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Integrability of Calogero-Moser Spin System

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1992
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Abstract

One-dimensional quantum particle system with spins is considered. The Hamiltonian of the system (Calogero-Moser spin system) is

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N} \frac{4a^2 - a - a\sigma_j\cdot\sigma_k}{(x_j - x_k)^2},$$

where $p_j = -i\partial/\partial x_j$ and σ_j is the Pauli spin operator associated with j-th particle. We prove the integrability of the model through the quantum inverse scattering method. By introducing the annihilation and creation-like operators from the Lax operator, we construct the ground state. The wave function is of Jastrow-type. Further, we discuss a generalization of the model to SU(M) spin case.

1 Introduction

Recently much attention has been paid to long-range interaction models. In particular, the relevances to the quantum Hall effect and high T_c superconductivity have been discussed.

It is well known that the 1-dimensional quantum N-body system,

$$H = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N} V(x_j - x_k), \tag{1.1}$$

is integrable when the potential $V(r)$ is $g/r^2, g/\sin^2 r$ and $\wp(r)$. Here $\wp(r)$ is the Weierstrass $\wp$ function[1,2,3]. Such integrable system is called the Calogero-Moser model.

Spin systems with long-range interactions have been considered on a 1-dimensional lattice. Haldane and Shastry showed that the Gutzwiller wave function $\Psi_G = \exp(i\pi\sum x_j)\prod_{i<j}\sin^2(x_i - x_j)$ is an exact eigenstate of the Heisenberg antiferromagnetic model with $1/r^2$ interaction,

$$H = J\sum_{j<k}\frac{\sigma_j\cdot\sigma_k}{\sin^2(x_j - x_k)}, \qquad J > 0, \tag{1.2}$$

where $\sigma_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$ are Pauli spins, and the periodic boundary condition is imposed[4,5]. It was suggested that excited states can be described in terms of semionic spin-1/2 spinons, and lead to a representation of the Wess-Zumino-Witten conformal field theory[6].

In this paper we consider a hybrid of the Haldane-Shastry model and the

Calogero-Moser model,

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\sum_{\substack{j,k=1 \\ j\neq k}}^{N} \frac{4a^2 - a - a\sigma_j \cdot \sigma_k}{(x_j - x_k)^2}, \tag{1.3}$$

$$p_j = -i\frac{\partial}{\partial x_j}. \tag{1.4}$$

Here a is a parameter for the interaction and σ_j is the spin operator for j-th particle satisfying the commutation relations

$$[\sigma_j^\alpha, \sigma_k^\beta] = \sum_{\gamma=x,y,z} 2i\delta_{jk}\epsilon_{\alpha\beta\gamma}\sigma_j^\gamma, \tag{1.5}$$

where δ_{jk} is the Kronecker's delta. We call (1.3) the Calogero-Moser spin system, since particles with spins are not confined on the lattice. This system can be considered as a quantum realization of the Calogero-Moser model with internal degrees of freedom[7].

In section 2, we prove the integrability of the Calogero-Moser spin system. Although the Hamiltonian (1.3) has been studied in [8,9], the proof based on the quantum inverse scattering method is new. In section 3, we construct the ground state and evaluate the energy by introducing "annihilation" and "creation" operators from the Lax operator L. The ground state wave function is of the Jastrow-type and the ground state energy is simply obtained from a bilinear form of the Hamiltonian. The last section is devoted to summary and discussions. We point out that the method in this paper is extended to include SU(M) spin model.

2 Integrability

To prove the integrability, we shall use the quantum inverse scattering method for particle systems. Namely we introduce operator-valued $N \times N$ matrices L and M (Lax pair) such that the Lax equation

$$\begin{aligned} \dot{L}_{jk} &\equiv i[H, L_{jk}] \\ &= i[L, M]_{jk} = i\sum_{l=1}^{N}(L_{jl}M_{lk} - M_{jl}L_{lk}), \end{aligned} \tag{2.1}$$

is equivalent to the equation of motion generated by a Hamiltonian H under consideration. For spin system such as

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\sum_{\substack{j,k=1 \\ j\neq k}}^{N}\left(4a^2 - a - a\sigma_j\cdot\sigma_k\right)h(x_j - x_k), \tag{2.2}$$

we choose the Lax pair as

$$\begin{aligned} L_{jk} &= p_j\delta_{jk} + ia(1-\delta_{jk})f(x_j - x_k)(1+\sigma_j\cdot\sigma_k), \\ M_{jk} &= a(1-\delta_{jk})g(x_j - x_k)(1+\sigma_j\cdot\sigma_k) \\ &\quad + a\delta_{jk}\sum_{\substack{s=1 \\ s\neq j}}^{N} z(x_j - x_s)(1+\sigma_j\cdot\sigma_s), \end{aligned} \tag{2.3}$$

where we suppose that

$$f(-x) = -f(x), \qquad g(-x) = g(x), \qquad z(-x) = z(x). \tag{2.4}$$

We substitute L and M (2.3) into the Lax equation (2.1). After a lengthy calculation we obtain the following functional equations for $f(x), g(x)$ and $z(x)$:

$$\begin{cases} g(x) = f'(x) \\ f(x)g(y) - f(y)g(x) = f(x+y)\left\{z(x) - z(y)\right\} \\ f(x)g(-x) - f(-x)g(x) = z'(x) \\ z(x) = h(x) \end{cases} \tag{2.5}$$

The third equation of (2.5) is an additional condition to the case of the classical Calogero-Moser model[3]. Functional equations for the classical Calogero-Moser model have been solved (see, for instance, [10]). Similarly, we can solve (2.5) and find that the function $f(x)$ has a form

$$f(x) = \begin{cases} \frac{1}{x}, \\ \frac{\alpha}{\sin(\alpha x)}, \alpha\cot(\alpha x), \\ \frac{\alpha}{\mathrm{sn}(\alpha x)}, \alpha\frac{\mathrm{cn}(\alpha x)}{\mathrm{sn}(\alpha x)}, \alpha\frac{\mathrm{dn}(\alpha x)}{\mathrm{sn}(\alpha x)}, \end{cases} \tag{2.6}$$

where $\mathrm{sn}(x), \mathrm{cn}(x), \mathrm{dn}(x)$ are the Jacobi's elliptic functions and α is a constant.

In the classical case, it is easy to derive from the Lax operator L a set of integrals of motion $\{J_n\}$. A formula is given by

$$J_n = \frac{1}{n}\mathrm{Tr}L^n. \tag{2.7}$$

For the quantum case, it is not so straightforward to obtain conserved operators from the Lax pair because of the noncommutability of the matrix elements. In general, using the "time evolution" operator U satisfying

$$[U, H] = MU, \tag{2.8}$$

we have from (2.1) and (2.8)

$$[H, U^{-1}LU] = 0. \tag{2.9}$$

Thus, conserved operators $\{I_n\}$ which correspond to classical integrals of motion $\{J_n\}$, may be written as

$$I_n = \frac{1}{n}\mathrm{Tr}\left(U^{-1}L^nU\right). \tag{2.10}$$

For the case $h(x) = 1/x^2$, we find a simple method to construct conserved operators. From (2.3) with $g(x) = -1/x^2$ and $z(x) = 1/x^2$, we have

$$\sum_j M_{jk} = \sum_k M_{jk} = 0. \tag{2.11}$$

Then, we can readily check that

$$[H, \sum_{j,k=1}^{N} (L^n)_{jk}] = \sum_{j,k} [L^n, M]_{jk} = 0. \tag{2.12}$$

That is, conserved operators $\{I_n\}$ are given by

$$I_n = \frac{1}{n} \sum_{j,k=1}^{N} (L^n)_{jk}, \qquad n = 1, 2, \cdots, N. \tag{2.13}$$

A formula (2.13) is new. First three of them read as :

$$I_1 = \sum_{j,k} L_{j,k} = \sum_j p_j, \tag{2.14}$$

$$I_2 = \frac{1}{2} \sum_{j,k} (L^2)_{jk} \equiv H, \tag{2.15}$$

$$\begin{aligned} I_3 &= \frac{1}{3} \sum_{j,k} (L^3)_{jk} \\ &= \frac{1}{3} \sum_j p_j^3 + \sum_{j,k}{}' \frac{4a^2 - 3a + (2a^2 - a)\sigma_j \cdot \sigma_k}{(x_j - x_k)^2} p_j \\ &\quad + \frac{1}{3} \sum_{l,m,n}{}' (-a^3 + a^2) \frac{(\sigma_l \times \sigma_m) \cdot \sigma_n}{(x_l - x_m)(x_m - x_n)(x_n - x_l)} \\ &\quad + \frac{1}{6} \sum_{j,k,m,n}{}' a^3 \frac{x_n - x_m}{(x_j - x_m)(x_m - x_k)(x_k - x_n)(x_n - x_j)} (\sigma_m \times \sigma_n) \cdot \sigma_k \end{aligned} \tag{2.16}$$

Here $\sum'$ means any two variables does not coincide. In general, $\{I_n\}$ has a form

$$I_n = \frac{1}{n} \sum_j p_j^n + \cdots. \tag{2.17}$$

Therefore, I_ns are independent. Involutiveness of $\{I_n\}$ is explained as follows. From the Jacobi's identity we know that $J_{mn} \equiv [I_m, I_n]$ commutes with H and that J_{mn} is a conserved operator. However, (2.17) implies that J_{mn} does not have a term, $\frac{1}{l}\sum p_j^l$. Then, $[I_n, I_m] \equiv 0$. A direct calculation for showing $[I_n, I_m] = 0$ is left for a future problem.

A set of independent conserved operators $\{I_n\}$ and their involutiveness prove the integrability of the model (1.3).

3 Ground State Energy

In this section we shall restrict our discussion to the case $h(x) = 1/x^2$ where the Hamiltonian (2.2) is nothing but (1.3). We may apply the same method to other cases. The ground state has also been discussed in [9], but here we shall give a formulation based on the quantum inverse scattering method.

We introduce a set of operators $\{h_j\}$,

$$h_j^\dagger = \sum_{k=1}^{N} L_{kj} = p_j - ia \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{1+\sigma_j\cdot\sigma_k}{x_j - x_k}, \tag{3.1}$$

$$h_j = \sum_{k=1}^{N} L_{jk} = p_j + ia \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{1+\sigma_j\cdot\sigma_k}{x_j - x_k}. \tag{3.2}$$

The operators $h_j^\dagger$ and h_j are hermitian conjugate each other. These operators may be considered as creation and annihilation operators. This interpretation becomes clear when we add harmonic potentials $\frac{1}{2}\omega^2\sum x_i^2$ to the model (1.3) without spoiling the integrability. They satisfy the commutation relations ,

$$[h_j, h_k] = [h_j^\dagger, h_k^\dagger] = 0, \tag{3.3}$$

$$[h_j, h_k^\dagger] = 2M_{jk}. \tag{3.4}$$

In terms of these operators, the Hamiltonian H is simply expressed as

$$H = \frac{1}{2}\sum_{j=1}^{N} h_j^\dagger h_j. \tag{3.5}$$

Note that (3.5) has a bilinear form (or quadratic form) and then H is a non-negative operator.

To obtain the ground state wave function, we consider the state $|\psi_g\rangle$ which is annihilated by the "annihilation" operator h_j,

$$h_j|\psi_g\rangle = 0, \qquad j = 1, \cdots, N. \tag{3.6}$$

Equation (3.6) is explicitly written as follows,

$$\left(\frac{\partial}{\partial x_j} - a\sum_{\substack{k=1\\k\neq j}}^{N} \frac{1+\sigma_j\cdot\sigma_k}{x_j - x_k}\right)|\psi_g\rangle = 0. \tag{3.7}$$

It is interesting to observe that (3.7) is the Knizhnik-Zamolodchikov equation in the conformal field theory [11]. Obviously $|\psi_g\rangle$ is an eigenstate of the Hamiltonian, $H|\psi_g\rangle = 0$, and therefore we conclude that the state $|\psi_g\rangle$ is the ground state.

Solutions of (3.7) are given by the Jastrow-type wave functions. The solution has a form

$$|\psi_g\rangle = \prod_{j<k}(x_j - x_k)^{2a}\cdot|\chi\rangle, \tag{3.8}$$

where $|\chi\rangle$ can be represented as a hypergeometric-type integral [12]. For the ferromagnetic case, $|\chi\rangle$ reduces to

$$|\chi\rangle \equiv |\chi_F\rangle = |\uparrow\rangle\otimes|\uparrow\rangle\otimes\cdots\otimes|\uparrow\rangle. \tag{3.9}$$

For the antiferromagnetic case, the state $|\chi_{AF}\rangle$ corresponds to Anderson's RVB (resonating-valence-bond) state [13]. For $a = 1/4$, the interaction term in (1.3) becomes purely spin-spin interactions and, as seen from (3.8), the particles behave like "semions".

The generalizations to the case of periodic boundary conditions, and the relations between $|\psi_g\rangle$ and Haldane-Shastry model (1.2) will be discussed in a separate paper.

4 Summary and Discussions

We have shown that the quantum Calogero-Moser model can be generalized to the quantum integrable particle system with spins (Calogero-Moser spin system). Also we have introduced a bilinear form of the Hamiltonian and constructed the exact ground state. We point out that this transformation of the Hamiltonian is essential for the system to have the Jastrow-type wave functions as eigenstates. For excited states the asymptotic Bethe ansatz (ABA) method seems to be useful[14]. Since there exist solutions besides ABA solutions, the further study must be done for a complete understanding of this model.

The method presented in this paper can be easily extended to the case of SU(M) spin model. The similar extension has been found in [9]. We use the permutation operator P_{ij} in spin space, $P_{ij}|\cdots\sigma_i\cdots\sigma_j\cdots\rangle = |\cdots\sigma_j\cdots\sigma_i\cdots\rangle$. For SU(2) case, the permutation operator P_{ij} can be written as $P_{ij} = (1 + \sigma_i \cdot \sigma_j)/2$. The Lax equation (2.1) is satisfied by choosing L, M and H as,

$$L_{jk} = p_j\delta_{jk} + ia(1-\delta_{jk})f(x_j - x_k)P_{jk},$$

$$M_{jk} = a(1-\delta_{jk})g(x_j - x_k)P_{jk} + a\delta_{jk}\sum_{s\neq j} z(x_j - x_s)P_{js}, \tag{4.1}$$

$$H_{SU(M)} = \frac{1}{2}\sum_j p_j^2 + \frac{1}{2}\sum_{j\neq k} a\,(a - P_{jk})\,h(x_j - x_k). \tag{4.2}$$

The functional equations for $f(x), g(x), z(x)$ and $h(x)$ are the same as (2.5). The existence of the Lax pair, L and M, guarantees the integrability of the model.

Acknowledgements

The authors thank P.P.Kulish and T.Nagao for stimulating discussions.

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