PRINT-93-0121

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An Obstruction to the Integrability of a Class of Non-Linear Wave Equations by 1-Stable Cartan Characteristics

Edward D Fackerell[††§]
David Hartley[†]
Robin W Tucker[‡]

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School of Physics and Materials
Lancaster University
Lancaster LA1 4YB
United Kingdom

Lancaster U.

May 1992

Abstract

We examine in detail the Cauchy problem for a class of non-linear hyperbolic equations in two independent variables. This class is motivated by the analysis of the dynamics of a line of non-linearly coupled particles by Fermi, Pasta and Ulam and extends the recent investigation of this problem by Gardner and Kamran. We find conditions for the existence of a 1-stable Cartan characteristic of a Pfaffian exterior differential system whose integral curves provide a solution to the Cauchy problem. The same obstruction to involution is exposed in Darboux's method of integration and the two approaches are compared. A class of particular solutions to the obstruction is constructed.

†† e-mail: fackerell_e@maths.su.oz.au

§ Permanent address: School of Mathematics and Statistics, University of Sydney, NSW, Australia

† e-mail: D.Hartley@uk.ac.lancaster

‡ e-mail: R.W.Tucker@uk.ac.lancaster

Introduction

In 1955 Fermi, Pasta and Ulam [1] investigated the dynamics in one dimension of a chain of non-linearly interacting particles [1] and were surprised at the non-ergodic behavior of the system. A continuum version of this model was considered by Kruskal who discovered that solutions of the partial differential equation developed a discontinuity in the first derivative of the longitudinal displacement after a finite time. This behavior was confirmed and clarified by Zabusky [2] and Lax [3] using classical techniques of analysis. The problem has recently been reconsidered using a modern approach by Gardner and Kamran [4]. The problem continues to be of interest for a number of reasons. Non-linear hyperbolic partial differential equations in two independent variables provide a viable means of testing new techniques of analytic integration. Such equations often exhibit interesting phenomena unique to non-linear systems, such as shock waves and solitons. Such behavior is of particular interest in physics where solitonic solutions are being considered as simplified models of elementary particles. The re-parameterisation invariant relativistic string and its generalisations has been extensively studied in the context of a unified model of the basic forces of nature and its non-linear modifications may be relevant for further developments in this area.

The analysis of the realistic behavior of a continuous material string is non-trivial [5]. To define the dynamical system one must specify the material stress tensor and a constitutive relation to relate the tension to the material strain. The traditional approach [6] succeeds in decoupling the longitudinal and transverse equations of motion by assuming small vibrations for a medium obeying a Hooke's law. For non-linear viscoelastic strings undergoing large amplitude vibrations such assumptions are unwarranted.

The Fermi-Pasta-Ulam longitudinal chain and its various continuum limits define a class of hyperbolic non-linear wave equations, characterised by a function κ, of the form

$$\partial_y^2 z - \kappa^2(\partial_x z)\partial_x^2 z = 0 \tag{1}$$

for the displacement z as a function of position x and time y.

It is the purpose of this note to treat the Cauchy problem for this equation. We begin with the method of generalised characteristics pioneered by Cartan and developed by Gardner [7]. In particular we expand on the recent work by Gardner and Kamran who analyse the Cauchy problem in terms of 1-stable vector fields on a locus of a bundle of 2-jets. We show that for most functions κ, this method meets an obstruction to the Cauchy evolution of initial data by a 1-stable vector field. The same problem is analysed using Darboux's method of integration and the same obstruction is encountered albeit in a somewhat different setting. We comment on the relative merits of these two approaches for integrating equation (1).

The Cauchy Problem

We recall briefly the formulation of the Cauchy problem for (1) in the language of exterior differential systems.

Define $\mathcal{S}$ to be the exterior ideal generated by the Pfaffian system

$$\begin{aligned} &dz - p\,dx - q\,dy \\ &dp - r\,dx - s\,dy \\ &dq - s\,dx - \kappa^2(p)\,r\,dy \end{aligned} \tag{2}$$

on the locus Σ_7 of $J^2(\{x,y\} \mapsto \{z\})$ with coordinates $\{x,y,z,p,q,r,s\}$. The solutions of (1) may be regarded as 2-dimensional integral manifolds of $\mathcal{S}$: immersions $C : U \subset \mathbf{R}\times\mathbf{R} \to \Sigma_7$ such that $C^*\mathcal{S} = 0$ and $C^*(dx \wedge dy) \neq 0$. Define Cauchy data for C to be the 1-dimensional integral manifold $C_0 : I \subset \mathbf{R} \to \Sigma_7$ of $\mathcal{S}$ such that $C(U) \subset C_0(I) \times \mathbf{R}$ and $C_0^*\mathcal{S} = 0$.

The Cauchy problem for (1) is then to find a 2-dimensional integral manifold

$$C : \{\sigma, \tau\} \mapsto \{x,y,z,p,q,r,s\} \tag{3}$$

of $\mathcal{S}$ such that

$$C(\sigma, 0) = C_0(\sigma) \qquad \forall \sigma \in I.$$

In some cases, such a map may be determined by the integral curves of a vector field X on Σ_7; each curve starting at $C_0(\sigma)$ at $\tau = 0$:

$$C(\sigma, \tau) = exp_{C_0(\sigma)}(\tau X) \tag{4}$$

If this holds, then the Cauchy problem is solved by the ODE system determining the integral curves of X. According to [7] the Cauchy problem for $\mathcal{S}$ with Cauchy data C_0 has this solution provided:

(a) X is an $l-$stable vector field

(b) X is transverse to the Cauchy data

(c) $C_0^*\mathcal{D}_X^l(\mathcal{S}) = 0$.

In this theorem the Pfaffian systems $\mathcal{D}_X^k(\mathcal{S}), \quad k \geq 0$ are defined recursively by

$$\mathcal{D}_X^0(\mathcal{S}) = \mathcal{S}, \qquad \mathcal{D}_X^k(\mathcal{S}) = \{\mathcal{D}_X^{k-1}(\mathcal{S}), \quad \mathcal{L}_X\mathcal{D}_X^{k-1}(\mathcal{S})\} \tag{5}$$

in terms of the Lie derivative. The vector field X is said to be $l-$stable for $\mathcal{S}$ if it lies in the annihilator $\mathcal{S}^{\perp}$ of $\mathcal{S}$, each system $\mathcal{D}_X^k(\mathcal{S})$ has constant rank and there exists an $l \geq 0$ such that

$$\mathcal{D}_X^l(\mathcal{S}) = \mathcal{D}_X^{l+1}(\mathcal{S}) \tag{6}$$

where

$$\mathcal{S}^{\perp} \equiv \{X | i_X \mathcal{S} = 0\}.$$

These conditions imply that X is a Cauchy vector field for $\mathcal{D}_X^l(\mathcal{S})$:

$$X \in Char\{\mathcal{D}_X^l(\mathcal{S})\} \equiv \{X | X \in \mathcal{D}_X^l(\mathcal{S})^{\perp},\ i_X d\mathcal{D}_X^l(\mathcal{S}) \subset \mathcal{D}_X^l(\mathcal{S})\} \tag{7}$$

1-Stability

In this section we exploit the above theorem to derive a quasilinear system of pde's that determine a solution to the Cauchy problem for (1). A basis for the space $\mathcal{S}^{\perp}$ is

$$\{X_1, X_2, X_3, X_4\} = \{\partial_x + p\partial_z + s\partial_q + r\partial_p, \partial_y + q\partial_z + r\kappa^2\partial_q + s\partial_p, \partial_r, \partial_s, \} \tag{8}$$

so we seek a Cauchy vector of the form

$$X = \sum_{i=1}^{4} h_i X_i$$

in terms of functions h_i on Σ_7. Then

$$\mathcal{D}_X^1(\mathcal{S}) = \mathcal{D}_X^0(\mathcal{S}) \cup d_1 \tag{9}$$

where

$$d_1 \equiv i_X d\mathcal{S} \quad mod \quad \mathcal{S}$$

$$= \{h_1\, dr + h_2\, ds - h_3\, dx - h_4\, dy,$$

$$h_1\, ds + h_2\, \kappa^2\, dr + 2h_2 r^2 \kappa \partial_p \kappa\, dx - h_4\, dx - 2\kappa\partial_p\kappa\, r^2 h_1\, dy - h_3 \kappa^2\, dy\}$$

Thus $\mathcal{D}_X^1(\mathcal{S})$ has maximal rank 5. One could proceed with a stability analysis of X keeping $\mathcal{D}_X^1(\mathcal{S})$ with maximal rank. However following Gardner a simplification is expected if we seek stability of a vector field X satisfying the property

$$\{X | X \in \mathcal{S}^{\perp},\ i_X\, d\mathcal{J} \in \mathcal{S}\} \tag{10}$$

for some subsystem $\mathcal{J}$ of $\mathcal{S}$. Gardner and Kamran [4] define a subsystem in terms of a particular 2-dimensional maximal isotropic subspace of a tensor on Σ_7, constructed from the elements in $\mathcal{S}$. Such a vector X then defines an irregular integral element [8] of the differential system obtained by closing $\mathcal{S}$. Thus we look for a drop in rank of $\mathcal{D}^1_X(\mathcal{S})$ from 5 to 4. This can be achieved by choosing:

$$\{h_1 = -\kappa, h_2 = 1, h_3 = -\lambda, h_4 = \kappa\lambda + 2\kappa r^2 \partial_p \kappa\} \tag{11}$$

where we have normalised X to agree with the $\mathcal{J}$-vector of [4]. The space d_1 is now 1 dimensional and not contained in $\mathcal{S}$ so X is not 0-stable. We next compute $\mathcal{D}^2_X(\mathcal{S})$. For an arbitrary function λ the space $d_2 = i_X d_1 \bmod \mathcal{D}^1_X(\mathcal{S})$ is 1 dimensional. However if, and only if, λ satisfies the quasilinear system:

$$\begin{gathered} 2\kappa(\partial_r\lambda + \kappa\partial_s\lambda) + 5\kappa r\partial_p\kappa - s\partial_p\kappa = 0 \\ 2\kappa(\kappa\partial_x\lambda + \partial_y\lambda) + 2\kappa(\kappa p + q)\partial_z\lambda + 2\kappa(\kappa r + s)(\partial_p\lambda + \kappa\partial_q\lambda) \\ +4\kappa^2\partial_p\kappa r^2\partial_s\lambda + (5\kappa\partial_p\kappa r - \partial_p\kappa s)\lambda + 4\kappa r^3(\kappa\partial_{pp}\kappa + \partial_p\kappa^2) = 0 \end{gathered} \tag{12}$$

then $d_2 = \{0\}$ and X is a 1-stable vector field.

If such a vector exists, its integral curves offer a solution to the Cauchy problem in terms of the Cauchy maps generated by the functions $\{\Phi(\sigma), \Psi(\sigma)\}$:

$$C_0 : \sigma \mapsto \{x = \sigma, y = 0, z = \Phi, p = \partial_\sigma\Phi, q = \Psi, r = \partial_{\sigma\sigma}\Phi, s = \partial_\sigma\Psi\}. \tag{13}$$

Condition (c) of the theorem in the last section then provides the initial conditions for $C_0^*\lambda$:

$$C_0^*\lambda = (C_0^*\kappa)\partial_{\sigma\sigma\sigma}\Phi - \partial_{\sigma\sigma}\Psi. \tag{14}$$

It is to the problem of the integrability of the system (12) that we now turn.

Integrability Condition

To investigate the integrability of the first order system (12), it is convenient to express it as an exterior system. Since there are two first-order equations in one unknown function of 7 independent variables, the system will be expressed as a locus Σ_{13} of $J^1(\mathbf{R}^7 \to \mathbf{R})$. It also makes the calculations more tractable if a coordinate transformation is performed. Let

$$\begin{aligned} u &= q - K(p) \\ v &= (s + \kappa r)/2 \\ w &= (s - \kappa r)/2 \end{aligned} \tag{15}$$

where $K(p)$ is such that $K'(p) = \kappa(p)$. After the transformation (15), the equations (12) can be expressed as an exterior system on the locus Σ_{13} of $J^1(\{x, y, z, p, u, v, w\} \mapsto \{\lambda\})$, by pulling back the contact system

$$\mathcal{Q} = \{\mathrm{d}\lambda - \lambda_x \mathrm{d}x - \lambda_y \mathrm{d}y - \lambda_z \mathrm{d}z - \lambda_p \mathrm{d}p - \lambda_u \mathrm{d}u - \lambda_v \mathrm{d}v - \lambda_w \mathrm{d}w\} \tag{16}$$

with the map $\gamma : \Sigma_{13} \to J^1(\mathbf{R}^7 \to \mathbf{R})$ defined by the equations

$$\begin{aligned} \lambda_v &= \partial_p \kappa(3w - 2v)/\kappa^2 \\ \lambda_w &= \big(2(\kappa\partial_{pp}\kappa - \partial_p\kappa^2)(v - w)^3 + \partial_p\kappa^2 v w^2 + \kappa^2\partial_p\kappa(2v - 3w)\lambda \\ &\quad + \kappa^3(\kappa\lambda_x + \lambda_y + (\kappa p + K + u)\lambda_z + 2v\lambda_p)\big)/\kappa^2\partial_p\kappa w(v - w). \end{aligned} \tag{17}$$

The integrability of equations (12) is then equivalent to the involution of the system $\gamma^*\mathcal{Q}$ with respect to $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}p \wedge \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w$. To check the involutivity, the exterior system can be prolonged to the Grassmann bundle $G(\Sigma_{13})$ of 7-planes over Σ_{13}. The set of 7-planes in $G(\Sigma_{13})$ which annul the 1- and 2-forms in $\{\gamma^*\mathcal{Q}, \gamma^*\mathrm{d}\mathcal{Q}\}$ forms a subvariety of $G(\Sigma_{13})$. If this subvariety does not project surjectively onto Σ_{13}, then there are further equations relating the variables in Σ_{13} implicit in the system (17). In that case, if there are m new equations, they must be appended to (17), defining a new locus Σ_{13-m} and a map $\gamma : \Sigma_{13-m} \to J^1(\mathbf{R}^7 \to \mathbf{R})$, and the process repeated.

Of course the manipulations involved in the procedure outlined above very rapidly exceed the scope of hand calculations. A computer algebra programme called AMEND has been developed at Lancaster [9] for the REDUCE computer algebra system to perform the prolongation calculation and check the surjectivity of the projection. If the projection is not surjective, then AMEND returns the new equations to be satisfied.

Using AMEND, the system $\gamma^*\mathcal{Q}$ is readily seen not to be involutive, but to require the additional equation

$$\begin{aligned}\lambda_p = \big((2\kappa\partial_{pp}\kappa - 5\partial_p\kappa^2)(-v^2w + 3vw^2 - 2w^3) + \kappa^2\partial_p\kappa w\lambda \\ - \kappa^3(\kappa\lambda_x + \lambda_y + (\kappa p + K + u)\lambda_z)\big)/2\kappa^3 w.\end{aligned} \tag{18}$$

The new system defined by equations (16), (17) and (18) can also be treated by AMEND, and gives an extra equation which can be solved for λ_x. Further repetitions of the process yield equations for λ_y and λ_z.

Incorporating these extra equations into the map γ gives a new exterior system $\gamma^*\mathcal{Q}$ on a locus Σ_9 of $J^1(\mathbf{R}^7 \to \mathbf{R})$. One further calculation with AMEND shows that the involutivity of this system also requires the vanishing of an extra equation which can be written

$$\frac{1}{E_2}\left[4E_4\frac{d^2E_4}{dp^2} - 4\left(\frac{dE_4}{dp}\right)^2 - \frac{\kappa'}{\kappa}E_4\frac{dE_4}{dp} - \frac{1}{\kappa}(50\kappa\kappa'' - 99\kappa'^2)E_4^2\right] = 0. \tag{19}$$

Here E_4 and E_2 are defined by

$$\begin{aligned}E_4 \equiv 16\kappa^4\kappa''\kappa^{iv} - 24\kappa'^2\kappa^3\kappa^{iv} - 16\kappa^4\kappa'''^2 + 32\kappa^3\kappa'\kappa''\kappa''' + 48\kappa^2\kappa'^3\kappa''' \\ -104\kappa^3\kappa''^3 + 372\kappa^2\kappa'^2\kappa''^2 - 630\kappa\kappa'^4\kappa'' + 315\kappa'^6\end{aligned} \tag{20}$$

$\kappa' \equiv \partial_p\kappa$ etc and

$$E_2 \equiv 2\kappa\kappa'' - 3\kappa'^2. \tag{21}$$

The division by E_2 is necessary to prevent a spurious factor being introduced into the sixth order equation (19). This time, the new equation does not involve either of the remaining dependent variables $\{\lambda, \lambda_u\}$. It is therefore a compatibility condition on the function κ. Unless this condition is satisfied, the 1-stability conditions (12) cannot be satisfied either.

Darboux Integration

Another possibility for the integration of hyperbolic partial differential equations in two independent variables by means of a Cauchy characteristic vector is given by Darboux's method of integration, although the connection with Cauchy characteristics is not usually brought out. In the older classical literature on Darboux's method, emphasis is laid on finding a so–called "intermediate integral" [10] which is in "involution" with the original partial differential equation, so that the solution space of the simultaneous equations made up by the original equation and the intermediate integral is identical to that of the original equation. For example, Liouville's equation, regarded as the locus

$$s = e^z \tag{22}$$

of $J^2(\{x,y\} \to \{z\})$ with local coordinates $\{x,y,z,p,q,r,s,t\}$ has two intermediate integrals derived from the invariants $\{x, r - \frac{1}{2}p^2\}$ on one of the Monge characteristics and $\{y, t - \frac{1}{2}q^2\}$ on the other, namely,

$$r - \frac{1}{2}p^2 = f(x) \tag{23}$$

and

$$t - \frac{1}{2}q^2 = g(y), \tag{24}$$

where f and g are any suitably smooth functions of their indicated arguments. The usual treatment then notes that the Pfaffian system

$$\begin{aligned} &dz - pdx - qdy \\ &dp - (f(x) + \tfrac{1}{2}p^2)dx - e^z dy \\ &dq - e^z dx - (g(y) + \tfrac{1}{2}q^2)dy \end{aligned} \tag{25}$$

is a completely integrable system that has the general solution

$$e^z = \frac{2F'(x)G'(y)}{[F(x) + G(y)]^2} \tag{26}$$

where f and F are related by

$$f(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2$$

and similarly g and G are related by

$$g(y) = \frac{G'''(y)}{G'(y)} - \frac{3}{2}\left(\frac{G''(y)}{G'(y)}\right)^2 .$$

The connection with Cauchy characteristics is brought out by the fact that if we search for irregular integral elements of the exterior differential system obtained by pulling back the contact system of $J^2(\{x,y\} \to \{z\})$ using the equations

$$\{s = e^z,\, r - \frac{1}{2}p^2 = f(x)\} \tag{27}$$

we find that there exists a Cauchy vector given by

$$X = \partial_x + p\partial_z + \left(f(x) + \tfrac{1}{2}p^2\right)\partial_p + e^z\partial_q + qe^z\partial_t. \tag{28}$$

This result is general, i.e., in every case of equations that are Darboux integrable there will always exist a Cauchy vector for the exterior differential system that corresponds to the original equation and an intermediate integral. The only problem then is how to find intermediate integrals. Probably the most convenient way to do this is to utilise the dual language of vector field systems that was first introduced by Vessiot [11]. Indeed, in two little–known papers [12] and [13] Vessiot gave a complete account of Darboux integrability for hyperbolic partial differential equations for a single unknown depending on two independent variables and showed how such integrability was tied up with the structure of 2 and 3–parameter Lie groups. Vessiot's work has been extended by Vassiliou [14] to the case of simultaneous hyperbolic partial differential equations for more than one unknown function of two independent variables provided all of the equations have the same Monge characteristics. More recently Vassiliou [15] and Grundland and Vassiliou [16] have discussed Darboux integrability in terms of vector field systems.

In the present case of the Fermi-Pasta-Ulam equation (1), a basis for the appropriate vector field system is given by (8). However, as pointed out by Vessiot, it is important to utilise the equivalent basis made up of two pairs of singular vector fields which in our case are given by

$$\mathcal{A} = \{A_0, A_1\} \tag{29}$$

with

$$A_0 = -\kappa\left(\partial_x + p\partial_z + r\partial_p + s\partial_q\right) + \partial_y + q\partial_z + s\partial_p + \kappa^2 r\partial_q + 2\kappa\kappa' r^2\partial_s,$$

$$A_1 = \partial_r - \kappa\partial_s$$

and

$$\mathcal{B} = \{B_0, B_1\}, \tag{30}$$

with

$$B_0 = \kappa\,(\partial_x + p\partial_z + r\partial_p + s\partial_q) + \partial_y + q\partial_z + s\partial_p + \kappa^2 r\partial_q + 2\kappa\kappa' r^2\,\partial_s,$$

$$B_1 = \partial_r + \kappa\partial_s$$

because these correspond to the Monge characteristics. In order for an intermediate integral to exist there must be at least 2 independent invariants of either $\mathcal{A}$ or $\mathcal{B}$. For purposes of comparison with the main results of this paper, it is sufficient to deal only with the $\mathcal{B}$ system. Furthermore, it is again useful to use the coordinate transformation (15), after which a suitable basis for $\mathcal{B}$ may be taken as

$$\tilde{B}_0 = \kappa\partial_x + \partial_y + (u + p\kappa + K)\partial_z + 2v\partial_p - \frac{\kappa'}{\kappa}w(v-w)\partial_w \tag{31}$$

and

$$\tilde{B}_1 = \partial_v. \tag{32}$$

It is obvious that $\mathcal{B}$ always has the invariant u. For Darboux integrability, $\mathcal{B}$ must have a second invariant. This will happen trivially if $\kappa' = 0$, in which case we have the elementary wave equation. Excluding this case, i.e., assuming $\kappa' \neq 0$, we examine the derived flag of $\mathcal{B}$ to find how many invariants $\mathcal{B}$ has. We readily find that if

$$2\kappa\kappa'' - 3\kappa'^2 = 0 \tag{33}$$

the derived flag stabilizes at degree 4, showing that $\mathcal{B}$ has 3 independent invariants. It is straightforward to show that in this case $\mathcal{A}$ also has 3 independent invariants. By a theorem of Vessiot, [12] this means that in this case the Fermi-Pasta-Ulam equation may be reduced to the elementary wave equation. Otherwise we find that for $\mathcal{B}$ to have 2 independent invariants and thereby to be Darboux integrable, κ must satisfy the equation $E_4 = 0$, i.e.,

$$\begin{aligned} 16\kappa^4\kappa''\kappa^{iv} - 24\kappa'^2\kappa^3\kappa^{iv} - 16\kappa^4\kappa'''^2 + 32\kappa^3\kappa'\kappa''\kappa''' + 48\kappa^2\kappa'^3\kappa''' \\ -104\kappa^3\kappa''^3 + 372\kappa^2\kappa'^2\kappa''^2 - 630\kappa\kappa'^4\kappa'' + 315\kappa'^6 = 0. \end{aligned} \tag{34}$$

However, this does not exhaust the possibilities for Darboux integration, since we may prolong $\mathcal{B}$ to $\mathcal{B}'$ on the third order jet bundle and find the condition that $\mathcal{B}'$ have 2 independent invariants. This was done with the aid of REDUCE, and we found that provided neither $E_2 = 0$ nor $E_4 = 0$, $\mathcal{B}'$ has two independent invariants and so is Darboux integrable provided the sixth order equation (19) is satisfied.

The interesting point to emerge is that the method of seeking 1-stable vector fields on the bundle of 2–jets is more effective than the method of Darboux integration, which can only achieve the same result after a prolongation to the third order jet bundle.

Comments

It is of interest to enquire about the solutions of (19). Physically motivated functions κ take the form

$$\kappa^2 = (1 + ap)^{2\nu} \tag{35}$$

for constant a and ν. Such an ansatz satisfies

$$\kappa'' = \gamma(\kappa')^2/\kappa \tag{36}$$

where $\gamma = (\nu - 1)/\nu$. In order to solve the Cauchy problem for (1) with such a κ, using a 1-stable vector field given by λ satisfying (12), it is necessary that

$$(\kappa')^{12}(2\gamma - 3)^3(2\gamma - 5)^2(4\gamma - 7)^2(4\gamma - 9)(6\gamma - 11) = 0 \tag{37}$$

which is obtained by substituting (36) into (19). However (19) admits a much larger class of solutions. Further solutions, for example, may be generated from (36) with the aid of the symmetries of the equation:

$$\begin{aligned} Y_1 &= \partial_p \\ Y_2 &= \kappa\partial_\kappa \\ Y_3 &= p\partial_p \\ Y_4 &= p^2\partial_p - 2p\kappa\partial_\kappa \end{aligned} \tag{38}$$

For such solutions, the Fermi-Pasta-Ulam system admits a 1-stable vector field that provides a solution to the Cauchy problem.

It would appear that (for this system) the involutivity analysis of the 1-stability condition is more straightforward in principle than the method of Darboux integration. In practice both approaches are computationally intensive and resort to automated algorithmic techniques appears essential. The results of this paper, however, show that the geometrical approach can lead to effective and powerful practical methods for analysing such non-linear problems.

Acknowledgments

DHH and RWT are grateful for the support of a BP Venture Research Fellowship and EDF for the hospitality provided by the School of Physics and Materials at Lancaster University. The authors found the symmetries (38) with a REDUCE programme written by Paul Kersten. They are grateful to Thomas Wolf for checking these with his REDUCE programme CRACK.

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