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EINSTEIN'S RADIATION FORMULA

AND

GENERAL RELATIVITY

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by

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Abstract

Einstein's radiation formula is supported by data although his theory is not self-consistent. Based on his radiation formula, a consistent theory is developed within the theoretical framework of general relativity. As if predicted by Einstein's remark, some modifications related to the source tensor are necessary. However, the radiation formula remains the same for the binary stars. Concurrently, a number of theoretical issues are clarified. It is determined that, because of radiation, the source tensor is not zero at vacuum. Anti-gravity coupling, discovered by Pauli as a possibility, is a necessary feature. Linearized gravity is valid only if the linearized conservation law is exact. The "gauge condition" is not applicable when self-gravitating is important.

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I. Introduction.

General relativity suggests the existence of gravitational waves. Although gravity waves have never been directly observed, there is indirect evidence which supports energy loss by gravitational radiation [1,2]. However, while Einstein's radiation formula is supported by the observed data [1], one should not consider this as a verification of Einstein's gravitational radiation theory (see §II) because his theory does not produce the radiation formula in a self-consistent manner [2,3]. In stead, one may first deduce the theoretical implications of the radiation formula. Then, based on these implications, one may attempt to develop a self-consistent theory to support the radiation formula. One may also expect that such a theory can be developed within the theoretical framework of general relativity.

In this paper, it will be shown that Einstein's radiation formula has important implications (see §III). It is concluded that, because of radiation, the source tensor at vacuum is necessarily not zero. Antigravity coupling, which is a possibility pointed out by Pauli [4], is a necessary feature of general relativity. The harmonic gauge is not applicable when self-gravitating is important. Moreover, Einstein's radiation formula can, indeed, be supported with a self-consistent theory within the theoretical framework of general relativity.

Einstein's radiation formula is based on his notion of energy-momentum components of the gravitational field t_{ab} [4], a pseudotensor. Since his theory is not covariant, doubts have been raised by Lorentz [5], Levi-Civita [6], and Einstein [7] himself. Moreover, it has been proven by Denisov et. al. [8] that the radiation formula is not an invariant; and the rate of energy emission, depending on the choice of the system of coordinates, may be positive or negative or zero. Thus, it seems, only a covariant theory can be self-consistent.

In the question of radiation, theorists seems to be divided into three schools. One school is happy with the radiation formula and accepts the limitations of Einstein's theory [9]. Another school develops a theory within the theoretical framework of general relativity [10]. But, some [11] have gone so far as to justify an alternative gravity theory. The crucial question, whether Einstein's radiation formula can be supported by a self-consistent theory, has not been answered.

It seems, there was little hope to develop a consistent theory within general relativity to support Einstein's formula because the linearized conservation law is a consequence of linearized gravity which is also the theoretical basis of deriving his formula (see §II). However, recently it is found that the gauge condition may not be applicable for waves [12]. In other words, linearized gravity is not reliable. Thus, it becomes clear that a previously seemingly impossible task would be feasible (see § III). Although a covariant theory may not produce the same radiation formula, it is sufficient to show that their rates of energy loss, on the time average, are the same.

II. Derivation of the Radiation Formula and Self-consistence.

To support Einstein's formula, it would be necessary to identify the problems first. Major steps of the derivation are presented. Much of the presentation is based on Wald's [2] and Yu's [3] analysis.

First, the non-linear Einstein's field equation reads

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = - K\, T_{ab} \,, \tag{1a}$$

where its source $K\, T_{ab}$ generally depends on the space-time metric g_{ab}. The harmonic coordinate condition [2,11] is

$$\frac{\partial}{\partial x^a}(|g|^{\frac{1}{2}} g^{ab}) = 0, \tag{1b}$$

where g is the determinant of the metric. For weak gravity, it is convenient to consider equations, which is expressed in terms of deviations γ_{ab} ($\equiv g_{ab} - \eta_{ab}$) from the flat metric. Then, eqs. (1a) and (1b) are respectively linearized to [2,13,14]

$$G_{ab}{}^{(1)} = - K\, T_{ab} \,, \tag{2a}$$

where

$$G_{ab}{}^{(1)} \equiv \frac{1}{2}\partial^c\partial_c\bar{\gamma}_{ab} - \partial^c\partial_{(b}\bar{\gamma}_{a)c} + \frac{1}{2}\eta_{ab}\partial^c\partial^d\bar{\gamma}_{cd} \;;$$

and

$$\partial^a\bar{\gamma}_{ab} = 0 \,, \tag{2b}$$

where

$$\bar{\gamma}_{ab} \equiv \gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma \,, \text{ and } \quad \gamma \equiv \eta^{ab}\gamma_{ab} \,.$$

The "transverse gauge" (2b) sufficiently reduces (2a) to

$$\frac{1}{2}\partial^c\partial_c\bar{\gamma}_{ab} = - K\ T_{ab}. \tag{2c}$$

It follows from eq. (2b) that the linearized conservation law,

$$\partial^a\ T_{ab} = 0\ , \tag{2d}$$

is necessarily <u>exact</u>. Note that eq. (2d) is also implied directly by eq. (2a) since $\partial^a G_{ab}{}^{(1)} \equiv 0$.

The effective stress-energy tensor of the gravitational field, valid to second order, is assumed to be [2]

$$t_{ab} = G_{ab}{}^{(2)}/K \tag{3a}$$

where

$$G_{ab}{}^{(2)} = G_{ab} - G_{ab}{}^{(1)}$$

and

$$\partial^a t_{ab} = \partial^a G_{ab}{}^{(2)}/K = -\partial^a T_{ab} = 0. \tag{3b}$$

Then, the rate of energy loss due to radiation is

$$\frac{dE}{dt} = -\int t_{a0}dS^a = \frac{-1}{K}\int G_{a0}{}^{(2)}dS^a \tag{3c}$$

where

$$E = \int t_{00}d^3x + \int T_{00}d^3x$$

where the first integral is the total energy associated with γ_{ab}. Note

that E is unchanged if a term $\partial^c\partial^d U_{acbd}$ is added to t_{ab} provided that $U_{acbd} = U_{[ac]bd} = U_{ac[bd]} = U_{bdac}$. Indeed, the terms quadratic in γ_{ab} of the Landau-Lifshitz "pseudotensor" differ from the expression (3a) for t_{ab} by such a term [2]. Note that eq. (3c) requires only $\int[\partial^a G_{ab}{}^{(2)} + \partial^a T_{ab}]d^3x = 0$.

To evaluate formula (3c), one must solve eq. (2c), and

$$\bar{\gamma}_{ab}\ (x^i,\ t) = -\frac{K}{2\pi}\int\frac{1}{R}T_{ab}[y^i,(t - R)]d^3y \tag{4a}$$

where

$$R^2 = \sum(x^i - y^i)^2$$

In the far field from the source, eq. (4a) can be approximated by

$$\bar{\gamma}_{ab} = -\frac{K}{2\pi}\frac{1}{r}\int T_{ab}[y^i,(t - R)]d^3y \tag{4b}$$

To establish relationship between different components of T_{ab}, let us consider the Fourier transform

$$\hat{T}_{ab}(\omega,\ x^i) = -\frac{1}{\sqrt{2\pi}}\int T_{ab}(t,\ x^i)e^{i\omega t}\ dt \tag{5a}$$

Then the linearized conservation law eq. (2d) implies

$$-i\omega\hat{T}_{0b}(\omega,\ x^i) = \sum_{a=1}^{3}\frac{\partial\hat{T}_{ab}}{\partial x^a} \tag{5b}$$

It follows from eq. (5b) that

$$\int\hat{T}^{jk}\ d^3x = -\frac{\omega^2}{2}\int\hat{T}^{00}\ x^k x^j\ d^3x \tag{5c}$$

Consequently,

$$\int T^{jk}\, d^3x = \frac{1}{2}\frac{\partial^2}{\partial t^2}\int T^{00}\, x^k x^j\, d^3x \cdot \tag{6a}$$

Substituting (6a) to formula (4b), one obtains,

$$\bar{\gamma}^{jk} = -\frac{K}{2\pi\, r}\frac{\partial^2}{\partial t^2}\int T^{00}\, x^k x^j\, d^3x \tag{6b}$$

Eq. (6) manifests that if the motion is periodic, then the metric is also periodic.

Then the rate of energy loss formula (3c) becomes

$$\frac{dE}{dt} = -\frac{G}{45}\,(\ddot{q}'_{kj}\,\ddot{q}'^{kj}) \tag{7}$$

where q_{jk} is the quadrupole moment tensor of the material system. Eq. (7) is the famed "quadrupole radiation" formula. In this derivation eq. (2c), eq. (2d), and eq. (3a) are the independent equations.

However, Einstein's theory is not self-consistent. As pointed out by Wald [2] and Yu [3] that the linearized conservation law eq. (2d) implies that "two stars would not orbit each other but would move on geodesics of the flat metric." This means $\ddot{q}'_{jk}$ is zero and therefore no gravitational radiation. The usual formula for the rate of change of orbital period has been derived assuming eq. (7) without reference to eq. (2d) and the analysis by Peters and Mathews [15] is based on Newtonian orbits. That derivation is illegetimate as eq. (7) has been derived from

eq. (2d). The claim that observed data from PSR1913 + 16 has verified Einstein's gravitational radiation theory is therefore groundless [3].

III. The Implications of Einstein's Radiation Formula.

From the viewpoint of linearized gravity, it is hopeless since eq. (2a) implies the linearized conservation law eq. (2d). However, if eq. (2c) can be justified directly, then the situation is very different.

In the derivation, the exact linearized conservation law (2d) is used only to obtain eq. (5b). Due to weak gravity, eq. (5b) can be derived with $\nabla^a T_{ab} = 0$. Note that the accuracy of $G_{ab}{}^{(2)}$, up to second order of deviations, remains the same. Thus, eq. (2d) is actually not needed.

However, eq. (2d) is due to the "gauge condition" (2b). Therefore, a necessary implication of the radiation formula is that eq. (2c) should be justifiable without eq. (2b). (Then, analysis by Peters and Mathews becomes valid.) *If gauge condition (2b) were valid, it should not lead to an unphysical consequence unless eq. (2c) is not valid.* Thus, another implication is that the gauge conditon (2b) is necessarily not valid if self-gravitating is important. This would mean that the gauge condition is not applicable for gravitational waves in general. It should be pointed out that the general applicability of the harmonic coordinate condition actually has never been established. Although there is an equation to calculate the required gauge vector, it remains to be shown that a physically valid solution would always exist [2,4,13,14]. Einstein's radiation formula provides a physical criterion for the applicability of the "harmonic gauge".

Einstein once remarked that the left-hand side of his equation was granite, but the right-hand side (the source) was sand. It will be shown that his radiation formula precisely verifies his remark. In other words, while his radiation formula supports the Einstein tensor, it necessarily implies modifications on the source tensor. The other theoretical implications of Einstein's radiation formula are the following:

1) At vacuum, the source tensor T_{ab} must <u>not</u> be zero because of gravitational radiation. It seems natural to assume that the source tensor is zero at vacuum. However, the radiation formula implies that this assumption is incorrect. If the metric is periodic, then obviously a certain time average of $G_{a0}{}^{(1)}$ is zero (see Appendix A). It follows from eq. (1a) that the time average of $-G_{a0}{}^{(2)}$ is the time average of KT_{a0}. Eq. (3c) implies that the time average of $G_{a0}{}^{(2)}$ is non-zero at vacuum. Therefore, the source tensor T_{ab} cannot be zero at vacuum. Physically, this is necessary since, as pointed out by Stephani [14], a radiation should carry energy-momentum.

2) t_{ab}, the effective stress-energy tensor of the gravitational field, is actually a <u>tensor</u>. The assumption that $G_{ab}{}^{(1)} = 0$ at vacuum is equivalent to

$$G_{ab} = G_{ab}{}^{(1)} + G_{ab}{}^{(2)} = Kt_{ab} \ . \tag{9a}$$

Therefore, t_{ab} is actually a tensor although it appears in eq. (3a) as a pseudotensor. This means that the covariant nature of general relativity is maintained.

3) The coupling of t_{ab} is anti-gravity. Eq.(9a) means that the factual assumption at vacuum is

$$T_{ab} = - t_{ab} \ . \tag{9b}$$

Eq. (9) means that the tensor t_{ab} has anti-gravity coupling. Pauli [4] pointed out that general relativity does not provide a physical interpretation for the sign of the gravitational coupling constant. Thus, in principle, anti-gravity coupling is allowed. Now, this is necessary due to the radiation formula. Moreover, anti-gravity coupling is supported by the fact that the Einstein tensor of a gravitational plane wave has, on the average, a different sign from that of the massive matter [12](see Appendix A).

4) Eq. (2c) is justifiable on a physical ground. To this end, for clarity, eq. (2c) is rewritten in the following form,

$$\partial^c \partial_c \overline{\gamma}_{ab} = -2KT_{ab}(m), \tag{10a}$$

where $T_{ab}(m)$ is the energy-stress tensor for massive matter. As mentioned above, the justification of eq. (10a) cannot be linearized gravity. Note that because $\partial^a G_{ab}{}^{(1)} \equiv 0$, the equation,

$$G_{ab}{}^{(1)} = -K \ T_{ab}(m), \tag{10b}$$

would imply the linearized conservation law. In fact, eq. (10b) is not a necessary condition for eq. (10a). For example, if the cause

of the metric is an electromagnetic plane wave, eq. (10a) is exact although eq. (10b) is not valid [12].

When self-gravitating is not important, eq. (10a) can be justified with linearized gravity. Since neither eq. (2b) nor eq. (10b) is necessary, eq. (10a) can be valid even self-gravitating must be considered. If eq. (10a) has a physical meaning which is generally valid for weak gravity, eq. (10a) would be justified from a physical ground. Eq. (10a) implies, as shown by Newton's law of gravity, that the mass distribution is the source of gravity. In vacuum, eq. (10a) implies that a gravity wave propagates with the speed of light. These physical implications should be independent of the range of validity of the mathematical method which produces eq. (10a). For example, if an electromagnetic plane wave is the source, then eq. (10a) is exact [12](see also Appendix A). It seems to me that observational confirmation of the radiation formula implies that the above justification is valid.

5) However, without the gauge condition, eq. (9a) and eq. (10a) imply that tensor t_{ab} becomes

$$t_{ab} = [G_{ab}{}^{(2)} + H_{ab}{}^{(1)}]/K \tag{11a}$$

where

$$H_{ab}{}^{(1)} = \frac{1}{2}(\eta_{ab}\partial^c L_c - \partial_b L_a - \partial_a L_b), \text{ and } \quad L_a = \partial^c \overline{\gamma}_{ca} .$$

Eq. (11a) implies that eq. (3c) would be modified. However, if the motion is periodic, on the average, the tensor t_{a0} remains essentially $G_{a0}{}^{(2)}/K$ as assumed.

6) Eq. (10a) and eq. (11a) imply that, in practice, Einstein equation must be extended to the following form,

$$R_{ab} - \frac{1}{2} Rg_{ab} = - K[T_{ab}(m) - t_{ab}(g)] = - KT_{ab} \qquad (11b)$$

where $T_{ab}(m)$ is the stress tensor for massive matter and $t_{ab}(g)$ is for the field energy. Now it is clear, if the stress tensor T_{ab} is zero at vacuum, then it is not possible to have gravitational radiation since $t_{ab} = 0$ (see also [10,16]). Note that eq. (11b) is similar to the suggestion [5,6] of Lorentz and Levi-Civita.

7) Eq. (11b) implies

$$\nabla^a T_{ab}(m) = \nabla^a t_{ab}(g) = 0. \qquad (11c)$$

Energy-momentum conservation requires that $T_{ab}(m)$ and $t_{ab}(g)$ are conserved separately. The gravitational force, as shown in the geodesics, has no separate term as the other force such as the Lorentz force. Thus, eq. (11c) is necessary.

8) Eq. (11) implies

$$G_{ab}{}^{(2)}/K + T(m)_{ab} = t_{ab} - G_{ab}{}^{(1)}/K \qquad (12a)$$

and

$$\partial^a G_{ab}{}^{(2)}/K + \partial^a T(m)_{ab} = \partial^a t_{ab} \simeq \nabla^a t_{ab} = 0 \qquad (12b)$$

because $\partial^a G_{ab}{}^{(1)} \equiv 0$. Note that $\partial^a t_{ab} = 0$ no longer implies that

$\partial^a T(m)_{ab} = 0$ (see eq. (3b)). Thus, $\partial^a G_{ab}{}^{(2)}$ related directly to the energy-momentum of matter. Therefore, it would be interest ing to compare eq. (12b) with $\nabla^a T(m)_{ab} = 0$. It follows from eq. (12b) that eq. (3c) implies that $\int \partial^a t_{ab}\, d^3x = 0$.

9) It follows from eq. (12b) that approximately

$$\frac{dE}{dt} = -\int t_{a0} dS^a = -\frac{1}{K}\int G_{a0}{}^{(2)} dS^a + \frac{1}{2K}\int (\partial_0 L_a + \partial_a L_0) dS^a \qquad (12c)$$

Note that the second integral comes from $\partial^a G_{ab}{}^{(1)} \equiv 0$. Since such a relation is independent of the physical process, from the view-point of physics, the second integral is irrelevant. Based on solution (6), calculation shows that the time average of the second integral is zero. Then eq. (12c) is reduced back to eq. (3c).

In summary, assuming Einstein's radiation formula is valid, as if predicted by Einstein's remark, modifications on the source tensor are necessary. Remarkably, this means that at vaccum the source tensor is not zero, and that its coupling is anti-gravity. Then, after the necessary modifications, the radiation formula remains the same. Thus, this modi-fication process is self-consistent.

IV. Conclusion and Discussion.

It is interesting that a self-consistent covariant theory can be developed to support Einstein's radiation formula. The source tensor T_{ab} at vacuum was controversial [10] since it was assumed to be zero. Now,

based on his formula, the source tensor is determined to be non-zero at vacuum. Consequently, t_{ab} must have anti-gravity coupling. Physically, this is meaningful since the field tensor t_{ab} is induced by $T_{ab}(m)$. Moreover, t_{ab} only appears to be a pseudotensor. Although eq. (10a) is not covariant, it is an approximation after the coordinate system has been chosen. Now, it seems, much of the severe criticism [11] of general relativity has become meaningless. They are as follows:

i) Einstein's quadrupole formula for gravitational radiation is not a corollary of general relativity.

ii) When a gravitational field and matter are taken in conjunction, the general theory of relativity has not, and cannot have, energy momentum conservation laws.

iii) It does not follow from general relativity, in principle, that a double star losses its energy by gravitational radiation.

iv) General relativity does not have the classical Newtonian limit and, hence, it does not satisfy a fundamental physical principle, that is the correspondence principle.

Some colleagues have linked the validity of "gauge" and linearized gravity with Einstein's radiation formula. The fact is that both linearized gravity and the "gauge condition" is inconsistent with the radiation formula. Either of them requires the approximate linearized conservation law to become exact (although such a conservation law does not assure the gauge condition [12]). Since self-gravitating is inconsistent with the linearized conservation law, the gauge condition (2b) cannot be valid for such cases.

From the viewpoint of physics, radiation should be independent of the mathematical method used. In this paper, the justification of the fundamental eq. (10a) is based on physical considerations. It seems, once the "linearized" equation is accepted, in this new derivation, the weakness related to linearized conservation law no longer exists. Damour [17] has concluded that nearly all aspects of approximation methods need to be thoroughly re-investigated. It seems that Einstein's radiation formula also supports Damour's conclusion.

While this analysis supports Einstein's radiation formula in a self-consistent manner, this theory may not be satisfactory to fundamentalists since it is based on Einstein's radiation formula. When there is no radiation, this analysis does not exclude that t_{ab} is zero at vacuum. This theory is incomplete since eq. (10a) is not exact as shown by Schwartzschild's solution. Nevertheless, in view of the absence of a commonly accepted theory, the present analysis may serve as an interim theory. It remains to develop an exact theory to handle more general cases. It is interesting to note that Yu's [10] Ω-field theory of gravitation implies a non-zero energy-momentum tensor at vacuum. It would be interesting to investigate whether Yu's theory supports Einstein's radiation formula.

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Appendix A: Radiation and Anti-Gravity Coupling.

In Einstein's theory, an unverified assumption seems to be that the source tensor is zero at vacuum. This assumption is not consistent with the understanding that radiation carries energy-momentum. It will be shown that under a very broad conditon this assumption is inconsist with his radiation formula. Thus, anti-gravity coupling of radiation is actually a necessary feature of Einstein's theory.

A main difficulty in general relativity is that, so far there is no exact solution for gravitational radiation generation. Conjectures based on linear theories are not necessarily valid [16]. However, recently there are exact solutions for gravitational waves which accompany electromagnetic plane waves [12]. It seems that the characteristics of such gravitational waves would facilitate understanding of gravitational waves in general. For simplicity, a circularly polarized plane wave, propagating in the z-direction, is selected as an example. An advantage of such an example is that rotational invariants with respect to the z-axis are constants.

Let a circularly polarized electromagnetic wave be,

$$A_x = \frac{1}{\sqrt{2}} A_0 \cos[\omega(t - z)], \quad A_y = \pm\frac{1}{\sqrt{2}} A_0 \sin[\omega(t - z)], \qquad \text{(A1)}$$

the transverse metric elements are of the following form,

$$g_{xx} = -1 - C_1 + B_\alpha \cos[2\omega(t - z) + \alpha],$$

$$g_{yy} = -1 - C_1 - B_\alpha \cos[2\omega(t - z) + \alpha],$$

$$g_{xy} = \pm B_\alpha \sin[2\omega(t - z) + \alpha] \ . \qquad \text{(A2)}$$

where

$$B_\alpha = KA_0{}^2[\cos \alpha]/2.$$

This is a circularly polarized gravitational wave. The constant C_1 is a small numbers for a not too strong wave. The source tensor is:

$$T_{lk} = P_l P_k g^{mn} A_m A_n - g^{mn}(\nabla_l A_m - \nabla_m A_l)(\nabla_k A_n - \nabla_n A_k), \qquad \text{(A3)}$$

where

$$P_x = P_y = 0, \ P_t = P_z = \omega$$

because the photon tensor must be included.

The invariants are: $(g_{xx} + g_{yy})$, $g_{tt}(= -g_{zz})$, $g_{tz}(= 0)$, R_{tt} $(= R_{zz} = - R_{tz})$, $T(E)_{tt}$ $(= T(E)_{zz} = - T(E)_{tz})$, T_{tt} $(= T_{zz} = -T_{tz})$, g and G $(= g_{xx}g_{yy} - g_{xy}^2)$. Note that g is the determinant of the metric, $T(E)_{lk}$ is the electromagnetic tensor, R_{lk} is the Ricci curvature tensor. The other components of $T(E)_{lk}$, T_{lk}, and R_{lk} are zero. Using equations (A1) and (A2), one obtains the constants,

$$R_{tt} = -KT_{tt} = 2\omega^2 B_\alpha{}^2/G \ , \ G = (1 + C_1)^2 - B_\alpha{}^2, \qquad \text{(A3)}$$

and

$$T(E)_{tt} = \omega^2 A_0{}^2(1 + C_1 - B_\alpha \cos \alpha)/2G \qquad \text{(A4)}$$

Note that the useful characteristics are:

1) T_{tt} is not zero if the wave amplitude B_α is not zero;

2) The coupling of T_{ab} is anti-gravity since R_{tt} is positive.

3) $g_{tk} = 0$ for $k = x, y, z$; but $T_{tz} = -T_{tt} \neq 0$;

4) $\gamma_{mn} \neq 0$ for $m,n = x, y$; but $T_{mn} = 0$;

5) $\partial^c\partial_c\bar{\gamma}_{ab} = 0$, but $\frac{1}{2}\partial^c\partial_c\bar{\gamma}_{ab} \neq - K\, T_{ab}$.

This shows that gravitational radiation is associated with anti-gravity coupling. Eq. (10a) is exact, but eq. (2c) is misleading. In particular, $g_{tz} = 0$ does not imply $T_{tz} = 0$. This example is interesting because it satisfies linearized gravity to the first order.

For a plane gravitational wave associated with massless particles, independent of the physical cause of the source tensor T_{lk}, the Einstein equation is reduced to a differential equation with respect to u [12],

$$[G'' - g_{xx}'g_{yy}' + (g_{xy}')^2 - G'(g'/2g)]/2G = -K\, T_{tt}. \qquad \text{(A5)}$$

Gravitational plane waves are generally believed to be associated with gravitons of spin two just as electromagnetic plane waves are associated with photons of spin one. From the viewpoint of mathematics, a circularly polarized gravitational monochromatic plane wave, independent of the physical origin of the source tensor, must have the (A2) form. Then, the above five characteristics would follow.

Now consider the case that, for $a \neq t$, $\bar{\gamma}_{ta} = 0$. This orthogonal condition is satisfied by the above example and many other metrics

[18]. Then, one has

$$G_{at}{}^{(1)} = H_{at}{}^{(1)} = -\frac{1}{2}(\partial_a L_t + \partial_t L_a)$$

$$= -\frac{1}{2}\partial^t(\partial^b \bar{\gamma}_{ba} + \partial_a \bar{\gamma}_{tt}). \qquad \text{(A6)}$$

Eq. (A6) implies that if the metric is periodic, then for each space point the time average of $G_{at}{}^{(1)}$ over a period is zero. On the other hand, if T_{ab} were zero at vacuum, then

$$G_{at}{}^{(1)} = - G_{at}{}^{(2)}. \qquad \text{(A7)}$$

Thus, the time average of $G_{at}{}^{(2)}$ is also zero. This in turn implies that $dE/dt = 0$ (i.e. of the third order of deviations) according to eq. (3c). If the metric is only almost periodic, the difficulty remains. For such a case, the time averge of $G_{at}{}^{(1)}$ is zero for a certain time interval which may depend on the space point considered. For the same time interval, the time average of $G_{at}{}^{(2)}$ is also zero. Since I (= dE/dt) and $dI/d\Omega$ are negative ($d\Omega$ is a solid angle), for some index a (= x,y,z) the time average of $G_{at}{}^{(2)}$ over a time interval is positive. If the motion is almost periodic, then the metric must also be almost periodic.

In conclusion, the above analysis shows that Einstein's radiation formula actually implies that the source tensor is not zero at vacuum if $\bar{\gamma}_{ta} = 0$ for $a \neq t$.