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On the consistency of the Nonsymmetric Gravitational Theory

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On the consistency of the Nonsymmetric Gravitational Theory. [1]

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Abstract

It has recently been claimed that the Nonsymmetric Gravitational Theory (NGT) is unphysical due to radiative instability. This claim that NGT predicts an infinite flux of negative energy from gravitational waves is here proven to be false by demonstrating that the flux of gravitational radiation from an isolated source in NGT is positive definite. The reasons why erroneous results were arrived at are also discussed.

1 Introduction

Recently Damour, Deser and McCarthy [1] claimed that the Nonsymmetric Gravitational Theory (NGT) [2] is unphysical due to radiative instability. They claim that NGT necessarily possesses curvature coupled ghost modes and bad asymptotic behaviour. To be considered viable any theory of gravitation must satisfy two physical requirements:

1. Correct Newtonian limit;
2. No negative gravitational energy modes (no ghost poles).

The former has not been disputed, but the latter was claimed by DDM to be violated even in the case without sources. I intend to show here that their assertion is incorrect due to not taking into

[1] This presentation draws freely not only from the work done in collaboration with N. J. Cornish and J. W. Moffat, but also from their independent research.

account physical boundary conditions and the correct asymptotic flatness limits.

In the work done since the initial DDM's claim, it has been shown that in the radiative solutions of NGT the leading order contribution to the flux of gravitational radiation is the same as in General Relativity (GR) [3]. The same result was shown to follow from the De Witt style, first order expansion about a curved GR background of the vacuum field equations for an isolated, axisymmetric, time dependent radiating NGT source [4]. (Notably, the same method was used by DDM in their claim of proving the contrary.) Also, an NGT generalization of the GR work by Bondi, van der Burg & Metzner [5] was obtained, giving an *exact* axisymmetric, time dependent radiative solution. By using the same method, a completely analogous result was arrived at — NGT has good radiation properties, just like GR [6]. Moreover, an analysis of the properties of gravitational radiation for the NGT field equations with sources now exists [7], extending beyond the vacuum case the assertion that the flux of gravitational waves, in NGT, including the possible contribution from the antisymmetric fields, is positive definite for an arbitrary source.

However, the analysis presented here will be limited to the vacuum case. That was the case for which the initial claim of untenability was made and I will keep my response within its bounds.

2 The theory

Leaving aside a rigorous introduction to NGT that can be found elsewhere (e.g. [2]), let us recall that NGT is based on a general (not necessarily symmetric) metric and an affine connection:

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]}, \tag{1}$$

$$W^{\lambda}_{\alpha\beta} = W^{\lambda}_{(\alpha\beta)} + W^{\lambda}_{[\alpha\beta]}. \tag{2}$$

(Throughout this presentation parentheses and square brackets enclosing indices stand for symmetrization and antisymmetrization, respectively.)

The NGT Lagrangian without sources takes the form:

$$\begin{aligned} \mathcal{L}_{NGT} &= \sqrt{-g}g^{\mu\nu}R_{\mu\nu}(W) \\ &= \sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma) + \frac{2}{3}\left(\sqrt{-g}g^{[\nu\mu]}\right)_{,\nu}W_{\mu}, \end{aligned} \tag{3}$$

with

$$W^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \frac{2}{3}\delta^{\lambda}_{\mu}W_{\nu},$$

$$
\begin{aligned}
W_\beta &= W^\lambda_{[\beta\lambda]} = \frac{1}{2}\left(W^\lambda_{\beta\lambda} - W^\lambda_{\lambda\beta}\right) \\
\Gamma_\beta &= \Gamma^\lambda_{[\beta\lambda]} = 0,
\end{aligned}
$$

and g the determinant of $g_{\mu\nu}$.

The NGT Ricci tensor is defined as:

$$
R_{\mu\nu}(W) = W^\beta_{\mu\nu,\beta} - \frac{1}{2}(W^\beta_{\mu\beta,\nu} + W^\beta_{\nu\beta,\mu}) - W^\beta_{\alpha\nu}W^\alpha_{\mu\beta} + W^\beta_{\alpha\beta}W^\alpha_{\mu\nu}, \tag{4}
$$

and

$$
R_{\mu\nu}(\Gamma) = \Gamma^\beta_{\mu\nu,\beta} - \frac{1}{2}(\Gamma^\beta_{(\mu\beta),\nu} + \Gamma^\beta_{(\nu\beta),\mu}) - \Gamma^\beta_{\alpha\nu}\Gamma^\alpha_{\mu\beta} + \Gamma^\beta_{(\alpha\beta)}\Gamma^\alpha_{\mu\nu}. \tag{5}
$$

Finally, the NGT vacuum field equations can be expressed as:

$$
\begin{aligned}
R_{(\mu\nu)}(\Gamma) &= 0, && \text{(6a)} \\
R_{[\mu\nu]}(\Gamma) &= \frac{2}{3}W_{[\nu,\mu]}, && \text{(6b)} \\
(\sqrt{-g}g^{[\mu\nu]})_{,\nu} &= 0, && \text{(6c)} \\
g_{\mu\nu,\lambda} - g_{\alpha\nu}\Gamma^\alpha_{\mu\lambda} - g_{\mu\alpha}\Gamma^\alpha_{\lambda\nu} &= 0. && \text{(6d)}
\end{aligned}
$$

These field equations represent 12 independent equations for the 12 independent field variables $g_{\mu\nu}$ (there exist four arbitrary coordinate transformations: $dx'^\mu = (\partial x'^\mu/\partial x^\alpha)dx^\alpha$, which can be used to remove 4 of the 16 $g_{\mu\nu}$'s).

Equation (6b) can be decomposed into the two sets of equations:

$$
R_{\{[\mu\nu],\sigma\}}(\Gamma) \equiv R_{[\mu\nu]}(\Gamma)_{,\sigma} + R_{[\nu\sigma]}(\Gamma)_{,\mu} + R_{[\sigma\mu]}(\Gamma)_{,\nu} = 0, \tag{7}
$$

and

$$
R_{[\mu\nu]}(\Gamma)^{;\nu} = \frac{2}{3}W_{[\nu,\mu]}{}^{;\nu}, \tag{8}
$$

where ; denotes covariant differentiation with respect to the connection $\Gamma^\lambda_{\mu\nu}$. Equations (6c) and (7) are constrained, in turn, by the identities

$$
\begin{aligned}
\left(\sqrt{-g}g^{[\mu\nu]}\right)_{,\nu,\mu} &= 0, && (9) \\
\epsilon^{\mu\nu\sigma\rho}R_{\{[\mu\nu],\sigma\}}(\Gamma)_{,\rho} &= 0. && (10)
\end{aligned}
$$

The field equations are further constrained by the four Bianchi identities,

$$
[\sqrt{-g}g^{\alpha\nu}G_{\rho\nu}(\Gamma) + \sqrt{-g}g^{\nu\alpha}G_{\nu\rho}(\Gamma)]_{,\alpha} + [\sqrt{-g}g^{\mu\nu}]_{,\rho}G_{\mu\nu}(\Gamma) = 0, \tag{11}
$$

where $G_{\mu\nu}(\Gamma) = R_{\mu\nu}(\Gamma) - 1/2 g_{\mu\nu} R(\Gamma)$.

If we employ the compatibility condition (6d) to eliminate Γ in favour of $g_{\mu\nu}$, equations (6a), (6c) and (7) represent 18 equations for $g_{\mu\nu}$. Taking into account the six identities (9), (10) and (11), this set of equations provides 12 independent field equations for the 12 independent field variables, $g_{\mu\nu}$. At no stage have we had to refer to the auxiliary condition (8). The vector W_μ corresponds to a Lagrange multiplier and as such *does not* describe propagating dynamical degrees of freedom.

3 The claim and the "proof"

DDM proceed with the de Witt type analysis in the weak field limit. Treating NGT as a theory with two independent coupling constants (an Einsteinian one and a purely NGT one), they consider an expansion in both. (In particular, if all orders of the expansion in $g_{GR(\mu\nu)}$ are kept, the theory has to reduce to GR in the zeroth NGT order.) If we take the weak NGT field approximation:

$$g_{\mu\nu} = g_{GR(\mu\nu)} + \epsilon h_{\mu\nu}, \tag{12}$$

then, to the first order in $h_{[\mu\nu]}$ and all orders in $g_{GR(\mu\nu)}$, one gets from (6):

$$\bar{R}_{\mu\nu}(g_{GR}) = 0, \tag{13a}$$

$$\bar{D}^\nu h_{[\mu\nu]} = 0, \tag{13b}$$

$$\bar{D}^\alpha F_{\mu\nu\alpha} - 4\bar{R}^\alpha{}_\mu{}^\beta{}_\nu(g_{GR}) h_{[\alpha\beta]} = \frac{4}{3} W_{[\nu,\mu]}, \tag{13c}$$

with $F_{\mu\nu\alpha}$ being the cyclic curl of $h_{[\mu\nu]}$ and $\bar{D}^\alpha$ the background covariant derivative.

As was the case for equations (6), the above set of equations can be solved for $h_{[\mu\nu]}$ without specifying the Lagrange multiplier W_μ. This, in fact, is the way to solve a set of field equations that includes an auxiliary field (which has no physical boundary conditions of its own): to eliminate that field from the equations.

However, in the gauge $\partial^\alpha W_\alpha = 0$, DDM proceed to take the divergence of (13c) which gives the wave equation:

$$\bar{D}^\mu \bar{D}_\mu W_\nu = -3\bar{D}^\mu(\bar{R}^\alpha{}_\mu{}^\beta{}_\nu(g_{GR}) h_{[\alpha\beta]}). \tag{14}$$

They argue that this is an inhomogeneous wave equation for W_μ, so W_μ has $1/r$ fall off in the wave zone. They then go on to argue that inserting this information back into (13c) drives $h_{[\mu\nu]}$ to have bad asymptotic behaviour.

DDM claim, in addition, that NGT has a negative energy flux of gravitational radiation in the wave zone. The rate of energy loss due to gravitational radiation is given by

$$\frac{dE}{dt} = -R^2 \oint t^{0i} \hat{n}_i d\Omega, \tag{15}$$

where the integration is over a sphere of radius R in the wave zone, $\hat{n}_i$ is an outward pointing unit vector, and $t^{\mu\nu}$ is the stress-energy tensor given in [1] (with some notational differences):

$$\begin{aligned} t^{\mu\nu} = & \left(\frac{1}{2}F^{\mu\alpha\beta}F^{\nu}{}_{\alpha\beta} - \frac{1}{12}g_{GR}{}^{\mu\nu}F^2\right) \\ + & \left(\frac{4}{3}h^{[\mu\alpha]}f^{\nu}{}_{\alpha} + \frac{4}{3}h^{[\nu\alpha]}f^{\mu}{}_{\alpha} - \frac{2}{3}g_{GR}{}^{\mu\nu}h_{[\alpha\beta]}f^{\alpha\beta}\right) \\ + & \left(3\bar{R}^{(\mu}{}_{\gamma\alpha\beta}h^{[\nu)\alpha]}h^{[\gamma\beta]} - g_{GR}{}^{\mu\nu}\bar{R}_{\gamma\alpha\delta\beta}h^{[\gamma\delta]}h^{[\alpha\beta]}\right) \\ - & \bar{D}_\beta\bar{D}_\alpha\left(h^{[\alpha(\mu]}h^{[\nu)\beta]}\right), \end{aligned} \tag{16}$$

where $f_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu$.

Now DDM argue that only the first term in the above expression leads to positive energy. They notice that the third term vanishes at infinity like $1/r^3$, while the fourth (being a total divergence) cannot contribute to the integral (15) and has no bearing on the energy-momentum flux. However, they maintain that the positivity of the second term cannot be established and state: "thus the proof of positivity of energy for normal fields fails here". This (as yet mild) statement gets somewhat strengthened in the conclusions of [1], where it reads that NGT *necessarily* contains negative radiative energy modes and is physically untenable.

Let us stress here that the failure to prove the positivity of gravitational energy, and DDM *did fail* to do this, is by no means equivalent to proving that the energy is negative. DDM's arguments cannot be used to prove that $h_{[\mu\nu]}$ *always* has bad asymptotic behaviour and that this *guarantees* negative gravitational energy modes. (A detailed discussion of the inconsistencies of DDM's analysis can be found in [3].)

Let us also note that if the analysis of the problem is conducted in a physically consistent way, taking into account the boundary conditions, DDM's way of arguing *can* be used to prove the positivity of gravitational energy in NGT.

4 The counter-claim and the proof

Any physical theory can be seen as consisting of its mathematical structure (i.e. symmetries and differential equations) and its boundary conditions, which establish its physical content. The former

allow solutions with both good and bad asymptotic behaviour. It is the role of the latter to decide which of those solutions are physically acceptable.

A trivial (but educational) example is the dissipation equation in the heat conduction problem that, on its own, allows for both damped and growing solutions. Here, as always, it is the boundary conditions that establish which solutions belong in the physical theory.

For the isolated radiating body the physical boundary condition is

$$h_{[\mu\nu]}h^{[\mu\nu]} \longrightarrow 0 \tag{17}$$

at future null infinity. Imposing the boundary condition on the future null infinity (the past null infinity) corresponds to ensuring that the field contains only outgoing radiation (only incoming radiation). A physically viable model could contain either of the two, but *not both* at the same time.

The discussion of gravitational radiation only makes sense for such spacetimes [5].

It is clear from the previous work [4, 6] that imposing the boundary condition (17) on the axisymmetric radiating system leads to solutions with good asymptotic behaviour. As we will see shortly, the same condition of asymptotic flatness ensures that DDM's analysis necessarily leads to the conclusion that the flux of gravitational radiation in NGT is finite and positive.

Let us prove our counter-claim in the weak GR field approximation, since at future null infinity the spacetime has the same characteristics as Minkowskian spacetime.

As follows from the discussion in the preceding section, the question of the finiteness and positivity of the energy flux critically hinges on the finiteness and positivity of the second term on the right–hand side of the equation (16).

In other words: if $W_{[\nu,\mu]}$ $(f_{\mu\nu})$ does *not* contribute in the wave zone, then the flux is positive definite.

In a weak field approximation obtained from expanding $g_{\mu\nu}$ about the Minkowski spacetime metric, $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{18}$$

where $\epsilon \ll 1$, the field equations take the form to lowest order:

$$\Box h_{(\mu\nu)} - h_{(\nu\sigma),\mu}{}^{,\sigma} - h_{(\mu\sigma),\nu}{}^{,\sigma} + h_{,\mu,\nu} = 0, \tag{19}$$

$$h_{[\mu\beta]}{}^{,\beta} = 0, \tag{20}$$

$$\Box h_{[\mu\nu]} = \frac{4}{3}W_{[\nu,\mu]}, \tag{21}$$

where $\Box = \partial^\mu \partial_\mu$ and $h = \eta^{\alpha\beta} h_{\alpha\beta}$.

Let us now define the longitudinal and transverse projection operators:

$$P^L_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box}, \quad P^T_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}, \tag{22}$$

and make the general decomposition:

$$h_{[\mu\nu]} = \alpha_{[\mu,\nu]} + \epsilon_{\mu\nu\kappa\lambda} \beta^{\kappa,\lambda},$$

where $\alpha_\mu{}^{,\mu} = 0$. Now

$$h^{TT}_{[\mu\nu]} = P^{T\alpha}_\mu P^{T\beta}_\nu h_{[\alpha\beta]} = \epsilon_{\mu\nu\kappa\lambda} \beta^{\kappa,\lambda}, \tag{23}$$

$$h^{LL}_{[\mu\nu]} = P^{L\alpha}_\mu P^{L\beta}_\nu h_{[\alpha\beta]} = 0, \tag{24}$$

$$h^{LT}_{[\mu\nu]} = P^{L\alpha}_{[\mu} P^{T\beta}_{\nu]} h_{[\alpha\beta]} = \alpha_{[\mu,\nu]}. \tag{25}$$

Also, using the gauge condition $W_\alpha{}^{,\alpha} = 0$, one obtains

$$W^{TT}_{[\mu,\nu]} = W^{LL}_{[\mu,\nu]} = 0, \quad W^{LT}_{[\mu,\nu]} = W_{[\mu,\nu]}. \tag{26}$$

Thus, the field equations to linear order are:

$$h^{LT}_{[\mu\beta]}{}^{,\beta} = 0, \tag{27}$$

$$\Box h^{TT}_{[\mu\nu]} = 0, \tag{28}$$

$$\Box h^{LT}_{[\mu\nu]} = \frac{4}{3} W_{[\nu,\mu]}. \tag{29}$$

Equation (27) then tells us that $\Box \alpha_\mu = 0$, and inserting this into (29) we find

$$W_{[\mu,\nu]} = 0$$

to lowest order.

Thus, curl of W does *not* contribute in the wave zone if physical boundary conditions are applied. The flux of gravitational radiation in the wave zone is finite and positive. There are *no* negative gravitational radiation modes.

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