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Variational Principle for Classical Particle Trajectories in Spaces with Torsion

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Abstract

To comply with recent developments of path integrals in spaces with curvature and torsion we find the correct variational principle for the classical trajectories. Although the action depends only on the length, the trajectories are *autoparallels* rather than geodesics due to the effects of a new torsion force.

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1) Recently it was shown [1,2] that in order to obtain a consistent [3] path integral for particles moving in spaces with curvature and torsion, the classical trajectories have to be autoparallels rather than geodesics. Whereas geodesics are described by

$$\ddot{q}^{\nu} + \bar{\Gamma}_{\alpha\beta}{}^{\nu}\dot{q}^{\alpha}\dot{q}^{\beta} = 0, \tag{1}$$

with Christoffel symbols $\bar{\Gamma}_{\mu\nu}{}^{\lambda}$, the equations of motion of autoparallels are

$$\ddot{q}^{\nu} + \Gamma_{\alpha\beta}{}^{\nu}\dot{q}^{\alpha}\dot{q}^{\beta} = 0, \tag{2}$$

where $\Gamma_{\mu\nu}{}^{\lambda}$ are the affine connections (related to the Christoffel via $\Gamma_{\mu\nu\lambda} = \bar{\Gamma}_{\mu\nu\lambda} + S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu}$ where $S_{\mu\nu\lambda}$ is the torsion tensor).

his result agrees with Einstein's equivalence principle according to which the correct dynamical equations in a noneuclidean space may be derived by transforming the free-particle equations of motion in flat space $\ddot{x}^{i}(t) = 0$ to arbitrary curvilinear coordinates $q^{\mu}(t)$ via some local transformation

$$dx^{i} = e^{i}{}_{\mu}(q)dq^{\mu}. \tag{3}$$

The transformed equations are postulated to describe the motion also in spaces in which the metric

$$g_{\mu\nu}(q) = e^{i}{}_{\mu}(q)e^{i}{}_{\nu}(q) \tag{4}$$

belongs to a non-euclidean geometry. In the presence of torsion, the equations of motion (3) go over into (2), not (1).

It is remarkable that Einstein's equivalence principle *cannot* be applied to the action of the free particle

$$\mathcal{A} = \int_{t_a}^{t_b} dt \frac{M}{2}\dot{x}^{2}(t). \tag{5}$$

Under the transformation (3) it goes over into the curvilinear form

$$\mathcal{A} = \int_{t_a}^{t_b} dt \frac{M}{2} g_{\mu\nu}(q(t))\dot{q}^{\mu}(t)\dot{q}^{\nu}(t) \tag{6}$$

If this is extremized in the standard way [4], one finds the euqations of motion (1), i.e., the the wrong trajectories in the presence of torsion.

The purpose of this note is to resolve this conflict by exhibiting a new variational principle in spaces with torsion where action (6) yields autoparallels after all.

2) The important point to realize is that in order to obtain the correct trajectories from the transformed action (6), the variations $\delta q^\mu(t)$ have to be performed in accordance with *independent* variations $\delta x^i(t)$ in the flat space which vanish at the endpoints (see Fig. 1a). The paths in the two spaces are related by the integral equation

$$q^\mu(t) = q^\mu(t_a) + \int_{t_a}^{t} e_i{}^\mu(q(t'))\dot{x}^i(t')dt' \tag{7}$$

where $e_i{}^\mu(q)$ is the inverse matrix of $e^i{}_\mu(q)$ in (3). It is easy to derive from this an explicit, albeit nonlocal, equation between $\delta x^i(t)$ and $\delta q^\mu(t)$. For this we introduce the auxiliary quantities

$$\delta_h q^\mu \equiv e_i{}^\mu(q)\delta x^i \tag{8}$$

which are equivalent to the independent variations $\delta x^i(t)$ to be performed on the system. From (3) we find the relation

$$\delta\dot{q}^\mu(t) = \frac{d}{dt}\delta_h q^\mu + 2\dot{q}^\nu S_{\nu\lambda}{}^\mu \delta_n q^\lambda - \delta\Delta^\lambda \Gamma_{\lambda\nu}{}^\mu \dot{q}^\nu \tag{9}$$

where

$$\delta\Delta^\mu \equiv \delta q^\mu - \delta_h q^\mu. \tag{10}$$

Using (7) we see that

$$\delta\dot{q}^\mu(t) = \frac{d}{dt}\delta q^\mu(t) \tag{11}$$

so that (9) can be rewritten as

$$\frac{d}{dt}\delta\Delta = -\delta\Delta^\lambda \Gamma^\mu_{\lambda\nu}\dot{q}^\nu + 2\dot{q}^\nu S_{\nu\lambda}{}^\mu \delta_h q^\lambda. \tag{12}$$

This differential equation can be integrated to

$$\delta\Delta(t) = \int_{t_a}^{t} dt' U_{tt'} \ \sigma(t') \tag{13}$$

where

$$U_{tt'} = T \exp\left[-\int_{t'}^{t} G(t'')dt''\right]. \tag{14}$$

We have used vector and matrix notation with

$$G_\mu{}^\lambda(t) = \Gamma_{\mu\nu}{}^\lambda(q(t))\dot{q}^\nu(t) \tag{15}$$

$$\sigma^\mu(t) = 2\dot{q}^\nu(t) S_{\nu\lambda}{}^\mu(q(t))\delta_h q^\lambda(t). \tag{16}$$

If the space is free of torsion, then $\sigma(t) \equiv 0$ and hence $\delta\Delta(t) \equiv 0$. In this case the variations $\delta q^\mu(t)$ coincide with the independent ones $\delta_h q^\mu(t)$. We shall therefore refer to $\delta_h q^\mu(t)$ as the *holonomic* variations of the path $q^\mu(t)$. In the presence of torsion, on the other hand, the variations $\delta q^\mu(t)$ differ from $\delta_h q^\mu(t)$.

Let us calculate the variation of the action which is

$$\delta A = M \int_{t_a}^{t_b} dt \left(g_{\mu\nu}\dot{q}^\nu \delta\dot{q}^\mu + \frac{1}{2}\delta q^\mu \partial_\mu g_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta \right) \tag{17}$$

The most direct way of deriving from δA the equations of motion is by inserting an infinitesimal local "knock" variation

$$\delta_h q^\mu(t) = \epsilon^\mu \delta(t - t_0). \tag{18}$$

In the torsion-free case with $\delta q^\mu(t) \equiv \delta_h q^\mu(t)$ we obtain

$$\delta A \equiv \delta_h A = -\epsilon^\mu M g_{\mu\nu}(\ddot{q}^\nu + \bar{\Gamma}_{\alpha\beta}{}^\nu \dot{q}^\alpha \dot{q}^\beta). \tag{19}$$

Setting this equal to zero produces at once the equations of motion which are those of geodesics, i.e., the correct particle trajectories in the absence of torsion.

In the presence of torsion, the holonomic "knock" variation (18) leads via (13) - (16) to the total variation

$$\delta q(t) = \epsilon\delta(t - t_0) + \delta\Delta(t) \tag{20}$$

with

$$\delta\Delta(t) = \Theta(t - t_0)U_{tt_0}\sigma(t_0) \tag{21}$$

Note that the term $\delta\Delta(t)$ is nonlocal. It has the important effect that while $\delta q^\mu(t_a) = 0$, the variation at the endpoint $\delta q^\mu(t_b) = b^\mu$ is nonzero. The vector b^μ corresponds to the Burgers vector in the physics of dislocations [4]. The situation is illustrated in Fig. 1c.

The nonlocal term modifies the usual derivation of the equations of motion. The time derivative of $\delta\dot{q}(t)$ is

$$\delta\dot{q}(t) = \epsilon\dot{\delta}(t - t_0) + \delta(t - t_0)\Sigma(t_0)\epsilon - G\delta\Delta(t) \tag{22}$$

where $\Sigma_\mu{}^\lambda(t) = 2S_{\mu\nu}{}^\lambda(q(t)\dot{q}^\nu(t)$. Inserting $\delta q^\mu(t)$ and $\delta\dot{q}^\mu(t)$ into (17) we obtain $\delta A = \delta_h A + \delta_a A$, with the anholonomic deviation $\delta_a A$ of the action

$$\delta_a A = -\epsilon^\mu 2MS_{\mu\alpha\beta}(q(t_0))\dot{q}^\alpha(t_0)\dot{q}^\beta(t_0) \tag{23}$$

originating from the second term in (22). The contribution of $-G\delta\Delta(t)$ in (22) cancels exactly the contribution of the term $\delta\Delta^\mu\frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta$ which originates from the second term in δA (17), because of (15) and the identity $\frac{1}{2}\partial_\mu g_{\alpha\beta} - \Gamma_{\mu\{\alpha\beta\}} \equiv 0$ which follows directly from the definitions $g_{\alpha\beta} = e^i{}_\alpha e^i{}_\beta$ and $\Gamma_{\mu\nu}{}^\lambda = e_i{}^\lambda\partial_\mu e^i{}_\nu$ (it expresses the fact that the covariant derivative of the metric tensor vanishes, $D_\mu g_{\alpha\beta} \equiv 0$.).

Hence the total variation of the action is

$$\begin{aligned}\delta A &= -M\epsilon^\mu g_{\mu\nu}\left[\ddot{q}^\nu + \left(\bar{\Gamma}_{\alpha\beta}{}^\nu - 2S^\nu_{\alpha\beta}\right)\dot{q}^\alpha\dot{q}^\beta\right] \\ &= -M\epsilon^\mu g_{\mu n}\left(\ddot{q}^\nu + \Gamma_{\alpha\beta}{}^\nu\dot{q}^\alpha\dot{q}^\beta\right),\end{aligned} \tag{24}$$

the second line following from the first via the identity $\bar{\Gamma}_{\alpha\beta}{}^\nu = \Gamma_{\{\alpha\beta\}}{}^\nu + 2S^\nu_{\{\alpha\beta\}}$. Setting $\delta A = 0$ gives the autoparallels as the equations of motions which is what we wanted to demonstrate.

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[3] There are several consistency requirements, the most stringent one being that when solving the path integral of the hydrogen atom via the Kustaanheimo-Stiefel transformation, the spectrum should be correct. This transformation is nonholonomic and leads to a space with curvature and torsion.

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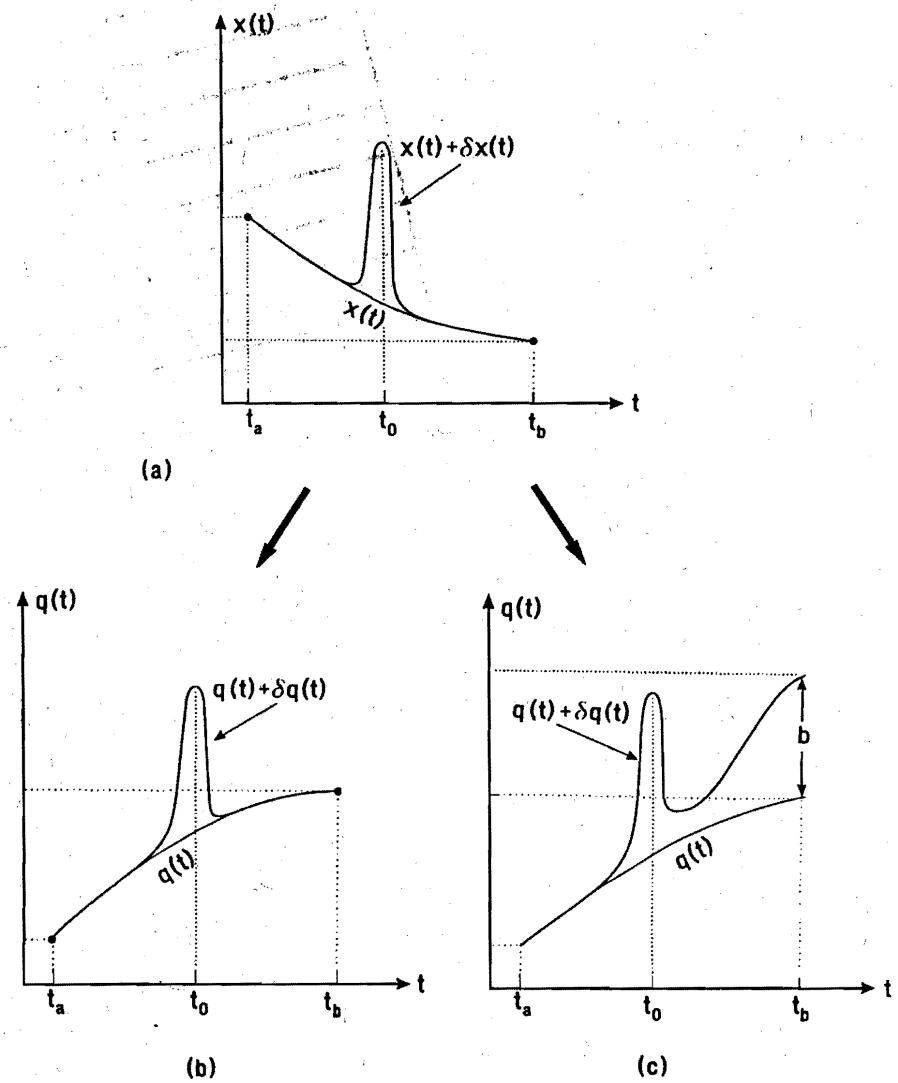
FIGURES

In the usual holonomic case, the paths $x(t)$ and $x(t) + \delta x(t)$ shown in (a) are mapped into paths $q(t)$ and $q(t) + \delta q(t)$ shown in (b). In the nonholonomic case with $S^{\nu}_{\alpha\beta} \neq 0$, however, they go over into $q(t)$ and $q(t) + \delta q(t)$ shown in (b) with a failure to close at t_b which is analogous to the Burgers vector in a solid with dislocations.