Area-preserving diffeomorphisms of the open membranes and the w_∞ algebra

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The area-preserving algebras are examined for a square and an annulus. It is shown that they contain the w_∞ algebra as their subalgebra.

1. Introduction

The quantum-mechanical theories of extended objects have recently been studied extensively. In particular one-dimensional extended objects, strings, have received much attention as a candidate for the unification theory. The properties of other extended objects such as membranes or, more generally, p-branes are not fully known, since the non-linearities of the theories cannot be gauged away. However these extended objects attract us very much and we have had several researches along this line. For instance, the question whether membranes contain massless states was investigated. Kikkawa and Yamasaki showed, calculating the Casimir energy, that the bosonic membrane does not yield massless particles [1]. For supermembranes Bars, Pope and Sezgin verified the existence of massless states [2]. The action of a membrane is

$$S = -\frac{1}{T}\int d^3\sigma\sqrt{-\det\partial_i X^\mu \partial_j X_\mu}, \tag{1}$$

where T is a constant, $\sigma^i (i = 0, 1, 2)$ are the world-volume coordinates, and $X^\mu(\sigma)(\mu = 0, \cdots, D-1)$ gives the embedding of the world-volume in a flat D-dimensional space-time. This is the analogous extension of a relativistic particle action and decides the motion of a membrane with any different topology. To obtain a solution concretely we fix the gauge freedom. As such we take up the light cone gauge. For this gauge there exists a residual symmetry in the membrane action. It is induced by a group of area-preserving diffeomorphisms of the two-dimensional membrane surface. The area-preserving algebras of a sphere [3], a torus [4], a Klein bottle and a real projective plane [5] were found. The algebras of the torus and the Klein bottle admit a central extension [4],[5], while those of the sphere and the real projective plane do not [5],[6]. On the other hand Bars obtained the quantum algebra for area-preserving diffeomorphisms by defining an operator product expansion and concluded that the algebra does not have any anomalies [7].

Up to the knowledge of the present author all the membranes which have been studied are restricted to be closed. Hence, in the present paper, we work out the area-preserving diffeomorphisms of *open membranes.*

2. the open membrane

Classically the action(1) is equivalent to

$$S=-\frac{1}{T}\int d^3\sigma\sqrt{-g}(g^{ij}\partial_i X^\mu\partial_j X_\mu-1), \tag{2}$$

where g_{ij} is a metric of the world-volume. We hereafter take up this action. Next we introduce an ADM parametrization of g_{ij} in terms of a shift-vector $N^a(a=1,2)$, a lapse function $N^0=N_0$ and a 2-metric h_{ab} [8]:

$$\begin{aligned}
g_{00} &= -(N_0)^2+N_aN_bh^{ab},\ g_{0a}=g_{a0}=h_{ab}N^b=N_a,\\
g_{ab} &= h_{ab},\\
g^{00} &= -(N_0)^{-2},\ g^{0a}=g^{a0}=N^a(N_0)^{-2},\\
g^{ab} &= h^{ab}-N^aN^b(N_0)^{-2},
\end{aligned} \tag{3}$$

where $h^{ab}h_{bc}=\delta^a_c$. Then we have

$$\sqrt{-g}=N_0\sqrt{h}. \tag{4}$$

With these new variables the action(2) is written as

$$\begin{aligned}
S &= -\frac{1}{T}\int d^3\sigma\sqrt{h}\left\{-N_0-\frac{1}{N_0}\partial_0X^\mu\partial_0X_\mu+\frac{2N^a}{N_0}\partial_0X^\mu\partial_aX_\mu\right.\\
&\quad\left.+\left(N_0h^{ab}-\frac{N^aN^b}{N_0}\right)\partial_aX^\mu\partial_bX_\mu\right\}.
\end{aligned} \tag{5}$$

We develop the Hamiltonian formulation according to Dirac's algorism [9]-[11]. The momenta conjugate to N_i, h_{ab} and X^μ are

$$\pi^i\equiv\frac{\partial\mathcal{L}}{\partial\dot{N}_i}\approx 0, \tag{6}$$

$$\pi^{ab}\equiv\frac{\partial\mathcal{L}}{\partial\dot{h}_{ab}}\approx 0, \tag{7}$$

$$P_\mu\equiv\frac{\partial\mathcal{L}}{\partial(\partial_0X^\mu)}=\frac{2\sqrt{h}}{TN_0}(\partial_0X_\mu-N^a\partial_aX_\mu). \tag{8}$$

Equations(6) and (7) become the primary constraints. The total Hamiltonian is given by

$$\mathcal{H}_T=N^iH_i+v_a\pi^a+v_{ij}\pi^{ij}, \tag{9}$$

where

$$\begin{aligned}
H_0 &\equiv \frac{T}{4\sqrt{h}}P^\mu P_\mu-\frac{\sqrt{h}}{T}+\frac{\sqrt{h}}{T}h^{ab}\partial_aX^\mu\partial_bX_\mu,\\
H_a &\equiv P^\mu\partial_aX_\mu,
\end{aligned}$$

and v_a and v_{ij} are Lagrange multipliers. The consistency conditions for the primary constraints provide the secondary constraints,

$$H_i\approx 0, \tag{10}$$

$$\phi_{ab}\equiv h_{ab}-\partial_aX^\mu\partial_bX_\mu\approx 0, \tag{11}$$

and no further constraints exist. The proper linear combination of the constraints(6), (7), (10) and (11) leads to the first and second class constraints. The first class constraints are

$$H_0'\equiv H_0-\frac{T}{\sqrt{h}}\partial_a\partial_bX^\mu P_\mu\pi^{ab}\approx 0, \tag{12}$$

$$H_a'\equiv H_a-2\partial_ch_{ab}\pi^{bc}-2h_{ac}\partial_b\pi^{bc}+\partial_ah_{bc}\pi^{bc}\approx 0, \tag{13}$$

$$\pi^i \approx 0. \tag{14}$$

The second class constraints are

$$\pi^{ab} \approx 0, \tag{15}$$

$$\phi_{ab} = h_{ab} - \partial_a X^\mu \partial_b X_\mu \approx 0. \tag{16}$$

These first class constraints generate the gauge transformations.

Now we fix the gauge freedom by the following gauge conditions:

$$G_1 = X^+ - P_0^+ \sigma^0 \approx 0, \tag{17}$$

$$G_2 = P^+ - P_0^+ \approx 0, \tag{18}$$

$$G^0 = (N^0)^2 - h \approx 0, \tag{19}$$

$$G^a = N^a \approx 0, \tag{20}$$

where

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{D-1}), \tag{21}$$

$$P^+ = \frac{1}{\sqrt{2}}(P^0 + P^{D-1}), \tag{22}$$

and P_0^+ is the constant mode of P^+. The gauge conditions (17)-(20) turn the first class constraints(12)-(14) into second class constrains. However the constraint

$$\epsilon^{ab}\partial_b H'_a \approx 0 \tag{23}$$

is still first class ($\epsilon^{12} = -\epsilon^{21} = 1$). This constraint(23) generates the area-preserving diffeomorphisms

$$\delta X^\mu = \eta^i \partial_i X^\mu, \tag{24}$$

where the parameters η^a satisfy

$$\eta^0 = 0, \tag{25}$$

$$\partial_a \eta^a(\sigma^b) = 0 \tag{26}$$

(see [3], [7] and [10]). Now we discuss the area-preserving diffeomorphisms of the open membrane ($-\frac{\pi}{2} \le \sigma^a \le \frac{\pi}{2}$). The particular solution of Eq.(26) is

$$\eta^a = \epsilon^{ab}\partial_b \Lambda(\sigma^c), \tag{27}$$

where $\Lambda(\sigma^c)$ is a σ^0-independent arbitrary function. Then the generator of the area-preserving diffeomorphisms is represented as

$$L_\Lambda = \int d^2\sigma \Lambda \epsilon^{ab} \partial_b H'_a. \tag{28}$$

Here we impose the open membrane boundary conditions

$$\partial_a X^\mu(\sigma) = 0, \quad for\ \sigma^b = -\frac{\pi}{2}, \frac{\pi}{2}, \quad (a \ne b), \tag{29}$$

to give a vanishing surface term. If we introduce the Dirac bracket with the second class constraints, we can set the second class constraints strongly equal to zero [11]. Instead of doing so we use the constraints to eliminate $X^+, P^+, X^-, P^-, N_0, \pi^0, N_a, \pi^a, h_{ab}$ and π^{ab}; we consider the reduced phase space made up of canonical pairs(X^I, P^I) ($I = 1 \cdots D-2$). From Eqs.(13), (15), (17) and (18) we obtain

$$\partial_a X^- = \frac{1}{P_0^+} P^I \partial_a X^I. \tag{30}$$

This constraint determines X^- in terms of the canonical pair (X^I, P^I). The integrability condition for it has the form

$$\epsilon^{ab}\partial_a X^I \partial_b P^I = 0. \tag{31}$$

This is equivalent to the first class constraint(23); the integrability condition(31) generates the area-preserving diffeomorphisms. Therefore the generator(28) reduces to

$$L_\Lambda = \int d^2\sigma \Lambda \varepsilon^{ab}\partial_a X^I \partial_b P^I, \quad I = 1, \cdots, D-2. \tag{32}$$

In the first place we study the square $(-\frac{\pi}{2} \le \sigma^1, \sigma^2 \le \frac{\pi}{2})$ as an example of the open membrane. We expand the parameter $\Lambda(\sigma)$ in a power series as

$$\Lambda(\sigma) = \sum_{k_1,k_2} \Lambda_{(k_1,k_2)}(\sigma^1)^{k_1}(\sigma^2)^{k_2}, \quad k_1, k_2 \in Z. \tag{33}$$

On inserting Eq.(33) into Eq.(32), we have

$$\begin{aligned} L_\Lambda &= \int d^2\sigma \sum_{k_1,k_2} \Lambda_{(k_1,k_2)}(\sigma^1)^{k_1}(\sigma^2)^{k_2}\varepsilon^{ab}\partial_a X^I \partial_b P^I \\ &= \sum_{k_1,k_2} \Lambda_{(k_1,k_2)} L_{(k_1,k_2)}, \end{aligned} \tag{34}$$

where

$$L_{(k_1,k_2)} = \int d^2\sigma (\sigma^1)^{k_1}(\sigma^2)^{k_2}\varepsilon^{ab}\partial_a X^I \partial_b P^I. \tag{35}$$

Since X^I and P^I are independent in the Hamiltonian formulation, we further impose the boundary conditions

$$\partial_a P^I(\sigma) = 0, \quad for\ \sigma^b = -\frac{\pi}{2}, \frac{\pi}{2}, \quad (a \ne b). \tag{36}$$

Equations(8) with the gauge constraints (19) and (20) lead to the fact that Eqs.(36) are the conditions for $\partial_0 X^I$. The Poisson brackets of these generators $L_{(k_1,k_2)}$ are calculated as

$$\left\{L_{(k_1,k_2)}, L_{(l_1,l_2)}\right\} = (k_1 l_2 - k_2 l_1) L_{(k_1+l_2-1, k_2+l_2-1)}, \tag{37}$$

with the conditions (29) and (36). In particular we take up a subset of the basis of polynomial functions

$$(\sigma^1)^{m+i-1}(\sigma^2)^{i-1}, \quad i \ge 2, \tag{38}$$

and define the generators corresponding to the basis(38) as

$$L_m^{(i)} \equiv \int d^2\sigma (\sigma^1)^{m+i-1}(\sigma^2)^{i-1}\varepsilon^{ab}\partial_a X^I \partial_b P^I. \tag{39}$$

These generators obey the following algebra:

$$\left\{L_m^{(i)}, L_n^{(j)}\right\} = [(j-1)m - (i-1)n] L_{m+n}^{(i+j-2)}, \tag{40}$$

which is the w_∞ algebra [12],[13]. This algebra contains the Virasoro algebra

$$\left\{L_m^{(2)}, L_n^{(2)}\right\} = (m-n) L_{m+n}^{(2)}, \tag{41}$$

and admits the central extension only in this Virasoro sector

$$\left\{L_m^{(2)}, L_n^{(2)}\right\} = (m-n) L_{m+n}^{(2)} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{42}$$

In the next place we analyze another open membrane, an annulus. This is homeomorphic to a cylinder. Thus we can choose the basis set of functions as

$$e^{in_1\sigma^1}(\sigma^2)^{n_2}, \quad n_1, n_2 \in Z, \tag{43}$$

where $0 \le \sigma^1 \le 2\pi$ and $-\frac{\pi}{2} \le \sigma^2 \le \frac{\pi}{2}$. In a similar way the parameter $\Lambda(\sigma)$ is expanded as

$$\Lambda(\sigma) = \sum_{n_1,n_2} \Lambda_{(n_1,n_2)} e^{in_1\sigma^1}(\sigma^2)^{n_2}, \tag{44}$$

and the generators $L_{(n_1,n_2)}$ are defined by the relation

$$L_{(n_1,n_2)} \equiv \int d^2\sigma e^{in_1\sigma^1}(\sigma^2)^{n_2}\varepsilon^{ab}\partial_a X^I \partial_b P^I. \tag{45}$$

The Poisson brackets of these generators result in

$$\left\{L_{(n_1,n_2)}, L_{(m_1,m_2)}\right\} = i(n_1 m_2 - n_2 m_1) L_{(n_1+m_2, n_2+m_2-1)}, \tag{46}$$

with the boundary conditions(29) and (36). We confine ourselves to a subset of the basis(43),

$$-ie^{im\sigma^1}(\sigma^2)^{k-1}, \quad k \geq 2, \tag{47}$$

just in the same way as in the square. Then the generators are given by

$$L_m^k = \int d^2\sigma[-ie^{im\sigma^1}(\sigma^2)^{k-1}\epsilon^{ab}\partial_a X^I \partial_b P^I]. \tag{48}$$

With Eq.(46) we obtain the algebra

$$\{L_m^k, L_n^l\} = [m(l-1) - n(k-1)]L_{m+n}^{k+l-2}. \tag{49}$$

This algebra is also equivalent to the w_∞ algebra.

3. Discussion

We have found that the area-preserving algebras for the square and the annulus contain the w_∞ algebra as their subalgebra. Pope, Romans and Shen conjectured that the w_∞ algebra might be connected with membrane theories [14]. The w_∞ algebra is in fact deduced, by the present author, from the area-preserving algebra acting on the square as well as on the annulus. This result is in conformity with their expectations.

In closed membranes some authors examined the relation between the algebra of area-preserving diffeomorphisms and the algebra of $SU(n)$ as n tends to infinity. There are two inequivalent $n \to \infty$ limits of $SU(n)$ algebras, called $SU(\infty)$ and $SU_+(\infty)$. The area-preserving algebra of the sphere is equivalent to $SU_+(\infty)$ [3],[15] and that of the torus is equivalent to $SU(\infty)$ [16]. Moreover the algebra for the Klein bottle can be represented as a limit of $SO(2n)$ algebra and that for the projective plane may be obtained as a limit of $SO(2n-1) \times SO(2n+1)$, $USp(2n) \times USp(2n+2)$ or $USp(2n)$ algebra [5].

Since the non-linearities of free membranes cannot be gauged away, they contain interactions and can, therefore, change their own topologies [17],[18]. Suppose, as an example, that the open square-shaped membrane changes into a spherical membrane. Then the area-preserving algebra of the square, i.e., w_∞ algebra, turns into $SU_+(\infty)$. We can discuss other membranes with different topologies in a similar way. Therefore the area-preserving algebras are connected with one another by the change of topologies.

Recently many authors have noted that the symmetries of area-preserving diffeomorphisms exist in the string theory [19], the conformal affine Toda model [20], the Quantum Hall effect [21] and so on (see also ref.[22]). Thus it is an interesting problem that we relate the membrane theory and these theories through the area-preserving diffeomorphisms.

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