Journal of The Physical Society of Japan
Vol. 61, No. 10, October, 1992, pp. 3425-3427

Integrability of the Quantum Calogero-Moser Model

Hideaki UJINO, Kazuhiro HIKAMI and Miki WADATI

Department of Physics, Faculty of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113

(Received August 14, 1992)

The integrability of the quantum Calogero-Moser model is proved through the quantum inverse scattering method. By use of the Lax operator L, a new formula for the conserved operators is found. The formula suggests that $\mathrm{Tr}\, L^n$, which are conserved in the classical case, are not necessarily conserved operators. The Hénon-type conserved operators are also presented.

A one-dimensional particle system in which all the particles interact through inverse square pairwise potentials is called the Calogero-Moser model.[1-3] Integrability of the classical case was proven by Moser.[3] The quantum version had been solved by Calogero[1] and Sutherland.[2]

In this letter, we shall prove the integrability of the quantum Calogero-Moser model by explicitly constructing a set of independent conserved operators. The Hamiltonian of the N-particle quantum Calogero-Moser model reads

$$H=\frac{1}{2m}\sum_{j=1}^{N}p_j^2+\frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{g}{(x_j-x_k)^2}, \tag{1}$$

$$p_j=-i\hbar\frac{\partial}{\partial x_j}, \tag{2}$$

where m is the mass, g the coupling constant and $\hbar$ the Planck constant. Hereafter, we shall set $m=1$ and $\hbar=1$ to simplify the notation.

The quantum inverse scattering method for the N-particle system on a line is introduced as follows. Let L and M be operator-valued $N\times N$ matrices. We choose L and M (a Lax pair) such that the Lax equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L_{jk}&\equiv i[H, L_{jk}]\\ &=i[L, M]_{jk}=i\sum_{l=1}^{N}(L_{jl}M_{lk}-M_{jl}L_{lk}),\end{aligned} \tag{3}$$

is equivalent to the equation of motion generated by the Hamiltonian H.

For the quantum Calogero-Moser model, we find that L and M are given by

$$L_{jk}=p_j\delta_{jk}+ia(1-\delta_{jk})\frac{1}{x_j-x_k}, \tag{4}$$

$$M_{jk}=-a(1-\delta_{jk})\frac{1}{(x_j-x_k)^2}+a\delta_{jk}\sum_{l\neq j}\frac{1}{(x_j-x_l)^2}, \tag{5}$$

where a is a constant related to the coupling constant g by

$$g=a^2-a. \tag{6}$$

In fact, it is easy to show that substitution of L and M of (4) and (5) into the Lax equation yields the Heisenberg equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}p_j=2g\sum_{l\neq j}\frac{1}{(x_j-x_l)^3},\quad j=1, 2, \cdots, N. \tag{7}$$

We note that (6) contains the quantum effect: with $\hbar$, relation (6) reads as $g=a^2-\hbar a$. Since the matrix elements of L are operators, a construction of conserved operators is not obvious.

We present a novel approach to constructing conserved operators. A key observation is that M_{jk} in (5) satisfies

$$\sum_{j=1}^{N}M_{jk}=\sum_{j=1}^{N}M_{kj}=0,\quad k=1, 2, \cdots, N. \tag{8}$$

Then, we can readily check that

$$\left[H, \sum_{j,k=1}^{N}(L^n)_{jk}\right]=\sum_{j,k=1}^{N}[L^n, M]_{jk}=0, \tag{9}$$

that is, a set of conserved operators $\{I_n\}$ is

given by

$$I_n=\frac{1}{n}\sum_{j,k=1}^{N}(L^n)_{jk},\quad n=1,2,\cdots,N. \tag{10}$$

Formula (10) is reported for the first time. Inclusion of the factor $1/n$ in I_n is merely for convenience. We may decompose the summation in (10) into

$$I_n=\frac{1}{n}\sum_{j=1}^{N}(L^n)_{jj}+\frac{1}{n}\sum_{j\neq k}(L^n)_{jk}. \tag{11}$$

From explicit form of the L-operator (4), we know that $L_{jk}=-L_{kj}$ for $j\neq k$ and L_{jj} are operators. With these conditions, the second term (11) need not vanish. This shows that Tr L^n are not necessarily conserved operators, while in the classical case, Tr L^n are integrals of motion.

We see from the formula (10) that $\{I_n\}$ are functionally independent. Then, the set of independent conserved operators $\{I_n\}$ proves the integrability of the quantum Calogero-Moser model (1).

It is instructive to write down the explicit forms of conserved operators. The first four are

$$\begin{aligned}
I_1&=\sum_{j,k}L_{jk}=\sum_j p_j,\\
I_2&=\frac{1}{2}\sum_{j,k}(L^2)_{jk}\equiv H,\\
I_3&=\frac{1}{3}\sum_{j,k}(L^3)_{jk}=\frac{1}{3}\sum_j p_j^3+\frac{1}{3}g\sum_{j,k}{}'\left\{p_j\frac{1}{x_{jk}^2}+\frac{1}{x_{jk}}p_k\frac{1}{x_{jk}}+\frac{1}{x_{jk}^2}p_k\right\},\\
I_4&=\frac{1}{4}\sum_{j,k}(L^4)_{jk}=\frac{1}{4}\sum_j p_j^4+\frac{1}{4}g\sum_{j,k}{}'\left\{p_j^2\frac{1}{x_{jk}^2}+p_j\frac{1}{x_{jk}}p_k\frac{1}{x_{jk}}+p_j\frac{1}{x_{jk}^2}p_j+\frac{1}{x_{jk}}p_k^2\frac{1}{x_{jk}}\right.\\
&\quad\left.+\frac{1}{x_{jk}}p_k\frac{1}{x_{jk}}p_j+\frac{1}{x_{jk}^2}p_j^2\right\}+\frac{1}{2}g^2\sum_{j,k,l}{}'\frac{1}{x_{jk}^2}\frac{1}{x_{kl}^2}+\frac{1}{4}(g^2-g)\sum_{j,k}{}'\frac{1}{x_{jk}^4}.
\end{aligned} \tag{12}$$

Here and hereafter, we use $x_{jk}=x_j-x_k$ and so on, and Σ' means that no two summation indices coincide. As we expected, only g (not a) appears in $\{I_n\}$. We have confirmed that I_1, I_2 and I_3 are the same as Tr L, $(1/2)$ Tr L^2 and $(1/3)$ Tr L^3, respectively, with the relation $g=a^2-a$. The last term in I_4 should be noted. With m and $\hbar$, the coefficient is $(1/4)(g^2-\hbar^2 g/m)$, which becomes $(1/4)g^2$ in the classical case.

The above conserved operators can be cast into the Hénon-type forms:[4]

$$\begin{aligned}
J_1&=I_1=\sum_j p_j,\\
J_2&=\frac{1}{2!}(I_1^2-2I_2)=\frac{1}{2}\sum_{j,k}{}'p_jp_k-\frac{1}{2}g\sum_{j,k}{}'\frac{1}{x_{jk}^2},\\
J_3&=\frac{1}{3!}(I_1^3-6I_1I_2+6I_3)=\frac{1}{6}\sum_{j,k,l}{}'p_jp_kp_l-\frac{1}{2}g\sum_{j,k,l}{}'p_j\frac{1}{x_{kl}^2},\\
J_4&=\frac{1}{4!}(I_1^4-12I_2I_1^2+24I_3I_1+12I_2^2-24I_4)\\
&=\frac{1}{24}\sum_{j,k,l,n}{}'p_jp_kp_lp_n-\frac{1}{4}g\sum_{j,k,l,n}{}'p_jp_k\frac{1}{x_{ln}^2}+\frac{1}{8}g^2\sum_{j,k,l,n}{}'\frac{1}{x_{jk}^2x_{ln}^2}.
\end{aligned} \tag{13}$$

The Hénon-type conserved operators have the advantage that orderings of the various factors in each term are not essential.

The expressions in (13) agree with the ones obtained from the expansion

$$\det|L_{jk}-\lambda\delta_{jk}|=(-\lambda)^N+\sum_{n=1}^{N}(-\lambda)^{N-n}J_n, \tag{14}$$

with $g=a^2$. Calogero, Ragnisco and Marchioro[5] proved that

$$[H, \det |L_{jk}-\lambda\delta_{jk}|]=0. \tag{15}$$

(Rigorously speaking, their potential is $1/\sinh^2 x$ instead of $1/x^2$, but the discussion is the same.) To carry out the proof of (15), they modified the Lax equation (3) to

$$[H, L_{jk}]=\frac{1}{2}\sum_{l=1}^{n}\{L_{jl}M_{lk}+M_{lk}L_{jl}-M_{jl}L_{lk}-L_{lk}M_{jl}\}. \tag{16}$$

However, the symmetrized commutation on the r.h.s. of (16) requires a justification from the viewpoint of the quantum inverse scattering method. It is extremely interesting that both approaches give the same Hénon-type conserved operators $\{J_n\}$, although the Lax equation is interpreted differently and the relation between g and a is different.

In conclusion, we have presented a new formula (10) for independent conserved operators and thus proved the integrability of the quantum Calogero-Moser model (1). This analysis can be extended to the cases with spins.[6]

References

1) F. Calogero: J. Math. Phys. **10** (1969) 2197; **12** (1971) 418.
2) B. Sutherland: J. Math. Phys. **12** (1971) 246, 251.
3) J. Moser: Adv. Math. **16** (1975) 197.
4) M. Hénon: Phys. Rev. B**9** (1974) 1921.
5) F. Calogero, O. Ragnisco and C. Marchioro: Lett. Nuovo Cimento. **13** (1975) 383.
6) K. Hikami and M. Wadati: preprint, 1992.

Note added in proof—The results in this letter hold for the trigonometric case:

$$L_{jk}=p_j\delta_{jk}+ia(1-\delta_{jk})\cot x_{jk},$$

$$M_{jk}=-a(1-\delta_{jk})\frac{1}{\sin^2 x_{jk}}+a\delta_{jk}\sum_{l\neq j}\frac{1}{\sin^2 x_{jl}},$$

$$H=\frac{1}{2}\sum_{j=1}^{N}p_j^2+\frac{1}{2}g\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{1}{\sin^2 x_{jk}}.$$