SCALE INVARIANCE AND THE BWS QUANTIZATION FOR POWER LAW ONE-DIMENSIONAL POTENTIAL WELLS

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Abstract

We start by computing the classical periods of motion $\tau(E)$ for a particle under the influence of a potential well of the form $U(x) = \alpha|x|^\nu$, with both ν and α positive real constants. Assuming the reflection convention at the origin, we can extend our results to the cases where both ν and α are negative real constants. We also analyse the scale invariance exhibited by these potentials using dimensional arguments directly on the classical equations of motion as well as the more powerful Lie method, appropriate for studying one-parameter symmetry groups of differential equations. The action variables $J(E)$ are obtained from $\tau(E)$ and we reobtain the Bohr-Wilson-Sommerfeld quantization rule for the energy spectrum of all the above potentials. An interpretation of the results is given in the light of semiclassical arguments.

March, 1992

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1. Introduction

Since the advent of Quantum Mechanics in the nineteen twenties, a lot of exact calculations have been made in the context of atomic and molecular physics in many different approaches. A reason is that this is a way of checking whether a new approach is correct or when an alternative approach may shed some light into the physical meaning of some results, etc.

However, despite the great variety of mathematical methods supporting Quantum Mechanics calculations, a considerable number of problems still remain unsolved. To deal with such cases, it is unavoidable to start with approximate solutions. The kind of approximation one should try will strongly depend on what kind of calculation one is interested in. Regarding energy spectra, semiclassical methods give in general very good results[1] (the same cannot always be said for the corresponding wave functions). In particular, for those problems where the computation of action variables are not quite involved, the Bohr-Wilson-Sommerfeld (BWS) quantization rule is extremely convenient for computing atomic energy levels, yielding many times exact results, as in the Coulomb case[2], or even non-spherical symmetrical potentials like the Hartmann potential[3].

In this paper, essentially of a pedagogical nature, we shall study many aspects of power law one-dimensional potential wells: we first consider a family of confining potentials $U(x) = \alpha|x|^\nu$, with both α and ν real positive constants, and then we consider the family of (singular) potentials $U(x) = \alpha|x|^{-\nu}$, $x > 0$, with ν and α being a positive and a negative real constant respectively. The latter is of a non-confining nature, since for $E > 0$ non-bounded motions are allowed.

We will start by computing exactly the classical periods of motion τ (a more precise meaning for this concept will be given later on). We will also show two different methods for obtaining the functional dependence of τ in terms of the total energy E: one based on dimensional arguments directly applied to the classical equation of motion and the other, using the Lie method[4], well suited for obtaining one-parameter symmetry groups of differential equations.

Although for the above family of confining potentials the WKB spectra have already been computed[5], as well as some numerical comparison between WKB and the BWS quantization has been made[6], we shall reobtain the BWS quantization rule using a slightly different approach, based on the results for the classical periods of motion. We

shall generalize our discussion to include the singular potentials $U(x) \propto \frac{1}{|x|^\nu}$, $\nu > 0$ ($x > 0$), but, as we shall see, the BWS quantization rule will work only for the cases where $\nu < 2$ (which fortunately includes the "one-dimensional Coulomb problem").

As far as we know, exact analytical results have already been found only for a very few particular cases within the above families of potentials, including the usual harmonic oscillator ($\nu = 2$ in the first family), the one-dimensional Coulomb potential ($\nu = 1$ in the second family), and a few others. Even though there is a lack of analytical solutions for an arbitrary ν, the periods of motion can be computed exactly. Hence, the action variables $J(E)$ follow imediatly from the relation $\frac{dJ(E)}{dE} = \tau(E)$, making possible the use of the BWS quantization rule for obtaining the corresponding semiclassical energy spectra.

This paper is organized as follows: in section **2** we sketch briefly the evaluation of the periods of motion in terms of E. In section **3** we show that had we been interested only in the functional dependence of $\tau(E)$, we could have used only dimensional arguments and asked for a scale invariance of the poblem. In this section we also discuss the scale invariance from the point of view of the Lie method. In section **4** we compute the action variables and proceed with the BWS quantization to get the associated energy spectra. A connection between the classical results for the periods $\tau(E)$ and the quantum spectra is also presented. Section **5** is left for the conclusions and final remarks.

2. Exact Evaluation of the Classical Periods of Motion

In this section, we shall compute exactly the periods of motion of a test particle of mass m under the influence of the potential $U(x) = \alpha|x|^\nu$ (in fact, $U(x)$ is the potential energy) for any $\nu \neq 0$. Of course, if $\nu > 0$ ($\nu < 0$), we must assume $\alpha > 0$ ($\alpha < 0$) in order to get a potential well and to make sense to talk about periodic motion. In the case of negative ν and α, we shall also assume the reflection convention at the origin and the motion must be considered only in the region $x > 0$.

We call periodic motions those solutions $x(t)$ satisfying the condition $x(t+\tau) = x(t)$ for any t and a finite τ. The minimal τ is naturally called the period of motion, since both the position and velocity of the particle repeat their respective values whenever a time interval equal to τ has passed.

Let us start then by considering a particle of mass m in a general potential $U(x)$ that allows only bounded motions for some range of the total energy E. It follows from

the energy conservation theorem that

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E - U(x))}. \tag{1}$$

As a consequence, the expression for the period of motion may be written as

$$\tau(E) = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}}, \tag{2}$$

where x_1 and x_2 are the turning points determined by the algebraic equation $E = U(x)$. Let us remark that although the period of a one-dimensional motion is obviously a function of the energy, this result is not so direct in three dimensions. However, it can be shown that if a motion is actually a periodic motion, and assuming certain conditions of differentiability, the associated period will be a function only of the energy[7].

Considering first the confining family potentials of the form $U(x) = \alpha|x|^\nu$, with both α and ν positive, we can write

$$\tau(A) = A^{-\frac{\nu}{2}}\sqrt{\frac{8m}{\alpha}}\int_0^A \left(1 - \frac{x^\nu}{A^\nu}\right)^{-\frac{1}{2}} dx, \tag{3}$$

where A is the amplitude of the oscillations and we used that $E = U(A) = \alpha A^\nu$. Observe that, for a given potential, once we have fixed the amplitude A, the energy E is automatically fixed. Hence, there is no problem in working with A instead of E, and this is what we shall do in the following calculations. Only when the energy is made explicitly necessary, shall we rewrite our results in terms of E.

In order to make explicit the functional dependence of $\tau(A)$ we make in (3) the following change of variables $u = \frac{x^\nu}{A^\nu}$, from which

$$\tau(A) = A^{1-\frac{\nu}{2}}\sqrt{\frac{8m}{\alpha}}\frac{I(\nu)}{\nu}, \tag{4}$$

where $I(\nu)$ is just the numerical factor $I(\nu) = \int_0^1 (1-u)^{-\frac{1}{2}} u^{\frac{1}{\nu}-1} du$.

Interesting physical informations can actually be obtained directly from the functional dependence $\tau(A)$, so that for some purposes the computation of the numerical factor $I(\nu)$ can be avoided[8] (we will come back to this point in the next section).

The computation of the periods of motion requires the exact evaluation of $I(\nu)$. Recalling the integral representation of the Euler gamma funcion[9] $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1}\, du$, we identify $I(\nu)$ as the beta function $B(\frac{1}{2}, \frac{1}{\nu})$, where $B(p,q) \equiv \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Hence, the periods of motion are given by

$$\tau(A) = \frac{A^{1-\frac{\nu}{2}}}{\nu}\sqrt{\frac{8\pi m}{\alpha}}\,\frac{\Gamma(\frac{1}{\nu})}{\Gamma(\frac{1}{\nu}+\frac{1}{2})}, \tag{5}$$

where we used that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Substituting $A = (\frac{E}{\alpha})^{\frac{1}{\nu}}$ into (5), we get

$$\tau(E) = \left(\frac{1}{\alpha}\right)^{\frac{1}{\nu}}\frac{\sqrt{8m\pi}}{\nu}\,\frac{\Gamma(\frac{1}{\nu})}{\Gamma(\frac{1}{\nu}+\frac{1}{2})}\,E^{\frac{2-\nu}{2\nu}} \tag{6}$$

in agreement with Landau's result[10] if we identify $\alpha = A$ and $\nu = n$.

Observe that $\nu = 2$ is the only particular case for which the period of motion is independent of the energy[11]. Besides this, choosing as usual $\alpha = \frac{1}{2}m\omega^2$ (with $\nu = 2$), equation (6) leads to the familiar result $\tau = \frac{2\pi}{\omega}$. Another well known example is given by the falling of a body under an uniform gravitational field (neglecting the resistance of the air). The time for a test mass, initially at rest at height h, is simply given by $\tau_{\text{fall}} = \frac{1}{4}\tau$. Substituting into (5) $\alpha = mg$, $\nu = 1$ and $A = h$ we readly get $\tau_{\text{fall}} = \sqrt{\frac{2h}{g}}$, a result known since Galilei's time.

Now, let us turn our attention to the non-confining potentials given by $U(x) = \frac{\alpha}{|x|^\nu}$, with $x > 0$, $\alpha < 0$ and $\nu > 0$. There is then a singularity at the origin, and then we cannot say that the particle undergoes a periodic motion for a given $E < 0$, because it would have necessarily to have crossed the origin.

However, despite this singularity at the origin, a particle which is initially at rest and at a finite distance from the origin, reaches the origin after a finite time interval. This fact naturally suggests that with a slight modification of our problem, a definition of period of motion will be possible. Then, we shall assume the usual reflection convention[12] at the origin, that is, that when the test particle reaches the origin it is instantaneously reflected and then, we shall define the period of motion as twice the time interval necessary for a particle, initially at rest at a finite distance from the origin, to get the origin, that is

$$\tau(A) = 2\int_0^A \frac{dx}{\sqrt{\frac{2}{m}\left(E+\frac{K}{x^\nu}\right)}} = \sqrt{\frac{2m}{K}}\int_0^A x^{\frac{\nu}{2}}\left(1-\frac{x^\nu}{A^\nu}\right)^{-\frac{1}{2}}dx, \tag{7}$$

where, since for this family $\alpha < 0$, we set $\alpha = -K$, $K > 0$ and $E = -\frac{K}{A^\nu}$. Observe that, due to the reflection convention, there is a difference of a factor 2 between definitions for the periods expressed by (7) and (3) respectively.

Making the same change of variables as before ($u = \frac{x^\nu}{A^\nu}$), we get after some manipulations

$$\tau(A) = \frac{A^{1+\frac{\nu}{2}}}{\nu}\sqrt{\frac{2m\pi}{K}}\,\frac{\Gamma(\frac{1}{\nu}+\frac{1}{2})}{\Gamma(\frac{1}{\nu}+1)}, \tag{8}$$

where for this case we identified the integral representation of $B(\frac{1}{\nu}+\frac{1}{2},\frac{1}{2})$. This formula generalizes Landau's result for the singular potentials $U(x) \propto -\frac{1}{|x|^\nu}$, $(x > 0)$ and $\nu > 0$.

We first remark that a kind of one-dimensional Kepler's third law can be obtained from (8). Substituting $K = GMm$ and $\nu = 1$ into (8) it is easy to show that

$$\frac{\tau^2}{A^3} = \left(\frac{3}{4}\right)^2\frac{4\pi^2}{GM}, \tag{9}$$

which exhibits the same property as Kepler's third law in three dimensions, that is, $\frac{\tau^2}{A^3} \propto \frac{1}{GM}$.

Secondly, looking at equation (8), we readily conclude that, in contrast with the confining potentials discussed previously, there is no possibility for a period to be independent of A, since $1+\frac{\nu}{2}$ is always positive for $\nu > 0$. This result, in fact, could have been foreseen by the following reasons: all these potentials allow non-bounded motions for $E \geq 0$. Hence, as $E \to 0_-$, one should already expect the period to increase and tend to ∞. This must happen for all these non-confining potentials.

It is also interesting to note that the power ν determines qualitatively the behaviour of $\tau(A)$. For instance, for $\nu > 2$, as A increases τ decreases. For $0 < \nu < 2$, although the potentials are still of a confining nature, the period increases as A increases. In fact, $\tau \to \infty$ as $A \to \infty$. And finally, for $\nu = 2$, (harmonic oscillator), the period always remains the same no matter how large A is. We shall see in section 4 the quantum mechanical analog of these behaviours.

3. Scale Invariance and the Lie Method

If we were only interested in finding out the functional dependence of $\tau(A)$, it would not be really necessary to compute the numerical factor $I(\nu)$. This fact suggests that this functional dependence may also be obtained through more general arguments. In fact, the functional dependence of $\tau(A)$ will be related to a scale invariance exhibited by the corresponding classical equations of motion. Using Lagrangian Mechanics, this functional dependence is obtained in the book by Loomis and Sternberg[13], and reobtained by Gordon[14] through the use of theorems of differential geometry.

Here, we shall present two alternative derivations: the first one will be a more intuitive method where dimensional analysis is used directly in the classical equations of motion, and the second one, the mentioned Lie method. We remark that we shall work with classical equations of motion, and we will never go out from Newtonian Mechanics.

To show this explicitly, let us consider the classical equations of motion for the family of confining potentials

$$\ddot{x}(t) + \nu\alpha x^{\nu-1} = 0 \quad ; \quad \text{for} \quad x > 0. \tag{10}$$

By a scale transformation in the variables t and x we understand

$$x \longrightarrow \lambda x \quad ; \quad t \longrightarrow \beta t, \tag{11}$$

where both λ and β are real positive constants.

Now, we search for a scale invariance of equation (10). This amounts to ask what relation between λ and β must exist in order that the substitution of (11) into (10) leads to the same equation, that is, to $f(\lambda,\beta;\nu)(\ddot{x} + \nu\alpha x^{\nu-1}) = 0$.

This means that, if $g(x,t) = 0$ is a solution of (10), $g(\lambda x, \beta t)$ will be also a solution. Let us then find out the relation between λ and β that verifies this assertion.

Substituting the scale transformations (11) into (10) we get

$$\frac{\lambda}{\beta^2}\ddot{x} + \lambda^{\nu-1}\nu\alpha x^{\nu-1} = 0. \tag{12}$$

In order to be able to factorize the same constant factor from both terms on the l.h.s. of (12), λ and β must be related as $\beta = \lambda^{1-\frac{\nu}{2}}$. This means that the equation of motion (10) is actually invariant under the scale transformations

$$x \longrightarrow \lambda x \quad ; \quad t \longrightarrow \lambda^{1-\frac{\nu}{2}} t. \tag{13}$$

As a consequence, whenever a physical quantity with the dimension of distance is scaled with λ, all physical quantities posessing the dimension of time must scale with $\lambda^{1-\frac{\nu}{2}}$. Since the period of motion and the amplitude have the dimensions of time and distance respectively, the only way this happens is if $\tau(A) \propto A^{1-\frac{\nu}{2}}$, as in equation (5).

It is worth noting that t in (13) does not change for $\nu = 2$. Then, the period of motion is independent of the amplitude, which is equivalent to the scale invariance of (10) where the time does not scale at all. In other words, if $x(t)$ is a solution, $\lambda x(t)$ is

also a solution. This corresponds to the fact that the classical equation for the harmonic oscillator is linear.

A complete analogous analysis for the non-confining potentials $U(x) = -\frac{K}{|x|^\nu}$, with $x > 0$, $K > 0$ and $\nu < 0$, leads to $\tau(A) \propto A^{1+\frac{\nu}{2}}$. It is interesting to observe that this result (and the same can be said to the family of confining potentials) can be used to construct physical quantities $\mathcal{I}(\tau, A)$ that are actually invariant under scale transformation using well known results of dimensional analysis[15]. For instance, for the non-confining (singular) potentials these invariants must depend on the period and amplitude in the very special combination $\mathcal{I}(\tau, A) = \mathcal{I}(\frac{\tau^2}{A^{2+\nu}})$. The "one-dimensional" Kepler problem corresponds to the choice $\nu = 1$, and hence, we see that Kepler's third law, $\tau^2 \propto A^3$, is nothing but a manifestation of this scale invariance. One could then think of the relation $\tau^2 \propto A^{2+\nu}$ as the "generalization" of Kepler's third law for these one-dimensional potentials[16].

We finish this section by showing that the above scale invariance could also have been analysed with the powerful Lie method[4], appropriate for searching one-parameter symmetry groups of differential equations.

In order to study the symmetries of a classical equation of motion of order n under infinitesimal coordinate transformations of the form $t \longrightarrow t + \varepsilon\xi(x,t)$; $x \longrightarrow x + \varepsilon\eta(x,t)$, which contain of course the scale transformations as a very particular case, one considers the vector field in R^2, $X = \xi(x,t)\partial/\partial t + \eta(x,t)\partial/\partial x$, whose local flow gives the above mentioned infinitesimal transformations. In order to determine the symmetries of a differential equation of order n we need to know how velocities, and accelerations $u^{(k)}$ of any order k, transform. This is carried out by introducing the n-prolongation $X^{(n)}$ which is a vector field in $R \times T^n R$ projecting on X and preserving the contact distribution[4]. In the case at hand, since the differential equations are always of second order, we will need only the second prolongation $X^{(2)}$ of X.

The n-prolongation $X^{(n)}$ is given by (see e.g. H. Stephani[4])

$$X^{(n)} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \eta^{(1)}\frac{\partial}{\partial u^{(1)}} + \ldots + \eta^{(n)}\frac{\partial}{\partial u^{(n)}}, \tag{14}$$

where $u^{(1)}$ is the velocity, $u^{(2)}$ the acceleration, and so on, $u^{(k)}$ the higher order accelerations, and the $\eta^{(k)}$ are defined by the recurrence rule

$$\eta^{(k+1)} = \frac{d\eta^{(k)}}{dt} - u^{(k+1)}\frac{d\xi}{dt}, \tag{15}$$

with $\eta^{(0)} = 0$. Since we are looking for a scale invariance, for which $t \to e^{\epsilon} t \approx (1+\varepsilon)t$ and $x \to e^{\gamma\epsilon} x \approx (1+\gamma\varepsilon)x$, we shall put $\xi(x,t) = t$; $\eta(x,t) = \gamma x$.

A simple calculation shows that in our case,

$$\eta^{(1)} = (\gamma-1)u^{(1)} \equiv (\gamma-1)v \quad ; \quad \eta^{(2)} = (\gamma-2)u^{(2)} \equiv (\gamma-2)a, \tag{16}$$

where $v = u^{(1)}$ and $a = u^{(2)}$, so that the second prolongation of X is simply given by

$$X^{(2)} = t\frac{\partial}{\partial t} + \gamma x\frac{\partial}{\partial x} + (\gamma-1)v\frac{\partial}{\partial v} + (\gamma-2)a\frac{\partial}{\partial a}. \tag{17}$$

The differential equation (10) is represented by the three-dimensional surface Σ in $R \times T^2R$ defined by the equation $a = \alpha\nu x^{\nu-1}$ and the condition for X to generate an infinitesimal symmetry of (10) is that the flow of $X^{(2)}$ preserves Σ, i.e., the tangency condition

$$\left\{X^{(2)}(a - \alpha\nu x^{\nu-1})\right\}_{a=\alpha\nu x^{\nu-1}} = 0, \tag{18}$$

an identity which leads, after simple calculations, to $\gamma = \frac{2}{2-\nu}$. This means, for an infinitesimal ε, that the equations of motion are invariant under

$$t \longrightarrow (1+\varepsilon)t \approx e^{\epsilon} t \quad ; \quad x \longrightarrow (1+\gamma\varepsilon)x \approx e^{\gamma\epsilon} x, \tag{19}$$

which correspond to the flow of $X = t\frac{\partial}{\partial t} + \gamma x\frac{\partial}{\partial x}$. Of course a function $f(x,t)$ invariant under X is a solution of $Xf = 0$, which is solved by looking for first integrals of the characteristic system $\frac{dt}{t} = \frac{dx}{\gamma x}$, i.e., $f(x,t)$ is a function of $xt^{-\gamma}$.

As indicated before, the connection with our previous results, is based on the identification $\lambda = e^{\gamma\epsilon}$. Then, equations (19) are rewritten as $x \to \lambda x$; $t \to \lambda^{\frac{1}{\gamma}} t = \lambda^{1-\frac{\nu}{2}} t$, which are precisely transformations (13).

4. Bohr-Wilson-Sommerfeld Quantization

We will start by obtaining the action variables $J(E)$ from the expressions for the periods $\tau(E)$ and then we proceed with the BWS quantization rule for the above one-dimensional potentials. We shall also compare classical and quantum mechanical results and give a semiclassical interpretation for the connection between them.

Just to introduce notation and basic ideas, let us make a very brief review of the formalism to be used. There are many books which may provide the interested reader with

more details in the Hamilton-Jacobi formalism[17-19]. In particular, for one-dimensional motions see Percival and Richards[20]. Let us consider one-dimensional systems described by a Hamiltonian $H(q,p)$. The Hamilton-Jacobi equation is given by

$$H(q, \frac{\partial S}{\partial q}) + \frac{\partial S}{\partial t}(q,P,t) = 0. \tag{20}$$

A solution $S(q,P,t)$ of this equation is the generating function of a canonical transformation from the canonical variables (q,p) to the new ones (Q,P) in such a way that the transformed Hamiltonian is identically zero.

Since the Hamiltonian is assumed to be time independent, we can write $S(q,P,t) = W(q,P) - Et$, where $W(q,P,t)$ is called Hamilton principal function. With such a decomposition, the Hamilton-Jacobi equation becomes $H(q, \frac{\partial W}{\partial q}) = E$.

For physical systems describing periodic motions (either a rotation or a libration), it is extremely convenient to make a canonical transformation into a new set of canonical variables, called action-angle variables. In particular, using these variables we can obtain the frequencies of the system without solving the problem at all.

The action variable for a one-dimensional periodic motion is defined as $J \equiv \oint p dq$, while the associated angle variable is $\omega \equiv \frac{\partial W}{\partial J}(q,J)$, where we have written the Hamilton principal function in terms of J. Hamilton's equations take the form

$$\dot{J} = \frac{\partial H}{\partial \omega}(J) = 0 \quad ; \quad \dot{\omega} = \frac{\partial H}{\partial J}(J) \equiv \beta(J). \tag{21}$$

From the above equations of motion, we can see that J is actually a constant, and then a direct integration yields $\omega(t) = \beta t + \delta$. It can be shown that β is the frequency associated to the periodic motion of the variable $q(t)$[17], which can be found without solving the problem. We just define J, write the Hamiltonian as $H(J)$ and then compute $\beta = \frac{\partial H}{\partial J}$.

However, since we have already computed the periods of motion (remember that $\tau = \frac{1}{\beta}$), we shall use this relation in the opposite way, that is, instead of evaluating the integral $J \equiv \oint p dq$, we shall obtain $J(E)$ by a direct integration of

$$\frac{dJ(E)}{dE} = \tau(E). \tag{22}$$

Hence, for the confining potentials, after substituting (6) into (22) and integrating, we obtain

$$J(E) = \left(\frac{1}{\alpha}\right)^{\frac{1}{\nu}} \sqrt{8m\pi}\, \frac{\Gamma(\frac{1}{\nu}+1)}{\Gamma(\frac{1}{\nu}+\frac{3}{2})} E^{\frac{2+\nu}{2\nu}}, \tag{23}$$

where we used that $z\Gamma(z) = \Gamma(z+1)$ and the integration constant was chosen such that $J(E) = 0$ for $E = 0$.

Solving (23) for $E(J)$ we obtain

$$E(J) = \left\{ \frac{\alpha^{\frac{1}{\nu}} \Gamma(\frac{1}{\nu} + \frac{3}{2})\, J}{\sqrt{8m\pi}\, \Gamma(\frac{1}{\nu} + 1)} \right\}^{\frac{2\nu}{2+\nu}}. \tag{24}$$

Using now the BWS quantization rule, that is, substituting J by nh ($n = 1, 2, ...$), we get

$$E_n = C(\nu, \alpha) n^{\frac{2\nu}{2+\nu}}, \tag{25}$$

with an obvious definition for $C(\nu, \alpha)$. According to the Correspondence Principle, this formula is supposed to be a good approximation only for large quantum numbers. The energy levels given by (25) will coincide with those obtained by the WKB method if n is replaced by $n + \frac{1}{2}$ (see ref. [5]; for a comparison one must use that $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ and to identify $q = \nu$).

In order to avoid the constant $C(\nu, \alpha)$ in the final results, we can use the ratio of two successive energy levels. Hence, we can summarize our results by writing

$$\frac{E_{n+1}}{E_n} = \left(\frac{n+1}{n} \right)^{\frac{2\nu}{2+\nu}}. \tag{26}$$

Once just one energy level is fixed, all the others are automatically determined by (26).

The harmonic oscillator corresponds to $\nu = 2$, and $\alpha = \frac{1}{2} m\omega^2$. Substituting these values into (25) and computing $C(\nu, \alpha)$, we get the well known result $E_n = nh\omega$, which is correct for large n (that is, in the domain of the correspondence limit).

Looking at (25), we also see that $\Delta E_n \equiv E_{n+1} - E_n$ actually increases (decreases) with n for $\nu > 2$ ($0 < \nu < 2$), so that, the harmonic oscillator is the limiting case (within the family of potentials $U(x) \propto |x|^\nu$, with $\nu > 0$) between those for which ΔE_n increases with n and those for which ΔE_n decreases with n. Although not obvious, this fact is closely related to the functional dependence of the classical period of motion $\tau(E)$. In what follows we shall establish this connection based on semiclassical arguments.

Having in mind that $\frac{1}{\tau(E)} = \frac{dE}{dJ}$, we assume that the Correspondence Principle is valid (large quantum numbers n), and write

$$\frac{1}{\tau(E)} \propto \frac{dE_n}{dn}, \tag{27}$$

where we substituted J by nh and we are considering in this limit n roughly as a continous variable. The meaning of $\frac{dE_n}{dn}$ is such that if $\frac{dE_n}{dn}$ increases (decreases) with increasing n, it means that the spacing between two successive energy levels, $\Delta E_n = E_{n+1} - E_n$ becomes larger (smaller) and larger (smaller) as n increases.

Hence, looking at equation (27) we see that those potentials for which $\tau(E)$ decreases (increases) as E increases will have energy spacing ΔE_n that increases (decreases) as n increases. When $\tau(E)$ is independent of E the potentials will exhibit an equal spacing energy levels.

Let us now pass to analyse the case of the non-confining (singular) potentials. For this case, the total energy is given by $E = -\frac{K}{A^\nu}$, so that $A = \left(\frac{K}{-E}\right)^{\frac{1}{\nu}}$. Substituting this expression into (8) we get

$$\tau(E) = K^{\frac{1}{\nu}} \frac{\sqrt{2m\pi}}{\nu} \frac{\Gamma\left(\frac{1}{\nu} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{\nu} + 1\right)} (-E)^{-\frac{2+\nu}{2\nu}}. \tag{28}$$

Then, putting (28) into (22) and integrating we get the action variables

$$J(E) = K^{\frac{1}{\nu}} \frac{\sqrt{8m\pi}}{\nu - 2} (-1)^{\frac{2+\nu}{2\nu}} \frac{\Gamma\left(\frac{1}{\nu} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{\nu} + 1\right)} E^{-\frac{\nu-2}{2\nu}}. \tag{29}$$

Solving for $E(J)$ we have

$$E(J) = \left\{ (-1)^{\frac{2+\nu}{2\nu}} \left[\left(\frac{1}{K} \right)^{\frac{1}{\nu}} \frac{(\nu - 2)\Gamma\left(\frac{1}{\nu} + 1\right)}{\Gamma\left(\frac{1}{\nu} + \frac{1}{2}\right)} J \right]^{\frac{2\nu}{\nu-2}} \right\}. \tag{30}$$

Using (naively) the BWS quantization rule ($J = nh$, $n = 1, 2, ...$), we finally obtain

$$E_n = C'(\nu, K) n^{\frac{2\nu}{\nu-2}}, \tag{31}$$

with an obvious definition for $C'(\nu, K)$.

However, we see from (31) that E_n diverges as $n \to \infty$ for $\nu \geq 2$ (in contrast with the expected non-confining behaviour $E_n \to 0$ as $n \to \infty$). This displays already in a semiclassical approach, that in these cases the corresponding Hamiltonian operator is not bounded from below, leading to problems in the quantization procedure.

Fortunately, the "one-dimensional Coulomb" problem corresponds to $\nu = 1$. In this case, after a straightforward evaluation of $C'(\nu, K)$ we get for the one-dimensional Coulomb energy levels the following result

$$E_n = -\frac{m\pi^2 K^2}{2h^2} \left(\frac{1}{n^2} \right), \tag{32}$$

in full agreement (for large n) with previous calculations using other methods[21].

5. Conclusions and Final Remarks

In this paper we have covered an extensive material about power law one-dimensional potential wells, including both a classical and quantum mechanical (semiclassical) study of them. We started by computing the classical periods of motion for a given energy E and used this result for obtaining the corresponding action variables and then we used the BWS quantization rule to get the corresponding spectra. The quantum and classical results were compared and the connection between them was analysed under the light of semiclassical arguments. We have also discussed the scale invariance of these systems using simple arguments and the more powerful approach of the Lie method. Hence, we think this material can be of great pedagogical value for enriching classical and quantum mechanical undergraduate courses.

As a final remark, we would like to make a few comments on the case of isochronous problems (those ones where $\tau(E)$ is independent of E). The usual harmonic oscillator is the simplest example of an isochronous problem, for which the BWS quantization rule leads to equal spacing energy levels. However, this does not mean that the energy spacing ($\Delta E_n = E_{n+1} - E_n$) will have the same value for different isochronous problems. The energy spacing will depend, of course, on the exact value of the respective action variables. Let us see this explicitly in one example.

Pippard[8] showed (see also ref. **[10]**), that if a potential well $U(x)$ is deformed into another $\tilde{U}(x)$, in such a way that for any value U_0, the two solutions of $U(x) = U_0$ are separated one from each other the same value that the two solutions of $\tilde{U}(x) = U_0$, the corresponding periods of motion will be the same. An explicit example is the case of the potential $U(x) = \frac{1}{2}kx^2 + \frac{\beta}{2x^2}$, which can be obtained from a parabolic potential by shearing and so its corresponding period of motion is actually independent of E. This can be seen without solving the problem[8], but since there is a straightforward way of solving the corresponding equation of motion, we shall do it as an interesting exercise in a few steps.

Newton's second law for a particle of mass m under the influence of the above potential is simply $m\ddot{x} = -kx + \frac{\beta}{x^3}$. It can be shown that the solution for the previous equation is given by $x(t) = \sqrt{A\cos(2(\omega t + \alpha)) + E/k}$, where $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \frac{\beta}{2x^2}$ and

we defined $\omega^2 = \frac{k}{m}$, and A and α are two integration constants, which are determined by the initial conditions.

Clearly, the period of motion is given by $\tau = \frac{2\pi}{2\omega}$ (this is one half the period for the harmonic oscillator obtained with $\beta = 0$). Hence, the action variable is given by (see equation (22)) $J(E) = \frac{2\pi}{2\omega}E$, so that $E(J) = \frac{2\omega}{2\pi}J$. Using the BWS quantization rule, we get the following energy levels $E_n = 2n(\hbar\omega)$, which clearly shows that the equal energy spacing in this case ($\Delta E_n = 2\hbar\omega$) is twice as that for the harmonic oscillator case ($\beta = 0$). This result is in complete agreement (for large n) with previous calculations obtained through other methods[19].

Acknowledgment

One of us (C.F.) would like to thank to E. Martinez, A. Pacheco, J.V.G. Esteve, F. Marroquim, M.V. Cougo Pinto and A. Seguí for valuable comments and specially to J.L. Cortés for enlightning discussions. We also thank to L.C. Albuquerque for providing us with ref. **[5]**. This work was partially supported by Conselho Nacional de Desenvolvimento Cientifico e Tecnologico of Brazil (CNPq) and by CICYT of Spain. C.F. also thanks the Theoretical Physics Department of Zaragoza for hospitality.

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