PRINT-93-0120
Fermilab Library
0 1160 0021680 8

Equivalence of the Darboux and Gardner methods for integrating hyperbolic equations in the plane

David Hartley[†§]

Robin W Tucker[‡]

Philip A Tuckey[††]

School of Physics and Materials
Lancaster University Lancaster U.
Lancaster LA1 4YB
United Kingdom

December 1992

Abstract

The Gardner and Darboux methods for integrating systems of partial differential equations are described. A modern formulation of the latter is given using the language of exterior differential systems. The methods are shown to be equivalent for a general hyperbolic second order partial differential equation in one dependent and two independent variables: an arbitrary prolongation of the equation is integrable by Gardner's method if and only if the next prolongation is integrable by Darboux's method.

† e-mail: dhh@lavu.physics.lancaster.ac.uk

§ Address from January 1993: Max Planck Institute for Physics, Föhringer Ring 6, D-8000 Munich 40, Germany Munich Max Planck Inst.

‡ e-mail: rwt@lavu.physics.lancaster.ac.uk

†† e-mail: pat@lavu.physics.lancaster.ac.uk

Introduction

In a well-posed Cauchy problem for a partial differential equation (PDE) system, the solution is uniquely determined by the Cauchy data. Nevertheless, the construction of that solution generally involves further PDE systems. If these latter sytems cannot be solved, then the construction of the solution for the original Cauchy problem falters, even though that solution is known to exist and be unique. It is therefore interesting to ask under what conditions a Cauchy problem may be solved using ordinary differential equation (ODE) systems, since these are always formally integrable locally.

Several approaches to the integration of partial differential equations by such methods are available. This paper focusses on two: the Darboux method [1] and the stability analysis of Gardner [2,3,4].

Darboux's method is a generalisation of techniques developed by Monge and Ampère which construct additional equations, called intermediate integrals, which are compatible with the original PDE. Taken together, the original PDE and the intermediate integral may be solved by ODE systems.

Gardner's method takes a different approach that constructs a vector field, in a suitable space, which satisfies a certain stability property in relation to the PDE system. The integral curves of this vector field then develop the Cauchy data into a full solution.

The main result of this paper is that for a hyperbolic second order PDE in one dependent and two independent variables, the Darboux and Gardner methods are equivalent. This result is presented in the context of solving the Cauchy problem. It is interesting to note that the equivalence of the two methods does not apply to exactly the same PDE system for the hyperbolic equation. Rather, the Gardner method for an arbitrary prolongation of a PDE system is equivalent to the Darboux method for its next prolongation.

In this paper, the language of exterior differential systems [5] is used to present and derive the results. First, in section 1, the exterior systems formulation of the hyperbolic PDE is described, and some of the concepts and notation to be used in the remaining sections are introduced. Section 2 presents the Gardner method, essentially as described in [2]. A formulation of the Darboux method in the language of exterior systems is given in section 3, and applied specifically to the Cauchy problem to allow comparison with section 2. The main results are presented in section 4, and illustrated by examples in section 5.

Some of the definitions used in this paper have been adapted to the problem at hand. For more general definitions, see [5]. It has also been assumed throughout that no irregularities

or singularities arise, other than those specifically mentioned. Finally, no consideration has been given to global effects, the results are purely local.

1. Exterior systems for hyperbolic equations

A general second-order partial differential equation in one unknown function u of two independent variables x, y may be written

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{1.1}$$

where $u_x = \partial u/\partial x$ etc. This equation is said to be hyperbolic at points where

$$(F_{u_{xy}})^2 > 4F_{u_{xx}}F_{u_{yy}}. \tag{1.2}$$

It will be assumed that equation (1.1) is hyperbolic at all points of interest. The aim of this section is to express such a hyperbolic partial differential equation, and its prolongations, as exterior differential systems in a notation which will be convenient for subsequent developments.

Equation (1.1) can be regarded as specifying a locus in the 8-dimensional jet bundle $J^2(\mathbf{R}^2, \mathbf{R})$, where $\{x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\}$ are viewed as coordinates. Under the assumption that this locus is the image of a smooth map $i\colon M_7 \to J^2(\mathbf{R}^2, \mathbf{R})$ from some 7-dimensional manifold M_7, equation (1.1) can be rewritten as

$$i^*F = 0. \tag{1.3}$$

The exterior differential system S_7 on M_7 corresponding to the PDE (1.1) is

$$S_7 = i^*C, \tag{1.4}$$

where C is the contact system on $J^2(\mathbf{R}^2, \mathbf{R})$. In terms of the standard coordinates, the contact system is

$$C = \begin{cases} \theta^1 = \mathrm{d}u - u_x\mathrm{d}x - u_y\mathrm{d}y \\ \theta^2 = \mathrm{d}u_x - u_{xx}\mathrm{d}x - u_{xy}\mathrm{d}y \\ \theta^3 = \mathrm{d}u_y - u_{xy}\mathrm{d}x - u_{yy}\mathrm{d}y. \end{cases} \tag{1.5}$$

Unless explicitly mentioned, a set of forms or vectors and the ring which they generate will be synonymous. The **closure** $S_7 \cup \mathrm{d}S_7$ of the exterior system S_7 will be denoted $\bar{S}_7$.

Local solutions of the PDE (1.1) are in one-to-one correspondence with **integral manifolds** of S_7: smooth maps $h: U \subset \mathbf{R}^2 \to M_7$ satisfying $h^* S_7 = 0$ as well as the **independence condition** $h^*(\mathrm{d}x \wedge \mathrm{d}y) \neq 0$. The first step toward finding integral manifolds h is to find 2-planes which could be tangent to the image of h. The next step is to see whether a distribution of these 2-planes is integrable. Since $h^* S_7 = 0$ implies that $h^* \mathrm{d}S_7 = 0$, it is necessary to require that the 2-planes annul the closure $\bar{S}_7$. Accordingly, an **involutive integral element** of $\bar{S}_7$ is defined to be any 2-plane over M_7 which annuls the 1- and 2-forms in $\bar{S}_7$ but not the **independence form** $\Omega = \mathrm{d}x \wedge \mathrm{d}y$.

Integral manifolds h of S_7 on M_7 give rise to immersions $i \circ h: U \subset \mathbf{R}^2 \to J^2(\mathbf{R}^2, \mathbf{R})$ which are integral manifolds of the system $S_8 = C \cup \{F\}$, since $(i \circ h)^* = h^* i^*$. The closure $\bar{S}_8$ of $C \cup \{F\}$ may be expressed in a simple form by suitably adapting the coframe for the jet bundle, and this in turn will give rise to a simplified set of structure equations for S_7.

As is well known, the hyperbolic PDE (1.1) has two Monge characteristics: local curves on the space of independent variables along which the usual Cauchy data is insufficient to uniquely determine a solution. Such a curve in the space of independent variables will annul some 1-form

$$\xi = \alpha \mathrm{d}x + \beta \mathrm{d}y. \tag{1.6}$$

Any lack of uniqueness in the solution determined by the usual Cauchy data on a curve specified by ξ will reveal itself as some drop in rank of the closed system $\bar{S}_8$ modulo ξ.

In particular, the 2-forms in $\mathrm{d}C$ (mod C) are

$$\mathrm{d}C = \begin{cases} \mathrm{d}\theta^1 \simeq 0 \\ \mathrm{d}\theta^2 \simeq -\mathrm{d}u_{xx} \wedge \mathrm{d}x - \mathrm{d}u_{xy} \wedge \mathrm{d}y \\ \mathrm{d}\theta^3 \simeq -\mathrm{d}u_{xy} \wedge \mathrm{d}x - \mathrm{d}u_{yy} \wedge \mathrm{d}y \end{cases} \quad (\mathrm{mod}\ C). \tag{1.7}$$

To examine the two non-vanishing 2-forms modulo ξ, it is convenient to calculate

$$\mathrm{d}\theta^2 \wedge \xi \simeq (-\beta \mathrm{d}u_{xx} + \alpha \mathrm{d}u_{xy}) \wedge \mathrm{d}x \wedge \mathrm{d}y \quad (\mathrm{mod}\ C) \tag{1.8}$$

$$\mathrm{d}\theta^3 \wedge \xi \simeq (-\beta \mathrm{d}u_{xy} + \alpha \mathrm{d}u_{yy}) \wedge \mathrm{d}x \wedge \mathrm{d}y \quad (\mathrm{mod}\ C). \tag{1.9}$$

There is one other 3-form with a similar structure in the ideal generated by $\bar{S}_8$, namely

$$\mathrm{d}F \wedge \mathrm{d}x \wedge \mathrm{d}y = (F_{u_{xx}} \mathrm{d}u_{xx} + F_{u_{xy}} \mathrm{d}u_{xy} + F_{u_{yy}} \mathrm{d}u_{yy}) \wedge \mathrm{d}x \wedge \mathrm{d}y. \tag{1.10}$$

The 3-forms (1.8), (1.9) and (1.10) are linearly independent modulo $\bar{S}_8$ for most values of α and β. However, they become linearly dependent if the 1-forms

$$\begin{cases} -\beta \mathrm{d}u_{xx} + \alpha \mathrm{d}u_{xy} \\ \qquad\qquad -\beta \mathrm{d}u_{xy} \quad + \alpha \mathrm{d}u_{yy} \\ F_{u_{xx}} \mathrm{d}u_{xx} + F_{u_{xy}} \mathrm{d}u_{xy} + F_{u_{yy}} \mathrm{d}u_{yy} \end{cases} \tag{1.11}$$

become linearly dependent. The matrix of coefficients in (1.11) has the determinant

$$F_{u_{xx}} \alpha^2 + F_{u_{xy}} \alpha\beta + F_{u_{yy}} \beta^2 \tag{1.12}$$

which has 0, 1, or 2 real roots for α (or β) as $(F_{u_{xy}})^2$ is less than, equal to, or greater than $4F_{u_{xx}}F_{u_{yy}}$ – the usual conditions for the Monge characteristics of the PDE (1.1). When the determinant (1.12) vanishes, the number of linearly independent 1-forms in (1.11) reduces to two.

In the hyperbolic case (1.2), there are two distinct Monge characteristics, which will be labelled + and −. The corresponding 1-forms will be denoted ω_+ and ω_-. With $\xi = \omega_+$ the two 3-forms $\mathrm{d}\theta^2 \wedge \xi$ and $\mathrm{d}\theta^3 \wedge \xi$ of (1.8), (1.9) are linearly dependent modulo $C \cup \{\mathrm{d}F\}$. It follows that there must be some linear combination θ_+ of θ^2 and θ^3 such that

$$\mathrm{d}\theta_+ \wedge \omega_+ \simeq 0 \quad (\mathrm{mod}\ C \cup \{\mathrm{d}F\}). \tag{1.13}$$

This implies that

$$\mathrm{d}\theta_+ \simeq \pi_+ \wedge \omega_+ \quad (\mathrm{mod}\ C \cup \{\mathrm{d}F\}) \tag{1.14}$$

for some 1-form π_+. The same argument for the other characteristic gives another (independent) linear combination θ_- of θ^2 and θ^3 such that

$$\mathrm{d}\theta_- \simeq \pi_- \wedge \omega_- \quad (\mathrm{mod}\ C \cup \{\mathrm{d}F\}) \tag{1.15}$$

for some 1-form π_-.

Since the rank of the system (1.11) for $\xi = \omega_\pm$ is two, the 1-forms π_+ and π_- are independent linear combinations of $\mathrm{d}u_{xx}$, $\mathrm{d}u_{xy}$ and $\mathrm{d}u_{yy}$ modulo $C \cup \{\mathrm{d}F, \mathrm{d}x, \mathrm{d}y\}$. This completes the construction of a new coframe $\{\omega_+, \omega_-, \theta^1, \theta_+, \theta_-, \pi_+, \pi_-, \mathrm{d}F\}$ for $J^2(\mathbf{R}^2, \mathbf{R})$, in which the contact system is

$$C = \{\theta^1, \theta_+, \theta_-\} \tag{1.16}$$

and the structure equations of S_8 are given by $\mathrm{d}\theta^1 \simeq 0\,(\mathrm{mod}\ C)$ together with (1.14) and (1.15). Since ω_+ and ω_- are linearly independent combinations of $\mathrm{d}x$ and $\mathrm{d}y$, the independence form Ω can be taken as $\Omega = \omega_+ \wedge \omega_-$.

Pulled back using the map $i: M_7 \to J^2(\mathbf{R}^2, \mathbf{R})$, the 1-form $\mathrm{d}F$ vanishes by (1.3), but the remaining coframe elements survive to give a coframe for M_7 (since i is assumed to be injective). The M_7 1-forms $i^*\omega_+$, $i^*\omega_-$ and so on will be denoted with the same symbols as on $J^2(\mathbf{R}^2, \mathbf{R})$.

In this coframe, the exterior system S_7 (1.4) is

$$S_7 = \{\theta^1, \theta_+, \theta_-\} \tag{1.17}$$

and the structure equations are

$$\mathrm{d}S_7 = \begin{cases} \mathrm{d}\theta^1 \simeq 0 \\ \mathrm{d}\theta_+ \simeq \pi_+ \wedge \omega_+ \\ \mathrm{d}\theta_- \simeq \pi_- \wedge \omega_- \end{cases} \quad (\mathrm{mod}\ S_7). \tag{1.18}$$

Returning to the problem of finding integral elements, a 2-plane over M_7 may be specified by giving a basis of two vectors, or alternatively by giving a basis of five 1-forms for the annihilating subspace of the dual space. Any 2-plane over M_7 which does not annul the independence form $\Omega = \omega_+ \wedge \omega_-$ must have an annihilating space which is specified by the 1-forms

$$\begin{cases} \theta^1 - a\omega_+ - b\omega_- \\ \theta_+ - c\omega_+ - d\omega_- \\ \theta_- - e\omega_+ - f\omega_- \\ \pi_+ - g\omega_+ - h\omega_- \\ \pi_- - i\omega_+ - j\omega_- \end{cases} \tag{1.19}$$

for some values of the parameters a, b, ..., j. To determine whether such a 2-plane annuls the 1- and 2-forms in $\bar{S}_7$, all possible contractions with the two basis vectors can be tested for vanishing. Equivalently, the forms in $\bar{S}_7$ can be evaluated modulo the five basis one-forms above, and again tested for vanishing.

From the form of the system given in (1.17) and the structure equations (1.18) it can be seen that the 2-planes specified by (1.19) are integral elements of $\bar{S}_7$ if and only if $a = b = c = d = e = f = h = i = 0$. The parameters g and j are unconstrained, and it

is convenient to re-label them as p_+ and p_-. Thus any involutive integral element of $\bar{S}_7$ is determined by a Pfaffian system on M_7

$$S_9 = \begin{cases} \theta^1 \\ \theta_+ \\ \theta_- \\ \pi_+ - p_+\omega_+ \\ \pi_- - p_-\omega_- \end{cases} \tag{1.20}$$

for some values of p_+ and p_-. The parameters p_+ and p_- may therefore be used to coordinate the fibres of a 9-dimensional bundle M_9 of involutive integral elements of $\bar{S}_7$ over M_7. Using the bundle projection, the coframe on M_7 can be lifted to give a set of 7 independent 1-forms on M_9. These 1-forms will be denoted by the same symbols on both spaces, and a similar practise will be followed in the rest of this paper. Adding the 1-forms $\mathrm{d}p_+$, $\mathrm{d}p_-$ to this set completes a coframe for M_9. With these provisions, the system S_9 in (1.20) can be re-interpreted as a system on M_9 with $p_\pm$ as additional coordinates.

The system S_9 on M_9 is called the **first prolongation** of S_7 on M_7. Converting back to jet-bundle coordinates reveals that S_9 is just the pull-back of the contact system of the third jet bundle $J^3(\mathbf{R}^2, \mathbf{R})$ using the original PDE (1.1) and its derivatives with respect to x and y – the usual PDE notion of prolongation.

For S_9, the structure equations can be calculated for the most part from those for S_7, since the pullback and exterior differentiation operations commute. Thus

$$\begin{aligned} \mathrm{d}\theta^1 &\simeq 0 \\ \mathrm{d}\theta_+ &\simeq 0 \quad (\mathrm{mod}\ S_9) \\ \mathrm{d}\theta_- &\simeq 0 \end{aligned} \tag{1.21}$$

(of course, S_9 is constructed just so that this is true).

On M_7, $\mathrm{d}\pi_+$ is naturally expressed in terms of the M_7 coframe alone. Since pullback and exterior derivative commute, it follows that $\mathrm{d}\pi_+$ on M_9 can be expressed in terms of the same 1-forms: there are no terms containing $\mathrm{d}p_\pm$ factors. All "M_7" 1-forms are equivalent to combinations of ω_+ and ω_- modulo S_9, so

$$\mathrm{d}\pi_+ \simeq A\omega_+ \wedge \omega_- \quad (\mathrm{mod}\ S_9) \tag{1.22}$$

for some function A on M_9. Similar remarks hold for the forms π_-, ω_+ and ω_-. Consequently, there exist functions $A_\pm$ on M_9 such that

$$\begin{aligned} \mathrm{d}(\pi_+ - p_+\omega_+) &\simeq (A_+\omega_- - \mathrm{d}p_+)\wedge\omega_+ \\ \mathrm{d}(\pi_- - p_-\omega_-) &\simeq (A_-\omega_+ - \mathrm{d}p_-)\wedge\omega_- \end{aligned} \quad (\mathrm{mod}\ S_9). \tag{1.23}$$

Comparing the structure equations (1.21), (1.23) for S_9 with those (1.18) for S_7, it is evident they follow the same pattern. If $\{\theta^1, \theta_\pm\}$ are denoted collectively by $\{\theta^a\}$, $\{\pi_\pm - p_\pm\omega_\pm\}$ are re-labelled $\{\theta_\pm\}$, and $\{A_\pm\omega_\mp - \mathrm{d}p_\pm\}$ are re-labelled as $\{\pi_\pm\}$, then the structure equations for S_9 will take on exactly the same form as (1.18), with θ^a in place of θ^1. From this it follows that S_9 on M_9 may itself be prolonged to give a new system S_{11} on on 11-dimensional bundle M_{11} over M_9, with structure equations again following the same pattern. Clearly, this procedure may be repeated indefinitely.

In order to treat all cases at once, the result of an arbitrary number (possibly zero) of prolongations of S_7 on M_7, will be written as a system S on a manifold M, where

$$S = \{\theta^a, \theta_+, \theta_-\} \tag{1.24}$$

($a = 1, \ldots, n-6$ where $n = \dim M$) with structure equations

$$\mathrm{d}S = \begin{cases} \mathrm{d}\theta^a \simeq 0 \\ \mathrm{d}\theta_+ \simeq \pi_+\wedge\omega_+ \\ \mathrm{d}\theta_- \simeq \pi_-\wedge\omega_- \end{cases} \quad (\mathrm{mod}\ S). \tag{1.25}$$

The 1-forms $\{\theta^a, \theta_+, \theta_-, \pi_+, \pi_-, \omega_+, \omega_-\}$ form a coframe for M.

The first prolongation of S on M will be written as $S^{(1)}$ on a bundle $M^{(1)} \cong M\times\mathbf{R}^2$ over M, with

$$S^{(1)} = \begin{cases} \theta^a \\ \theta_+ \\ \theta_- \\ \theta_+^{(1)} = \pi_+ - p_+\omega_+ \\ \theta_-^{(1)} = \pi_- - p_-\omega_-. \end{cases} \tag{1.26}$$

The structure equations for $S^{(1)}$ are

$$\mathrm{d}S^{(1)} = \begin{cases} \mathrm{d}\theta^a & \simeq 0 \\ \mathrm{d}\theta_+ & \simeq 0 \\ \mathrm{d}\theta_- & \simeq 0 \\ \mathrm{d}\theta_+^{(1)} & \simeq \pi_+^{(1)} \wedge \omega_+ \\ \mathrm{d}\theta_-^{(1)} & \simeq \pi_-^{(1)} \wedge \omega_- \end{cases} \qquad (\mathrm{mod}\ S^{(1)}) \tag{1.27}$$

where there are functions $A_\pm$ on $M^{(1)}$ such that

$$\pi_\pm^{(1)} = A_\pm \omega_\mp - \mathrm{d}p_\pm. \tag{1.28}$$

The systems S and $S^{(1)}$ on M and $M^{(1)}$ will be the setting for the main discussion in section 4.

As mentioned earlier, there are characteristics associated with hyperbolic PDE's. These are not Cauchy characteristics, but Monge characteristics. For the exterior system S there are corresponding notions. The **associated vector space** of any exterior system E (not necessarily closed) is the set of vectors V satisfying $i_V E \simeq 0$ (mod E). The space of 1-forms annihilating these vectors, called the **associated Pfaffian system**, is denoted $A(E)$. The associated vector space is denoted $A(E)^\perp$. A **Cauchy characteristic** of E is a vector in $A(\bar{E})^\perp$, the associated vector space of the closure of E.

As expected, S has no Cauchy characteristics: $A(\bar{S})^\perp$ is empty. The Monge characteristics of the PDE correspond to 1-forms $\xi \in \{\omega_+, \omega_-\}$ such that the system $\bar{S} \cup \{\xi\}$ has a non-empty associated vector space $A(\bar{S} \cup \{\xi\})^\perp$. The coframe used here has been designed precisely so that these Monge characteristics are easy to write down. From the structure equations (1.25) it can be seen that they correspond to the forms $\xi = \omega_+$ and $\xi = \omega_-$. The associated vector spaces are seen to be

$$A(\bar{S} \cup \{\omega_\pm\})^\perp = \{\partial_{\omega_\mp}, \partial_{\pi_\mp}\}. \tag{1.29}$$

where the vectors denoted $\{\partial_{\theta^a}, \partial_{\theta_\pm}, \partial_{\pi_\pm}, \partial_{\omega_\pm}\}$ comprise the moving frame dual to $\{\theta^a, \theta_\pm, \pi_\pm, \omega_\pm\}$. The associated Pfaff systems, here called the **Monge characteristic systems**, will be denoted $S_\pm$, and are seen to be

$$S_\pm = S \cup \{\pi_\pm, \omega_\pm\}. \tag{1.30}$$

Similar considerations show the Monge characteristic systems of $S^{(1)}$ to be

$$S_{\pm}^{(1)} = S^{(1)} \cup \{\pi_{\pm}^{(1)}, \omega_{\pm}\}. \tag{1.31}$$

For the remainder of this paper, "characteristic" will be taken to mean "Monge characteristic" unless explicitly stated otherwise. By themselves, the Monge characteristics do not allow the integration of the original PDE, but they can be used to augment the exterior system in such a way that the resulting system does have a Cauchy characteristic, and so may be integrated more easily. This is the purpose of the Gardner and Darboux methods.

2. Gardner method

As mentioned in the introduction, the solution of the Cauchy problem for a PDE such as (1.1), although uniquely determined by the Cauchy data, generally still requires the solution of PDE systems for its construction. It is therefore interesting to ask under what circumstances the solution to the Cauchy problem may be determined by the easier method of solving an ODE system.

One approach to this question is the stability analysis of Gardner [2,3,4]. This method is not restricted to the hyperbolic PDE in the plane described in the previous section, or any PDE system at all, but applies to general Pfaffian exterior differential systems. The following definitions form the basis of the method.

Definition. *Let S be a Pfaffian exterior differential system on a manifold M, and let X be a vector field over M such that $X \in S^{\perp}$. Define the* ***stability systems*** *$D_X^l(S)$ for $l = 0, 1, 2, \ldots$ by the recursion relations*

$$\begin{aligned} D_X^0(S) &= S \\ D_X^l(S) &= D_X^{l-1}(S) \cup \mathcal{L}_X D_X^{l-1}(S) \qquad l \geqslant 1. \end{aligned} \tag{2.1}$$

The vector X is said to be ***k-stable*** *for some $k \geqslant 0$ if*

$$D_X^{k+1}(S) = D_X^k(S). \tag{2.2}$$

It is useful to re-formulate the recursion relations (2.1). From the definition, $X \in S^{\perp}$, so $i_X D_X^0(S) = 0$. Using $\mathcal{L}_X = i_X \mathrm{d} + \mathrm{d} i_X$, it follows that

$$D_X^1(S) = S \cup i_X \mathrm{d}S. \tag{2.3}$$

Furthermore, since $(i_X)^2 = 0$, (2.3) yields $i_X D^1_X(S) = 0$. By induction, the recursion relations (2.1) can then be re-written with the Lie derivative replaced by a differentiation and a contraction:

$$\begin{aligned} D^0_X(S) &= S \\ D^l_X(S) &= D^{l-1}_X(S) \cup i_X \mathrm{d} D^{l-1}_X(S) \qquad l \geqslant 1. \end{aligned} \tag{2.4}$$

It is worth noting that

$$S \subset D^l_X(S) \qquad l = 0, 1, 2, \ldots \tag{2.5}$$

$$i_X D^l_X(S) = 0 \quad l = 0, 1, 2, \ldots \tag{2.6}$$

From the re-written recursion relations (2.4) and the identity (2.6), it can be seen that a k-stable vector field X is a Cauchy vector for the Pfaffian system $D^k_X(S)$, that is,

$$\begin{aligned} i_X D^k_X(S) &= 0 \\ i_X \mathrm{d} D^k_X(S) &\subset D^k_X(S). \end{aligned} \tag{2.7}$$

The standard theorem on Cauchy characteristics then applies [5], so that an integral manifold of $D^k_X(S)$ can be extended along the integral curves of the Cauchy vector. However, from (2.5), any integral manifold of $D^k_X(S)$ is automatically an integral manifold of S. So an integral manifold of S which is also an integral manifold of $D^k_X(S)$ can be extended by the same Cauchy vector. These considerations give rise to the following theorem (the detailed proof may be found in [3,4]).

Theorem. *Let X be a k-stable vector field for S, $C: \Sigma \subset \mathbf{R}^m \to M$ an m-dimensional integral manifold of S, and assume that $D^k_X(S)$ has constant rank in a neighbourhood of the image of C. Then the integral curves of X determine an $(m+1)$-dimensional integral manifold $h: \Sigma \times \mathbf{R} \to M$ of S by*

$$h: (\sigma, \tau) \mapsto \exp_{C(\sigma)}(\tau X) \tag{2.8}$$

if and only if X is transversal to the image of C and

$$C^* D^k_X(S) = 0. \tag{2.9}$$

Remark. "Transversal" usually means that X should be nowhere tangent to the image of C, but, in the presence of an independence condition, it means that the projection of X onto the base manifold of the independent variables should be nowhere tangent to the

same projection of the image of C. In terms of the independence form Ω, the requirement is $C^*(i_X\Omega) \neq 0$.

This is the desired result. The solution h of the Cauchy problem is constructed by solving the ODE system determining the integral curves of X, using the Cauchy data specified by the integral manifold C.

The drawback of this method is that to find a k-stable vector field X, it is necessary to solve a quasi-linear PDE system implicit in the stability relation (2.2), subject to boundary conditions given by (2.9). In general, this is a non-trivial problem, but for a hyperbolic PDE in the plane, it may be circumvented using the results to be presented in section 4.

It should be emphasised that the concepts of a characteristic vector field and a k-stable vector field are independent, although characteristic vector fields may be useful candidates for k-stability analysis. For example, the 1-dimensional heat equation $u_t = u_{xx}$ may be solved for Cauchy data along $x = 0$ using a 1-stable characteristic vector, and for Cauchy data along $t = 0$ using a 1-stable non-characteristic vector. For Cauchy data on $t = 0$ this equation gives an example where a 1-stable characteristic vector X is nowhere tangent to the image of C, yet doesn't solve the Cauchy problem because it fails the independence condition.

3. Darboux method

Another approach to the integration of a hyperbolic PDE in the plane is offered by the method of Darboux. Actually, there are several different methods which go by this title, although they all have in common the search for Riemann invariants and intermediate integrals. The variant of Darboux integration described here is one of the weaker ones, leading to the integration of the PDE, but not necessarily in closed form. For descriptions of other "Darboux methods" see, for example [1].

The core of this method is to find a pair of functions which are constant on one of the characteristics for the PDE. From these, an intermediate integral can be constructed such that for any given Cauchy data, the original PDE taken together with the intermediate integral has a Cauchy vector. The Cauchy problem then reduces to the ODE system for finding integral curves of the Cauchy vector, just as in the previous section. Before giving the details of the Darboux approach, a few standard definitions are required.

Definition. *An* ***invariant*** I *of a Pfaffian system* S *is a function* $I\colon M \to \mathbf{R}$ *such that*

$$i_V \mathrm{d}I = 0 \qquad \forall V \in S^{\perp}. \tag{3.1}$$

It follows immediately from the definition that I is an invariant of S if and only if $\mathrm{d}I \in S$. So the task of finding invariants of a system S is the same as that of finding exact 1-forms in S. This is accomplished by calculating the derived flag.

Definition. *Given a Pfaffian system* S, *the* ***derived flag*** $\{S_{(0)} = S, S_{(1)}, S_{(2)}, \ldots\}$ *is defined by the recursion relation*

$$S_{(k+1)} = \{\theta \in S_{(k)} : \mathrm{d}\theta \simeq 0 \pmod{S_{(k)}}\} \quad k \geqslant 0. \tag{3.2}$$

Since $S_{(k+1)} \subseteq S_{(k)}$, this flag must eventually stabilise at some value of k, possibly with $S_{(k)}$ empty. The final system $S_{(k)}$ is completely integrable, as can be seen from the recursion relation (3.2), and will be the largest completely integrable subsystem of S. If non-empty, this $S_{(k)}$ may be integrated locally by means of an ODE system to give an algebraically equivalent set of exact 1-forms. Since this is the largest completely integrable subsystem of S, this procedure determines a complete set of independent invariants of S.

Returning to the discussion of Darboux integrability, let S be the exterior differential system (1.24) corresponding to an arbitrary prolongation of the hyperbolic PDE (1.1), and $S_{\pm}$ be the corresponding characteristic systems (1.30).

Definition. *A* ***Riemann invariant*** I *of* S *is an invariant of one of the characteristic systems* $S_{\pm}$.

The aim of the Darboux method is to find a Riemann invariant I of S such that the augmented system $S \cup \{\mathrm{d}I\}$ has a Cauchy vector solving the Cauchy problem for S. Without loss of generality, the discussion can be restricted to just one of the characteristic systems.

Lemma. *There is a one-to-one correspondence between the vectors in the characteristic vector space* $S_{-}^{\perp}$ *and equivalence classes in* S_{+} *modulo* S, *given by*

$$i_{S_{-}^{\perp}} \mathrm{d}S \simeq S_{+} \pmod{S}. \tag{3.3}$$

Proof. Let $V = \alpha\partial_{\omega_+} + \beta\partial_{\pi_+}$ be a general element of $S_-^\perp$. Since $i_V S = 0$, it follows from the structure equations (1.25) of S that $i_V \mathrm{d}S \simeq \{i_V \mathrm{d}\theta_+\}$ (mod S). But $i_V \mathrm{d}\theta_+ \simeq \beta\omega_+ - \alpha\pi_+$ (mod S), which is the general form of an element of S_+ modulo S. □

Consider the characteristic system S_+. By calculating its derived flag, it is possible to establish a complete set of independent Riemann invariants. But this may not be sufficient to integrate the PDE. There may not be any invariants at all, and even if they do exist, they may not be suitable, as will be seen shortly.

Lemma. *Let I be an invariant of S_+. The augmented system $S \cup \{\mathrm{d}I\}$ has a Cauchy vector X defined by*

$$i_X \mathrm{d}S \simeq \{\mathrm{d}I\} \quad (\mathrm{mod}\ S). \tag{3.4}$$

Proof. By the last lemma, there will be a vector $X \in S_-^\perp$ satisfying (3.4). Consequently, $i_X \mathrm{d}I = 0$ and $i_X S = 0$, since $(i_X)^2 = 0$ and $S_-^\perp \subset S^\perp$. Finally, $\mathrm{d}I$ is closed, so it follows from (3.4) that

$$i_X \mathrm{d}(S \cup \{\mathrm{d}I\}) = i_X \mathrm{d}S \simeq 0 \quad (\mathrm{mod}\ S \cup \{\mathrm{d}I\}). \tag{3.5}$$

Hence X is a Cauchy vector for $S \cup \{\mathrm{d}I\}$. □

Suppose that an of S_+ can be found, so that $S \cup \{\mathrm{d}I\}$ has a Cauchy vector X. Then the standard theorem on Cauchy characteristics can be applied again. As in the Gardner method, any integral manifold of $S \cup \{\mathrm{d}I\}$ is automatically an integral manifold of S, but in order that an integral manifold of S be an integral manifold of $S \cup \{\mathrm{d}I\}$, it is also necessary that $\mathrm{d}I$ vanish under pullback to the integral manifold of S. The discussion so far can be summarised in the following theorem, which should be compared with the similar theorem for the Gardner method.

Theorem. *Let I be an invariant of S_+, $X \in S_-^\perp$ the corresponding vector defined by (3.4), $C\colon \Sigma \subset \mathbf{R} \to M$ a 1-dimensional integral manifold of S, and assume that $S \cup \{\mathrm{d}I\}$ has constant rank in a neighbourhood of the image of C. Then the integral curves of X determine a 2-dimensional integral manifold $h\colon \Sigma \times \mathbf{R} \to M$ of S by*

$$h\colon (\sigma, \tau) \mapsto \exp_{C(\sigma)}(\tau X) \tag{3.6}$$

if and only if X is transversal to the image of C, and

$$C^* \mathrm{d}I = 0. \tag{3.7}$$

Remark. The comments about transversality made in the previous section apply equally well here.

The condition (3.7) and the transversality requirement can be analysed further to see what constraints they place on the invariant I. If the surface C is characteristic, that is, if the tangent vector to the image of C is in $S_{\pm}^{\perp}$, then $C^*S_{\pm} = 0$. If $C^*S_- = 0$, then $X \in S_-^{\perp}$ cannot be transversal to the image of C, and so no solution using an invariant of S_+ is possible. If, on the other hand, $C^*S_+ = 0$, then condition (3.7) is satisfied immediately, as is the transversality condition. Normally, however, C is not characteristic, and (3.7) imposes a real constraint upon I which will not be satisfied in general for a given I and a given C.

The Darboux method overcomes this difficulty by requiring some flexibility in I. If I_1 and I_2 are two independent Riemann invariants in S_+, then C^*I_1 and C^*I_2 are functions on Σ. If one of them, say, I_2 has $C^*\mathrm{d}I_2 \neq 0$, then by the inverse function theorem there exists locally a function g such that $C^*I_1 = -g \circ C^*I_2$. Taking the combination $I = I_1 + g(I_2)$ gives a new Riemann invariant, since $\mathrm{d}I = \mathrm{d}I_1 + g'\mathrm{d}I_2 \in S_+$. Furthermore, this new invariant satisfies $C^*I = 0$ and hence $C^*\mathrm{d}I = 0$. Such an invariant is called an **intermediate integral**. The necessity of finding two independent Riemann invariants of course restricts the class of problems which can be treated by this method.

The transversality requirement further constrains the intermediate integral I. Recall that the vector X defined by (3.4) is of the form $X = \alpha\partial_{\omega_+} + \beta\partial_{\pi_+}$, and the corresponding Riemann invariant is $\mathrm{d}I \simeq \beta\omega_+ - \alpha\pi_+ \pmod{S}$. The transversality requirement expressed in terms of the independence form Ω, $C^*(i_X\Omega) \neq 0$ demands that $\alpha \neq 0$. For the Riemann invariant I, this means that $\mathrm{d}I \notin S \cup \{\omega_+\}$.

The intermediate integral, when written out in canonical coordinates from the jet bundle, is a PDE in its own right. In terms of partial derivatives, the constraint $\mathrm{d}I \notin S \cup \{\omega_+\}$ states that the intermediate integral is a PDE of the same order in the derivatives as the (prolonged) hyperbolic PDE system. This is, strictly speaking, just the case to which the Darboux method applies [1]. If an intermediate integral of lower order can be found, then the Monge-Ampère method can be used. More importantly, an intermediate integral of lower order will still be an intermediate integral of the system when it is prolonged one less time. In other words, if a problem is in this Monge-Ampère class, then it would have been in the Darboux class on a lower-order jet bundle. The focus of this paper is the Darboux method, so the previous theorem can be refined to give a practical method for solving the Cauchy problem for arbitrary Cauchy data, as follows.

Theorem. *Let I_1, I_2 be a pair of independent Riemann invariants of S, with $\mathrm{d}I_1, \mathrm{d}I_2 \in S_+$, and $\mathrm{d}I_1$ or $\mathrm{d}I_2 \notin S \cup \{\omega_+\}$. Let $C\colon \Sigma \subset \mathbf{R} \to M$ be a 1-dimensional integral manifold of S which is non-characteristic, and suppose that $S \cup \{\mathrm{d}I_1, \mathrm{d}I_2\}$ has constant rank in a neighbourhood of the image of C. Then there exists an intermediate integral $I = I_1 + g(I_2)$ such that $C^*\mathrm{d}I = 0$, and a corresponding vector $X \in S_-^\perp$ defined by (3.4) whose integral curves determine a 2-dimensional integral manifold $h\colon \Sigma \times \mathbf{R} \to M$ of S by*

$$h\colon (\sigma, \tau) \mapsto \exp_{C(\sigma)}(\tau X). \tag{3.8}$$

Once again, the desired result is achieved. The solution h of the Cauchy problem is constructed by solving an ODE system for the integral curves of X, using the Cauchy data provided by C.

The advantage of the Darboux method is that *all* steps in the construction of the solution can be made (locally) by solving nothing worse than ODE systems. The calculation of a complete set of invariants and the final solution of the Cauchy problem are all accomplished by ODE systems. One disadvantage of the version of the method presented here is that the solution may involve implicit functions. Other means of exploiting intermediate integrals may overcome this problem at the expense of more stringent requirements on the invariants of the characteristics of S. As will be seen in the next section, another disadvantage is that calculations may have to be done on a higher-dimensional manifold than is the case with the Gardner method, making calculations less tractable.

4. Equivalence of Gardner and Darboux methods

Having presented both Gardner and Darboux methods, the main result of this paper is to show their equivalence for systems corresponding to hyperbolic partial differential equations in the plane. However, this equivalence does not hold within the same system; rather it holds between the Gardner method using 1-stable characteristic vectors for a system S and the Darboux method for its first prolongation $S^{(1)}$.

To establish this equivalence, the 1-stability criterion (2.2) will first be re-formulated for characteristic vectors entirely in terms of differential forms. This will enable the 1-stability criterion to be written as an exterior system. The integrability of this exterior system will then be shown to be equivalent to the conditions for the existence of a Riemann invariant. As before, only one set of characteristics will be used in the discussion, the results for the other set following by the same arguments.

Recall that the system S is

$$S = \{\theta^a, \theta_+, \theta_-\} \tag{4.1}$$

with structure equations

$$\mathrm{d}S = \begin{cases} \mathrm{d}\theta^a \simeq 0 \\ \mathrm{d}\theta_+ \simeq \pi_+ \wedge \omega_+ \\ \mathrm{d}\theta_- \simeq \pi_- \wedge \omega_- \end{cases} \pmod{S}. \tag{4.2}$$

The 1-forms $\{\theta^a, \theta_+, \theta_-, \pi_+, \pi_-, \omega_+, \omega_-\}$ form a coframe for M, and the characteristic systems are

$$S_\pm = S \cup \{\pi_\pm, \omega_\pm\}. \tag{4.3}$$

The 1-stability criterion can be expressed as an exterior system by use of the following lemma, which formulates the criterion in terms of differential forms.

Lemma. *The system S possesses a 1-stable vector $X = \partial_{\omega_+} + \lambda\partial_{\pi_+} \in S_-^\perp$ (where λ is some function on M) if and only if there exists a 1-form*

$$\phi \simeq \pi_+ - \lambda\omega_+ \pmod{S} \tag{4.4}$$

such that

$$\mathrm{d}\phi \simeq 0 \pmod{S \cup \{\phi\}}. \tag{4.5}$$

Proof. Recall that $S_-^\perp = \{\partial_{\omega_+}, \partial_{\pi_+}\}$. A general vector $X \in S_-^\perp$ will be a combination of these two vectors, but the coefficient of ∂_{ω_+} may be chosen to be 1 without loss of generality in view of the independence condition. Take, then

$$X = \partial_{\omega_+} + \lambda \partial_{\pi_+}. \tag{4.6}$$

The first stability system for X is $D_X^1(S) = S \cup i_X \mathrm{d}S = S \cup \{\pi_+ - \lambda\omega_+\}$, using the structure equations (4.2). The 1-stability condition is then

$$i_X \mathrm{d}(\pi_+ - \lambda\omega_+) \simeq 0 \pmod{S \cup \{\pi_+ - \lambda\omega_+\}}. \tag{4.7}$$

Modulo S, any 2-form must lie in the exterior algebra of $\{\pi_\pm, \omega_\pm\}$, so $\mathrm{d}(\pi_+ - \lambda\omega_+)$ can be written

$$\begin{aligned} \mathrm{d}(\pi_+ - \lambda\omega_+) &\simeq A\pi_+ \wedge \pi_- + B\pi_+ \wedge \omega_+ + C\pi_+ \wedge \omega_- \\ &\quad + D\pi_- \wedge \omega_+ + E\pi_- \wedge \omega_- + F\omega_+ \wedge \omega_- \pmod{S} \\ &\simeq (D - \lambda A)\pi_- \wedge \omega_+ + (F + \lambda C)\omega_+ \wedge \omega_- \\ &\quad + E\pi_- \wedge \omega_- \pmod{S \cup \{\pi_+ - \lambda\omega_+\}} \end{aligned} \tag{4.8}$$

where the coefficients B, D and F may contain first derivatives of the function λ. Using $i_X S = 0 = i_X(\pi_+ - \lambda\omega_+)$, this gives

$$i_X \mathrm{d}(\pi_+ - \lambda\omega_+) \simeq (F + \lambda C)\omega_- - (D - \lambda A)\pi_- \pmod{S \cup \{\pi_+ - \lambda\omega_+\}}. \tag{4.9}$$

So X is 1-stable if and only if

$$(F + \lambda C) = (D - \lambda A) = 0. \tag{4.10}$$

Substituting these conditions back into the expression (4.8), X is 1-stable if and only if

$$\mathrm{d}(\pi_+ - \lambda\omega_+) \simeq E\pi_- \wedge \omega_- \pmod{S \cup \{\pi_+ - \lambda\omega_+\}}. \tag{4.11}$$

But from the structure equations (4.2) for S, $\mathrm{d}\theta_- \simeq \pi_- \wedge \omega_- \pmod{S}$ so setting

$$\phi = \pi_+ - \lambda\omega_+ - E\theta_- \tag{4.12}$$

and noting that $S \cup \{\pi_+ - \lambda\omega_+\} = S \cup \{\phi\}$, (4.11) yields the result (4.5). □

Remark. Owing to the contraction, there is and can be no ω_+ term on the right-hand side of (4.9). Thus condition (4.9) still yields (4.10) if the calculation is done modulo

$S \cup \{\pi_+ - \lambda\omega_+, \omega_+\}$, that is, modulo S_+. This leads to the formulation of the 1-stability condition

$$\mathrm{d}\lambda \simeq \lambda i_X \mathrm{d}\omega_+ - i_X \mathrm{d}\pi_+ \pmod{S_+} \tag{4.13}$$

given in [4].

It is apparent from (4.10) that the 1-stability condition is a system of two first-order partial differential equations for λ as a function on M. This system can be written as an exterior system on the trivial bundle $M \times \mathbf{R}$ over M with λ as the fibre coordinate.

The aim is to find an immersion $f: M \to M \times \mathbf{R}$ such that the function $f^*\lambda$ on M is a solution of the 1-stability condition. Without loss of generality, it may be assumed that $Q \circ f = \mathrm{id}$, where Q is the projection $M \times \mathbf{R} \to M$. With this assumption, the 1-stability condition for f is

$$f^*\mathrm{d}\phi \simeq 0 \pmod{S \cup \{f^*\phi\}} \tag{4.14}$$

where $\phi \simeq \pi_+ - \lambda\omega_+ \pmod{S}$ is a 1-form on $M \times \mathbf{R}$ (and $f^*\phi$ is the 1-form on M denoted ϕ up to this point). The exterior system to be satisfied by f consists then of a single form of degree $n-1$ (where $n = \dim M$),

$$\{\theta^1 \wedge \ldots \wedge \theta^{n-6} \wedge \theta_+ \wedge \theta_- \wedge \phi \wedge \mathrm{d}\phi\}, \tag{4.15}$$

and the independence form is the volume form $\theta^1 \wedge \ldots \wedge \theta^{n-6} \wedge \theta_+ \wedge \theta_- \wedge \pi_+ \wedge \pi_- \wedge \omega_+ \wedge \omega_-$ on M. Before discussing this exterior system further, it is necessary to calculate $\mathrm{d}\phi \pmod{S \cup \{\phi\}}$.

Lemma. *On $M \times \mathbf{R}$, it is possible to choose a 1-form $\phi \simeq \pi_+ - \lambda\omega_+ \pmod{S}$ which satisfies*

$$\mathrm{d}\phi \simeq \psi \wedge \omega_+ \pmod{S \cup \{\phi\}} \tag{4.16}$$

where ψ is a non-vanishing 1-form on $M \times \mathbf{R}$ with $\psi \notin S_+$.

Proof. A similar argument to that used in the proof of the preceding lemma applies, with the following precautions. The function λ on M becomes an independent coordinate on $M \times \mathbf{R}$, so there is an extra term $\mathrm{d}\lambda \wedge \omega_+$ in the expression for $\mathrm{d}\phi$. The 1-forms π_+ and ω_+ on $M \times \mathbf{R}$ are the trivial pullbacks of the same forms on M, so their exterior derivatives have the same expressions on both spaces. Thus

$$\mathrm{d}\phi \simeq (D - \lambda A)\pi_- \wedge \omega_+ + (F + \lambda C)\omega_+ \wedge \omega_- - \mathrm{d}\lambda \wedge \omega_+ + E\pi_- \wedge \omega_- \pmod{S \cup \{\phi\}} \tag{4.17}$$

for some functions A, C, D, E and F on $M \times \mathbf{R}$. Again the term $E\pi_- \wedge \omega_-$ may be absorbed into ϕ (since it is only defined modulo S), leaving

$$\mathrm{d}\phi \simeq \psi \wedge \omega_+ \pmod{S \cup \{\phi\}} \tag{4.18}$$

where

$$\psi \simeq (D - \lambda A)\pi_- - (F + \lambda C)\omega_- - \mathrm{d}\lambda \pmod{S \cup \{\phi\}}. \tag{4.19}$$

The presence of the $\mathrm{d}\lambda$ term in ψ ensures that it is non-vanishing, and also that $\psi \notin S_+$. □

The exterior system of (4.15) may not be sufficient by itself to allow f to be determined. There may well be further first-order equations for f to be found among the integrability conditions from (4.15), and still further equations which arise from these. Nevertheless, if there is ultimately a solution, it must satisfy the following property.

Lemma. *Suppose that $\phi \simeq \pi_+ - \lambda\omega_+ \pmod{S}$ on $M\times\mathbf{R}$ has been chosen such that $\mathrm{d}\phi \simeq \psi \wedge \omega_+ \pmod{S \cup \{\phi\}}$ for some 1-form ψ on $M\times\mathbf{R}$. Then the 1-stability condition (4.14) has a solution if and only if*

$$f^*\psi \simeq 0 \pmod{S_+}. \tag{4.20}$$

Proof. Suppose that there exists an immersion $f\colon M \to M\times\mathbf{R}$ satisfying (4.14). Then since

$$f^*\mathrm{d}\phi \simeq f^*\psi \wedge \omega_+ \pmod{S \cup \{f^*\phi\}}, \tag{4.21}$$

it follows that

$$f^*\psi \simeq 0 \pmod{S \cup \{f^*\phi, \omega_+\}}. \tag{4.22}$$

But $S \cup \{f^*\phi, \omega_+\} = S \cup \{\pi_+, \omega_+\} = S_+$, leading to (4.20).

On the other hand, if (4.20) is satisfied, then so is (4.22), and thus

$$f^*\psi \wedge \omega_+ \simeq 0 \pmod{S \cup \{f^*\phi\}}, \tag{4.23}$$

from which (4.14) follows using (4.21). □

Since the immersion $f\colon M \to M\times\mathbf{R}$ defines a hypersurface, it is natural to expect that the 1-form ψ should be equivalent modulo S_+ to a completely integrable 1-form, which vanishes under f^*. However this is so if, and only if, there is not just one solution to (4.20), but a local foliation of $M\times\mathbf{R}$ by such solutions. The assumption that there is such a local foliation must therefore be made in passing from a 1-stable vector to a completely integrable 1-form on $M\times\mathbf{R}$. This assumption is entirely justified in a wide range of cases. For example, in the analytic case, the Cartan-Kuranishi and Cartan-Kähler theorems ensure that if there is a solution f whose image passes through some point in $M\times\mathbf{R}$, then there will be a solution passing through each point in a neighbourhood of the first.

With this assumption made, it follows immediately from the Frobenius theorem that if there is a solution f to (4.20) if and only if there is a local coordinate function $I\colon M\times\mathbf{R}\to\mathbf{R}$ with

$$f^*\mathrm{d}I = 0. \tag{4.24}$$

Equivalently, $\mathrm{d}I \propto \psi \pmod{S_+}$, or

$$\mathrm{d}I \in S_+ \cup \{\psi\} \quad \text{but} \quad \mathrm{d}I \notin S_+. \tag{4.25}$$

Now consider the prolongation $S^{(1)}$ on $M^{(1)}$ of S. Recall that

$$S^{(1)} = \begin{cases} \theta^a \\ \theta_\pm \\ \theta_\pm^{(1)} = \pi_\pm - p_\pm\omega_\pm \end{cases} \tag{4.26}$$

and $M^{(1)} \cong M\times\mathbf{R}^2$, while the structure equations are

$$\mathrm{d}S^{(1)} = \begin{cases} \mathrm{d}\theta^a & \simeq 0 \\ \mathrm{d}\theta_\pm & \simeq 0 \\ \mathrm{d}\theta_\pm^{(1)} & \simeq \pi_\pm^{(1)} \wedge \omega_\pm \end{cases} \pmod{S^{(1)}}, \tag{4.27}$$

revealing characteristic Pfaffian systems

$$S_\pm^{(1)} = S^{(1)} \cup \{\pi_\pm^{(1)}, \omega_\pm\}. \tag{4.28}$$

Given an exact form $\mathrm{d}I$ in $S_+ \cup \{\psi\}$, it is possible to lift it to give an exact form in $S_+^{(1)}$. Define a projection $\pi\colon M^{(1)} \cong M\times\mathbf{R}^2 \to M\times\mathbf{R}$ by

$$\pi = (\mathrm{id}, \rho) \text{ with } \rho\colon (p_+, p_-) \mapsto \lambda = p_+. \tag{4.29}$$

Then ϕ on $M\times\mathbf{R}$ lifts to $\pi^*\phi \simeq \pi_+ - p_+\omega_+ \pmod{S}$ so that the generator $\theta_+^{(1)}$ in $S^{(1)}$ can be chosen as

$$\theta_+^{(1)} = \pi^*\phi \tag{4.30}$$

without affecting the structure equations (4.27). Following this, and using (4.16),

$$\mathrm{d}\theta_+^{(1)} = \pi^*\mathrm{d}\phi \simeq \pi^*\psi \wedge \omega_+ \pmod{S \cup \{\theta_+^{(1)}\}}. \tag{4.31}$$

Comparing (4.31) with the structure equations (4.27) shows that the coframe for $M^{(1)}$ may be chosen so that

$$\pi_+^{(1)} = \pi^*\psi. \tag{4.32}$$

Lemma. *Suppose that* $\phi \simeq \pi_+ - \lambda\omega_+ \pmod{S}$ *on* $M\times\mathbf{R}$ *has been chosen so that* $\mathrm{d}\phi \simeq \psi\wedge\omega_+ \pmod{S\cup\{\phi\}}$. *Then there is a function* I *on* $M\times\mathbf{R}$ *such that* $\mathrm{d}I \in S_+\cup\{\psi\}$ *but* $\mathrm{d}I \notin S_+$ *if and only if there is a corresponding function* $I^{(1)}$ *on* $M^{(1)}$ *such that* $\mathrm{d}I^{(1)} \in S_+^{(1)}$ *but* $\mathrm{d}I^{(1)} \notin S^{(1)}\cup\{\omega_+\}$.

Proof. Suppose that there is some function I on $M\times\mathbf{R}$ with $\mathrm{d}I \in S_+\cup\{\psi\}$. By the definitions of S_+ and ϕ, $S_+\cup\{\psi\} = S\cup\{\pi_+,\omega_+,\psi\} = S\cup\{\phi,\omega_+,\psi\}$. So the lifted function $I^{(1)} = \pi^*I$ on $M^{(1)}$ satisfies $\mathrm{d}I^{(1)} = \pi^*\mathrm{d}I \in S\cup\{\pi^*\phi,\omega_+,\pi^*\psi\}$. But again, using (4.30) and (4.32), $S\cup\{\pi^*\phi,\omega_+,\pi^*\psi\} = S\cup\{\theta_+^{(1)},\omega_+,\pi_+^{(1)}\} \subset S\cup\{\theta_+^{(1)},\theta_-^{(1)},\omega_+,\pi_+^{(1)}\} = S_+^{(1)}$. That is, $\mathrm{d}I^{(1)} \in S_+^{(1)}$.

Conversely, if $I^{(1)}$ is a function on $M^{(1)}$, then there exists some function I on $M\times\mathbf{R}$ such that $I^{(1)} = \pi^*I$ if and only if $\mathcal{L}_V I^{(1)} = 0$ for all vertical vectors V. The set of vertical vectors is $\ker\pi_*$: those vectors V satisfying $\pi_*V = 0$. Since $I^{(1)}$ is a function, $\mathcal{L}_V I^{(1)} = i_V\mathrm{d}I^{(1)}$, so the lift condition becomes $\mathrm{d}I^{(1)} \in (\ker\pi_*)^\perp$. By definition, $i_{(\pi_*V)}\zeta = i_V(\pi^*\zeta)$ pointwise for any vector V and 1-form ζ, so $(\ker\pi_*)^\perp = \operatorname{im}\pi^*$ at each point in $M^{(1)}$. However, $S_+^{(1)} \subset \operatorname{im}\pi^*$ at each point in $M^{(1)}$, since there are no $\mathrm{d}p_-$ terms in $S_+^{(1)}$. Consequently, any invariant $I^{(1)}$ of $S_+^{(1)}$ is the lift $I^{(1)} = \pi^*I$ of some function I on $M\times\mathbf{R}$.

From this it follows that the differential 1-form $\mathrm{d}I^{(1)} = \pi^*\mathrm{d}I$. Considered differentially, and not pointwise, the image of π^* is slightly different. All of the 1-forms in $S_+^{(1)}$ are lifts of differential 1-forms on $M\times\mathbf{R}$, except for $\theta_-^{(1)}$, which has a p_- coefficient. So if $\mathrm{d}I^{(1)} \in S_+^{(1)}$, it cannot have a $\theta_-^{(1)}$ term. Hence any invariant $I^{(1)}$ of $S_+^{(1)}$ satisfies $\mathrm{d}I^{(1)} \in S\cup\{\theta_+^{(1)},\omega_+,\pi_+^{(1)}\}$. It follows that $\mathrm{d}I \in S\cup\{\phi,\omega_+,\psi\} = S_+\cup\{\psi\}$.

Following the same argument, but leaving out the ψ and $\pi_+^{(1)}$ elements, shows that $\mathrm{d}I^{(1)} \notin S^{(1)}\cup\{\omega_+\}$ if and only if $\mathrm{d}I^{(1)} \notin S\cup\{\theta_+^{(1)},\omega_+\}$, which is satisfied if and only if $\mathrm{d}I \notin S_+$. □

Remark. The condition $\mathrm{d}I^{(1)} \notin S^{(1)}\cup\{\omega_+\}$ means that $I^{(1)}$ is a Riemann invariant of highest order, as discussed in section 3.

The main result of this paper follows from the last two lemmas and the discussion in-between.

Theorem. *Let S be a Pfaffian exterior differential system on as manifold M representing an arbitrary prolongation of a hyperbolic PDE in one dependent and two independent variables, and let $S_\pm$ be the Monge characteristic systems of S. Let $S^{(1)}$ be the first prolongation of S, and $S^{(1)}_\pm$ be the corresponding Monge characteristic systems. Suppose that any solution of the 1-stability conditions provides a local foliation of $M\times\mathbf{R}$. Then S possesses a 1-stable vector $X \in S^\perp_\pm$ if and only if $S^{(1)}$ possesses a Riemann invariant $I^{(1)}$ in $S^{(1)}_\mp$ of highest order.*

Remark. Since the 1-stability solution $f: M \to M\times\mathbf{R}$ and the Riemann invariant $I^{(1)} = \pi^* I$ are related according to (4.24) by $f^*\mathrm{d}I = 0$, there is a one-to-one correspondence between 1-stability solutions (up to a constant) and Riemann invariants (up to functional dependence). So the number of independent Riemann invariants is one more than the number of degrees of freedom in the 1-stable vector field.

To apply this theorem to the Cauchy problem solutions offered by the two methods, it is necessary first to set the Cauchy problem up on both spaces in a consistent way. Let $C^{(1)}: \Sigma \subset \mathbf{R} \to M^{(1)}$ be a 1-dimensional integral manifold of $S^{(1)}$ which is non-characteristic, and let $P: M^{(1)} \to M$ be the bundle projection. Then $C: \Sigma \to \mathbf{R}$ defined by $C = P \circ C^{(1)}$ is a 1-dimensional integral manifold of S, since $C^*S = C^{(1)*}P^*S$ and $P^*S \subset S^{(1)}$. C is also non-characteristic. $C^{(1)}$ and C defined in this way specify the same Cauchy problem on the two spaces.

This gives rise to two different maps $\pi \circ C^{(1)}$ and $f \circ C$ from Σ to $M\times\mathbf{R}$, using the projection $\pi: M^{(1)} \to M\times\mathbf{R}$ of (4.29) and the immersion $f: M \to M\times\mathbf{R}$ of the 1-stability condition (4.14). The map $\pi \circ C^{(1)}$ is fixed once the Cauchy data are specified, while the map $f \circ C$ depends further on the immersion f. So there are, in general, two different images of Σ in $M\times\mathbf{R}$. For a given immersion f, the requirement (2.9) that the first stability system satisfy the Cauchy data is exactly the same as the condition that the two maps from Σ to $M\times\mathbf{R}$ coincide, as shown by the following lemma.

Lemma. *The first stability system $D^1_X(S) = S \cup \{\phi\}$ satisfies the Cauchy data, $C^*D^1_X(S) = 0$, if and only if the immersion $f: M \to M\times\mathbf{R}$ is such that $f \circ C = \pi \circ C^{(1)}$.*

Proof. Firstly, recall that it was assumed that $Q \circ f = \mathrm{id}$ where Q is the projection $M\times\mathbf{R} \to M$. Any function v on M lifts to give a function $w = Q^*v$ on $M\times\mathbf{R}$ which will be independent of λ. The pullback with f returns the original function: $f^*w = f^*Q^*v = v$. Noting that $Q \circ \pi = P$, it follows that $\pi^*w = \pi^*Q^*v = P^*v = P^*f^*w$. Using this information, the pullbacks of w to Σ via the two routes are $C^{(1)*}\pi^*w = C^{(1)*}P^*f^*w =$

C^*f^*w. That is, $C^{(1)*}\pi^* = C^*f^*$ for all functions on $M\times\mathbf{R}$ which are independent of λ. The only question left is whether they coincide for functions involving λ, and to answer this it is sufficient to test a single representative, say λ itself.

Secondly, since $C^*S = 0$ is already assumed, $C^*D^1_X(S) = 0$ is equivalent to $C^*\phi = 0$, or

$$C^*f^*(\pi_+ - \lambda\omega_+) = 0. \tag{4.33}$$

Finally, $C^{(1)*}\pi^*(\pi_+ - \lambda\omega_+) = C^{(1)*}(\pi_+ - p_+\omega_+) = C^{(1)*}\theta^{(1)}_+ = 0$, since $\theta^{(1)}_+ \in S^{(1)}$ and $C^{(1)*}S^{(1)} = 0$. So (4.33) is equivalent to

$$C^*f^*(\pi_+ - \lambda\omega_+) = C^{(1)*}\pi^*(\pi_+ - \lambda\omega_+). \tag{4.34}$$

But since $C^*f^* = C^{(1)*}\pi^*$ for functions lifted from M, it follows that they also coincide for 1-forms lifted from M. So, since C and $C^{(1)}$ are non-characteristic, (4.34) is satisfied if and only if

$$C^*f^*\lambda = C^{(1)*}\pi^*\lambda, \tag{4.35}$$

which proves the lemma. □

In view of the main theorem above, the Gardner and Darboux methods for solving the Cauchy problem are shown to be equivalent if the conditions $C^{(1)*}\mathrm{d}I^{(1)} = 0$ and $C^*D^1_X(S) = 0$ are equivalent once a 1-stability solution or Riemann invariant has been found. If no 1-stability solution or no Riemann invariant exists, then the theorem shows that both methods will fail. It is also not necessary to ask about the existence of two Riemann invariants here. In principle only one is required – so long as it happens to fit the Cauchy data. Of course, to be able to solve the PDE for arbitrary Cauchy data, two invariants are required. But the question here is whether the two methods are equivalent for any given Cauchy data. The answer is provided in the following theorem.

Theorem. *Let S be a Pfaffian exterior differential system on a manifold M representing an arbitrary prolongation of a hyperbolic PDE in one dependent and two independent variables, and let $S^{(1)}$ be the first prolongation of S. Suppose that any solution of the 1-stability conditions provides a local foliation of $M\times\mathbf{R}$. The Cauchy problem for S may be solved by Gardner's method using 1-stable Monge characteristic vectors if and only if the corresponding Cauchy problem for $S^{(1)}$ may be solved by Darboux's method.*

Proof. Take $C^{(1)}\colon\Sigma \to M^{(1)}$, $C\colon\Sigma \to M$ and $\pi\colon M^{(1)} \to M\times\mathbf{R}$ as before. Suppose that $f\colon M \to M\times\mathbf{R}$ is a solution of the 1-stability condition (4.14) and that $I^{(1)} = \pi^*I$ is a

corresponding Riemann invariant, so that $f^*\mathrm{d}I = 0$. Without loss of generality, $I^{(1)}$ may be chosen such that $f^*I = 0$.

By the last lemma, the boundary condition $C^*D^1_X(S) = 0$ for the first stability system is satisfied if and only if $f \circ C = \pi \circ C^{(1)}$. In the proof of the last lemma, if was shown that this last condition is achieved of and only if C^*f^* and $C^{(1)*}\pi^*$ coincide for some coordinate on the fibres of $M \times \mathbf{R}$. From equations (4.19) and (4.25) in the derivation of I, it is apparent that I can be used in place of λ as a local coordinate for the fibres of $M \times \mathbf{R}$. So $f \circ C = \pi \circ C^{(1)}$ if and only if $C^{(1)*}\pi^*I = C^*f^*I$. But $\pi^*I = I^{(1)}$ and $f^*I = 0$, so this is just $C^{(1)*}I^{(1)} = 0$, and the result follows immediately. □

5. Examples

A simple example is provided by the Euler-Poisson-Darboux equation

$$u_{xy} = -n\frac{u_x + u_y}{x + y}. \tag{5.1}$$

For any positive integer n, this equation first becomes Darboux-integrable on the n-th jet bundle $J^n(\mathbf{R}^2, \mathbf{R})$. For example, take $n = 3$. The initial system on $M = M_7 \subset J^2(\mathbf{R}^2, \mathbf{R})$ is

$$S = \begin{cases} \theta^1 = \mathrm{d}u - u_x\mathrm{d}x - u_y\mathrm{d}y \\ \theta_+ = \mathrm{d}u_x - u_{xx}\mathrm{d}x + 3\dfrac{u_x + u_y}{x + y}\mathrm{d}y \\ \theta_- = \mathrm{d}u_y + 3\dfrac{u_x + u_y}{x + y}\mathrm{d}x - u_{yy}\mathrm{d}y \end{cases} \tag{5.2}$$

with structure equations

$$\mathrm{d}S = \begin{cases} \mathrm{d}\theta^1 \simeq 0 \\ \mathrm{d}\theta_+ \simeq \left(-\mathrm{d}u_{xx} - 3\dfrac{u_{xx}(x+y) - 4(u_x + u_y)}{(x+y)^2}\mathrm{d}y\right) \wedge \mathrm{d}x \\ \mathrm{d}\theta_- \simeq \left(-\mathrm{d}u_{yy} - 3\dfrac{u_{yy}(x+y) - 4(u_x + u_y)}{(x+y)^2}\mathrm{d}x\right) \wedge \mathrm{d}y \end{cases} \pmod{S}. \tag{5.3}$$

The system is seen to be already in the required form, with the remaining coframe elements being

$$\begin{aligned}
\omega_+ &= \mathrm{d}x \\
\omega_- &= \mathrm{d}y \\
\pi_+ &= -\mathrm{d}u_{xx} - 3\frac{u_{xx}(x+y) - 4(u_x+u_y)}{(x+y)^2}\mathrm{d}y \\
\pi_- &= -\mathrm{d}u_{yy} - 3\frac{u_{yy}(x+y) - 4(u_x+u_y)}{(x+y)^2}\mathrm{d}x.
\end{aligned} \tag{5.4}$$

The problem is symmetrical in x and y, so the results for both Monge characteristics will be similar. The Monge characteristic system $S_+ = S \cup \{\pi_+, \omega_+\}$ is

$$S_+ = \begin{cases} \mathrm{d}u - u_y\mathrm{d}y \\ \mathrm{d}u_x + 3\dfrac{u_x+u_y}{x+y}\mathrm{d}y \\ \mathrm{d}u_y - u_{yy}\mathrm{d}y \\ \mathrm{d}u_{xx} + 3\dfrac{u_{xx}(x+y) - 4(u_x+u_y)}{(x+y)^2}\mathrm{d}y \\ \mathrm{d}x \end{cases} \tag{5.5}$$

The derived flag for S_+ stabilises at the fourth derived system $\{\mathrm{d}x\}$, so the only Riemann invariant is x. This is not suitable for integrating the PDE since an intermediate integral $I = x$ would violate the independence condition. So the PDE (5.1) with $n = 3$ is not Darboux integrable on the second jet bundle.

However, the stability analysis is a different story. The first stability system is $S \cup \{\phi\}$ where

$$\begin{aligned}
\phi &= \pi_+ - \lambda\omega_+ \\
&= -\mathrm{d}u_{xx} - 3\frac{u_{xx}(x+y) - 4(u_x+u_y)}{(x+y)^2}\mathrm{d}y - \lambda\mathrm{d}x
\end{aligned} \tag{5.6}$$

with exterior derivative

$$\begin{aligned}
\mathrm{d}\phi \simeq \mathrm{d}x \wedge \Bigg(&\lambda_{u_{yy}}\mathrm{d}u_{yy} + \Bigg[\lambda_y + u_y\lambda_u - 3\frac{u_x+u_y}{x+y}\lambda_{u_x} + u_{yy}\lambda_{u_y} \\
&- \left(3\frac{u_{xx}}{x+y} - 12\frac{u_x+u_y}{(x+y)^2}\right)\lambda_{u_{xx}} + 15\frac{u_{xx}}{(x+y)^2} - 60\frac{u_x+u_y}{(x+y)^3} + 3\frac{\lambda}{x+y}\Bigg]\mathrm{d}y\Bigg).
\end{aligned} \tag{5.7}$$

So the 1-stability conditions are

$$\begin{aligned}
&\lambda_y + u_y\lambda_u - 3\frac{u_x+u_y}{x+y}\lambda_{u_x} + u_{yy}\lambda_{u_y} - \left(3\frac{u_{xx}}{x+y} - 12\frac{u_x+u_y}{(x+y)^2}\right)\lambda_{u_{xx}} \\
&\quad = -15\frac{u_{xx}}{(x+y)^2} + 60\frac{u_x+u_y}{(x+y)^3} - 3\frac{\lambda}{x+y} \\
&\lambda_{u_{yy}} = 0.
\end{aligned} \tag{5.8}$$

These two equations generate a series of first-order integrability conditions

$$\begin{aligned}
&\lambda_{u_y} = 0 \\
&\lambda_u - \frac{3}{x+y}\lambda_{u_x} + \frac{12}{(x+y)^2}\lambda_{u_{xx}} = \frac{60}{(x+y)^3} \\
&\lambda_{u_x} - \frac{8}{x+y}\lambda_{u_{xx}} = -\frac{60}{(x+y)^2} \\
&\lambda_{u_{xx}} = \frac{15}{x+y},
\end{aligned} \tag{5.9}$$

where each equation in (5.9) is the integrability condition for the preceding one with the first equation in (5.8).

Once (5.8) has been simplified using (5.9), the general solution is easily found to be

$$\lambda = 15\frac{u_{xx}}{x+y} + 60\frac{u_x}{(x+y)^2} + 60\frac{u}{(x+y)^3} + \frac{k(x)}{(x+y)^3}, \tag{5.10}$$

with an arbitrary function $k(x)$. This solution may be used to construct a Cauchy vector for $S \cup \{\phi\}$, and the function k may be determined from the Cauchy data. Thus the Euler-Poisson-Darboux equation (5.1) with $n = 3$ can be integrated on the second jet bundle using the Gardner method.

Prolonging to the third jet bundle, the system S in (5.2) becomes

$$S^{(1)} = \begin{cases}
\theta^1 = \mathrm{d}u - u_x\mathrm{d}x - u_y\mathrm{d}y \\
\theta_+ = \mathrm{d}u_x - u_{xx}\mathrm{d}x + 3\dfrac{u_x+u_y}{x+y}\mathrm{d}y \\
\theta_- = \mathrm{d}u_y + 3\dfrac{u_x+u_y}{x+y}\mathrm{d}x - u_{yy}\mathrm{d}y \\
\theta_+^{(1)} = \mathrm{d}u_{xx} + p_+\mathrm{d}x + 3\dfrac{u_{xx}(x+y) - 4(u_x+u_y)}{(x+y)^2}\mathrm{d}y \\
\theta_-^{(1)} = \mathrm{d}u_{yy} + 3\dfrac{u_{yy}(x+y) - 4(u_x+u_y)}{(x+y)^2}\mathrm{d}x + p_-\mathrm{d}y
\end{cases} \tag{5.11}$$

with structure equations

$$
\mathrm{d}S^{(1)} = \begin{cases} \mathrm{d}\theta^1 & \simeq 0 \\ \mathrm{d}\theta_+ & \simeq 0 \\ \mathrm{d}\theta_- & \simeq 0 \\ \mathrm{d}\theta_+^{(1)} & \simeq \left(\mathrm{d}p_+ + 3\dfrac{p_+(x+y)^2 + 5u_{xx}(x+y) - 20(u_x+u_y)}{(x+y)^3}\mathrm{d}y\right) \wedge \mathrm{d}x \\ \mathrm{d}\theta_-^{(1)} & \simeq \left(\mathrm{d}p_- + 3\dfrac{p_-(x+y)^2 + 5u_{yy}(x+y) - 20(u_x+u_y)}{(x+y)^3}\mathrm{d}x\right) \wedge \mathrm{d}y \\ & \qquad (\mathrm{mod}\ S^{(1)}). \end{cases} \tag{5.12}
$$

Once again, the system is already in the required form. The derived flag of the Monge characteristic system $S_+^{(1)} = S^{(1)} \cup \{\pi_+^{(1)}, \omega_+\}$ stabilises at the fifth derived system, which is

$$
\begin{cases} \mathrm{d}p_+ - \dfrac{60}{(x+y)^3}\mathrm{d}u - \dfrac{60}{(x+y)^2}\mathrm{d}u_x - \dfrac{15}{x+y}\mathrm{d}u_{xx} \\ \qquad + 3\dfrac{p_+(x+y)^2 - 10u_{xx}(x+y) - 20u_x}{(x+y)^3}\mathrm{d}y \\ \mathrm{d}x. \end{cases} \tag{5.13}
$$

There are now two Riemann invariants,

$$
\begin{aligned} I_1 &= p_+(x+y)^3 - 15u_{xx}(x+y)^2 - 60u_x(x+y) - 60u \\ I_2 &= x \end{aligned} \tag{5.14}
$$

for $S_+^{(1)}$, and I_1 contains p_+, which is a third derivative coordinate, so it has the same order of derivatives as the PDE system represented by $S^{(1)}$. So Darboux integration on the third jet bundle succeeds. Comparison of the intermediate integral $I_1 + g(I_2)$ with the 1-stability solution (5.10) shows the close connection between the two, and illustrates the relation between independent Riemann invariants and degrees of freedom in the 1-stability solution.

A more complicated example is provided by the Fermi-Pasta-Ulam equation

$$
u_{yy} = \kappa(u_x)u_{xx}, \tag{5.15}
$$

where κ is a prescribed function of u_x. This equation was analysed in [6], where it was found to be Darboux integrable on the second jet bundle provided the function κ satisfies the constraint

$$\begin{aligned} 0 = E_4 = {} & 16\kappa^4\kappa''\kappa^{iv} - 24\kappa'^2\kappa^3\kappa^{iv} - 16\kappa^4\kappa'''^2 + 32\kappa^3\kappa'\kappa''\kappa''' + 48\kappa^2\kappa'^3\kappa''' \\ & - 104\kappa^3\kappa''^3 + 372\kappa^2\kappa'^2\kappa''^2 - 630\kappa\kappa'^4\kappa'' + 315\kappa'^6 \end{aligned} \tag{5.16}$$

where $\kappa' \equiv \mathrm{d}\kappa/\mathrm{d}u_x$ *etc*. However, it was found that equation (5.15) is integrable on the second jet bundle using a 1-stable characteristic vector in more general cases where κ satisfies the weaker constraint

$$0 = E_6 = \frac{1}{E_2}\left[4E_4\frac{\mathrm{d}^2E_4}{\mathrm{d}u_x{}^2} - 4\left(\frac{\mathrm{d}E_4}{\mathrm{d}u_x}\right)^2 - \frac{\kappa'}{\kappa}E_4\frac{\mathrm{d}E_4}{\mathrm{d}u_x} - \frac{1}{\kappa}(50\kappa\kappa'' - 99\kappa'^2)E_4{}^2\right] \tag{5.17}$$

where $E_2 = 2\kappa\kappa'' - 3\kappa'^2$. On prolonging to the third jet bundle, it was found that (5.15) is Darboux integrable there under the same weaker constraint (5.17).

Acknowledgements

The authors are grateful to E Fackerell for early discussions on this work.

References

[1] A R Forsyth, *Theory of Differential Equations* Cambridge University Press (1906) Vol V,VI.

[2] R B Gardner, Comm. Pure and Applied Math. **22** (1969) 587–596.

[3] R B Gardner, Comm. Pure and Applied Math. **22** (1969) 597–626.

[4] R B Gardner, N Kamran, *Characteristics and the Geometry of Hyperbolic Equations in the Plane* J. Diff. Equations (1992) To Appear.

[5] R L Bryant, S S Chern, R B Gardner, H L Golddschmidt, P A Griffiths, *Exterior Differential Systems* (Springer Verlag New York Inc. 1991).

[6] E Fackerell, D Hartley, R W Tucker, *An Obstruction to the Integrability of a Class of Non-Linear Wave Equations by 1-Stable Cartan Characteristics* J. Diff. Equations (1993) To Appear.