfies $ds^2 = 0$. For a light in the z-direction (i.e. $dx = dy = 0$), one obtains

$$ds^2 = du\,dv + H du^2 = 0, \quad \text{and} \quad \frac{dz}{dt} = -\frac{1 + H}{1 - H} \tag{13}$$

is the backward light speed while light speed is 1 in a vacuum. Then, ***relativistic causality*** requires,

$$H \leq 0; \tag{14}$$

and if dt is time-like, $-1 < H$. Obviously, condition (14) is not satisfied by metric (10). This analysis further confirms the conclusion that currently there is no physical solution for an electromagnetic plane wave, and causality will not be satisfied unless the source tensor is modified [5].

Moreover, the fact that metric (10) is not physical, can also be shown from the viewpoint of gravitational forces. For an electromagnetic plane wave, from eq. (11) it is obvious that the origin of the x-y plane can be chosen arbitrarily. Then, it follows that, for a massive point-like particle, the gravitational force is also arbitrary since it depends on the origin of the x-y plane. Note that $\Gamma'_{11} = (1/2)\partial H/\partial t$. Clearly, the arbitrariness is due to the existence of resulting parameters (the choice of origin) which are not related to the cause (an electromagnetic plane wave). Thus, the principle of causality is violated, and such a violation is the root of the violation of relativistic causality. This illustrates that, in general relativity, relativistic causality and the principle of causality are not unrelated.

4. Conclusion and Discussion

Currently, some relativists are unable to distinguish physics from mathematics, and many exact solutions are unphysical [4,6]. It is often said that a coordinate system is arbitrary because it may not be an inertial system. However, this arbitrariness is only among physically valid coordinates [2]. It is often said that coordinates have no meaning. This only means that without the metric g_{ik} there is no physical space [9]. Although there is no absolute time, a time coordinate is, in several aspects [12], distinct from a space coordinate. Thus, it is not valid in physics to consider that there is no real distinction between the space and

time coordinates, just as there is no real difference between any two space coordinates.

In special relativity, the space–time metric is an invariant. A Galilean transformation is not physical because it, although compatible with Einstein's equation, is incompatible with this invariance and the constancy of light speeds in a vacuum. In general relativity, the metric is no longer necessarily invariant under coordinate transformations because the non–inertial coordinates are included. However, as shown, relativistic causality and the principle of equivalence are physical requirements which are independent of Einstein's equation. The light cone condition is inadequate to ensure relativistic causality. And the Galilean transformation, which associates with absolute time, remains incompatible with experiments.

Mathematically, a Riemann metric (with a correct signature), at any point, can be considered as locally flat since it is diffeomorphic to a Minkowski space for a small region. In current theory, there is an ***implicit*** physical assumption that such a diffeomorphism is physically valid. On the other hand, the principle of equivalence ***requires*** that such a diffeomorphism is obtained through a free falling coordinate system. However, as shown, such a requirement may ***not*** be satisfied, and therefore the implicit assumption may not be valid. In other words, the principle of equivalence puts a restriction on physical coordinates. As pointed out by Weinberg [13], the mathematical viewpoint based on geometry may not be adequate.

It was believed that the harmonic coordinate conditions were generally applicable because Hilbert [14] shows that solutions are not unique and the required gauge can be obtained from a differential equation [1,3]. It was also believed that every mathematical coordinate system is valid in physics and that a gauge can be arbitrarily chosen. These beliefs have been proven to be incorrect because a gauge may not be compatible with relativistic causality and therefore the principle of equivalence. Thus, covariance is considered among a smaller set of coordinate systems. Moreover, a physical coordinate system may not exist for a specific problem [5]. This would mean that a search for a valid source remains to be completed [7].

It is interesting to note that Einstein [15] is probably the first who discovered (in 1936) a problem related to gauge. Moreover, in view of the fact that the current notion of gauge is inadequate, the harmonic gauge is probably a problem that Einstein's radiation formula is not derived in a self–consistent manner [3,16]. While relativistic causality is a necessary condition, it remains to clarify whether the satisfaction of the principle of equivalence is generally sufficient for a physical coordinate system.

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