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Topological Effects on the Physics of the Standard Model*

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1 Four Dimensional Gauge Theories and Instantons

1.1 Notation

Nonabelian gauge theories deal with matrix-valued vector potentials which can be decomposed with respect to a basis of the Lie algebra of a gauge group G:[1]

$$A_\mu(x) = A^a_\mu(x) T^a \ . \tag{1}$$

The group generators are in my notation [1] the (for SU(N) N^2-1) antihermitean, traceless matrices T^a obeying the normalization condition and algebra

$$T^{a\dagger} = -T^a \ , \ \ \mathrm{tr} T^a T^b = -\tfrac{1}{2}\delta^{ab} \tag{2}$$

$$\left[T^a, T^b\right] = f^{abc} T^c \tag{3}$$

with f^{abc} the totally antisymmetric, real structure constants.

The covariant derivative, field strength tensor and Lagrangean of the Yang–Mills field are given by

$$\nabla_\mu = \partial_\mu + A_\mu \text{ acting on a representation of } G, \tag{4}$$

$$D_\mu = \partial_\mu + [A_\mu, .] = T^a\left(\delta^{ac}\partial_\mu + f^{abc}A^b_\mu\right) \text{ acting on a rep. of the Lie algebra} \tag{5}$$

$$F_{\mu\nu} \equiv F^a_{\mu\nu} T^a := [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \tag{6}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{7}$$

$$\mathcal{L}_{YM} = -\tfrac{1}{4} F^{\mu\nu\,a} F^a_{\mu\nu} = \tfrac{1}{2}\mathrm{tr} F^{\mu\nu} F_{\mu\nu} \ . \tag{8}$$

Note that we have scaled the potentials so that the coupling constant is absorbed into A_μ in order to simplify notation. To make contact with the conventions used in perturbation theory (eg. [2]), one should substitute $-i\frac{\lambda^a}{2}$ for T^a, $-igA^a_\mu\frac{\lambda^a}{2}$ for A_μ, where λ^a are the (hermitean) Gell–Mann matrices, and in addition replace $F^{\mu\nu\,a}$ by $-igF^{\mu\nu\,a}$. The Lagrangean density (8) remains unchanged.

Under a gauge transformation $g(x) \in G$ at a point x in spacetime, the fields transform as

$$A_\mu \ \rightarrow \ {}^g A_\mu := g^{-1}\left(A_\mu + \partial_\mu\right) g \tag{9}$$

$$F_{\mu\nu} \ \rightarrow \ {}^g F_{\mu\nu} := g^{-1} F_{\mu\nu} g \ , \tag{10}$$

[1]Summation over repeated indices is understood, as is the use of the natural system of units $\hbar = c = 1$.

which shows that the Lagrangean (8) remains unchanged.

The equations of motion (transforming covariantly under gauge transformations)

$$D_\mu F^{\mu\nu} = 0 = \partial_\mu F^{\mu\nu\,a} + f^{abc} A^b_\mu F^{\mu\nu\,c} \tag{11}$$

show that, due to the self-coupling in the second term, the theory is not free even in the absence of matter. Indeed, in most what follows we will not bother with matter fields.

From the definition (6) of the field strength tensor one finally obtains the Bianchi identity:

$$\epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0 \ . \tag{12}$$

1.2 Canonical Quantization

As in Maxwell theory, a straightforward quantization of nonabelian gauge theories is impossible due to the absence of a momentum conjugate to A^a_0:

$$\frac{\partial \mathcal{L}_{YM}}{\partial \dot{A}^a_0} = 0 \tag{13}$$

There is a variety of ways to handle this problem: In QED, one introduces a "transversal" Dirac function in order to obtain canonical commutation relations which are consistent with Gauß's law $\vec{\partial}\cdot\vec{E} = 0$ [3], but this procedure obscures the physics in Yang–Mills theory since from Gauß's law the transversality of the gauge bosons does not follow (see Section 1.3). Technically even more involved is a constraint quantization following Dirac [4].

If we do not want to use the path integral formalism, the simplest way of quantization is to perform a classical gauge transformation yielding the Weyl gauge $A_0 = 0$ before quantizing [1]. One finds for the momentum conjugate to $\vec{A}(\vec{x})$ the chromoelectric field

$$\Pi^a_i = \frac{\partial \mathcal{L}_{YM}}{\partial \dot{A}^{i\,a}} = -F^a_{0i} = -\dot{A}^a_i = -E^a_i \ , \tag{14}$$

and therefore postulates the canonical equal time commutation relations

$$\left[A^a_i(\vec{x}), \Pi^b_j(\vec{y})\right] = i\delta_{ij}\delta^{ab}\delta^{(3)}(\vec{x}-\vec{y}) = \left[E^b_j(\vec{y}), A^a_i(\vec{x})\right] \ , \tag{15}$$

$$\left[A^a_i(\vec{x}), A^b_j(\vec{y})\right] = 0 = \left[\Pi^a_i(\vec{x}), \Pi^b_j(\vec{y})\right] \ . \tag{16}$$

The Hamiltonian equations of motion obtained from the Hamilton operator

$$H = \frac{1}{2}\int d^3x \left[E^a(\vec{x})E^a(\vec{x}) + \frac{1}{2}F^a_{ij}(\vec{x})F^a_{ij}(\vec{x})\right] \tag{17}$$

reproduce the generalized Ampère's law as the spatial components of (11)

$$i\,[H, A_i^a(\vec{x})] = \dot{A}_i^a(\vec{x}) = E_i^a(\vec{x}) \ , \ i\,[H, E_i^a(\vec{x})] = \dot{E}_i^a(\vec{x}) = (D_j F^{ji})^a\,(\vec{x}) \qquad (18)$$
$$\Rightarrow D_\mu F^{\mu i}(\vec{x}) = 0 \ ,$$

but the time component of (11), the generalized Gauß's law $G(\vec{x}) := \vec{D} \cdot \vec{E}(\vec{x}) = 0$, is absent, as it is an equation at fixed time.

Note that the resulting theory (without Gauß's law) has its own right, but it is not clear whether it is renormalizable, and Lorentz invariance is surely lost. Rather than imposing it, one regains Gauß's law by the following considerations:

Going to the Weyl gauge before quantization does not fix the gauge completely: One can still perform residual, time independent gauge transformations, in particular infinitesimal ones

$$\delta\vec{A}(\vec{x}) = \vec{D}\beta(\vec{x}) + \mathcal{O}(\beta^2) \qquad (19)$$

which are symmetries of H. Since

$$i\left[\int d^3y\ \beta^a(\vec{y})G^a(\vec{y}), A_i(\vec{x})\right] = \delta A_i(\vec{x}) \ , \qquad (20)$$

$$i\left[H, \int d^3x\ \beta^a(\vec{x})G^a(\vec{x})\right] = 0 \ , \qquad (21)$$

Gauß's law is the generator of the infinitesimal gauge transformations and commutes with the Hamilton operator. It also obeys the commutation relations of group generators:

$$i\left[G^a(\vec{x}), G^b(\vec{y})\right] = f^{abc}G^c(\vec{x})\delta^{(3)}(\vec{x} - \vec{y}) \qquad (22)$$

which means that there exist in general only as many independent constants of motion associated with the G^a's as there are linearly independent matrices T^a which can be diagonalized simultaneously, namely $N - 1$ in SU(N).

One can think of the G^a's as generators of a symmetry of H we just discovered, without any reference to the Lagrangean (8) we started with. Imposing as a constraint on physical states

$$G^a(\vec{x})\ |\ \text{phys} >= 0 \ , \qquad (23)$$

one regains Gauß's law and therefore the complete quantum theory of the Lagrangean (8). Note that since $[H, G^a(\vec{x})] = 0$, the sector of physical states is invariant under time development.

All topological effects of the quantum theory can be uncovered by looking at the Gauß's law operator in a theory which is carefully quantized in this way, as can be seen from experience [1].

An analogy of the above situation is known from rotation invariant Hamilton operators in quantum mechanics. In the s-wave sector, the angular momentum operators $\vec{J}$ as generators of this symmetry have to annihilate the states one allowes for:

$$\vec{J}\ |\ \text{s-wave} >= 0 \ . \qquad (24)$$

Setting $\vec{J} = 0$ is inconsistent since its components do not commute with each other. In contradistinction to this example, the Gauß's law operators have a continuous spectrum and hence in looking at their zero eigenvalues one obtains non-normalizable states.

A note on the procedure: We first quantized the theory and t en imposed the constraint on physical states. In general, reversing this order will yield a different result to order $\hbar$, none of the two ways being a priori right or wrong.

Furthermore it is not trivial that choosing the Weyl gauge and quantizing commute with each other. Again, one example for that is the rotation invariant Hamilton operator in quantum mechanics [4]: Quantizing first yields a centrifugal barrier proportional to $j(j+1)/r^2$, while first going to polar coordinates one misses the barrier. It is only reintroduced if one observes that the momentum conjugate to r, $-i\partial/\partial r$ is not hermitean, and the true canonical momentum is $-i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$. If both procedures do not commute, the transformation eliminating A_0 would induce a curvature, and the momentum $-E_i^a$ would not be self-adjoint as is the case for the central force potential. Instead, one would have to hermitize it, $\Pi_i^a = -E_i^a + f_i^a(\vec{A})$, so that the components of the chromoelectric field do not commute with each other, thus revealing the curvature in the "Christoffel symbols" $f_i^a(\vec{A})$.

In both cases, one prefers to take the procedure for which one regains the classical theory for $\hbar \to 0$. The problem is that one doesn't know whether – due to confinement – a classical limit to the quantum Yang–Mills theory exists at all. However, in QED the classical limit exists and – what is more – one can show that all quantization methods yield the same result. One therefore can expect this to hold in Yang–Mills theory, too. At least the induction of a curvature by the Weyl gauge can be ruled out, since ghosts decouple in the path integral version when choosing an axial gauge.

1.3 The Schrödinger Representation

In the Schrödinger representation,

$$-E_i^a(\vec{x}) = \Pi_i^a(\vec{x}) = -i\frac{\delta}{\delta A_i^a(\vec{x})} \ , \tag{25}$$

one obtains as fixed time Schrödinger equation for energy eigenstates

$$\int d^3x \left[-\frac{1}{2}\frac{\delta^2}{\delta A_i^a(\vec{x})\delta A_i^a(\vec{x})} + \frac{1}{4}F_{ij}^a(\vec{x})F_{ij}^a(\vec{x})\right]\Psi_E\left[\vec{A}\right] = E\Psi_E\left[\vec{A}\right] \ , \tag{26}$$

and Gauß's law constraint (23) on physical states reads

$$\left[\partial_i\frac{\delta}{\delta A_i^a(\vec{x})} + f^{abc}A_i^b(\vec{x})\frac{\delta}{\delta A_i^c(\vec{x})}\right]\Psi_{\text{phys}}\left[\vec{A}\right] = 0 \ . \tag{27}$$

In the abelian theory ($f^{abc} = 0$) one considers $\Psi\left[\vec{A}\right]$ to be a functional of the Fourier transform of $\vec{A}(\vec{x}) = \vec{A}_T(\vec{x}) + \vec{\partial}A_L(\vec{x})$ decomposed into its transverse ($\vec{\partial}\cdot\vec{A}_T(\vec{x}) = 0$) and longitudinal part[2]. Gauß's law reads after applying the chain rule

$$k_ik_iA_L(\vec{k}) = 0 \ , \tag{28}$$

and hence physical states can be an arbitrary functional of the transverse components of $\vec{A}$ only, independent of its longitudinal degrees of freedom. This can also be seen from the fact that an abelian gauge transformation $A_i(\vec{x}) \to A_i(\vec{x}) + \partial_i\beta(\vec{x})$ leaves the transverse components untouched and changes only the longitudinal ones. Therefore the choice of the Coulomb gauge for free QED is unavoidable in the Hamiltonian formulation. In Yang-Mills theories, the Coulomb gauge is no natural choice since from Gauß's law (27) one cannot conclude that the wave functional depends on $\vec{A}_T$ only.

Free QED can even be solved this way [1]: Looking at the Schrödinger equation

$$\frac{1}{2}\int d^3x \left[-\frac{\delta^2}{\delta A_i(\vec{x})\delta A_i(\vec{x})} + A_i(\vec{x})h_{ij}A_i(\vec{x})\right]\Psi_E\left[\vec{A}\right] = E\Psi_E\left[\vec{A}\right] \ , \tag{29}$$

$$h_{ij} := -\vec{\partial}^2\delta_{ij} + \partial_i\partial_j \ , \tag{30}$$

[2]We neglect the zero mode of $\vec{A}$.

one constructs the gauge invariant ground state in analogy to the harmonic oscillator as

$$\Psi_0\left[\vec{A}\right] \propto \exp -\frac{1}{2}\int d^3xd^3y\ A_i(\vec{x})\omega_{ij}(\vec{x},\vec{y})A_j(\vec{y}) \tag{31}$$

$$\propto \exp -\frac{1}{4}\int d^3xd^3y\ F^{ij}(\vec{x})\frac{1}{\sqrt{-\vec{\partial}^2}}F^{ij}(\vec{y}) \ , \tag{32}$$

$$\omega_{ij}(\vec{x},\vec{y}) := h_{ij}\int d^3k\ e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{1}{|\vec{k}|} = -\frac{2}{\pi^2|\vec{x}-\vec{y}|^4}\left(\delta_{ij} - 2\frac{x_i}{|\vec{x}|}\frac{y_j}{|\vec{y}|}\right) \tag{33}$$

with the infinite vacuum energy $E_0 = \frac{1}{2}\text{tr}\omega$.

Since $\Psi_0\left[\vec{A}\right]$ depends on transverse fields only, Gauß's law is automatically satisfied, and the vacuum state of the free theory is unique. One can now construct excited states like the one photon state

$$\Psi_1\left[\vec{A}\right] := A_i^T(\vec{p})\Psi_0\left[\vec{A}\right] \ , \ A_i^T(\vec{p}) = \left(\delta_{ij} - \frac{p_ip_j}{\vec{p}^2}\right)\int d^3x\ e^{i\vec{p}\cdot\vec{x}}A_j(\vec{x}) \ . \tag{34}$$

The Schrödinger representation offers an alternative way to derive Gauß's law. As remarked above, states should be invariant against infinitesimal spatial gauge transformations: $\Psi\left[\vec{A} + \vec{D}\beta\right] \stackrel{!}{=} \Psi\left[\vec{A}\right]$, so that expanding around $\Psi\left[\vec{A}\right]$ yields

$$\int d^3x\,(D_i\beta)^a\frac{\delta}{\delta A_i^a}\Psi\left[\vec{A}\right] = 0 \ , \tag{35}$$

and one recovers (27) after partial integration.

1.4 Large Gauge Transformations and the ϑ Angle

Gauß's law (23) as generator of infinitesimal gauge transformations annihilates physical states, and therefore physical states are invariant under infinitesimal gauge transformations and all gauge transformations that can be built up by iterating infinitesimal ones, called *small gauge transformations.* The question arises whether all gauge transformations are small or whether there exist *large* gauge transformations, i.e. if there are solutions to (26,27) which obey Gauß's law but are not gauge invariant:

$$\Psi\left[^g\vec{A}\right] \neq \Psi\left[\vec{A}\right] \ . \tag{36}$$

Let's turn to the question of boundary conditions for the fields. Assuming the absence of monopoles, all position dependent observables should vanish faster than $\frac{1}{|\vec{x}|^2}$

for $|\vec{x}| \to \infty$. This means that going to spatial infinity one finds a unique physical vacuum. Strictly speaking, the vector potentials have only to approach a pure gauge configuration at spatial infinity, but one can show that there exists always a regular gauge transformation after which

$$\lim_{|\vec{x}|\to\infty} |\vec{x}|\vec{A}(\vec{x}) = 0 \ , \tag{37}$$

simultaneously reducing the set of possible gauge transformations to those that do not violate this condition:

$$\lim_{|\vec{x}|\to\infty} g(\vec{x}) = \text{const.} \tag{38}$$

These boundary conditions have been used to derive (20,21).

The last requirement identifies all points at spatial infinity so that g is uniquely defined there, and one compactifies the Euclidean space R^3 to the sphere S^3 when considering g.

One may investigate whether the maps $g(\vec{x}) : S^3 \to G$ can be decomposed into different classes. All maps in a given class can be deformed into each other and differ only by small gauge transformations. The classes are separated by topologically nontrivial, large gauge transformations. The set of all classes clearly forms a group, called the third homotopy group of G, $\Pi_3(G)$ [5]. If $\Pi_3(G) = 1$, as is the case in QED, only small gauge transformations exist, and all of these can be continuously deformed to the map $S^3 \to 1$. For any semisimple Lie group G, particularly for SU(N), it has been shown that $\Pi_3(G) = Z$, the additive group of integers, and hence large gauge transformations do exist. One can indeed show the existence of large gauge transformations without bothering with such topological considerations [6, 7], as we will explain now.

There exists a functional of $\vec{A}$ which satisfies Gauß's law but is not gauge invariant, known as the integral over the Chern–Simons three form:

$$\begin{aligned} W\left[\vec{A}\right] &= -\frac{1}{16\pi^2}\int d^3x\, \varepsilon^{ijk}\mathrm{tr}\left[A_i\left(F_{jk} - \frac{2}{3}A_jA_k\right)\right] = \\ &= -\frac{1}{8\pi^2}\int d^3x\, \varepsilon^{ijk}\mathrm{tr}\left[A_i\left(\partial_jA_k + \frac{2}{3}A_jA_k\right)\right] \ . \end{aligned} \tag{39}$$

Since

$$\frac{\delta W\left[\vec{A}\right]}{\delta A_i^a(\vec{x})} = \frac{1}{16\pi^2}\varepsilon^{ijk}F_{jk}^a(\vec{x}) + \frac{1}{16\pi^2}\int d^3y\, \varepsilon^{ijk}\partial_j\left[\delta^{(3)}(\vec{x}-\vec{y})A_k^a(\vec{y})\right] \tag{40}$$

and the surface term vanishes due to (38), $W\left[\vec{A}\right]$ fulfills Gauß's law (27) because of the Bianchi identity (12):

$$D_i\frac{\delta W\left[\vec{A}\right]}{\delta A_i^a(\vec{x})} = 0 \ . \tag{41}$$

On the other hand,

$$\begin{aligned} W\left[{}^g\vec{A}\right] - W\left[\vec{A}\right] &= n(g) - \frac{1}{8\pi^2}\int d^3x\, \varepsilon^{ijk}\partial_i\mathrm{tr}\left[(\partial_jg)g^{-1}A_k\right] \ , \\ n(g) &:= \frac{1}{24\pi^2}\int d^3x\, \varepsilon^{ijk}\mathrm{tr}\left[(g^{-1}\partial_ig)(g^{-1}\partial_jg)(g^{-1}\partial_kg)\right] \ , \end{aligned} \tag{42}$$

where with the boundary conditions (37,38) the surface term vanishes again. $n(g)$ is in general a nonzero integer and corresponds to the winding number of the map $g : S^3 \to G$, as can be seen most easily for G =SU(2)$\cong S^3$. As one can imagine, there are infinitely many ways to map spheres on spheres which are not continuously deformable into each other and can be labeled by the number of times one sphere is wrapped around the other. This winding number is additive :

$$n(g_1g_2) = n(g_1) + n(g_2) + \text{ a vanishing surface term} \ . \tag{43}$$

As an example, one representative of each class can be obtained by considering the following gauge transformations obeying the boundary conditions (37,38), where σ^i are the Pauli matrices which for SU(N) only have to be embedded into the higher groups:

$$g(\vec{x}) = \exp i\vec{\sigma}\cdot\frac{\vec{x}}{|\vec{x}|}f(|\vec{x}|) \ : \ f(0) = 0 \ , \ \lim_{|\vec{x}|\to\infty} f(|\vec{x}|) = n\pi \ . \tag{44}$$

Assuming physical states to be eigenstates of all unitary operators $\Omega_n[\beta]$ implementing gauge transformations $g_n(\vec{x}) = e^{i\beta(\vec{x})}$ of winding number n, we see that

$$\Omega_n[\beta]\Psi\left[\vec{A}\right] = \Psi\left[{}^{g_n}\vec{A}\right] = e^{-i\vartheta n(g_n)}\Psi\left[\vec{A}\right] \ , \tag{45}$$

because $\Omega_0[\beta]$ describes a gauge transformation generated by (a succession of) infinitesimal ones, and hence $\Psi\left[\vec{A}\right]$ is invariant under it by virtue of Gauß's law (23). ϑ is the Yang–Mills vacuum angle [6, 7], a new, hidden parameter in the quantum theory, which has been derived without any approximations here. Its effects will be examined in greater detail later.

It is tantalizing to observe that $\exp(\pm 8\pi^2 W\left[\vec{A}\right])$ solves the non-abelian functional Schrödinger equation (26) with zero eigenvalue (even in QED). Unfortunately, this solution is divergent for large $\vec{A}$ and hence not normalizable[3]. On top of that, it lacks any physical meaning; yet one can use it to show that the gauge invariant state

$$\Phi\left[\vec{A}\right] := e^{i\vartheta W[\vec{A}]}\Psi\left[\vec{A}\right] \quad : \quad \Omega_n[\beta]\Phi\left[\vec{A}\right] = \Phi\left[\vec{A}\right] \tag{46}$$

is an eigenstate to the same energy eigenvalue as the original state and obeys a Schrödinger equation which reads:

$$\int d^3x \left[\left(-i\frac{\delta}{\delta A_i^a(\vec{x})} + \frac{\vartheta}{16\pi^2}\varepsilon^{ijk}F_{jk}^a(\vec{x})\right)^2 + \frac{1}{2}F_{ij}^a(\vec{x})F_{ij}^a(\vec{x})\right]\Phi\left[\vec{A}\right] = E\Phi\left[\vec{A}\right] \ . \tag{47}$$

By that, one moved the ϑ angle from the state to a Hamilton operator which can be obtained from the Lagrangean

$$\begin{aligned}\int d^3x\, \mathcal{L}_\vartheta &= \int d^3x\, \mathcal{L}_{YM} - \frac{\vartheta}{16\pi^2}\int d^3x\, \varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left[F^{\mu\nu}F^{\rho\sigma}\right] = \\ &= \int d^3x\, \mathcal{L}_{YM} + \vartheta\frac{d}{dt}W\left[\vec{A}\right] \ ,\end{aligned} \tag{48}$$

where in order to derive the last line one used that the Chern-Simons term is related to the Chern-Pontryagin density [5] via

$$\frac{1}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left[F^{\mu\nu}F^{\rho\sigma}\right] = \frac{1}{8\pi^2}\partial_\mu\varepsilon^{\mu\nu\rho\sigma}\mathrm{tr}\left[A_\nu\left(\partial_\rho A_\sigma + \frac{2}{3}A_\rho A_\sigma\right)\right] \tag{49}$$

and that the surface terms at spatial infinity do not contribute due to (37,38). Therefore one can make three observations:

(i) The ϑ angle can be removed from the gauge variant states $\Psi\left[\vec{A}\right]$ making them gauge invariant (46), but only on the expense of breaking the invariance of the Lagrangean under large gauge transformations, changing $\mathcal{L}_{YM}$ to $\mathcal{L}_\vartheta$ by adding a Lorentz invariant, but P and T violating term.

The additional term in (48) is independent of the choice $A_0 = 0$, and therefore the occurence of the angle ϑ does not depend on choosing the Weyl gauge before quantization. It is a new, unremovable hidden parameter in the theory, and no principle is known which requires it to be zero. The unique classical Yang-Mills theory gives rise to a ϑ-family of quantum theories.

[3]Compare to all $E \neq (n+\frac{1}{2})\omega$ - solutions of the quantum mechanical harmonic oscillator: They also diverge for large x.

(ii) There is no remnance of the Yang-Mills angle in the equations of motion, nor in the Hamilton operator obtained from $\mathcal{L}_\vartheta$ via the procedure described above, as long as one writes it in terms of the vector potential and chromoelectric field. Yet since under a large, time dependent gauge transformation $\int d^3x\ \mathcal{L}_\vartheta$ changes by a total time derivative $\vartheta\frac{d}{dt}n(g)$, gauge invariant quantum states acquire a phase in the temporal developement between two states that are connected by g_n, as is familiar from quantum mechanics.

(iii) The previous point is connected with the fact that the momentum conjugate to $A_i^a(\vec{x})$ in $\mathcal{L}_\vartheta$ is no longer $-E_i^a(\vec{x})$ (14), but (cf. (47))

$$\Pi_i^a(\vec{x}) = -E_i^a(\vec{x}) - \frac{\vartheta}{16\pi^2}\varepsilon^{ijk}F_{jk}^a(\vec{x}) \ . \tag{50}$$

Therefore the components of the electric field do not commute with each other, and a connection is introduced in the physical Hilbert space thus revealing its nonzero curvature.

1.5 QED in Two Dimensional Spacetime

There is an intriguing example of the occurence of a new hidden parameter [8, 9, 1, 11] in two dimensions. The Hamilton operator and Gauß's law of QED are in the Schrödinger representation given by (cf. (26,27)):

$$H = \frac{1}{2}\int dx\ E^2(x) = -\frac{1}{2}\int dx\ \frac{\delta^2}{\delta A(x)\delta A(x)} \ , \tag{51}$$

$$\frac{d}{dx}\frac{\delta}{\delta A(x)}\Psi[A] = 0 \ . \tag{52}$$

Therefore, $\Psi[A]$ is a function of the zero mode of A only:

$$\Psi[A] = f\left(\int_{-\infty}^{\infty} dx\ A(x)\right) \ . \tag{53}$$

The wave functional solving both the Schrödiner equation and Gauß's law is

$$\Psi[A] = \exp -iE_0\int dx\ A(x) \ , \tag{54}$$

where applying $E(x) = i\frac{\delta}{\delta A(x)}$ (25) shows that E_0, due to Gauß's law the only observable, is the zero mode of the electric field. The energy density is finite and given by $\frac{1}{2}E_0^2$.

In analogy to the discussion above, compactifying the space R^1 to S^1 by requiring all field fluctuations to vanish at spatial infinity[4] amounts to the following boundary condition on the gauge transformations allowed:

$$e^{-i\Lambda(\infty)} = e^{-i\Lambda(-\infty)} \ . \tag{55}$$

Again, $e^{-i\Lambda(x)}$ has a well defined value at spatial infinity.

We again ask whether there exist large gauge transformations, i.e. transformations which are not generated by Gauß's law. The mappings $g(x) : S^1 \to U(1) \cong S^1$ decompose obviously into different classes, labeled by the number of times one circle winds around the other. Hence, under a gauge transformation in QED

$$A(x) \to A(x) - \frac{d}{dx}\Lambda(x) \ , \tag{56}$$

the zero mode

$$\int dx\ A(x) \to \int dx\ A(x) - \Delta\Lambda \ , \ \Delta\Lambda := \Lambda(\infty) - \Lambda(-\infty) = 2\pi n \ , \ n \in Z \tag{57}$$

changes by 2π times the winding number n (55). If $n \neq 0$, the unitary operator implementing the gauge transformation is not

$$\exp i \int dx\ (\frac{d}{dx}E(x))\Lambda(x) \ , \text{but} \ \ \Omega[\Lambda] = \exp -i \int dx E(x)\frac{d}{dx}\Lambda(x) \ , \tag{58}$$

because the surface term in which the two expressions differ cannot be dropped.

The effect of such gauge transformations on $\Psi[A]$ can easily be calculated:

$$\Omega_n[\Lambda]\Psi[A] = e^{-in\vartheta}\Psi[A] \ , \ \vartheta := 2\pi E_0 \ . \tag{59}$$

So the ϑ angle emerges as a constant electric background field which cannot be changed within the theory since $[H, E_0] = 0$, and whose different values therefore separate different worlds.

The operator which is invariant under small gauge transformations, but changes under large ones is the zero mode of the vector potential (57), cf. (39):

$$W[A] = \frac{1}{2\pi}\int dx\ A(x) \ : \ \Omega_n[\Lambda]W[A]\Omega_n^\dagger[\Lambda] = W[^\Lambda A] = W[A] + n \ . \tag{60}$$

[4]Note that one may not demand physical observables to vanish at infinity since then $E_0 = 0$ and the wave functional (54) is 1.

In order to construct the Schrödinger equation for gauge invariant states, cf. (46),

$$\Phi[A] := e^{\frac{i\vartheta}{2\pi}\int dx\ A(x)}\Psi[A] \ , \tag{61}$$

one has to move the ϑ angle to the Hamiltonian and Lagrangean (cf. (48)):

$$\mathcal{L}_\vartheta = \frac{1}{2}E^2(x) - \frac{\vartheta}{2\pi}E(x) \ . \tag{62}$$

The momentum conjugate to $A(x)$ is given by

$$\Pi(x) := \frac{\partial\mathcal{L}_\vartheta}{\partial\dot{A}(x)} = \dot{A}(x) + \frac{\vartheta}{2\pi} \ . \tag{63}$$

Since $\mathcal{L}$ changes by a total time derivative under these operations, there is again no remnance of ϑ in the equations of motion, yet physical states acquire a phase under time developement.

If one incorporates fermions into the theory,

$$\mathcal{L} = \mathcal{L}_\vartheta + \bar{\psi}\,(i\gamma^\mu\nabla_\mu - m)\,\psi \ , \tag{64}$$

one notes that in the Schwinger model ($m = 0$) $\mathcal{L}$ changes under a chiral redefinition of the fermionic fields due to the axial anomaly (see Section 1.8) ($\gamma^5 = -\gamma^{5\dagger}$) [10, 11, 1]:

$$\psi \to e^{\alpha\gamma^5}\psi \ : \ \mathcal{L} \to \mathcal{L} + \frac{\alpha}{\pi}E(x) \ . \tag{65}$$

Since it can be eliminated by re-defining the fermionic fields $2\alpha = \vartheta$, the Yang–Mills vacuum angle is physically irrelevant in that case.

Yet as soon as $m \neq 0$, this chiral redefinition is impossible and the ϑ angle is physical [8, 9], giving the value of the background electric field, on which e.g. the number of stable particles and the spacing between successive isosingulet states crucially depend.

If one would embed two dimensional QED into a larger theory, the background field might be determined by the new theory, dynamically fixing ϑ; but no such mechanism has been found so far.

1.6 A Physical Picture of ϑ Vacua and Instantons

Before deriving the axial anomaly in four dimensions and showing that the value of ϑ is unobservable in QCD in the presence of massless fermions by the same mechanism as

in twodimensional QED, we compare the situation in QCD with a well-known quantum mechanical example.

A physical picture of the vacuum ϑ angle [12, 13, 6, 7] emerges when one looks at a particle in a periodic potential (Figure 1):

$$L = \tfrac{1}{2}\dot{x}^2 - V(x) \ , \ \ H = \tfrac{1}{2}p^2 + V(x) \tag{66}$$

$$V(x+a) = V(x) \ , \ \ p = \dot{x} \ . \tag{67}$$

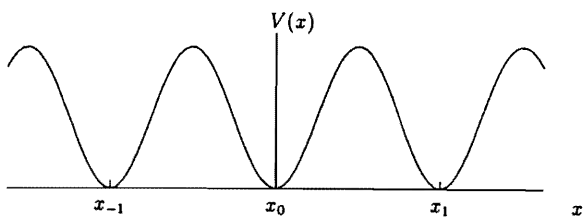


Figure 1: Particle in a Periodic Potential

The discrete displacement as implemented by the translation operator

$$\Omega_n \ : \ \Omega_n x \Omega_n^\dagger = x + na \ , \ n \in Z \tag{68}$$

is a symmetry of the system. Ω_n should be compared to the operator $\Omega_n[\beta]$ implementing large gauge transformations in the physical Hilbert space of QCD. The infinite degeneracy of the classically stable "ground state" solutions at $x_n \ : \ V(x_n) = 0$ corresponds to an infinite number of classical gauge field configurations $\vec{A}(\vec{x}) = g_n^{-1}\vec{\partial} g_n$ which are "pure gauge" and therefore have zero kinetic and potential energy but are topologically distinct from the trivial vacuum $\vec{A} = 0$ because of their nonzero winding numbers n.

In the interpretation of Floquet's (Bloch's) Theorem via the tight binding approximation of solid state physics, this degeneracy is removed in Quantum Mechanics by a nonzero tunneling probability from one x_n-"vacuum" to another. If the wave function $\Psi_n(x)$ is an approximate solution of least energy to one well of the potential, localized around the n-th minimum x_n, the superposition

$$\Psi_\vartheta(x) = \sum_n e^{-in\vartheta}\Psi_n(x) \tag{69}$$

is an eigenfunction to Ω_n (cf. 45)

$$\Omega_n \Psi_\vartheta(x) = e^{-in\vartheta}\Psi_\vartheta(x) \ , \tag{70}$$

and the ground state energy now depends on the Bloch momentum ϑ.

How can one describe the tunneling process just sketched in classical mechanics? Of course, there exists no classical zero energy solution which interpolates between different classical minima. Yet going to imaginary time $t \to -i\tau$, one interchanges the rôle of Hamiltonian and Lagrangean

$$L \to L_I = \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + V(x) \tag{71}$$

$$H \to H_I = \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 - V(x) \tag{72}$$

and thus obtains a classical solution in imaginary time

$$\frac{\partial x}{\partial \tau} = \pm\sqrt{2V(x)} \tag{73}$$

that maintains zero energy throughout the interpolation between two different classical vacua x_n , x_m. Such a solution is called "instanton".

The instanton action is given by

$$S_I = \int d\tau L_I = \int_{x_n}^{x_m} dx \ \sqrt{2V(x)} \tag{74}$$

which is closely connected to the tunneling amplitude through the potential barrier in real time as given by the WKB approximation

$$P_{n\to m}^{\mathrm{WKB}} \propto \exp - \int_{x_n}^{x_m} dx \ \sqrt{2V(x)} \ . \tag{75}$$

In Yang Mills theory, instantons are classical solutions of least energy interpolating between two classical vacua of different winding number, localized both in space and time, as explicit construction shows. They can be constructed [12] in the same way as above by going to imaginary time and solving

$$H_I = \tfrac{1}{2}\int d^3x \left[\vec{E}^a(\vec{x})\vec{E}^a(\vec{x}) - \tfrac{1}{2}F_{ij}^a(\vec{x})F_{ij}^a(\vec{x})\right] \tag{76}$$

$$\Rightarrow \ \ F_{\mu\nu}^a = \pm\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma\,a} \ . \tag{77}$$

So instantons are classical (anti)selfdual solutions to the Euclidean Yang Mills equations with zero energy.

The tunneling amplitude between two vacua which can only be connected by a large gauge transformation of winding number n is (74,75)

$$\exp - \int d\tau L_I(\tau) = \exp - \int d^4x \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} \mathrm{tr} F^{\mu\nu} F^{\rho\sigma} = \exp - \frac{8\pi^2}{g^2} |n| \ , \tag{78}$$

where we reintroduced the coupling constant g as described in Section 1.1. Note the interplay between the instanton action, the Chern–Pontryagin density (49) and the winding number of the gauge transformation g_n.

A word of caution is in order here:

The analogy between the tunneling process in solid state physics and the connection of different classical QCD vacua by instantons should not be pushed too far. After all, the occurence of a physically measurable Bloch momentum is connected to Ω_n being a "physical" transformation, namely implementing spatial displacement. The gauge transformation $\Omega_n[\beta]$ is unobservable. The Bloch momentum can also be changed, while there is – as indicated – no way to change the vacuum ϑ angle, which moreover becomes physically irrelevant in certain situations e.g. the chiral limit, as has been hinted on in the previous section and we shall see now.

1.7 The Axial Anomaly

In the two dimensional example we gave in Section 1.5 it was shown that there exists a connection between the chiral symmetry of massless fermions and the ϑ-angle.

In this section we continue to discuss topological aspects of the standard model with a more detailed analysis of the chiral symmetry [1]. Therefore we consider the quark sector of a four-dimensional gauge theory. The Lagrangean density is

$$\mathcal{L}_{\text{quark}} = \bar{\psi} i \left(\partial\!\!\!/ + A\!\!\!/ \right) \psi \ , \tag{79}$$

where A^a_μ describes a nonabelian background gauge field.

On the classical level this Lagrangean has the global chiral symmetry

$$\psi \to e^{\alpha\gamma_5} \psi \ , \qquad \bar{\psi} \to \bar{\psi} e^{\alpha\gamma_5} \qquad \left(\gamma_5^\dagger = -\gamma_5 \right) \ . \tag{80}$$

Since $\{\gamma_5, \gamma_\mu\} = 0$ we get for the classical theory $\mathcal{L}_{\text{quark}} \to \mathcal{L}_{\text{quark}}$ under this transformation.

The Noether current connected to the chiral symmetry is

$$j_5^\mu = i \bar{\psi} \gamma^\mu \gamma_5 \psi \tag{81}$$

which is classically conserved

$$\partial_\mu j_5^\mu = 0 \ . \tag{82}$$

This can easily be verified to be a consequence of the equation of motion

$$i \left(\partial\!\!\!/ + A\!\!\!/ \right) \psi = 0 \ . \tag{83}$$

For a quantum theory the situation is different. Expressions like $\mathcal{L}_{\text{quark}}$ in (79) or j_5^μ in (81) are not well defined. The product of two field operators at the same space-time point is singular and requires regularisation. This is most easily seen from the quantization relation

$$\{\psi(x), \psi^\dagger(y)\}_{x^0=y^0} = \delta^{(3)}(\vec{x} - \vec{y}) \ . \tag{84}$$

The regularisation may be carried out using point splitting. However it has to be done carefully since the introduction of a further parameter may spoil the symmetries of the theory. Nevertheless it is possible to regularize the theory in a way that the local gauge invariance is maintained. This is necessary since the gauge symmetry is a fundamental intrinsic property of the theory and it should not be spoiled.

The requirement of keeping the gauge symmetry restricts the freedom how to regularize. Therefore one has to take into account that other, less important symmetries may be violated within the regularization procedure. For such a symmetry the corresponding currents are not conserved. The symmetry is said to be broken by an anomaly. One example is the axial symmetry which is spoiled by quantization according to the axial anomaly.

Since j_5^μ has no gauge group label it is a gauge singlet current. In this sense we call the axial anomaly also "abelian anomaly".

We will proceed with a discussion of this anomaly. We take the expectation value of j_5^μ with respect to the perturbative fermionic vacuum.

$$\langle 0_F | j_5^\mu(x) | 0_F \rangle \equiv \langle j_5^\mu(x) \rangle = \langle \bar{\psi}(x) i \gamma^\mu \gamma_5 \psi(x) \rangle \equiv \langle \bar{\psi}(x) \Gamma_5^\mu \psi(x) \rangle \tag{85}$$

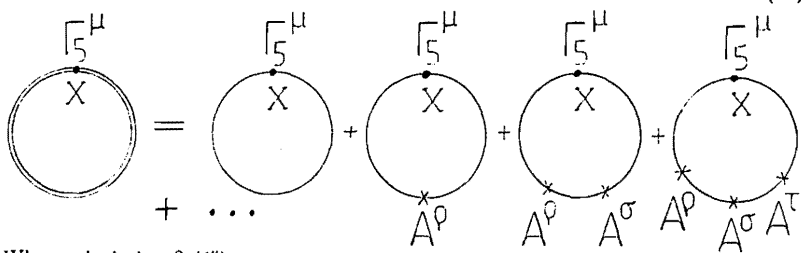
This can be done without loosing information since we expect the result for $\partial_\mu j_5^\mu$ to have no fermion operator component[5]. The result may be regarded as the amplitude for a quark to interact at the space-time point x with Γ_5^μ, to propagate in the gauge background field and to return to x[6]. The background field coupling can be treated as

[5] This is confirmed by the path integral approach, which gives the same result as our calculations (see eg. [14, p. 100]).

[6] Or close to it, when we are applying point splitting to regularize the theory.

two point interaction. So the propagation in the background field may be calculated perturbatively to get a power series in the backgroud field A_μ.

$$\langle j_5^\mu(x)\rangle = \langle \bar\psi(x)\,\Gamma_5^\mu \psi(x)\rangle = \tag{86}$$



When calculating $\partial_\mu\langle j_5^\mu\rangle$ we recognize that the first term on the right hand side does not contribute since it is x-independent[7]. The second term does not contribute since it is linear in A and we expect the background field to be invariant under charge conjugation. If the background field stems from the Feynman integral of a physical theory this property is guaranteed[8]. With the same reasoning the third order term in A vanishes, as does every odd order in A.

With each interaction of the background field one gets an extra fermion propagator $S_F = 1/(\not p + i\varepsilon)$ and the amplitude becomes more convergent. So, by power counting, terms of fourth and higher order are finite. Their amplitudes can not contribute to $\partial_\mu\langle j_5^\mu\rangle$ since j_5^μ is classically conserved order by order, and for finite amplitudes we can apply the classical result.

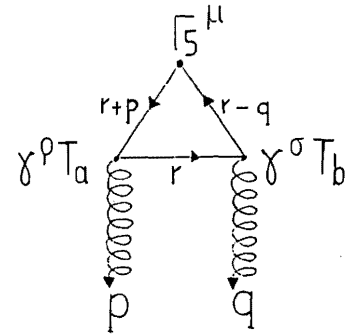
Therefore the only diagram that can give rise to a non-vanishing $\partial_\mu\langle j_5^\mu\rangle$ is the second order contribution in A_μ. We will focus on it in the following. Its contribution to $\langle j_5^\mu\rangle$ is given by (the trace goes over color as well as spinor indices)

$$\begin{aligned}\langle j_5^\mu(x)\rangle_{A^2} &= i\int d^4z_1 d^4z_2\ \mathrm{tr}\left[\Gamma_5^\mu S_F(x-z_1)\not A(z_1)\,S_F(z_1-z_2)\not A(z_2)\,S_F(z_2-x)\right]\\ &= \int d^4z_1 d^4z_2 A_\rho^a(z_1)\,A_\sigma^b(z_2)\int d^4p\,d^4q\,e^{i(p+q)x}e^{ipz_1}e^{iqz_2}T_{ab}^{\mu\rho\sigma}(p,q)\ ,\end{aligned} \tag{87}$$

[7]To be precise each of the lower order contributions of $\langle j_5^\mu\rangle$ is singular. Therefore it gets a more complicated x- and A_μ-dependence as a consequence of a gauge invariant regularization prescription. For details see [20], [17].

[8]From experiments we know that, in contrary to the two dimensional theory (s. below), the charge conjugation symmetry is not dynamically broken.

where $T_{ab}^{\mu\rho\sigma}(p,q)$ is given by the triangle graph



+ crossed graph

$$\begin{aligned}T_{ab}^{\mu\rho\sigma} &= -ie^2\int\frac{d^4r}{(2\pi)^4}\ \mathrm{tr}\ \gamma^\mu\gamma_5\frac{1}{\not r+\not p+i\varepsilon}\gamma^\rho T_a\frac{1}{\not r+i\varepsilon}\gamma^\sigma T_b\frac{1}{\not r-\not q+i\varepsilon}\\ &+ \quad (\rho\leftrightarrow\sigma\ ,\quad p\leftrightarrow q)\\ &= \frac{1}{2}\delta_{ab}ie^2\int\frac{d^4r}{(2\pi)^4}\ \mathrm{tr}\ \gamma^\mu\gamma_5\frac{1}{\not r+\not p+i\varepsilon}\gamma^\rho\frac{1}{\not r+i\varepsilon}\gamma^\sigma\frac{1}{\not r-\not q+i\varepsilon}\\ &+ \quad (\rho\leftrightarrow\sigma\ ,\quad p\leftrightarrow q)\ \equiv\ -\frac{1}{2}\delta_{ab}T^{\mu\rho\sigma}(p,q)\ .\end{aligned} \tag{88}$$

The integral is linearly divergent which reflects the fact that $j_5^\mu(x)$ was not properly regularized. This has the consequence that a shift in the integration variable $r\to r+a$ changes the value of the (finite part of the) integral by a surface term[9]. This can easily be seen in a one dimensional analogon: Consider the integral

$$\Delta(a) = \int_{-\infty}^{\infty}\left(f(x+a)-f(x)\right)dx \tag{89}$$

where f is an analytic function.

We expand $f(x+a)$ in a Taylor series at the point x and perform the integral with the result

$$\Delta(a) = a\left(f(\infty)-f(-\infty)\right)+\frac{a^2}{2}\left(f'(\infty)-f'(-\infty)\right)+\ldots\ . \tag{90}$$

[9]The Feynman ruls do not describe how to introduce the loop integration variable. Each of the choices $r+a$ are a priory possible.

If the integral would be convergent or at most logarithmically divergent then, of course, $0 = f(\pm\infty) = f'(\pm\infty) = \ldots$ and the integral vanishes.

However if the integral is linearly divergent we only have $0 = f'(\pm\infty) = f''(\pm\infty) = \ldots$ and we get for $\Delta(a)$ the surface contribution

$$\Delta(a) = a\left(f(\infty) - f(-\infty)\right) \tag{91}$$

which is in general non zero.

The same applies to the four-dimensional integral $T^{\mu\rho\sigma}$. The surface term can be calculated quite easily. We start with the first part of $T^{\mu\rho\sigma}$ and get

$$\begin{aligned}\Delta_1(a) &= -ie^2\int\frac{d^4r}{(2\pi)^4}\ \mathrm{tr}\ \gamma^\mu\gamma_5\left(\exp\left(a_\alpha\frac{\partial}{\partial r_\alpha}\right)-1\right)\frac{1}{\not r+\not p+i\varepsilon}\gamma^\rho\frac{1}{\not r+i\varepsilon}\gamma^\sigma\frac{1}{\not r-\not q+i\varepsilon}\\ &= -ie^2a_\alpha\int\frac{d^4r}{(2\pi)^4}\frac{\partial}{\partial r_\alpha}\left(1+\mathcal{O}\left(r^{-1}\right)\right)\mathrm{tr}\ \gamma^\mu\gamma_5\frac{1}{\not r+\not p+i\varepsilon}\gamma^\rho\frac{1}{\not r+i\varepsilon}\gamma^\sigma\frac{1}{\not r-\not q+i\varepsilon}\ .\end{aligned}$$

Now we Wick-rotate to Euclidean space-time ($t \to ix_4$) and use "one quarter" of the four-dimensional Gauß theorem

$$\int_M d^4r\frac{\partial}{\partial r_\alpha}f(r) = \int_{\partial M} d\sigma^\alpha f(r) \tag{92}$$

where ∂M is the boundary of M (which is the sphere $S^3(R)$ in our case) and $\int d\sigma^\alpha$ is the α-component of the surface integral. We get ($R \to \infty$)

$$\begin{aligned}\Delta_1(a) &= \frac{e^2a_\alpha}{(2\pi)^4}\int_{S^3(R)} d\sigma^\alpha\ \mathrm{tr}\ \gamma^\mu\gamma_5\frac{1}{\not r+\not p}\gamma^\rho\frac{1}{\not r}\gamma^\sigma\frac{1}{\not r-\not q}\\ &= \frac{e^2a_\alpha}{(2\pi)^4}\int d\Omega^\alpha\ \mathrm{tr}\ [\gamma^\mu\gamma_5\gamma^\nu\gamma^\rho\gamma^\delta\gamma^\sigma\gamma^\beta]R_\nu R_\delta R_\beta/R^3\\ &= -\frac{e^2a_\alpha}{(2\pi)^4}\int d\Omega^\alpha 4\varepsilon^{\mu\nu\rho\sigma}\frac{R_\nu}{R}\ .\end{aligned}$$

Now we introduce polar coordinates and let the north pole point into the α-direction. Since $\int d\Omega^\alpha R_\nu/R$ is zero if $\alpha \neq \nu$ we get

$$\Delta_1(a) = -\frac{e^2a_\nu}{\pi^3}\varepsilon^{\mu\nu\rho\sigma}\int_0^\pi d\vartheta\sin^2\vartheta\cos\vartheta\cdot\cos\vartheta = -\frac{e^2a_\nu}{8\pi^2}\varepsilon^{\mu\nu\rho\sigma}\ .$$

The crossed term gives the same result so that the total surface term is

$$\Delta(a) = -\frac{e^2a_\nu}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}\ . \tag{93}$$

The vector Ward identities which enssure gauge invariance have the form

$$p_\rho T^{\mu\rho\sigma}(p,q) = 0\ , \qquad q_\sigma T^{\mu\rho\sigma}(p,q) = 0\ . \tag{94}$$

The chiral Ward identity which is connected to the chiral symmetry is

$$(p+q)_\mu T^{\mu\rho\sigma}(p,q) = 0\ . \tag{95}$$

Gauge invariance is one of the most fundamental principles of QCD and in fact there exists a choice of the integration variable $r + a$ that enssures (94) (namely $a = -2p$ [20, p. 122]), but for any other a the gauge symmetry is spoiled by the surface term (93).

Unfortunately we need different a's to assure (94) and (95). So it is impossible to have both gauge symmetry and chiral symmetry. We choose (94) to hold and get a correction on the right hand side of (95) [16], [17]:

$$(p+q)_\mu T^{\mu\rho\sigma}(p,q) = -\frac{e^2}{2\pi}\varepsilon^{\rho\sigma\mu\nu}p_\mu q_\nu\ . \tag{96}$$

If this is plugged into (88) we get

$$\langle\partial_\mu j_5^\mu\rangle = \frac{\varepsilon^{\mu\nu\rho\sigma}}{8\pi^2}\langle\mathrm{tr}\ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\rangle$$

The right hand side equals $1/8\pi^2\cdot\langle\mathrm{tr}\ F^*{}_{\mu\nu}F^{\mu\nu}\rangle$ for the following reason: The third order term in A_μ vanishes since we have invariance under charge conjugation. The fourth order term is proportional to $\varepsilon^{\mu\nu\rho\sigma}\mathrm{tr}\ ([A_\mu, A_\nu][A_\rho, A_\sigma]) = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}A^a_\mu A^b_\nu A^c_\rho A^d_\sigma f_{abe}f_{cde}$ where f are the structure constants of the SU(3) group. One can now use the total antisymmetry of f and the Jacobi identity to show that the last expression is zero.

Thus we have motivated the final result

$$\partial_\mu j_5^\mu = \frac{1}{8\pi^2}\mathrm{tr}\ F^*{}_{\mu\nu}F^{\mu\nu}\ . \tag{97}$$

It can be shown that there are no other contributions to the anomaly as for example virtual gluon effects[10] [17], [21]. The expression on the right hand side of (97) has a topological interpretation: It is just twice the four-dimensional Pontryagin density.

[10]The fermion loop becomes more convergent with every internal gluon line and the intergrations over the gluon lines do not contribute to the anomaly.

Although the axial current is not conserved we can carry on by constructing a conserved current. Due to (49) we have

$$\mathrm{tr}\ F^*{}_{\mu\nu}F^{\mu\nu} = 4\partial_\mu\varepsilon^{\mu\nu\rho\sigma}\ \mathrm{tr}\ \left(\frac{1}{2}A_\nu\partial_\rho A_\sigma + \frac{1}{3}A_\nu A_\rho A_\sigma\right) \tag{98}$$

and therefore

$$\partial_\mu J_5^\mu = 0 \tag{99}$$

with

$$J_5^\mu = j_5^\mu - \frac{1}{2\pi^2}\varepsilon^{\mu\nu\rho\sigma}\ \mathrm{tr}\ \left(\frac{1}{2}A_\nu\partial_\rho A_\sigma + \frac{1}{3}A_\nu A_\rho A_\sigma\right)\ . \tag{100}$$

The conserved charge Q_5 of J_5^μ is

$$\begin{aligned} Q_5 &= \int d^3r\left(j_5^0 - \frac{1}{2\pi^2}\varepsilon^{ijk}\ \mathrm{tr}\ \left(\frac{1}{2}A_i\partial_j A_k + \frac{1}{3}A_iA_jA_k\right)\right) \\ &= \int d^3r\left(j_5^0 + 2W(A)\right) \end{aligned} \tag{101}$$

where $W(A)$ is the Chern Simons three-form which was already defined in (39).

Q_5 consists of two pieces, a gauge invariant fermion contribution coming from j_5^μ and an anomalous term constructed from the gauge potentials. This term has the immediate consequence that neither J_5^μ nor Q_5 are invariant under topological non-trivial (large) gauge transformations Ω_n (under (small) gauge transformations that are smoothly connected to unity they are still invariant). Q_5 changes by two times the winding number.

$$\Omega_n Q_5 \Omega_n^{-1} = \Omega_5 - 2n \tag{102}$$

The commutator algebra of the Hamilton operator H, Q_5 and Ω_n is

$$[H,Q] = 0\ , \qquad [H,\Omega_n] = 0\ , \qquad [\Omega_n, Q_5] = 2n\Omega_n\ . \tag{103}$$

Since the ϑ-angle is defined by (45)

$$\Omega_n|\vartheta\rangle = e^{-i\vartheta n}|\vartheta\rangle \tag{104}$$

we conclude that Q_5 acts as a shift operator for ϑ:

$$e^{i\frac{\vartheta'}{2}Q_5}\psi(\vartheta) = \psi(\vartheta + \vartheta') \tag{105}$$

Since H and Q_5 can be diagonalized simultaneously, applying $e^{i\frac{\vartheta'}{2}Q_5}$ can not change the energy eigenvalue of an energy eigenstate ψ. Therefore the energy spectrum does not

depend on ϑ. The value of the ϑ-angle is physically irrelevant. If on the other hand fermions are massive, Eq. (97) and all successive equations acquire a mass correction and we can not argue that the ϑ-angle has no physical consequences.

The same result may be obtained in a functional integral formulation. If one decides to have massless fermions and to translate the ϑ-dependence from the wavefunctions to the Lagrangean one gets the action (48)

$$Z_\vartheta = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A_\mu^a \exp\left(i\int dx\mathcal{L}_\vartheta(x)\right)\ , \tag{106}$$

where

$$\mathcal{L}_\vartheta = \frac{1}{2g^2}\ \mathrm{tr}\ [F_{\mu\nu}F^{\mu\nu}] - \frac{\vartheta}{16\pi^2}\ \mathrm{tr}\ [F^*{}_{\mu\nu}F^{\mu\nu}] + i\bar{\psi}(\not{\partial} + \not{A})\psi\ . \tag{107}$$

Redefining the fermionic integration variables according to the chiral transformation law (80) $\mathcal{L}_\vartheta$ remains unaffected, but we get a contribution from the integration measure. This contribution corresponds to the anomaly, and we get

$$Z_\vartheta \to Z_{\vartheta+2\alpha}\ . \tag{108}$$

Since we just substituted our integration variables, Z_ϑ does not change. Therefore $Z_\vartheta = Z_{\vartheta+2\alpha}$ has to be independent of ϑ. So we can conclude that in the presence of massless fermions the ϑ-angle is no physical parameter.

1.8 The Two Dimensional Analogon

Let us come back to the two dimensional example QED_{1+1} that was already discussed in Sec. 1.5.

In a two dimensional space-time, the Dirac spinors become two-component objects. The Dirac matrices may be chosen to be the Pauli matrices

$$\gamma^0 = \sigma^1\ , \qquad \gamma^1 = i\sigma^2\ , \qquad \gamma_5 = -i\sigma^3\ . \tag{109}$$

It is a particular property of two dimensions that axial vectors are dual to vectors

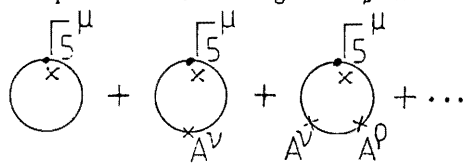
$$\Gamma_5^\mu = i\gamma^\mu\gamma_5 = \varepsilon^\mu{}_\nu\gamma^\nu\ , \qquad \varepsilon^{01} = 1 = -\varepsilon_{01} \tag{110}$$

and therefore the axial vector current is dual to the vector current

$$j_5^\mu = \varepsilon^\mu{}_\nu j^\nu\ . \tag{111}$$

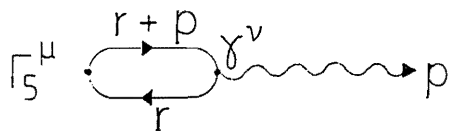
Let us consider the fermionic sector of two dimensional QED. We start to calculate the divergence of the chiral current in the same way as in the four-dimensional case.

$\langle j_5^\mu(x)\rangle$ can be expanded in a power series of the background A_μ field:

$\langle j_5^\mu(x)\rangle =$ 

$+ \cdots$ (112)

In two dimensional QED we can not use charge conjugation to simplify the result since the symmetry under charge conjugation is dynamically broken. This can be concluded most easily from the existence of a constant electric field E_0 (Eq. (54)), which is incompatible with the symmetry under charge conjugation. However, the A_μ^2-order is already convergent enough not to produce an anomaly. Instead of the triangle graph we get the relevant contribution from

$T^{\mu\nu}(p) =$ 

. (113)

We are using the duality between axial vectors and vectors to obtain

$$T^{\mu\nu}(p) = \varepsilon^\mu{}_\rho \Pi^{\rho\nu}(p) \tag{114}$$

where $\Pi^{\rho\nu}(p)$ is the vacuum polarization tensor. Its space-time structure is determined by the requirement of gauge invariance

$$p_\rho \Pi^{\rho\nu}(p) = p_\nu \Pi^{\rho\nu}(p) = 0 \tag{115}$$

to be of the form

$$\Pi^{\rho\nu}(p) \propto g^{\rho\nu} - \frac{p^\rho p^\nu}{p^2} \ . \tag{116}$$

Therefore we have

$$p_\nu T^{\mu\nu}(p) = 0 \ , \tag{117}$$

but the Ward identity related to the chiral symmetry,

$$p_\mu T^{\mu\nu}(p) \propto p_\mu \varepsilon^{\mu\nu} \ , \tag{118}$$

does not vanish.

So we regain the result that gauge symmetry can be maintained, while the axial symmetry is broken on the quantum level.

The result of a detailed calculation is [10], [18]

$$\partial_\mu j_5^\mu = -\frac{1}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu} = -\frac{1}{\pi}F^* \ , \tag{119}$$

where we have once more absorbed the electromagnetic charge e in the A_μ field. The anomaly is now given by twice the two dimensional Pontryagin density.

Therefore it is again possible to define a conserved current J_5^μ and its time independent charge Q_5

$$J_5^\mu = j_5^\mu + \frac{1}{\pi}\varepsilon^{\mu\nu}A_\nu \ , \tag{120}$$

$$\begin{aligned} Q_5 &= \int dx \left(j_5^0(x) + \frac{1}{\pi}A_1(x)\right) \\ &= \int dx j_5^0(x) + 2W(A) \end{aligned} \tag{121}$$

In the Feynman path integral approach a chiral re-definition (80) of the fermionic integration variables amounts, due to the measure, to a new term in the Lagrangean

$$\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}A_\mu e^{i\int\mathcal{L}} \to \int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}A_\mu e^{i\int(\mathcal{L}+\frac{\vartheta}{2\pi}F^*)} \tag{122}$$

which coincides with (65). Thus QED with massless fermions in 1+1 dimensions (the Schwinger model) has no physically relevant ϑ-angle [18], [19].

Let us close this section with a remark that is specific to a two dimensional theory. If we contract the gauge field equation

$$\partial_\mu F^{\mu\nu} = e^2 j^\nu \tag{123}$$

with $\varepsilon_{\nu\rho}$ and use the antisymmetry of $F_{\mu\nu}$ we get

$$\partial^\mu F^* = e^2 j_5^\mu \ . \tag{124}$$

The divergence of this equation yields

$$\Box F^* = e^2 \partial_\mu j_5^\mu = -\frac{e^2}{\pi} F^* \ . \tag{125}$$

Thus the gauge field acquires the topological mass $m^2 = e^2/\pi$.

Whereas in three-dimensional space-time there exists another topological mechanism for vector meson mass generation (see below), no similarly elegant result has yet been established in four dimensions.

1.9 Conclusions of the First Part

(i) For a long time it appeared that QCD possesses too much symmetry. An additional chiral $U(1)$ symmetry would predict that there would be a particle degenerate with the pion, but no such particle exists [23]. Now we have recognized that the chiral symmetry is broken by an anomaly and the $U(1)$ problem has dissolved [13].

(ii) If the theory includes massless fermions the ϑ-angle is unphysical.

But physical fermions are not massless and the ϑ-angle is supposed to remain observable. For $\vartheta \neq 0$ CP-invariance is violated, but in QCD the experiments require that $\vartheta = 0$ and CP is not violated (measurements of the electric dipole moment of the neutron give $\vartheta \leq 10^{-9}$ [22]).

No principle is known that insures the vanishing of ϑ. In fact the situation is even more complicated: If we suppose that the fermion masses arise from spontaneous symmetry breaking then we would expect that the fermion mass matrix in the QCD Lagrangean would point in an arbitrary CP direction $\bar{\psi} M_1 \psi + \bar{\psi} \gamma_5 M_2 \psi$. One can remove the M_2-term by a chiral transformation. But this induces, due to the anomaly, a $\mathrm{tr}[F^*{}_{\mu\nu} F^{\mu\nu}]$-term giving rise to a ϑ-angle. This angle has to be canceled by the "initial" ϑ-angle in the Lagrangean in order not to yield CP-violating effects[11].

This problem is not unlike that of the cosmological constant which is a parameter that in principle is present, but experiments force it to be zero.

(iii) In the electroweak sector of the standard model couplings to $\gamma^\mu(1-i\gamma_5)$ are present due to the coupling of only left-handed fermions to the weak charged currents. The requirement of renormalizability forces the theory to avoid the anomalies in the gauge current (anomalies may not occour in subdiagrams where the axial current couples to internal lines). This is only possible if the quarks and leptons balance in number. In particular the existence of a top-quark is demanded.

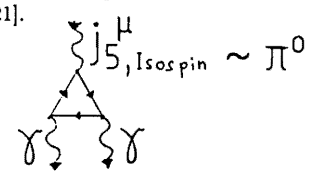
[11] Or there exists a reason why even in the presence of massive fermions the ϑ-angle is unphysical and not CP-violating.

(iv) In the standard model the baryon number current acquires an anomaly [24]. The decay rate is controled by $\mathrm{tr}[F^*{}_{\mu\nu} F^{\mu\nu}]$. There are two mechanisms for baryon decay known:

The first involves tunnelling. The tunnelling rate is given by the exponential of the instanton action (in a semiclassical description). But exp(− instanton action)=$\exp(-8\pi^2/g^2)$ is a negligible small number ($\approx 10^{-122}\mathrm{year}^{-1}$) [13].

The second mechanism is connected to 't Hooft-Polyakov monopoles [25]. The magnitude of this effect is still controversial (but it seems to suffice) and moreover an experimental evidence for monopoles is still missing.

(v) The hypothesis of partial conservation of flavour $SU(2)$ axial vector currents (PCAC) implies, in the absence of anomalies, that a massless neutral pion can not decay into two photons [15]. But the physical pion does decay with a width of about 7.9 eV. This large number can only be understood with the axial anomaly [16], [17], [21].



Moreover one gets the result that the width depends on the number of quark colors. The best agreement with the experiment is achieved for $N = 3$ colors. The remaining discrepancy of about 10% can be understood as an effect due to the non-zero pion mass.

Therefore the anomaly allows an experimental determination of the number of colors.

2 High-Temperature Quantum Chromodynamics

In Section 1 we were discussing more or less settled physics, i.e. work that has been done during the eighties. Now we would like to come to talk about some current research in QCD. In this section, we are going to show you the connections between QCD at high temperature (QCD well in the deconfined, chirally symmetric region) and a three-dimensional topolgical field theory: the nonabelian Chern-Simons (CS) theory. More explicitly, we want to show you that the generating functional of the so-called hard thermal loops in QCD is the eikonal of the nonabelian CS theory. These connections have been established recently by several people [26], [27], [29]. They are relevant for the nonabelian generalization of the Kubo formula as well as for a gauge-invariant description of Landau damping in the quark-gluon plasma at high temperature.

First of all, we would like to give you a short introduction to thermal field theory. For details, see, for example, [30], [31].

2.1 Temperature Green Functions

The objects of study in a field theory at finite temperature are the temperature n-point correlation (or Green) functions

$$G_n(x_1,\ldots,x_n) := \langle \phi(x_1)\ldots\phi(x_n)\rangle \tag{126}$$

where the x_i are elements of Minkowski space, and the $\phi(x_i)$ are the generic fields of the theory in the Heisenberg picture. The angle brackets denote thermal average within the canonical ensemble

$$\langle\ldots\rangle := \frac{\mathrm{tr}(e^{-\beta H}\ldots)}{\mathrm{tr}e^{-\beta H}}\,. \tag{127}$$

Here, H is the Hamiltonian of the theory, and β^{-1} represents the inverse temperature in natural units that we are going to use for the rest of the talk.

Depending on the boundary conditions chosen to solve the equations of motion, one defines *various* Green functions. For example, $\langle T\phi(x)\phi(y)\rangle$ gives the *time-ordered* two-point function, whereas $\theta(x^0-y^0)\langle[\phi(x),\phi(y)]\rangle$ defines the *retarded commutator* two-point function.

The set of all these n-point Green functions, e.g. in momentum space representation,

$$G_n(p_1,\ldots,p_n) := \int d^4x_1\ldots d^4x_n e^{i(p_1x_1+\ldots p_nx_n)}G_n(x_1,\ldots,x_n) \tag{128}$$

with *real* p_i and *real* x_i, contains all the physical information about the system at finite temperature. But, as a matter of fact, perturbation theory within this description is rather difficult. A simpler perturbation theory can, however, be established on accomplishing the following *unphysical continuation*: one allows the time arguments x_i^0 to be complex valued. For Bose fields, it can be shown that — for analyticity reasons of the n-point functions — they have to be periodic in the imaginary time direction

$$\phi(x^0,\vec{x}) = \phi(x^0-i\beta,\vec{x})\,. \tag{129}$$

Similarly, fermionic fields $\psi(x^0,\vec{x})$ have to obey antiperiodic boundary conditions:

$$\psi(x^0,\vec{x}) = -\psi(x^0-i\beta,\vec{x})\,. \tag{130}$$

Note that these boundary conditions are the essential differences between field theory at zero and field theory at finite temperature; the equations of motions do not differ except for a thermal average, of course, in the latter case. This extension to complex values of

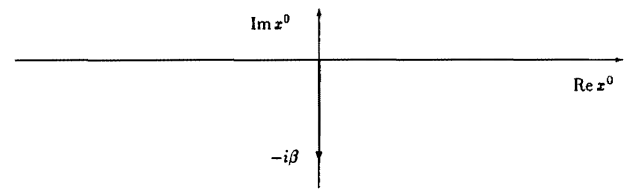


Figure 2: Time contour in the ITF

x^0 is certainly not unique. In the so-called *imaginary-time formalism* (ITF) one restricts x^0 to the imaginary axis in the complex x^0-plane, i.e. $x^0 \in [0,-i\beta]$ (cf. Fig. 2).

This can — for Bose fields — be interpreted as a transition from the Minkowski space-time manifold $\mathbb{R}^3\times\mathbb{R}$ to the new space-time manifold $\mathbb{R}^3\times S^1$.

Besides the ITF scheme, another popular choice for the complex time-path contour is shown in Fig. 3. This is just one of infinitely many possibilities (the choice depends on the parameter σ) of setting up the *real time formalism* (RTF) using the time-path contour method. Choosing $\sigma = 1/2$ provides equivalence with yet another formulation of field theory at finite temperature, called *thermo field dynamics* [32], [33]. In actual calculations, one always considers the limit $t_0\to\infty$. The advantage of the RTF over the ITF is that perturbation theory can be defined with Green functions depending solely

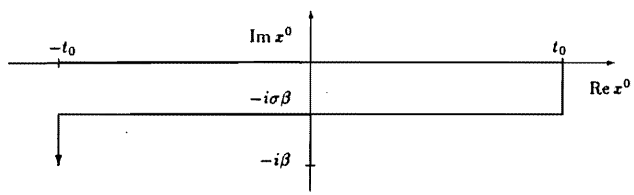


Figure 3: Time contour in the RTF with $\sigma = 1/2$

on *real* time arguments. Thus one does not have the problem of a backward continuation from purely imaginary times to purely real, hence physical, time arguments.

In turn, perturbation theory is a bit more cumbersome as, for example, the RTF two-point function is a 2×2 matrix.

2.2 Imaginary-Time Formalism

In the imaginary-time formalism, perturbation theory corresponds to the well-kwown Dyson-Feynman series with the integration over p_0 replaced by an infinite sum

$$\int \frac{dp_0}{2\pi} \to iT \sum_{n\in\mathbb{Z}} . \tag{131}$$

The usual time-ordering along the real x^0-axis is converted into an imaginary-time-ordering down the imaginary x^0-axis. That is, later times are positioned *below* earlier times (cf. Fig. 2). Furthermore, all the Green functions are unique, because the inverse d'Alembertian $\Box^{-1}$ is unique on $\mathbb{R}^3 \times S^1$.

(Anti-)periodicity in position space on the interval $[0, -i\beta]$ provides for discrete imaginary energies $p_0 = 2\pi i n T$ (for bosons) and $p_0 = 2\pi i T(n + \frac{1}{2})$ for fermions, $n \in \mathbb{Z}$, in momentum space. These discrete energies are (proportional to) the so-called *Matsubara frequencies.*

At this point, it is interesting to take a look at the high-temperature limit. In position space, the time interval $[0, -i\beta]$ shrinks down to a point when $T \to \infty$, since then $\beta = 1/T \to 0$. Hence we lose the time dimension and end up with a three-dimenional field theory:

$$\mathbb{R}^3 \times S^1 \stackrel{T\to\infty}{\longrightarrow} \mathbb{R}^3 . \tag{132}$$

In momentum space, the same result can be deduced by looking at some generic perturbation theoretic diagram. Let the boson propagator have the form

$$D(p) = \frac{i}{p_0^2 - \vec{p}^{\,2} - m^2}, \quad \text{where} \quad p_0 = 2\pi i n T, \tag{133}$$

while a fermion propagator be

$$S(p) = \frac{i}{\gamma_0 p_0 - \vec{\gamma}\cdot\vec{p} - m}, \quad \text{where} \quad p_0 = 2\pi(n + \tfrac{1}{2})iT. \tag{134}$$

So our diagram might be something like

$$iT \sum_n \int \frac{d^3p}{(2\pi)^3}\, e\, \frac{i}{-4\pi^2 n^2 T^2 - \vec{p}^{\,2} - m^2}\, e \ldots, \tag{135}$$

where e denotes the coupling constant. In the limit $T \to \infty$, all modes with $n \neq 0$ decouple — they behave like very heavy particles. Only the zero mode survives, and so we are left with

$$\int \frac{d^3p}{(2\pi)^3}\, e\sqrt{T}\, \frac{1}{\vec{p}^{\,2} + m^2}\, e\sqrt{T} \ldots \tag{136}$$

This is exactly what one would find in a field theory on a Eulidean space of one dimension less. Moreover, fermion contributions are obviously subdominant since the energy modes in the fermion propagator never vanish. Taking the infinite temperature limit in this way means, in the end, setting external $p_0 = 2\pi n i T \stackrel{n\equiv 0}{=} 0$. A more detailed treatment must, however, allow for a high-temperature limit with fixed, nonvanishing external p_0 in order to be able to continue back to real energies. But, even in the case p_0 is kept finite one has a problem. Namely, does one try to continue backward

$$2\pi n T \to -i p_0, \tag{137}$$

one immediately notices that this continuation is not unique. I.e., from a single Euclidean Green function one can obtain *several* Minkowski Green functions. Which one to take depends on the physical setting.

As a rule, the ITF represents the natural scheme for calculating static quantities like the effective potential.

2.3 Hard Thermal Loops

By transferring the QCD Feynman rules for $T = 0$ to finite temperature in a naive way one gets a confusing infra-red limit: on mass-shell both the sign and the magnitude of the gluon damping rate appear to be gauge dependent. Braaten and Pisarski [34] have argued that whenever a quantity is calculated perturbatively in a hot nonabelian gauge theory, sooner or later an infinite subset of diagrams nominally of higher order in the loop expansion contribute to the same order in the coupling constant g. These higher-loop diagrams have to be isolated and resummed into an effective expansion which includes all effects to leading order in g. This resummation technique is necessary to get, even at one loop, gauge invariant results.

More explicitly, hard thermal loops are the ones with exceptional (soft) external momenta

$$\text{both } p_0 \text{ and } |\vec{p}| \text{ of order } gT \tag{138}$$

and large (hard) internal momenta

$$k_0 \text{ and/or } |\vec{k}| \text{ of order } T\,. \tag{139}$$

The need for resummation can be seen from a simple example. Look at the one- and two-loop contributions to the gluonic self energy depicted in Figs. 4 and 5.

$\Pi_2(p) =$

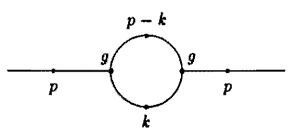


Figure 4: The one-loop self energy contribution $\Pi_2(p)$

Let us write them as

$$\Pi_2(p) \equiv \int d^4k \Pi_2(k,p) \tag{140}$$

$$\Pi_4(p) \equiv \int d^4k \Pi_4(k,p)\,. \tag{141}$$

$\Pi_4(p) =$

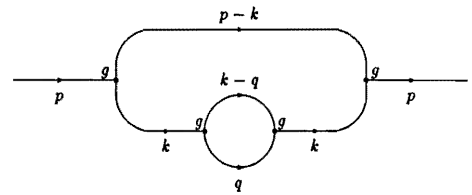


Figure 5: The two-loop self energy contribution $\Pi_4(p)$

One can then easily derive for the following quotient ($D(k)$ symbolizes the free gluon propagator)

$$\frac{\Pi_4(k,p)}{\Pi_2(k,p)} = \text{`}\Pi_2(k)D(k)\text{'}\,. \tag{142}$$

For small k, $\Pi_2(k)$ is known to behave like g^2T^2. Hence

$$\frac{\Pi_4(k,p)}{\Pi_2(k,p)} \overset{\text{small } k}{\simeq} \frac{g^2T^2}{k^2}\,. \tag{143}$$

Hence, for soft internal momentum $k \sim gT$, the two contributions are of the same order in g. Stated in a slightly different way, the fourth order diagram contains second order contributions.

2.4 The Kubo Formula

A recent application of hard thermal loops is the generalization of Kubo's formula of linear response theory to nonabelian gauge theories. In this subsection, we shall follow mainly reference [35].

Before tackling the nonabelian case, let me first remind you of Kubo's formula within quantum electrodynamics.

The behavior of electromagnetic fields in a plasma of charged particles is described by the polarization tensor $\Pi^{\mu\nu}(x,y)$ which is the two-point current correlation function

$$\Pi^{\mu\nu}(x,y) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \Pi^{\mu\nu}(k) = -i\langle j^\mu(x) j^\nu(y)\rangle\,. \tag{144}$$

Perturbatively, this is a one-charged-particle-loop diagram with two external photon lines.

The real part of this tensor describes phenomena such as Debye screening and propagation of plasma waves; the imaginary part describes the damping of fields in the plasma (Landau damping). If one integrates out the charged fields in a functional integral for the theory, the polarization tensor naturally emerges as the thermal average of the time-ordered product of two currents. However, there are situations where the response of the plasma to the electromagnetic field is described as the average of the retarded commutator of currents. To see this in the case of QED with fermion current $J^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x)$ and interaction Lagrangean $\mathcal{L}_{\rm int} = -J^\mu A_\mu$, one calculates J^μ in an expansion in small gauge fields A_ν. The equation of motion for this theory is

$$\partial_\nu F^{\nu\mu}(x) = J^\mu(x). \tag{145}$$

$J^\mu(x)$ is related to the scattering operator $S[A] = T\exp[-i\int d^4x\, A_\mu(x)j^\mu(x)]$ (here $j^\mu(x) \equiv e\bar{\psi}_{\rm I}(x)\gamma^\mu\psi_{\rm I}(x)$ and the subscript I denotes the interaction picture) in the following way:

$$J^\mu(x) = iS^{-1}\frac{\delta S[A]}{\delta A_\mu(x)}. \tag{146}$$

The rhs of (146) is ready for an epansion in A. The result up to linear order is

$$J^\mu(x) = j^\mu(x) - i\int d^4y\theta(x^0-y^0)[j^\mu(x), j^\nu(x)]A_\nu(y) + \mathcal{O}(A^2). \tag{147}$$

Using this in (145) and taking the thermal average with the unperturbed density matrix $e^{-H_0/T}$, one arrives at the *Kubo formula*

$$\partial_\nu F^{\nu\mu}(x) = \int d^4y\, \Pi_{\rm R}^{\mu\nu}(x,y)A_\nu(y) \tag{148}$$

where $\Pi_{\rm R}^{\mu\nu}(x,y) = -i\theta(x^0-y^0)\langle[j^\mu(x), j^\nu(x)]\rangle$. Hence, the average of the retarded commutator is the appropriate function for the situation where we perturb the plasma by the field and ask how the field evolves.

Now, we discuss the relationship between the time-ordered and the retarded response functions, $\Pi_{\rm T}^{\mu\nu}$ and $\Pi_{\rm R}^{\mu\nu}$. The real part is the same for both of them

$$\mathrm{Re}\,\Pi_{\rm T}^{\mu\nu} = \mathrm{Re}\,\Pi_{\rm R}^{\mu\nu}. \tag{149}$$

This has long been familiar, see e.g. [36]. Here we shall concentrate on their imaginary parts. A large-T calculation to one-loop order yields for their imaginary parts

$$\mathrm{Im}\,\Pi_{\rm R}^{\mu\nu} \simeq \frac{k_0T^2}{12}P^{\mu\nu} \tag{150}$$

as well as

$$\mathrm{Im}\,\Pi_{\rm T}^{\mu\nu} = \mathrm{Im}\,\Pi_{\rm R}^{\mu\nu} + \frac{T^3}{6}P^{\mu\nu}, \tag{151}$$

where

$$P^{\mu\nu} = -k^2\theta(-k^2)\frac{6\pi}{|\vec{k}^2|}\left(\frac{1}{3}P_1^{\mu\nu} + \frac{1}{2}P_2^{\mu\nu}\right), \tag{152}$$

$$P_1^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \tag{153}$$

$$P_2^{\mu 0} = P_2^{0\nu} = 0,\ P_2^{ij} = \delta^{ij} - \frac{k^ik^j}{\vec{k}^2}. \tag{154}$$

This relationship between $\Pi_{\rm T}^{\mu\nu}$ and $\Pi_{\rm R}^{\mu\nu}$ can be understood in the following way. $\Pi_{\rm R}^{\mu\nu}$, being retarded, obeys a spectral representation of the form [37]

$$\Pi_{\rm R}^{\mu\nu} = \Pi_{\rm sub}^{\mu\nu} + \int dk_0'\frac{\rho^{\mu\nu}(k_0',\vec{k})}{k_0'-k_0-i\epsilon} \tag{155}$$

for some spectral function $\rho^{\mu\nu}(k)$. $\Pi_{\rm sub}^{\mu\nu}$ is a 'subtraction term' that can arise in the real part of $\Pi_{\rm R}^{\mu\nu}$. For $\Pi_{\rm T}^{\mu\nu}$, we then have [37]

$$\Pi_{\rm T}^{\mu\nu} = \Pi_{\rm sub}^{\mu\nu} + \int dk_0'\frac{\rho^{\mu\nu}(k_0',\vec{k})}{k_0'-k_0-i\epsilon} + 2\pi i f(k_0)\rho^{\mu\nu}(k_0,\vec{k}), \tag{156}$$

where

$$f(k_0) = \frac{1}{e^{k_0/T}-1}. \tag{157}$$

The bosonic distribution function $f(k_0)$ appears because $\Pi_{\rm T}^{\mu\nu}$ is ultimatively part of the bosonic (*i.e.* photon) propagator, and also because it is given by the thermal average of the T-product of two bosonic operators, viz. the two currents j_μ and j_ν. The essence of our results (150) and (151) is that the high-temperature spectral function is

$$\rho^{\mu\nu}(k) = \frac{k_0T^2}{12\pi}P^{\mu\nu} \tag{158}$$

and the difference in the high-temperature behavior between $\mathrm{Im}\,\Pi_{\rm R}^{\mu\nu}$ and $\mathrm{Im}\,\Pi_{\rm T}^{\mu\nu}(\mathcal{O}(T^2)$ vs. $\mathcal{O}(T^3))$ is attributed to the presence in the latter of $2\pi f(k_0)\rho^{\mu\nu}$, which according to (157) and (158) tends to $\frac{1}{6}T^3P^{\mu\nu}$.

Our result for the retarded function $\Pi_R^{\mu\nu}$ agrees with various previous calculations [36]. It is noteworthy, that these early calculations in the Soviet literature, based on the Boltzmann and Vlasov transport equations of kinetic theory, are here regained in quantum field theory at one-loop order.

Another correlation function that is frequently considered is the imaginary-time one. It too is given by a dispersive integral

$$\Pi_{\text{im-t}}^{\mu\nu} = \Pi_{\text{sub}}^{\mu\nu} + \int dk_0' \frac{\rho^{\mu\nu}(k_0', \vec{k})}{k_0' - \omega_n}, \; \omega_n = 2\pi i n T. \tag{159}$$

Because the external energy ω_n is temperature dependent in imaginary time, it makes sense to speak of high-temperature behavior only for the $n = 0$ mode, effectively reducing dimensionality to three, where the spectral function enforces an $\mathcal{O}(T^2)$ large-T behavior.

In QED, the one-loop calculations at finite temperature are useful since higher-order contributions are down by the coupling e. This is related to the following consideration of the effective action $\Gamma_{\text{high-}T}[A]$ that produces the Kubo formula as the corresponding equation of motion

$$\Gamma_{\text{high-}T}[A] = -\frac{1}{4}\int d^4x F^{\mu\nu}(x)F_{\mu\nu}(x) - \frac{1}{2}\int d^4x\, d^4y A_\mu(x)\Pi_R^{\mu\nu}(x,y)A_\nu(y). \tag{160}$$

The important fact is that the expression above is *gauge invariant*. The polarization tensor of QCD ($= SU(N)$ gauge theory with N_F flavors of fermions in the fundamental representation) at finite temperature is related to the one of QED simply by factors

$$\Pi_{ab}^{\mu\nu} = (N + \frac{1}{2}N_F)\delta_{ab}\,\Pi_{\text{QED}}^{\mu\nu}. \tag{161}$$

But, in the nonabelian case, Eq. (160) is no longer gauge invariant. The reason is, as one might already expect from the foregoing discussion of hard thermal loops, that higher-order contributions in the coupling must be taken into account. Hence, the task is to find the correct effective action $\Gamma_{\text{high-}T}[A]$ of QCD giving us the generalized Kubo formula.

2.5 Analysis of Hard Thermal Loops in QCD

One way of analyzing hard thermal loop contributions is the calculation of the corresponding Feynman diagrams (e.g. [28]). Another one is to use a high-temperature action and to require gauge invariance for it. The gauge invariance condition relates

the high-T QCD to a three-dimensional Yang-Mills theory with topological mass term, a theory worked out about ten years ago [38]. The high-temperature action is deduced in different publications (e.g. [34], [28]):

$$\Gamma_{\text{high-}T}[A] = \frac{1}{2}\int d^4x\, \text{tr}F^{\mu\nu}F_{\mu\nu} + \left(N + \frac{1}{2}N_F\right)\frac{T^2}{12\pi}\Gamma[A]. \tag{162}$$

In the following, light-like vectors are used

$$Q_\pm^\mu := \frac{1}{\sqrt{2}}(1, \pm\hat{q}), \quad \hat{q}^2 = 1 \tag{163}$$

$$A_\pm := Q_\pm^\mu A_\mu = \left(A_0 \pm \hat{q}^i A_i\right) \tag{164}$$

and an angular integration $\int d\Omega_{\hat{q}}$ over the directions of the unit vector $\hat{q}$. The temperature-independent term $\Gamma[A]$ has the form:

$$\Gamma[A] = 2\pi\int d^4x\, A_0^a A_0^a + \int d\Omega_{\hat{q}}\, W(A_+). \tag{165}$$

Gauge invariance for the action requires gauge invariance for $\Gamma[A]$:

$$\delta\Gamma[A] = \delta\left[2\pi\int d^4x\, A_0^a A_0^a\right] + \delta\left[\int d\Omega_{\hat{q}}\, W(A_+)\right] = 0 \tag{166}$$

$$\rightarrow (\partial_+ + \partial_-)\, A_+^a + \partial_+\frac{\delta W(A)}{\delta A_+^a} + f^{abc}A_+^b\frac{\delta W}{\delta A_+^c} = 0 \tag{167}$$

$$\begin{aligned} \Rightarrow \quad & \partial_+\frac{\delta}{\delta A_+^a}\left[W(A_+) + \frac{1}{2}\int d^4x\, A_+^{a'}A_+^{a'}\right] + \\ & + f^{abc}A_+^b\frac{\delta}{\delta A_+^c}\left[W(A_+) + \frac{1}{2}\int d^4x\, A_+^{a'}A_+^{a'}\right] - (-)\partial_- A_+^a = 0. \end{aligned} \tag{168}$$

Calling the term in the square bracket S, the gauge invariance condition gets the form

$$\partial_1\frac{\delta}{\delta A_1^a}S - \partial_2 A_1^a + f^{abc}A_1^b\frac{\delta}{\delta A_1^c}S = 0. \tag{169}$$

S is an integrated functional of the fields, so we set by analogy with Hamilton-Jacobi theory

$$A_2^a = \frac{\delta}{\delta A_1^a} S\,, \tag{170}$$

what gives for gauge invariance condition

$$\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c = 0. \tag{171}$$

With this constraint for the A–fields we can relate the high-temperature QCD to a topic of topological field theory the Chern Simons theory. In the eighties [38] the CS–Lagrangean

$$\Omega(A) = -\frac{1}{8\pi^2}\epsilon^{ijk}\mathrm{tr}\left(\partial_i A_j A_k + \frac{2}{3} A_i A_j A_k\right) \tag{172}$$

was used as topological mass term for three-dimensional Yang-Mills theories

$$\mathcal{L} = \frac{1}{2}\mathrm{tr} F^{\mu\nu} F_{\mu\nu} + 8\pi^2 m \Omega(A) \tag{173}$$

with equations of motion

$$\mathcal{D}_\mu F^{\mu\nu} + \frac{m}{2}\epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0 \tag{174}$$

where m is a gauge invariant mass which, in nonabelian theories, is quantized so that

$$e^{i8\pi^2 m} = 1 \Longrightarrow m = n/4\pi. \tag{175}$$

So we have a topological massive gauge theory with multi-valued action. But with the quantization condition the phase exponential of the action remains gauge invariant.

In order to see that this theory resembles much of the hard thermal loop analysis in QCD, we must relate $\Gamma[A]$ to the CS functional $\Omega[A]$. This can be done by the constraint (171), which arises also in CS theory.

2.6 Pure Chern-Simons Theory

$$\mathcal{L}_{\mathrm{CS}} = 8\pi^2 k \Omega[A] \tag{176}$$

because $\mathcal{L}$ is a volume form, the corresponding action (here integration in 2+1 dimensions) is independent of a metric. So we are dealing with a topological field theory, a framework which is used in mathematics to investigate the topology of low-dimensional manifolds [41].

The equations of motion are

$$\epsilon^{\alpha\mu\nu} F_{\mu\nu} = 0. \tag{177}$$

For doing canonical quantization we choose the condition $A_0 = 0$, then we have

$$\mathcal{L}_{CS} = \frac{k}{2}\epsilon^{ij} \dot{A}_i^a A_j^a. \tag{178}$$

Now we can choose phase space variables by a method for first order Lagrangeans of the form $\mathcal{L} = \omega^{ij}\dot{q}_i p_j$, so we have $A_1^a = q^a$, $A_2^a = p^a$. By Legendre transformation we get a vanishing Hamiltonian

$$H = 0, \tag{179}$$

which causes trivial equations of motion

$$\dot{A}_i = 0. \tag{180}$$

The 0-component equation of motion does not involve a time derivative. It is merely the Gauß's law constraint

$$\epsilon^{ij} F_{ij} = 0 \tag{181}$$

giving the generator for gauge transformations

$$G^a(\mathbf{x}) = -\frac{k}{2}\epsilon^{ij} F_{ij}^a(\mathbf{x}). \tag{182}$$

We now implement the constraint as a relation for the quantum states, what means **quantization before solving of constraints** [39].

The antisymmetric matrix in the Lagrangean determines the symplectic structure of our theory and establishes the phase space commutation relations

$$\left[A_i^a(\mathbf{x})\,,\, A_j^b(\mathbf{y})\right] = \frac{i}{k}\epsilon_{ij}\delta^{ab}\delta(\mathbf{x}-\mathbf{y}). \tag{183}$$

Since $H = 0$ all the dynamics is in the constraint

$$G^a(\mathbf{x})|\Psi\rangle = 0, \tag{184}$$

where $|\Psi\rangle$ are the physical states and

$$i\left[G^a(\mathbf{x})\,,\, G^b(\mathbf{y})\right] = f^{abc} G^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \tag{185}$$

the algebra of the constraints, which follows the Lie algebra of the gauge transfomation group.

For the realization of the quantum theory we have to get an irreducible representation of an algebra of observables, which consists of functions on the phase space. The irreducibility is obtained by choosing a polarization, what means stating what is p and what is q and what is the argument of the wave functions (there is not a single unique polarization, but quantization should be independent of which one is taken). Here we choose Cartesian polarization

$$A_1^a \equiv \phi^a \ , \ A_2^a = \frac{1}{ik}\frac{\delta}{\delta\phi^a} \tag{186}$$

$$|\Psi\rangle \leftrightarrow \Psi(\phi) \tag{187}$$

the phase space variables are represented as

$$A_1^a(\mathrm{x})|\Psi\rangle \leftrightarrow \phi^a(\mathrm{x})\Psi(\phi) \tag{188}$$

$$A_2^a(\mathrm{x})|\Psi\rangle \leftrightarrow \frac{1}{ik}\frac{\delta}{\delta\phi^a(\mathrm{x})}\Psi(\phi). \tag{189}$$

The constraint implies an equation for the physical states $\Psi(\phi)$

$$\begin{aligned} &G^a(\mathrm{x})|\Psi\rangle = 0 \Longleftrightarrow \\ &\left(\partial_1\frac{\delta}{\delta\phi^a(\mathrm{x})} + f^{abc}\phi^b(\mathrm{x})\frac{\delta}{\delta\phi_c(\mathrm{x})} - ik\partial_2\phi^a(\mathrm{x})\right)\Psi(\phi) = 0. \end{aligned} \tag{190}$$

For finding a solution of this equation we use the WKB-method. Our Lagrangean has the form

$$\begin{aligned} \mathcal{L}_{CS} &= \frac{k}{2}\epsilon^{ij}\dot{A}_i^a A_j^a - H_{CS} \\ &= \frac{k}{2}\dot{A}_1^a A_2^a - \frac{k}{2}\dot{A}_2^a A_1^a \end{aligned} \tag{191}$$

$$\Rightarrow kA_2^a\dot{A}_1^a \equiv p\dot{q}. \tag{192}$$

The WKB-eikonal of ordinary quantum mechanics is defined as

$$\psi(q) = e^{i\int^q dq' p(q')}. \tag{193}$$

Analogously, we state for the CS-WKB-eikonal

$$\Psi(\phi) = e^{i\int^{\phi^a} \mathcal{D}A_1 kA_2^a(A_1)}. \tag{194}$$

To get $A_2^a(A_1)$, we use the zero curvature condition for the A-fields

$$\partial_1 A_2^a + f^{abc}A_1^b A_2^c = \partial_2 A_1^a. \tag{195}$$

Now we take a solution for the Gauß's law constraint (190) of the form

$$\Psi(\phi) = e^{iW(\phi)}. \tag{196}$$

In the polarization chosen above we get

$$\left(\partial_1\frac{\delta}{\delta\phi^a(\mathrm{x})} + f^{abc}\phi^b(\mathrm{x})\frac{\delta}{\delta\phi_c(\mathrm{x})} - ik\partial_2\phi^a(\mathrm{x})\right)\Psi(\phi) = 0 \tag{197}$$

$$\Rightarrow \partial_1\frac{\delta W(\phi)}{\delta\phi^a(\mathrm{x})} + f^{abc}\phi^b(\mathrm{x})\frac{\delta W(\phi)}{\delta\phi^c(\mathrm{x})} = k\partial_2\phi^a(\mathrm{x}). \tag{198}$$

Comparing this with (195) gives

$$\frac{\delta W(\phi)}{\delta\phi^a(\mathrm{x})} = kA_2^a(\mathrm{x}) \tag{199}$$

By identifying S from (169) with W and ϕ with A_1 and comparing the constraints (171) of the hard thermal loop analysis and of the CS-theory (195) we make the conclusion that the hard thermal loop generating function is given by the WKB-eikonal of CS-theory [40].

What still remains to do, is to construct the phase W that means solving the 'quantum' constraint. Questions of representation theory of symmetries on quantum states arise here. These are nontrivial and represent a source of anomalies. For the solution of the constraint we use a two-step strategy

(i) determine $e^{i\int_{\mathrm{x}}\lambda^a(\mathrm{x})G^a(\mathrm{x})}\Psi(\phi)$

(ii) demand $e^{i\int_{\mathrm{x}}\lambda^a(\mathrm{x})G^a(\mathrm{x})}\Psi(\phi) = \Psi(\phi) \Longleftarrow$ Gauß's law.

Whereas the Gauß's law represents the infinitesimal action of the Lie algebra on the states, (i) and (ii) is the action of the Lie group (λ^a are the gauge parameters).

For the exponent we get

$$\int_{\mathbf{x}} \lambda^a G^a = i \int_{\mathbf{x}} \lambda^a \left(\partial_1 \frac{\delta}{\delta \phi^a} + f^{abc} \phi^b \frac{\delta}{\delta \phi^c} \right) - k \int_{\mathbf{x}} \phi^a \partial_2 \lambda^a. \tag{200}$$

We define

$$G \equiv G_\phi + 2k \int_{\mathbf{x}} \mathrm{tr} \phi \partial_2 \lambda, \tag{201}$$

where G_ϕ should only transform the argument

$$e^{iG_\phi} \Psi(\phi) = \Psi(\phi^g) \tag{202}$$

with $g = e^\lambda \in$ gauge group and

$$\phi^g \equiv g^{-1} \phi g + g^{-1} \partial_1 g. \tag{203}$$

By gauge transformation the wave functional picks up a phase

$$\begin{aligned} \Rightarrow e^{iG} \Psi(\phi) &= e^{iG} e^{-iG_\phi} \Psi(\phi^g) \\ e^{iG} e^{-iG_\phi} &= e^{-2\pi i \alpha_1(\phi;g)}. \end{aligned} \tag{204}$$

for α_1 one gets [39]

$$\alpha_1(\phi; g) = -\frac{k}{2\pi} \int_{\mathbf{x}} \mathrm{tr} \left(2\phi \partial_2 g g^{-1} + g^{-1} \partial_1 g g^{-1} \partial_2 g \right) + 4\pi k \int_{\mathbf{x}} \omega^0(g). \tag{205}$$

The ω arising in α_1 is a total derivative and has the form

$$\partial^\mu \omega_\mu := \omega(g) := \frac{1}{24\pi^2} \epsilon^{\alpha\beta\gamma} \mathrm{tr} \left(g^{-1} \partial_\alpha g g^{-1} \partial_\beta g g^{-1} \partial_\gamma g \right) \tag{206}$$

the ω-term in α_1 represents the winding number of the gauge transformation g, so $\int_{\mathbf{x}} \omega^0(g)$ is multi-valued, but this is innocuous when CS-quantization condition

$$4\pi k = \text{integer} \tag{207}$$

is fulfilled.

Conclusion:

From the quantum mechanical transformation law

$$\begin{aligned} e^{iG} \Psi(\phi) &= e^{-2\pi i \alpha_1(\phi;g)} \Psi(\phi^g) \\ &= \Psi(\phi) \Longleftarrow \text{Gauß's law} && (208) \\ \Psi(\phi^g) &= e^{2\pi i \alpha_1(\phi;g)} \Psi(\phi) && (209) \\ |\Psi(\phi^g)|^2 &= |\Psi(\phi)|^2 && (210) \end{aligned}$$

follows that α_1 fulfills the cocycle condition

$$\alpha_1(\phi; g) = \alpha_1(\phi; g\tilde{g}) - \alpha_1(\phi^g; \tilde{g}) \tag{211}$$

and so is a 1-cocycle [42]. Such objects arise in quantum mechanics, if a symmetry transformation is represented not only by shifting the argument of the wave functions, but also giving them a phase (e.g. quantum mechanical representation of Galileo boosts). The response of the action to this implementation of gauge symmetry is a change by a total derivative

$$L(A^g) - L(A) = \frac{d}{dt} 2\pi \alpha_1, \tag{212}$$

what indicates a residual symmetry of the theory.

Solution:

Explicit construction of states obeying (209). To this end, we write

$$\Psi(\phi) = e^{2\pi i \alpha_0(\phi)} \psi(\phi) \tag{213}$$

and seek a quantity $\alpha_0(\phi)$ called a 0-cochain that satisfies

$$\alpha_0(\phi^g) - \alpha_0(\phi) = \alpha_1(\phi; g). \tag{214}$$

Then (213) solves (209) with gauge invariant $\psi(\phi)$

$$\psi(\phi^g) = \psi(\phi). \tag{215}$$

If Eq. (214) holds the 1-cocycle α_1 is trivial–it is a coboundary. It is possible [39], to construct such an α_0 which trivializes α_1

$$\alpha_0(\phi) = 4\pi k \int_{\mathbf{x}} \omega^0(h) - \frac{k}{2\pi} \int_{\mathbf{x}} \mathrm{tr}(\phi h^{-1} \partial_2 h), \tag{216}$$

where h is defined by

$$\phi \equiv h^{-1}\partial_1 h. \tag{217}$$

The wave functional is single-valued provided $4\pi k$ = integer.

The Hilbert space is one-dimensional when no gauge invariant functionals of ϕ can be constructed (e.g. physical plane). For that the explicit physical states are given by

$$\Psi(\phi) = N e^{2\pi i \alpha_0(\phi)} \tag{218}$$

$$\text{with } A_i\Psi(\phi) = h^{-1}\partial_i h \Psi(\phi) \ . \tag{219}$$

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