Fermilab Library
0 1190 0011396 3
DUP
41
PRINT-93-0327

NONLINEAR WAVES AND SOLITONS IN MOLECULAR CLOUDS

Fred C. Adams and Marco Fatuzzo

Physics Department, University of Michigan
Ann Arbor, MI 48109

to appear in The Astrophysical Journal
submitted: 20 May, 1992; revised: 17 July, 1992

1992

ABSTRACT

We begin a study of nonlinear wave phenomena in molecular clouds. These clouds exhibit highly nonlinear structure that is often described in terms of "clumps" and "filaments" which are bouncing around, twisting, and colliding within the cloud. These clouds are important because they ultimately produce the initial conditions for the star formation process. Our motivation is to explore the possibility that *solitons* (i.e., spatially localized, single hump wave entities which often exhibit remarkable stability) can live in these molecular clouds and produce their observed structure. In this paper we focus on the case of one spatial dimension and we show that a rich variety of nonlinear waves can exist in molecular cloud fluid systems (where self-gravity is included). We show that in the absence of magnetic fields no true soliton solutions are allowed, although highly nonlinear waves (whose crests become widely spaced and thus soliton-like) do exist. For clouds with embedded magnetic fields, we derive a model equation which describes the behavior of wave phenomena; this model equation allows solutions which correspond to nonlinear waves, solitons, and topological solitons. We briefly consider the stability of these wave entities and discuss the possible role they play in molecular cloud dynamics.

Subject headings: hydromagnetics – wave motions – interstellar: molecules – stars: formation

1. INTRODUCTION

Molecular clouds are important objects in our galaxy. These clouds comprise a substantial fraction of the gas in the galaxy, they provide the initial conditions for star formation, and they are interesting astrophysical objects in their own right. Molecular clouds are now fairly well-observed (see, e.g., the recent review of Blitz 1992). In particular, the *IRAS* satellite has mapped these clouds at far-infrared wavelengths and has thus provided us with detailed column density maps (see, e.g., Wood, Myers, & Daughtery 1999). Such studies indicate that molecular clouds are highly nonlinear structures, where the observed dynamic range in column density is typically $\sim 10^3$ and larger. Although observations clearly indicate the presence of nonlinear behavior, very little theoretical work along these lines has been done. The goal of this paper is to correct this situation by beginning a study of nonlinear wave phenomena in molecular clouds.

Previous theoretical work has focused on the smallest size (and mass) scales, namely, the scales of molecular cloud cores, which are the actual sites of star formation. This work has studied the collapse of such cores (e.g., Shu 1977; Terebey, Shu, & Cassen 1984), and the formation of these cores through the process of ambipolar diffusion (e.g., Shu 1983; Nakano 1985; Mouschovias 1976, 1978; Lizano & Shu 1989; see also the review of Shu, Adams, & Lizano 1987). On the larger size scales (larger than $\sim$ 1pc), relatively little theoretical work has been done. The observed structure in molecular clouds on these size scales is sufficiently complicated that even describing, much less theoretically predicting, the cloud structure is a nontrivial task (see, e.g., Scalo 1990; Myers 1991; Houlahan & Scalo 1992; Wiseman & Adams 1992; Adams 1992). The linear stability of magnetically supported clouds has been studied (Langer 1978; Pudritz 1990; see also Dewar 1970), but such studies do not address the problem of nonlinear structure in these clouds. One notable paper which includes nonlinear effects (Elmegreen 1990) studies collisions of Alfvén waves in the absence of self-gravity and suggests a wavelike origin for clumpy structure. A study of magnetoacoustic waves in nonuniform media has also recently been completed (Fatuzzo and Adams 1992).

In this present work we begin a study of nonlinear waves and solitons in molecular clouds. The study of nonlinear waves and solitons began during the last century (see, e.g., the classic papers by Russell 1844 and Riemann 1858) and has been one of the most important developments in classical physics in the last few decades (see, e.g., Infeld & Rowlands 1990 and references therein). Although this subject can have an important impact on our understanding of structure in molecular clouds, it has not yet been adequately explored in this context (or for any self-gravitating fluid). Since this subject is rather large and can rapidly become mathematically complicated, this present paper will concentrate on nonlinear waves and solitons in one spatial dimension. The study of nonlinear waves in higher dimensions (and other related topics) will be addressed in later papers.

Solitons are a particular kind of nonlinear wave and will occupy much of our attention. Since many different definitions of solitons and related entities exist in the literature (e.g., Whitham 1974; Coleman 1985; Rajaraman 1987; Drazin & Johnson 1989; and Infeld & Rowlands 1990), we begin by presenting working definitions of our own: We use

the term *stationary wave* to mean any wave structure that is a function of the variable $\xi = x - v_0 t$ only (for one spatial dimension). We use the term *soliton* to mean any solution of a nonlinear wave equation (or system of equations) which (a) represents a stationary wave, and (b) is localized in space so that the wave form either decays or approaches a constant at spatial infinity. This working definition is closest to what Coleman (1985) refers to as "classical lumps" and is more general than many in the literature, where it is often also required that (c) the wave form can interact strongly with other solitons and retain its identity. Solitons which also satisfy requirement (c) must have extraordinary stability in order to pass through each other and, after emerging from the collision, retain their initial forms. Such solitons are very rare; wave entities which satisfy the first two requirements and not (c) are much more common.

It is easy to see that solitons, if they exist, could be very important in understanding the structure of molecular clouds. One can imagine that a giant molecular cloud complex consists of a large collection of solitons bouncing around, colliding, and then emerging (perhaps even intact!) to collide again. In this paper, however, we show that true soliton solutions to the equations of motion do not exist for the case of one-dimensional molecular clouds with no magnetic fields. On the other hand, the nonlinear model equations for clouds with magnetic fields do exhibit soliton behavior. In addition, even in the case of no magnetic fields, the nonlinear stationary waves can be *nearly* soliton-like. We quantify these statements below.

This paper is organized as follows. We begin in §2 with a discussion of nonlinear stationary waves in molecular clouds with no magnetic fields. In §3, we generalize the calculation to include the effects of a magnetic field. In particular, we derive model equations which can be easily solved and which include most of the essential physics. We generalize the discussion even further in §4, where we discuss whether or not soliton solutions can exist. We show that clouds without magnetic fields cannot exhibit true soliton behavior, whereas the model equations for clouds with magnetic fields do in fact allow soliton solutions. In §5, we discuss briefly the stability of the waves. We conclude (in §6) with a summary of our results, a comparison to observations of molecular clouds, and a discussion of future research. Additional formal results (e.g., an estimate of the wavelength of nonlinear waves and a group of scaling transformations which leave the equations of motion invariant) are presented in the appendices.

2. ONE-DIMENSIONAL WAVES WITH NO MAGNETIC FIELDS

In this section, we study stationary nonlinear waves in molecular clouds without magnetic fields. In one spatial dimension, the equations of motion for the molecular cloud fluid reduce to the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial \psi}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial^2 \psi}{\partial x^2} = 4\pi G\rho. \tag{2.3}$$

For a molecular cloud fluid on the size scales of interest (1 – 30 pc), we take the pressure to have two contributions, i.e.,

$$p = a_s^2 \rho + p_0 \log(\rho/\rho_0), \tag{2.4}$$

where the first term corresponds to ordinary thermal pressure (a_s is the isothermal sound speed and is taken to be constant). The second term corresponds to a "turbulent" contribution to the pressure; this contribution is motivated by the empirical finding that observed line-widths vary with density according to $\Delta v \propto \rho^{-1/2}$ (see, e.g., Larson 1981; Myers 1983; Dame et al. 1986; Myers 1987; and especially Myers & Fuller 1992). The turbulent motions are generally supersonic, but subalfvénic. The particular functional form given in equation [2.4] results from interpreting the observed linewidth as an effective transport speed, i.e., $\Delta v = v_{\rm turb} = (\partial p_{\rm turb}/\partial \rho)^{1/2} = (p_0/\rho)^{1/2}$, where the last equality follows from the observed correlation; the turbulent term in equation [2.4] is then obtained by integration (Lizano & Shu 1989; see also Myers & Fuller 1992). Notice that we are implicitly assuming that the turbulent motions are sufficiently small and isotropic so that they produce a pressure as described by equation [2.4]. These assumptions are consistent with available observations; in fact, an equation of state of the form [2.4] which includes a nonthermal contribution is *more* consistent with the data than a purely isothermal model (see Myers & Fuller 1992 for further discussion).

We now want to combine equations [2.1 – 2.4] to produce a single nonlinear differential equation for the density ρ. We first non-dimensionalize all quantities according to

$$u \rightarrow u/a_s, \tag{2.5a}$$

$$\rho \rightarrow \rho/\rho_0, \tag{2.5b}$$

$$x \rightarrow kx \qquad \text{where} \qquad k^2 = 4\pi G \rho_0 / a_s^2, \tag{2.5c}$$

$$t \rightarrow k a_s t, \tag{2.5d}$$

$$\kappa \equiv \frac{p_0}{\rho_0 a_s^2}. \tag{2.5e}$$

The value of p_0 is determined by observational considerations, which suggest that $p_0 = 1 \rightarrow 7 \times 10^{-11}$ dyne cm^{-2} for "typical" molecular clouds (see Myers & Goodman 1988; Solomon et al. 1987). In this study we are primarily interested in spatial size scales of 1 – 30 pc where number densities are ~100 – 1000 cm^{-3}. For this range of density and a sound speed of $a_s = 0.20$ km/s (appropriate for a temperature T=10 K), the expected values of κ lie in the range 6 – 50. The wavenumber k is just the inverse of the Jean's length that would result if there were only thermal pressure. Since κ is typically large compared to unity, $x = 1$ corresponds to a fraction of the effective Jean's length, which is equal to $(1+\kappa)^{1/2}$ for our assumed equation of state (see Appendix A).

Here, we will study the class of nonlinear waves known as stationary waves. For these waves, the fluid variables are functions of the quantity

$$\xi = x - v_0 t, \tag{2.6}$$

where v_0 is the (nondimensional) speed of the wave. Next, we introduce a new velocity variable

$$v = u - v_0, \tag{2.7}$$

which is simply the speed of the fluid relative to the speed v_0 of the wave.

Combining all of the above results, we obtain the continuity equation in the form

$$v\rho_\xi + v_\xi \rho = 0, \tag{2.8}$$

where the subscripts denote the derivatives. The continuity equation can be integrated to obtain

$$\rho v = A = constant, \tag{2.9}$$

where the constant of integration A is the "Mach number" of the wave.

If we differentiate the force equation [2.2] with respect to x and use Poisson's equation [2.3] to eliminate the potential, we obtain the following:

$$\rho^3\left[vv_{\xi\xi} + v_\xi v_\xi\right] + (\rho+\kappa)\rho\rho_{\xi\xi} - (\rho+2\kappa)\rho_\xi\rho_\xi + \rho^4 = 0. \tag{2.10}$$

Next, we eliminate the velocity dependence by using the solution [2.9] and the remaining differential equation for the density ρ becomes

$$\rho\rho_{\xi\xi}\left[\rho^2 + \kappa\rho - A^2\right] + \rho_\xi\rho_\xi\left[3A^2 - \rho^2 - 2\kappa\rho\right] + \rho^5 = 0. \tag{2.11}$$

After a bit of manipulation, equation [2.11] can be integrated once to obtain

$$\frac{1}{2}\rho_\xi^2 = \rho^6\left[\rho^2 + \kappa\rho - A^2\right]^{-2}\left\{\beta - \rho - A^2/\rho - \kappa\log\rho\right\}, \tag{2.12}$$

where β is the constant of integration.

Equation [2.12] provides us with much information concerning traveling wave solutions to the fluid equations. Physically interesting solutions occur when $\rho_\xi^2 > 0$. For given values of A and β (and for a given κ), there will be a range of densities $[\rho_1, \rho_2]$ for which $\rho_\xi^2 > 0$. The maximum value ρ_2 corresponds to the density at the crest of the wave, whereas the minimum density ρ_1 is the density at the wave trough. The present analysis is complicated by the presence of a singularity in equation [2.12]. The quantity ρ_ξ^2 blows up when $\rho^2 + \kappa\rho = A^2$. This condition has one positive root which we denote as ρ_C:

$$\rho_C = \left[\kappa^2/4 + A^2\right]^{1/2} - \kappa/2. \tag{2.13}$$

It is straightforward to show (see Appendix B) that the singularity always occurs within the allowed range of densities, i.e.,

$$\rho_1 \le \rho_C \le \rho_2. \tag{2.14}$$

Furthermore, the singular value ρ_C corresponds to the density where the function in brackets is a maximum. In addition, it can be shown (Appendix C) that the singularity

is removable in the sense that we can integrate over it to obtain a continuous wave profile $\rho(\xi)$.

For a given value of A (and a given κ), there exists a minimum value of β for which a (real) range of densities $[\rho_1, \rho_2]$ exists:

$$\beta_{min} = \rho_C + A^2/\rho_C + \kappa \log[\rho_C], \tag{2.15}$$

where ρ_C is defined by equation (2.13). For values of β near (but larger than) β_{min}, the range of allowed densities is small in the sense that $|\rho_2 - \rho_1|/\rho_C \ll 1$. Thus, the density variations of the wave are small and hence $\beta \to \beta_{min}$ corresponds to the linear limit of this class of waves. The quantity $\beta - \beta_{min}$ provides a measure of the degree of nonlinearity of the wave.

The parameter space available for nonlinear waves is fairly large: we get to specify κ (which determines the equation of state) and the constants of integration A and β. As an example, we take $\kappa = 10$ (which is consistent with observations as discussed above) and $A = 3$ (which is $\sim$1 km/s in dimensionful units if the sound speed is 0.33 km/s). For this choice of parameters, we find a family of nonlinear waves corresponding to varying choices of β. The phase diagram for this family of waves is shown in Figure 1. Waves profiles $\rho(\xi)$ for varying values of β are shown in Figure 2. For this example, $\beta_{min} = 9.81$. Notice that as β increases, the waves become increasingly nonlinear and the wavelength also increases (see Appendix D); the wave crests thus become widely separated and hence localized like solitons.

3. ONE-DIMENSIONAL WAVES WITH MAGNETIC FIELDS

In this section, we generalize our discussion of stationary nonlinear waves in molecular clouds to include the effects of magnetic fields on the wave motion. Molecular clouds are lightly ionized, i.e., the ratio of ions to neutrals is very small, typically $\sim 10^{-6}$. As a result, essentially all of the mass resides in the neutral component. However, only the ionized component is directly coupled to the magnetic field; the neutral component is affected by the field indirectly through ion-neutral collisions.

3.1 Equations of Motion for the Neutral Species

The equations of motion for the neutral species are almost the same as before:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{3.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial \psi}{\partial x} + \eta \rho_i (u - u_i) = 0, \tag{3.2}$$

$$\frac{\partial^2 \psi}{\partial x^2} = 4\pi G \rho\,, \tag{3.3}$$

where we have neglected the gravitational contribution of the ions in the Poisson equation [3.3]. Here, quantities without subscript labels refer to the neutral component, whereas

subscripted quantities refer to the ionized component. The new (and final) term in equation [3.2] arises from the friction exerted on the neutral component of the fluid by the ionized component, which is in turn coupled to the magnetic field. The parameter η is the drag coefficient, which we will take to have a constant value of $\eta = 3.5 \times 10^{13}$ cm^3 g^{-1} s^{-1} (Draine, Roberge, & Dalgarno 1983; see also the discussion of Shu 1992).

3.2 Ion Evolution: Voyage into the Neutral Sea

The equations of motion for the ionized component take the form

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}(\rho_i u_i) = 0, \tag{3.4}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{1}{\rho_i}\frac{\partial p_i}{\partial x} + \frac{\partial \psi}{\partial x} = \eta\rho(u - u_i) + \frac{1}{4\pi\rho_i}\left[(\nabla \times \mathbf{B}) \times \mathbf{B}\right] \cdot \hat{x}, \tag{3.5}$$

where $\hat{x}$ is the unit vector in the x direction. Notice that in using the continuity equation [3.4] for the ions, we are implicitly ignoring any chemical effects. The equations of motion for the magnetic field can be written

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (u_i \hat{x} \times \mathbf{B}), \tag{3.6}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{3.7}$$

So far we have left the magnetic field $\mathbf{B}$ expressed as a vector. It is straightforward to show that the x-component of the field must remain constant (by the "no monopole equation" [3.7]) and furthermore that only the perpendicular (to the x-axis) component enters into the dynamics. Without loss of generality, we can thus take the magnetic field to be of the form

$$\mathbf{B} = B\hat{y}. \tag{3.8}$$

The equation of motion [3.5] for the ion component can be simplified greatly by noticing that the two terms on the right hand side of the equation are typically several orders of magnitude larger than the other terms (see the discussion of Shu 1992). Thus, to a high degree of approximation, the drift speed $(u - u_i)$ is produced by the Lorentz force driving the ions through the sea of neutrals and the equation of motion for the ions becomes

$$\eta\rho(u - u_i) = \frac{1}{4\pi\rho_i}\frac{\partial}{\partial x}(B^2). \tag{3.9}$$

3.3 The Stationary Wave Approximation

We now look for stationary wave solutions for the above system of differential equations. We invoke the non-dimensionalization procedure of equation [2.5] and the definition of ξ given by equation [2.6]. In this case, the continuity equations reduce to

$$\rho v = A, \tag{3.10}$$

$$\rho_i v_i = A_i, \tag{3.11}$$

where we have defined $v = u - v_0$ and $v_i = u_i - v_0$. In the stationary wave approximation, the condition [3.6] which expresses field freezing also assumes the form of a continuity equation

$$v_i B_\xi + B v_{i\xi} = 0, \tag{3.12}$$

which can be integrated to obtain

$$v_i B = \alpha, \tag{3.13}$$

where α is a constant. Notice that if we divide equation [3.13] by equation [3.11], we obtain the result B/ρ_i = constant, which expresses the fact that when we squeeze the ion fluid together (in one dimension) the magnetic field strength increases by the same factor. Notice also that we have converted the field strength to its nondimensional form: $B \to B/B_0$, where B_0 is a reference field strength (typically $\sim 10 - 30$ μG; see, e.g., Myers & Goodman 1988; Goodman et al. 1989).

The equation of motion for the neutral component is almost the same as in the field-free case. The only difference is the presence of the magnetic friction term in equation [3.2]. Proceding as in §2, we can differentiate the equation of motion [3.2] with respect to x to eliminate the gravitational potential and then use the continuity equations to eliminate the velocities. After a bit of algebra, we obtain

$$\rho\rho_{\xi\xi}\left[\rho^2 + \kappa\rho - A^2\right] + \rho_\xi\rho_\xi\left[3A^2 - \rho^2 - 2\kappa\rho\right] + \rho^5 + \tilde{\eta}A\rho^4\partial_\xi\left[\frac{\rho_i}{\rho}\right] = 0, \tag{3.14}$$

where $\tilde{\eta} = \eta\rho_0(4\pi G\rho_0)^{-1/2}$ is the nondimensional version of the drag coefficient.

We can now use the continuity equations derived above to simplify the ion equation of motion [3.9]. We eliminate the velocities $(u - u_i) = (v - v_i)$ and the field strength B in favor of the ion and neutral densities:

$$\frac{1}{\gamma}\rho_{i\xi} = A - A_i\frac{\rho}{\rho_i}, \tag{3.15}$$

where we have defined $\gamma \equiv 2\pi\zeta\tilde{\eta}A_i^2/\alpha^2$ and where the parameter $\zeta = \rho_0 a_s^2/B_0^2$ arises from the nondimensionalization of equation [3.9].

3.4 A Model Equation for Nonlinear Waves

We have thus far reduced our problem to two coupled nonlinear ordinary differential equations (namely [3.14] and [3.15]). Although it is straightforward to integrate these equations using standard numerical techniques, we want to procede as far as possible using analytic methods. In order to make further progress, we must make some sort of simplifying approximations. The bulk of the matter is contained in the neutral component; we are interested in the ionized component only in order to calculate the friction that the ions exert on the neutrals. This friction term is proportional to $\partial_\xi(\rho_i/\rho)$, which can be written in the form

$$\partial_\xi(\rho_i/\rho) = \frac{\gamma A}{\rho} - \frac{\gamma A_i}{\rho_i} - \frac{\rho_i}{\rho^2}\rho_\xi, \tag{3.16}$$

where we have used the ion equation of motion [3.14]. If we now ignore the third term (which will be small when γ is sufficiently large), we can rewrite this quantity as

$$\partial_\xi(\rho_i/\rho) = \frac{\gamma A}{\rho}\left[\frac{A\rho_i}{A_i}\right]^{-1}\left[\frac{A\rho_i}{A_i} - \rho\right] = \frac{\gamma A}{\rho}\frac{1}{\rho_F}(\rho_F - \rho), \tag{3.17}$$

where we have defined a reference density $\rho_F = A\rho_i/A_i$. If we make the further approximation that the reference density ρ_F can be taken to be constant, we obtain an easily manageable model equation which describes the motion of the neutral component:

$$\rho\rho_{\xi\xi}\left[\rho^2 + \kappa\rho - A^2\right] + \rho_\xi\rho_\xi\left[3A^2 - \rho^2 - 2\kappa\rho\right] + \rho^5 + \Gamma\rho^3\left[1 - \frac{\rho}{\rho_F}\right] = 0, \tag{3.18}$$

where we have collected all of the constants that determine the ion-neutral coupling into a single parameter $\Gamma = 2\pi\zeta\tilde{\eta}^2 A_i^2 A^2/\alpha^2$ (the parameter Γ is typically $\sim$1–10). Although this model equation is somewhat idealized, it does retain the essential physics of the friction term. In particular, the term is positive at low densities and becomes negative for sufficiently large densities. We emphasize, however, that our rather cavalier approach is motivated primarily to carry the problem as far as we can analytically (see Appendix E for a discussion of the validity of the model equation).

The model equation of motion [3.18] can be easily integrated to obtain

$$\begin{aligned}\frac{1}{2}\rho_\xi^2 = \rho^6\left[\rho^2 + \kappa\rho - A^2\right]^{-2}\Bigg\{&\beta - \rho - (\kappa - \Gamma/\rho_F)\log\rho - (A^2 + \Gamma\kappa/\rho_F - \Gamma)\frac{1}{\rho} \\ &+ \Gamma(\kappa + A^2/\rho_F)\frac{1}{2\rho^2} - \frac{\Gamma A^2}{3\rho^3}\Bigg\},\end{aligned} \tag{3.19}$$

where we have once again used β as the constant of integration. As we discuss in the following section, the solution [3.19] can exhibit many interesting types of nonlinear wave behavior.

4. SOLITONS IN ONE-DIMENSIONAL MOLECULAR CLOUDS

In this section, we discuss the existence and/or non-existence of solitons in molecular clouds for a variety of circumstances. In this discussion, we will use the methods of phase plane analysis as discussed at some length in Infeld & Rowlands (1990) and in Drazin & Johnson (1989). In this method, we look for traveling wave solutions to the equations of motion, i.e., we consider all fluid fields to be functions of $\xi = x - v_0 t$. We then try to reduce the system of equations to a single equation of the form

$$\frac{1}{2}\rho_\xi^2 = \mathcal{F}(\rho, C_j), \tag{4.1}$$

where the C_j are constants. Although we have used the symbol ρ as the field of choice in equation [4.1], one can, in principle, use any of the dependent variables. As we discuss

below, the properties of the function $\mathcal{F}$ determine the properties of the wave solutions $\rho(\xi)$ in a fairly simple way.

First of all, physically meaningful solutions require that $\mathcal{F} \geq 0$ (so that the solutions are real). In the present case, the field ρ is a density and must also be positive. Thus, physically relevant solutions exist when $\mathcal{F}(\rho)$ is positive over a range of postive densities. Next, we note that wave solutions exist when $\mathcal{F}(\rho)$ is positive between two zeroes of the function $\mathcal{F}$. The zeroes of $\mathcal{F}$ correspond to maximum and minimum densities, which are obviously required for wavelike behavior. Finally, we note that the nature of the zeroes of $\mathcal{F}$ determines the nature of the wavelike solutions. In particular, if $\mathcal{F}$ is positive between two simple zeroes, then (ordinary) nonlinear waves result. However, if $\mathcal{F}$ is positive between a simple zero and a double zero (i.e., a point where both $\mathcal{F}$ and $\partial\mathcal{F}/\partial\rho$ vanish), then a new type of solution – a soliton – can result. One way to see this is to expand equation [4.1] about the double zero, which we denote as ρ_0:

$$\rho_\xi^2 = (\rho - \rho_0)^2 \mathcal{F}''(\rho_0) + \mathcal{O}\left[(\rho - \rho_0)^3\right], \tag{4.2}$$

where $\mathcal{F}''(\rho_0) > 0$ because we are considering that case in which $\mathcal{F}$ is positive. Thus, as $\rho \to \rho_0$, we obtain

$$\rho - \rho_0 \sim \delta \exp\left[\pm\sqrt{\mathcal{F}''(\rho_0)}\xi\right], \tag{4.3}$$

where δ is a constant. Putting all of these results together, we see that a soliton consists of a single large hump of material and that the density smoothly goes to its asymptotic value ρ_0 as $\xi \to \pm\infty$. Another way to describe this behavior is to note that the wavelength of the solution diverges for a soliton (see Appendix D). Notice that this same sort of behavior also occurs if the function $\mathcal{F}$ has a higher order zero (i.e., the first $n-1$ derivatives of $\mathcal{F}$ vanish) with $\partial^n\mathcal{F}/\partial\rho^n > 0$. In this case the density still smoothly approaches its asymptotic value as $\xi \to \pm\infty$, although the approach is no longer exponential.

The remaining possibility of interest occurs when the function $\mathcal{F}$ is positive between *two* double (or higher order) zeroes of $\mathcal{F}$, say ρ_A and ρ_B. In this case, it can be shown that the solution $\rho(\xi)$ can approach one value (e.g., ρ_A) in the limit $\xi \to -\infty$ and the other value (ρ_B) in the limit $\xi \to \infty$. Solutions of this type are known as *kinks* or *topological solitons* † and arise frequently in the context of quantum field theory (e.g., Coleman 1985; Rajaraman 1987).

† The reader might wonder just what is "topological" about a "topological soliton". In the case considered here, e.g., we can divide the entire space of solutions into subspaces labeled by the values of $\rho(\xi)$ as $\xi \to \pm\infty$. Here we have four such subspaces: (ρ_A, ρ_A), (ρ_A, ρ_B), (ρ_B, ρ_A), and (ρ_B, ρ_B). These subspaces are disconnected subsets of the whole space of solutions in the usual topological sense (a solution belonging to a given subspace cannot be continuously distorted into a solution belonging to a different subspace). The space of solutions thus has a nontrivial topology (see, e.g., Coleman 1985 for further discussion).

4.1 A No-Soliton Theorem for Molecular Clouds without Magnetic Fields

In this section we demonstrate that no true soliton solutions exist for one dimensional field-free clouds, i.e., physical systems that obey the equations of motion [2.1 – 2.3]. We could begin with the solution [2.12], which is applicable only for the equation of state [2.4], and show that no solitons are allowed. However, one might suppose that different equations of state could lead to a different conclusion. We will therefore generalize this present discussion to include any barotropic equation of state, i.e., any equation of state of the form

$$p = p(\rho). \tag{4.4}$$

However, we must place mild further restrictions on the equation of state. We require that

$$\rho^2 \frac{\partial p}{\partial \rho} \to 0 \qquad \text{and} \qquad \rho p \to 0 \qquad \text{as } \rho \to 0, \tag{4.5}$$

which imply that the pressure does not become infinitely large in the low density limit $\rho \to 0$. Finally, we require that the function $p(\rho)$ can be expanded in a Taylor series about any density point $\rho \neq 0$. Given this class of equations of state, we now derive the main result of this subsection, which we state as a theorem for emphasis:

Theorem 1. *No soliton solutions exist for one dimensional molecular clouds without magnetic fields, i.e., the physical system defined by equations [2.1 – 2.3], [4.4], and [4.5].*

Proof: Following the same procedure outlined in §2, we derive the general equation of motion for a stationary wave:

$$\rho\rho_{\xi\xi}\Big[\rho^2\frac{\partial p}{\partial \rho} - A^2\Big] + \rho_\xi\rho_\xi\Big[3A^2 - \rho^2\frac{\partial p}{\partial \rho} + \rho^3\frac{\partial^2 p}{\partial \rho^2}\Big] + \rho^5 = 0, \tag{4.6}$$

which is written in nondimensional form. Fortunately, the first integral of equation [4.6] can still be found analytically:

$$\frac{1}{2}\rho_\xi^2 = \rho^6\Big[\rho^2\frac{\partial p}{\partial \rho} - A^2\Big]^{-2}\Big\{\beta - p(\rho) - A^2/\rho\Big\} \equiv \mathcal{F}(\rho), \tag{4.7}$$

where β is the constant of integration. Equation [4.7] is the generalization of the solution [2.12] for an arbitrary barotropic equation of state $p(\rho)$.

In order to show that no solitons exist for this physical system, we must show that the function $\mathcal{F}$ cannot be positive between a double (or higher order) zero and a simple zero of $\mathcal{F}$ (where $\mathcal{F}$ is given by equation [4.7]). We first show that no multiple zeroes exist other than (possibly) at $\rho = 0$. To show this, we first define

$$g(\rho) = \beta - p(\rho) - A^2/\rho, \tag{4.8}$$

and write the solution in the form

$$\frac{1}{2}\rho_\xi^2 = \mathcal{F}(\rho) = \rho^2 g(\rho)\Big(\frac{\partial g}{\partial \rho}\Big)^{-2}. \tag{4.9}$$

Now suppose that $\rho_0 \neq 0$ is a zero of $\mathcal{F}$. Then we must have $g(\rho_0) = 0$ *and* $\partial g/\partial\rho\ (\rho_0) \neq 0$ (for our given class of equations of state). We now write $\partial\mathcal{F}/\partial\rho$ in the form

$$\frac{\partial\mathcal{F}}{\partial\rho} = 2g\Big\{\rho\Big(\frac{\partial g}{\partial\rho}\Big)^{-2} - \rho^2\Big(\frac{\partial g}{\partial\rho}\Big)^{-3}\frac{\partial^2 g}{\partial\rho^2}\Big\} + \rho^2\Big(\frac{\partial g}{\partial\rho}\Big)^{-1} = \rho^2\Big(\frac{\partial g}{\partial\rho}\Big)^{-1}, \tag{4.10}$$

where all quantities are evaluated at $\rho = \rho_0$ and where we have used the fact that $g(\rho_0) = 0$ in obtaining the second equality (note that no competing infinities in the term in brackets are present). Since $\rho_0 \neq 0$ (by hypothesis) and $\partial g/\partial\rho$ is finite (for any physically reasonable equation of state), we find that $\partial\mathcal{F}/\partial\rho \neq 0$ for any possible zero $\rho_0 \neq 0$.

To complete the proof, we consider the possibility that $\rho = 0$ provides the required double or higher order zero. Let us expand the function $\mathcal{F}(\rho)$ about the point $\rho = 0$:

$$\mathcal{F} = A^{-4}\rho^5\Big\{\beta\rho - A^2 - \rho p - 2\rho^2\frac{\partial p}{\partial\rho} + \ldots\Big\}. \tag{4.11}$$

Given the constraints of equation [4.5], by inspection we see that $\mathcal{F}$ is always negative in a sufficiently small neighborhood of $\rho = 0$; thus, $\rho = 0$ does not provide the zero of $\mathcal{F}$ required for the existence of solitons.

Since both $\rho_0 \neq 0$ and $\rho_0 = 0$ fail to provide double (or higher order) zeroes of $\mathcal{F}$ with $\mathcal{F} > 0$, we conclude that no solitons exist for this system. Q.E.D.

4.2 Soliton Solutions to the Model Equation

In this section we show that the model equation [3.18] can exhibit a wide variety of nonlinear wave phenomena including soliton solutions. As discussed above, the types of allowed wave solutions depend on the properties of the function $\mathcal{F}(\rho)$, which, in this case, is given by the right hand side of equation [3.19]. In particular, we are interested in the nature of the zeroes of the function $\mathcal{F}(\rho)$. We begin by noting that the point $\rho = 0$ will always be a multiple zero, as shown by the expansion about $\rho = 0$

$$\mathcal{F}(\rho) = \rho^3 A^{-4}\Big\{-\frac{1}{3}\Gamma A^2 + \frac{1}{2}\Gamma\big(\frac{A^2}{\rho_F} - \frac{\kappa}{3}\big)\rho + \mathcal{O}(\rho^2)\Big\}. \tag{4.12}$$

The point $\rho = 0$ thus provides a triple zero of $\mathcal{F}$, but the function $\mathcal{F}$ is always negative in a sufficiently small neighborhood of $\rho = 0$. As a result, soliton behavior in which $\rho \to 0$ as $\xi \to \pm\infty$ cannot occur for the solution [3.19] of the model equation of motion.

We now consider (possible) multiple zeroes of $\mathcal{F}$ for some point $\rho_0 \neq 0$. To study this case, we write $\mathcal{F}(\rho)$ in the form

$$\mathcal{F}(\rho) = \rho^3[\rho^2 + \kappa\rho - A^2]^{-2} f(\rho), \tag{4.13}$$

$$f(\rho) \equiv \beta\rho^3 - \rho^4 - (\kappa - \frac{\Gamma}{\rho_F})\rho^3 \log\rho - (A^2 + \frac{\Gamma\kappa}{\rho_F} - \Gamma)\rho^2 + \frac{1}{2}\Gamma(\kappa + \frac{A^2}{\rho_F})\rho - \frac{1}{3}\Gamma A^2. \tag{4.14}$$

It is straightforward to show that a double (or higher order) zero of $\mathcal{F}(\rho)$ can occur at a point $\rho_0 \neq 0$ if and only if the function $f(\rho)$ has a double (or higher order) zero at ρ_0 (provided that the double zero of $f(\rho)$ does not occur at the singularity, i.e., provided that $\rho_0^2 + \kappa\rho_0 \neq A^2$). Our discussion is thus reduced to finding the nature of the zeroes of $f(\rho)$. We can simplify the situation even further by adopting the particular choice $\rho_F = \Gamma/\kappa$, which reduces the number of parameters in the problem and eliminates the logarithmic term from $f(\rho)$. Notice that the available parameter space is still quite large; we get to specify the constants κ, Γ, A, and β.

Now suppose that f has a double zero at a point $\rho_0 \neq 0$. Since f is a quartic polynomial, this supposition implies that f is of the form

$$f(\rho) = (\rho - \rho_0)^2\left[-\rho^2 + B\rho - C\right], \tag{4.15}$$

where B and C are constants which are related to the parameters appearing in equation [4.14]. By equating the coefficients in equations [4.14] and [4.15], we obtain consistency conditions of the form

$$\Gamma(\rho_0^2 + \kappa\rho_0 - A^2) = \rho_0(\kappa - \rho_0)(\rho_0^2 + \kappa\rho_0 - A^2), \tag{4.16}$$

$$\beta = \frac{\kappa}{2\rho_0^2}(\Gamma + A^2) - \frac{2\Gamma A^2}{3\rho_0^3} + 2\rho_0. \tag{4.17}$$

Since the soliton solution cannot have the double zero of $f(\rho)$ at the singularity, we must consider parameters such that $\rho_0^2 + \kappa\rho_0 \neq A^2$ and hence equation [4.16] reduces to $\Gamma = \rho_0(\kappa - \rho_0)$. In order to ensure that f (and hence $\mathcal{F}$) is positive in the neighborhood of the double zero, we need the further requirement that

$$\beta > 3\rho_0 + \Gamma A^2/3\rho_0^3. \tag{4.18}$$

Finally, we note that we have sufficient freedom in our choice of parameters that we can make the further restriction

$$\rho_0^2 + \kappa\rho_0 > A^2, \tag{4.19}$$

which ensures that the singularity (see equation [4.13]) does not lie in the range of densities taken by the wave form.

As an example, we take the choice of parameters $\kappa = 10$, $\Gamma = 9$, $A = 3$, $\beta = 38$, and hence $\rho_0 = 1$ and $\rho_F = 9/10$. The resulting function $\mathcal{F}(\rho)$ is shown in Figure 3a, and the corresponding wave profile is shown in Figure 3b. Notice that the wave profile of the soliton is very localized – most of the interesting part of the profile occurs within $|\xi| < 1$. As a second example, we consider the parameters $A = 1$, $\kappa = 10$, $\Gamma = 16$, $\beta = 23\frac{11}{12}$, and hence $\rho_0 = 2$ and $\rho_F = 1.6$. The resulting function $\mathcal{F}(\rho)$ is shown in Figure 4a, and the corresponding wave profile is shown in Figure 4b. Notice that the wave profile of this soliton is wider and lower than that of Figure 3. As we discuss in Appendix E, this second soliton solution is more consistent (compared to that of Fig. 3) with the approximations used in deriving the model equation.

4.3 Kink Solutions to a Limiting Form of the Model Equation

In this section we study kink solutions (topological solitons) to a limiting form of the model equation of motion. As noted above, kink solutions require that the function $\mathcal{F}(\rho)$ is positive between two double (or higher order) zeroes. Notice that since the quartic term (ρ^4) is negative in the function $f(\rho)$, we cannot have two double zeroes of f where $f > 0$ in between. In the previous section we showed that although the point $\rho = 0$ is a multiple zero of $\mathcal{F}$, the function is always negative in a neighborhood of $\rho = 0$. However, we can in principle reduce the model equation of motion to a limiting form in which $\rho = 0$ does provide a higher order zero about which $\mathcal{F}$ is positive. Let us simultaneously take the limits

$$\Gamma \to 0 \qquad \text{and} \qquad \rho_F \to 0, \tag{4.20}$$

in such a manner that the ratio approaches a constant, i.e.,

$$\frac{\Gamma}{\rho_F} \to \mu = \textit{constant}. \tag{4.21}$$

In this limit, the model equation of motion can show qualitatively new behavior. In particular, as we show below, kink solutions can be found.

Given the limit described above, we can find the first integral as before. We will begin by removing the logarithmic term from the analog of equation [3.19] by adopting the judicious choice of parameter $\mu = \kappa$. The solution then becomes

$$\mathcal{F}(\rho) = \rho^4\left[\rho^2 + \kappa\rho - A^2\right]^{-2}\left\{\beta\rho^2 - \rho^3 - (A^2 + \kappa^2)\rho + \frac{1}{2}\kappa A^2\right\}. \tag{4.22}$$

By inspection we see that $\rho = 0$ is now a fourth order zero of $\mathcal{F}$ and that $\mathcal{F}$ is positive in the neighborhood of $\rho = 0$. The point $\rho = 0$ thus provides the first of the two higher order zeroes required for kink solutions.

The second multiple zero must be a double zero of the cubic polynomial

$$f(\rho) \equiv \beta\rho^2 - \rho^3 - (A^2 + \kappa^2)\rho + \frac{1}{2}\kappa A^2. \tag{4.23}$$

If we denote the desired double zero as ρ_0, the polynomial $f(\rho)$ must be of the form

$$f(\rho) = (\rho - \rho_0)^2\left[b - \rho\right], \tag{4.24}$$

where b is a constant (and we must have $b > \rho_0$ in order to keep $f(\rho)$ positive). By equating coefficients in equations [4.23] and [4.24], we can derive consistency conditions as in the previous section. The first of these can be written

$$A^2(\rho_0 - \kappa) = \rho_0(\rho_0^2 - \kappa^2). \tag{4.25}$$

Notice that if $\kappa \neq \rho_0$, then equation [4.25] can be reduced to the condition $A^2 = \rho_0^2 + \kappa\rho_0$, i.e., the singularity occurs at the double zero of $f(\rho)$ and hence we do not obtain the

required double zero of $\mathcal{F}(\rho)$. As a result we must take $\rho_0 = \kappa$ as the relevant root of the consistency equation [4.25]. The remaining consistency equation then takes the simple form

$$\beta = \frac{A^2}{2\kappa} + 2\kappa\,. \tag{4.26}$$

The requirement that $\mathcal{F}$ is positive between the two multiple zeroes implies an additional constraint

$$\beta > 3\rho_0 = 3\kappa\,. \tag{4.27}$$

Given the above constraints, we can find kink solutions in which the allowed range of densities is $0 \leq \rho \leq \rho_0$. If we want to simplify the situation by eliminating the singularity from the allowed range of densities, we must impose the further constraint

$$A^2 > \rho_0^2 + \kappa\rho_0 = 2\kappa^2\,. \tag{4.28}$$

Notice that if both equations [4.26] and [4.27] are satisfied, then the constraint [4.28] is automatically satisfied also. In other words, for this kink solution, the singular value of the density (ρ_C) always lies outside the range of densities accessible to the wave profile.

As an example of this type of kink solution, we take $\kappa = 10$ as before. The first consistency condition then requires that $\rho_0 = 10 = \kappa$. The consistency condition [4.26] and the positivity constraint [4.27] can both be satisfied if we take $A = 10$ and $\beta = 40$. For this choice of parameters, the first integral $\mathcal{F}(\rho)$ is shown in Figure 5a and the corresponding wave profile $\rho(\xi)$ is shown in Figure 5b. Notice the asymmetry in the wave profile $\rho(\xi)$. This behavior results from the fact that the zero of $\mathcal{F}$ on the high density side is a double zero, whereas that on the low density side is a fourth order zero.

5. STABILITY OF THE WAVES

In this section we discuss the stability of nonlinear waves in molecular clouds. In the previous sections we have shown that stationary nonlinear wave solutions exist for one-dimensional clouds. Furthermore, we have shown that solitons (and even kinks) can exist under the proper circumstances. However, there is no guarantee that these wave solutions will be strictly stable to the growth of small perturbations. If the wave solutions are unstable, then the clumps which they represent can live for only a finite (and hopefully calculable) lifetime. We stress that unstable waves are not fatal for a description of substructure in molecular clouds, since the clouds themselves have a relatively short lifetime ($\sim 10^7$ yr, i.e., less than a sound crossing time – see, e.g., Blitz 1992). Note that the substructure within the cloud (the clumps) could be even shorter-lived. However, the timescale for instability cannot be shorter than the required lifetime of the clumps.

Unfortunately, a full treatment of the stability properties of these nonlinear waves is beyond the scope of this current paper. We can, however, obtain useful preliminary results. In particular, we show in this section that for clouds without magnetic fields the waves are always stable in the limit of short wavelength perturbations and that the waves are always unstable for perturbations with sufficiently large wavelengths. For the model equation, we also show that large scale perturbations can be stabilized with a sufficiently large coupling parameter Γ.

In order to discuss the issue of stability, we must return to the original time dependent formulation of the problem. We can simplify the situation somewhat by using $\xi = x - v_0 t$ as the spatial coordinate, i.e., we transform the equations of motion to a frame of reference which is traveling along with the unperturbed wave disturbance. Proceeding as in §2, we derive the general equations of motion in the form

$$\rho_t + \rho_\xi v + \rho v_\xi = 0, \tag{5.1}$$

$$v_{\xi t} + v v_{\xi\xi} + v_\xi v_\xi + \frac{\rho + \kappa}{\rho^2}\rho_{\xi\xi} - \frac{\rho + 2\kappa}{\rho^3}\rho_\xi \rho_\xi + \rho = 0, \tag{5.2}$$

where we have defined $v = u - v_0$ as before and where subscripts refer to differentiation. We now expand the fluid fields according to

$$\rho = \rho(\xi) + \delta(\xi, t), \tag{5.3}$$

$$v = v(\xi) + w(\xi, t), \tag{5.4}$$

where the functions $\rho(\xi)$ and $v(\xi)$ on the right hand sides of the equations are the unperturbed stationary solutions and where δ and w are to be considered as small perturbations. Next, we find the differential equations for the first order fields:

$$\delta_t + \partial_\xi(\rho w + v\delta) = 0, \tag{5.5}$$

$$w_{\xi t} + [wv]_{\xi\xi} + \left[1 + \frac{2(\rho + 3\kappa)}{\rho^4}\rho_\xi \rho_\xi - \frac{\rho + 2\kappa}{\rho^3}\rho_{\xi\xi}\right]\delta + \frac{\rho + \kappa}{\rho^2}\delta_{\xi\xi} - 2\frac{\rho + 2\kappa}{\rho^3}\rho_\xi \delta_\xi = 0, \tag{5.6}$$

where the subscript on the square brackets denotes differentiation. We can now state the first result of our stability analysis:

Result 1. *For one dimensional molecular clouds without magnetic fields, perturbations with small size scales are always stable in the limit $k \to \infty$.*

Proof: To prove this claim, it is sufficient to consider perturbations which have a size scale much smaller than the wavelength of the unperturbed wave. In this case, we can consider the fluid functions $\rho(\xi)$ and $v(\xi)$ which appear in equations [5.5] and [5.6] as slowly varying. For differential equations of the form [5.5] and [5.6] with constant cofficients, it is valid to expand the first order fields δ and w into fourier modes and to consider only one mode at a time, i.e., we take

$$\delta = \delta_{(k)} \exp[i(k\xi - \omega t)], \tag{5.7}$$

$$w = w_{(k)} \exp[i(k\xi - \omega t)], \tag{5.8}$$

where $\delta_{(k)}$ and $w_{(k)}$ are constants and where k is the wavenumber of the perturbation. The first order equations then take the form

$$(\rho_\xi + ik\rho)\, w_{(k)} + (v_\xi + ikv - i\omega)\, \delta_{(k)} = 0, \tag{5.9}$$

$$(k\omega - k^2 v + v_{\xi\xi} + 2ikv_\xi)\, w_{(k)} + \\ \left[1 - \frac{\rho+\kappa}{\rho^2}k^2 - \frac{\rho+2\kappa}{\rho^3}(2ik\rho_\xi + \rho_{\xi\xi}) + 2\frac{\rho+3\kappa}{\rho^4}\rho_\xi\rho_\xi\right]\delta_{(k)} = 0. \tag{5.10}$$

The above two equations can be combined to produce a dispersion relation which governs the stability/instability of the waves in this (constant coefficient) approximation. In this particular case, we are interested in perturbations with short wavelengths and hence large k. If we take the large k limit of equations [5.9] and [5.10], we obtain a dispersion relation of the form

$$D(\omega, k) = k\left[\rho^2\omega^2 - 2Ak\rho\omega + k^2(A^2 - \kappa\rho - \rho^2)\right] \\ - ik^2\rho_\xi\left[\rho + 3\kappa - 3A(A/\rho - \omega/k)\right] + \mathcal{O}(k) = 0, \tag{5.11}$$

where we have used the solution $v = A/\rho$ in obtaining this form. Notice that the second term (on the second line) is smaller than the first by one power of k. To leading order in k, the angular frequency $\omega = \omega_R$ is given by

$$\frac{\omega_R}{k} = \frac{A}{\rho} \pm [1 + \kappa/\rho]^{1/2}, \tag{5.12}$$

which is always a real number – the quantity in the square root is the square of the (local) sound speed and is always positive. Thus, perturbations in the short wavelength limit are always stable (to leading order in k) and propagate at the sound speed relative to the unperturbed flow (recall that $A/\rho = v$, the speed of the fluid relative to the rest frame of the wave). However, if we write

$$\omega = \omega_R + i\gamma, \tag{5.13}$$

we can solve for the leading order contribution to the growth rate γ:

$$\gamma = \pm\frac{\rho_\xi}{2\rho^2}\left\{3A \pm (\rho + 3\kappa)[1 + \kappa/\rho]^{-1/2}\right\}, \tag{5.14}$$

where we have used the leading order solution [5.12] for the real part of ω in obtaining this form. The growth rate γ can be positive or negative (corresponding to either growing or damped perturbations), but the size of γ is smaller than that of ω_R by one factor of k. The waves are thus truly stable only in the limit $k \to \infty$. Notice, however, that we cannot literally take the $k \to \infty$ limit; this discussion applies to size scales $1/k \ll 1$ pc (the size of a small "clump"), but still sufficiently large that the equation of state [2.4] is valid.

Result 2. *For one dimensional molecular clouds without magnetic fields, perturbations with sufficiently large size scales are always unstable.*

Proof: It is clear from physical considerations that this result must be true: On a sufficiently large size scale, the "waviness" of the solution can be ignored and the wave profile $\rho(\xi)$ can be conceptually replaced by a mean density $\langle\rho\rangle$. Standard Jeans analysis then implies that gravitational instability will occur for a sufficiently large size scale. In the following discussion, we show that this argument does indeed hold (and thus this proof provides a consistency check on our approach).

Let us define L to be the size scale of the perturbation (so that $k = 2\pi/L$ is the wavenumber) and let λ be the wavelength of the (unperturbed) nonlinear wave. We want to average all quantities in the problem over an intermediate length scale ξ_I, such that the ordering

$$L \gg \xi_I \gg \lambda \tag{5.15}$$

applies. Furthermore, we will take ξ_I to be an integer number of wavelengths (λ). For a given function f, we denote its average via

$$\frac{1}{\xi_I}\int_0^{\xi_I} f\, d\xi \equiv \langle f\rangle. \tag{5.16}$$

When we average the equations of motion over the interval $(0, \xi_I)$ we can pull the slowly varying functions δ and w out of the integral, e.g.,

$$\frac{1}{\xi_I}\int_0^{\xi_I} (\rho w_\xi + \rho_\xi w)\, d\xi = w_\xi\langle\rho\rangle + w\langle\rho_\xi\rangle + \ldots = w_\xi\langle\rho\rangle + \ldots, \tag{5.17}$$

where the dots ... indicate higher order terms due to the (slow) variation of w and w_ξ. In obtaining the second equality we have used the fact that the average of ρ_ξ vanishes. Using similar arguments for the other terms, the continuity equation [5.5] can be written

$$\delta_t + \langle\rho\rangle w_\xi + \langle v\rangle\delta_\xi = 0, \tag{5.18}$$

where we have ignored the higher order terms (in $1/L$). Similarly, the force equation [5.6] can be written in this same approximation in the form

$$w_{\xi t} + \langle v\rangle w_{\xi\xi} + (1 + P_1)\delta + P_2\delta_{\xi\xi} = 0, \tag{5.19}$$

where, for convenience, we have defined

$$P_1 \equiv 2\Big\langle \frac{\rho + 3\kappa}{\rho^4}\rho_\xi^2\Big\rangle \qquad \text{and} \qquad P_2 \equiv \Big\langle \frac{\rho + \kappa}{\rho^2}\Big\rangle. \tag{5.20}$$

We now assume that the perturbations can be written in terms of Fourier modes as in equations [5.7 – 5.8] and obtain a dispersion relation of the form

$$D(\omega, k) = k\Big[(\omega - \langle v\rangle k)^2 + \langle\rho\rangle\,(1 + P_1 - k^2 P_2)\Big] = 0. \tag{5.21}$$

The solution of equation [5.21] for ω in terms of k can be written

$$\omega = k\langle v\rangle \pm \left[\langle\rho\rangle(k^2 P_2 - 1 - P_1)\right]^{1/2}. \tag{5.22}$$

In comparing the above form to the usual dispersion relation for a uniform density cloud in one dimension (see Appendix A), we see that the quantity $\langle\rho\rangle P_2$ plays the role of the sound speed squared (note that P_2 is the average of a^2/ρ) and that the other terms arise from the nonzero fluid motions present in the waves. In this notation, the critical wavenumber k_c for gravitational instability takes the form

$$k_c^2 = \frac{1 + P_1}{P_2}. \tag{5.23}$$

In the $k \to 0$ limit, we obtain

$$\omega^2 + \langle\rho\rangle\,(1 + P_1) = 0, \tag{5.24}$$

which implies that ω becomes pure imaginary and therefore the waves become unstable.

Result 3. *For physical systems governed by the model equation of motion [3.18], which includes magnetic fields in a heuristic fashion, perturbations on large size scales can be stabilized with a sufficiently large coupling parameter* Γ.

Proof: In order to study the stability of the solutions, we must have a time-dependent form of the equations of motion (where magnetic fields are now included). Since the ion density and velocity have already been eliminated in obtaining the model equation itself, it is sufficient to consider only the netural density ρ and velocity v, which we expand according to equations [5.3] and [5.4]. The first order continuity equation [5.5] remains the same as before. In the force equation [5.2], before we take the first order perturbation, we must add a coupling term and obtain

$$v_{\xi t} + v v_{\xi\xi} + v_\xi v_\xi + \frac{\rho + \kappa}{\rho^2}\rho_{\xi\xi} - \frac{\rho + 2\kappa}{\rho^3}\rho_\xi\rho_\xi + \rho + \Gamma\left[\frac{1}{\rho} - \frac{1}{\rho_F}\right] = 0, \tag{5.25}$$

which implies a first order equation of the form

$$w_{\xi t} + [wv]_{\xi\xi} + \frac{\rho + \kappa}{\rho^2}\delta_{\xi\xi} - 2\frac{\rho + 2\kappa}{\rho^3}\rho_\xi\delta_\xi + \left[1 + \frac{2(\rho + 3\kappa)}{\rho^4}\rho_\xi\rho_\xi - \frac{\rho + 2\kappa}{\rho^3}\rho_{\xi\xi} - \frac{\Gamma}{\rho^2}\right]\delta = 0\,, \tag{5.26}$$

where we have again expanded the fluid fields according to equations [5.3] and [5.4]. If we redo the stability analysis (as outlined by equations [5.15 – 5.20]) with the inclusion of the magnetic coupling term, the dispersion relation becomes

$$D(\omega, k) = k\left[(\omega - \langle v\rangle k)^2 + \langle\rho\rangle\,(1 + P_1 - \Gamma\langle\frac{1}{\rho^2}\rangle - k^2 P_2)\right] = 0\,, \tag{5.27}$$

where the angular brackets denote the intermediate spatial average as before. The critical wavenumber for gravitational stability/instability is thus

$$k_c^2 = \frac{1}{P_2}\left[1 + P_1 - \Gamma\langle\frac{1}{\rho^2}\rangle\right], \tag{5.28}$$

which clearly shows the stabilizing effect of the magnetic field (compare equations [5.23] and [5.28]).

Notice that this model has the peculiar property that for sufficiently large coupling strength Γ (and for a range of values of the other parameters), the critical wavenumber becomes imaginary, i.e., the perturbations become stable on all length scales. In this regard our model equation departs from the true physical situation, where a sufficiently large scale perturbation will always be unstable (i.e., the magnetic Jean's length always exists – see, e.g., Strittmatter 1966; Mouschovias 1976). On the other hand, this stable feature of the model allows us to study wave propagation against a background which is not collapsing, a situation which apparently does occur in nearby molecular clouds. Notice also that, in practice, the parameter Γ must be very large to completely stabilize the system. If we use the soliton solutions presented in §4.2 to obtain a rough estimate of the values of P_1 and $\langle 1/\rho^2 \rangle$ appearing in equation [5.28], we find that a value of $\Gamma \sim 100 - 200$ is required to keep perturbations stable on all length scales. Notice, however, that this stability calculation is not strictly valid for solitons. We have assumed here that the ordering of equation [5.15] applies whereas the wavelength of a soliton diverges. More detailed stability considerations must be applied in the case of solitons (see, e.g., Pego & Weinstein 1992).

6. DISCUSSION

6.1 Summary of Results

In this paper we have begun a systematic exploration of nonlinear wave phenomena in molecular clouds. This present study has concentrated on the case of one-dimensional clouds. We have shown that a rich variety of nonlinear wave entities can live in these clouds and may thus provide some of the observed structure.

For the case of one-dimensional molecular clouds without magnetic fields, highly nonlinear waves can exist (see Figure 2). As the waves become more and more nonlinear, the wavelength increases. We have shown that these waves are stable to perturbations on small size scales (the $k \to \infty$ limit) and unstable to perturbations with sufficiently large size scales (the $k \to 0$ limit). For molecular clouds without magnetic fields, we have proved a "no soliton theorem" which shows that no true solitons can exist in these physical systems.

For molecular clouds which contain magnetic fields, we have derived a model equation which governs wave propagation. This model equation admits a rich variety of phenomena (see Figures 3 – 5) including nonlinear waves, solitons, and kinks (topological solitons). One interesting property of the model equation and its solutions is that for sufficiently large values of the coupling parameter Γ, the waves are stable even on the largest scale. Since the simplest physical arguments show that an effective Jeans length exists even in the presence of magnetic fields, this stability property of the model equation implies an apparent contradiction. However, observed molecular clouds are not collapsing on a free-fall time scale and are thus supported in some manner. The model equation also has this property and thus might be a reasonably realistic model of the true

astrophysical situation. In particular, the model equation allows us to study nonlinear wave propagation against a background which is not collapsing.

We have derived several formal results which are presented in the Appendices. We have derived the linear dispersion relation for clouds with the joint isothermal/logatropic equation of state (Appendix A). We have studied the properties of the singularity (Appendices B and C). For clouds without magnetic fields, we have found an estimate of the wavelength of nonlinear waves as a function of parameters (Appendix D). We have discussed the conditions under which the model equation of motion is valid (Appendix E) and have found a Lie group of scaling transformations which leave the model equation invariant (Appendix F).

6.2 Comparison with Observed Molecular Clouds

A direct comparison between observed molecular clouds and the theory developed in this paper is somewhat difficult at present. The theory has focused on one-dimensional dynamics, whereas observed molecular clouds maps show structure in either two dimensions (for column density maps) or in three dimensions (when line velocity information is included). Fortunately, however, some comparisons can be made. One way to describe cloud structure is to measure the mass fraction m of material above a given threshold density Σ as a function of threshold density (Adams 1992):

$$m(\rho;\Sigma) \equiv \frac{\int d^n\mathbf{x}\ \rho(\mathbf{x})\ \Theta\big[\rho(\mathbf{x}) - \Sigma\big]}{\int d^n\mathbf{x}\ \rho(\mathbf{x})}, \tag{6.1}$$

where $\rho(\mathbf{x})$ is a map of the cloud, Θ is a step function, and where the integrals are taken over the area (or volume) of the map. Observed *IRAS* maps of nearby molecular clouds (Wood et al. 1999) show smooth distributions of density as defined by equation [6.1] (see Wiseman & Adams 1992). Such profiles are similar to those produced by wave motions [Figures 2–5 show wave profiles $\rho(\xi)$ which imply corresponding smooth distributions $m(\rho;\Sigma)$] and are not consistent with the picture of high density "clumps" moving through a diffuse interclump medium.

Another hint that large scale wavelike entities occur in molecular clouds comes from observed periodic or nearly periodic structure in maps of these objects. For example, the 69μm continuum map of the star forming region NGC 6334 shows five regularly spaced condensations (see McBreen *et al.* 1979; see also Zinnecker, McCaughrean, & Wilking 1992). In fact, the structure of molecular clouds is often described in terms of "large scale filaments" which have "fragmented" to form "clumps" (see, e.g., de Geus, Bronfman, & Thaddeus 1990 for a description of the Ophiuchus clouds). The question of whether or not nonlinear waves and solitons can explain the observed "clumps" in these clouds will be answered when the present calculations are extended to higher spatial dimensions (see below).

6.3 Directions for Future Work

Although some progress has been made, much work remains to be done before nonlinear waves in molecular clouds are sufficiently well understood. In the one dimensional

problem, the stability of the waves remains somewhat unresolved. Although we have studied the stability of the system in the limits of small scale perturbations ($k \to \infty$) and large scale perturbations ($k \to 0$), a general stability analysis remains to be done. In particular, we would like to know the timescales on which these waves will change their form. Numerical simulations of wave propagation and/or more sophisticated mathematical techniques may be necessary to settle this issue.

The next issue which must be addressed is the collision and interaction of the waves. In other physical systems, soliton wave entities can remain intact during collisions. Waves with this property occur in such widely disparate systems as the single humped solitons in shallow water (see Infeld & Rowlands 1990) and the solitons of the sine-Gordon system from quantum field theory (see Rajaraman 1987). It will be interesting to study the interactions of the soliton solutions to the model equation of motion derived in this paper.

Another issue that we have not dealt with is the actual generation of nonlinear waves. We expect that a self-gravitating cloud, which is not in general in complete equilibrium, will tend to collapse and thereby generate a rich and complicated spectrum of wave motions (see, e.g., Arons & Max 1975). The self gravity of the cloud thus represents a more than adequate energy source for the waves; the process of star formation provides an additional energy source (e.g., Norman & Silk 1980). Only the most stable (longest lived) wave entities will survive to affect the observed cloud structure. In order to study the details of how these waves are generated, however, one must solve the full time-dependent form of the equations of motion as an initial value problem. We leave this task for future work.

An important limitation of this present study is that it includes only one spatial dimension. In order to describe realistic molecular clouds, the present calculations must be generalized to include two (and three) spatial dimensions. One standard trend from fluid dynamics that should be kept in mind is that systems which are stable in one dimension are often unstable in two dimensions. Suppose, for example, we consider a nonlinear wave propagating in the $\hat{x}$ direction. In many cases, the wave will break up into several units (clumps?) along the $\hat{y}$ direction. This kind of unstable behavior produces beautiful patterns in the case of surface waves on water (e.g., Saffman & Yuen 1985; Infeld & Rowlands 1990) and may play an important role in our understanding of the observed patterns in molecular clouds.

In summary, we have shown that a wide variety of nonlinear wave behavior, including solitons and kinks, can occur in one-dimensional molecular clouds. The extension of this work to include two spatial dimensions, instabilities, and interactions between waves provides an especially intriguing set of problems to explore in the future.

Acknowledgements

We would like to thank J. Langer, P. Myers, B. Shaw, R. Watkins, and J. Wiseman for useful discussions. We also thank the Institute for Theoretical Physics at the University of California in Santa Barbara, where portions of this work were carried out. This work was supported by NASA Grant No. NAGW–2802 and by funds from the Physics Department at the University of Michigan.

APPENDIX A: LINEAR DISPERSION RELATION

In this Appendix, we derive the linear dispersion relation appropriate for a self-gravitating fluid with the joint thermal/logatropic equation of state given by equation [2.4]. The linearized version of the fluid equations can be written

$$\rho_t^{(1)} + \rho^{(0)} u_x^{(1)} = 0, \tag{A1}$$

$$u_t^{(1)} + \frac{1}{\rho^{(0)}}[a_s^2 + p_0/\rho^{(0)}]\rho_x^{(1)} + \psi_x^{(1)} = 0, \tag{A2}$$

$$\psi_{xx}^{(1)} = 4\pi G \rho^{(1)}, \tag{A3}$$

where we have made the traditional approximation that the unperturbed potential $\psi^{(0)}$ can be neglected in this treatment (see, e.g., Binney & Tremaine 1987 for further discussion). After writing all of the first order quantities in the usual form

$$F = F_0 \exp[i(\omega t + kx)] \tag{A4}$$

and performing a bit of algebra, we obtain the relevant dispersion relation:

$$\omega^2 = a_s^2 k^2 (1 + \kappa) - 4\pi G \rho^{(0)}. \tag{A5}$$

APPENDIX B: PRESENCE OF THE SINGULARITY

In this Appendix, we show that the singular value of density (ρ_C) always lies within the range of allowed densities of propagation of the waves. We begin by defining a function h according to

$$h(\rho) = \beta - \rho - \frac{A^2}{\rho} - \kappa \log \rho, \tag{B1}$$

which is just the quantity in brackets in equation [2.12]. Thus, the requirement that $\rho_\xi^2 > 0$ corresponds to $h > 0$; the end points of the range of allowed densities (i.e., ρ_1 and ρ_2) are simply the zeroes of $h(\rho)$. By inspection we see that $h \to -\infty$ in the limits

$\rho \to \infty$ and $\rho \to 0$. Thus, the maximum of h must occur somewhere in between, i.e., when

$$\frac{dh}{d\rho} = -1 + \frac{A^2}{\rho^2} - \frac{\kappa}{\rho} = 0\,. \tag{B2}$$

Notice that we must have $\beta > \beta_{min}$ in order to have $h(\rho)$ positive at the maximum. The only solution to equation [B2] with positive density is

$$\rho_C = \left[\kappa^2/4 + A^2\right]^{1/2} - \kappa/2, \tag{B3}$$

which is the same as the singular value given by equation [2.13]. The singularity thus occurs at the maximum value of h; since the range of allowed densities is determined by the zeroes of h (provided they are real), the maximum value of h and hence the singularity will always fall within the allowed range. In order to complete the argument, we must show that the critical point ρ_C does correspond to a maximum of h. Consider the second derivative,

$$\frac{d^2h}{d\rho^2} = \rho^{-3}\left[\kappa\rho - 2A^2\right]. \tag{B4}$$

We show that the quantity in brackets is negative (and hence h is a maximum) at ρ_C as follows: By inspection

$$\kappa(\kappa^2 + 4A^2)^{1/2} < \kappa^2 + 4A^2. \tag{B5}$$

After subtracting the right hand side from both sides of the equation, we obtain

$$-\kappa^2 + \kappa(\kappa^2 + 4A^2)^{1/2} - 4A^2 < 0. \tag{B6}$$

Next, we divide by 2 and use the definition of ρ_C to obtain

$$\kappa\rho_C - 2A^2 < 0, \tag{B7}$$

which shows that the second derivative is indeed negative at the critical point ρ_C.

APPENDIX C: REMOVAL OF THE SINGULARITY

In this Appendix, we show that the singularity of equation [2.12] is removable in that we can integrate over it to obtain a continuous wave profile. Using equation [2.12] we obtain

$$\rho_\xi = \pm\sqrt{2}\rho^3\left[\rho^2 + \kappa\rho - A^2\right]^{-1}\left\{\beta - \rho - A^2/\rho - \kappa\log\rho\right\}^{1/2}. \tag{C1}$$

The wave profile is obtained by integrating the above equation with respect to ξ. Near the singular point ρ_C, let us expand according to

$$\rho = \rho_C + \delta, \tag{C2}$$

$$\delta_\xi = \pm\frac{\sqrt{2}\rho_C^3}{(2\rho_C + \kappa)}\left\{\beta - \rho_C - A^2/\rho_C - \kappa\log\rho_C\right\}^{1/2}\frac{1}{\delta} \equiv \pm C\frac{1}{\delta}. \tag{C3}$$

In the neighborhood of the singularity, the above equation can be integrated to obtain

$$\delta = \pm\left[\pm 2\mathcal{C}(\Delta\xi)\right]^{1/2}, \tag{C4}$$

where $\Delta\xi$ is measured from the singularity and the inner $\pm$ sign is chosen to keep the quantity in brackets positive. The solution thus changes from one branch of the square root to the other at the singularity.

APPENDIX D: AN APPROXIMATION FOR THE WAVELENGTH

In this Appendix we estimate the wavelength for the nonlinear waves considereed in this paper. We explicitly consider only clouds without magnetic fields (§2), although the generalized case including magnetic fields can be treated similarly. We begin by writing the first integral of the equation of motion in the form

$$\frac{1}{\sqrt{2}}\frac{d\rho}{d\xi} = \sqrt{\mathcal{F}(\rho)}. \tag{D1}$$

By definition, the wavelength λ is then given by

$$\lambda = 2\int_{\rho_1}^{\rho_2} \frac{d\rho}{\sqrt{2\mathcal{F}}}, \tag{D2}$$

where ρ_1 and ρ_2 determine the allowed range of densities (see §2). The factor of two in front of the integral arises because the wave profile samples the range of densities twice during a complete wave cycle. Using the known solution for nonlinear waves (see equation [2.12]), we can write the integral for λ in the form

$$\begin{aligned}\lambda =& \sqrt{2}\int_{\rho_1}^{\rho_C} \frac{1}{\rho}\left\{\beta - \rho - A^2/\rho - \kappa\log\rho\right\}^{-1/2}[A^2 - \rho^2 - \kappa\rho]\frac{d\rho}{\rho^2} \\ &- \sqrt{2}\int_{\rho_C}^{\rho_2} \frac{1}{\rho}\left\{\beta - \rho - A^2/\rho - \kappa\log\rho\right\}^{-1/2}[A^2 - \rho^2 - \kappa\rho]\frac{d\rho}{\rho^2},\end{aligned} \tag{D3}$$

where we have split the integral into two parts because we must take different signs for the square root on different sides of the singularity at ρ_C (see equation [2.13]). We have separated out one factor of $1/\rho$ in the integrands because the remaining factor forms a perfect differential. If we pull out one factor of $1/\rho$ from each integral by invoking the Mean Value Theorem, we can evaluate the remaining integrals and thereby obtain

$$\lambda = (2)^{3/2}\left\{\beta - \rho_C - A^2/\rho_C - \kappa\log\rho_C\right\}^{1/2}\left\{\frac{1}{\rho_a} + \frac{1}{\rho_b}\right\}, \tag{D4}$$

where $\rho_a \in [\rho_1, \rho_C]$ and $\rho_b \in [\rho_C, \rho_2]$. The expression [D4] is exact form some (unknown) values ρ_a and ρ_b in the given ranges. Although we are left with this uncertainty in how to estimate ρ_a and ρ_b, equation [D4] does illustrate the general behavior of the wavelength

as a function of parameters. In particular, for large constants of integration β, we see that $\lambda \sim \beta^{1/2}$.

Finally, we note that for the case of solitons (and kinks), the function $\mathcal{F}(\rho)$ has a double (or higher order) zero at either ρ_1 and ρ_2. As a result the integral in equation [D2] diverges for a soliton, i.e., solitons have an infinite wavelength.

APPENDIX E: VALIDITY OF THE MODEL EQUATION

In this Appendix we try to ascertain the conditions under which the model equation of motion (see §3) is valid. This model equation involves two principal approximations: (a) the ρ_ξ term is dropped altogether from equation [3.16], and (b) the ion density is replaced by an equivalent constant value. The second of these approximations will generally lead to quantitative, but not qualitative, differences in the behavior of the solutions. On the other hand, the first approximation is potentially more dangerous.

In order to estimate when the omitted term plays an important role in the dynamics, we must compare it to the terms which are kept. The omitted term, which we will denote as M, is given by

$$M \equiv -\tilde{\eta} A \rho_i \rho^2 \rho_\xi \,. \tag{E1}$$

This term will generally be small compared to the gravitational term. However, the most important term in the model equation for purposes of comparison is the coupling term K given by

$$K \equiv -\frac{\Gamma}{\rho_F} \rho^4, \tag{E2}$$

because this term allows the model equation to exhibit qualitatively different behavior than that of the field-free case. In particular, this term K allows for the existence of soliton solutions. The ratio of the terms given by equations [E1] and [E2] is simply

$$\mathcal{R} \equiv \frac{M}{K} = \frac{\tilde{\eta} A \rho_F}{\Gamma} \frac{\rho_i}{\rho} \frac{\rho_\xi}{\rho} \,. \tag{E3}$$

We expect the model equation to be a good approximation to the true physical system when the ratio $\mathcal{R}$ is small.

We now try to estimate the size of the ratio $\mathcal{R}$ for cases of interest. Let χ denote the average value of ρ_i/ρ, i.e., χ is the average ionization fraction (where the average is taken over the range of densities appearing in the wave profile). Next, we replace the quantity ρ_ξ/ρ by an appropriate average value, which we denote as $\langle \rho_\xi/\rho \rangle$. In this present discussion, in order to obtain values for the quantity $\langle \rho_\xi/\rho \rangle$ we simply average over the appropriate range of densities. Notice, however, that if we were to average over an entire wave cycle, the derivative ρ_ξ would be negative as often as positive and the average would vanish. We are thus overestimating the effect of this term. The definition of $\tilde{\eta}$ implies that $\tilde{\eta} = 7.5 \times 10^4 n_0^{1/2}$, where n_0 is the number density of the reference state (see equations [2.4] and [2.5]). Given these definitions, the ratio $\mathcal{R}$ can be written

$$\mathcal{R} = 7.5 \times 10^4 \, \chi \, n_0^{1/2} \, \frac{A \rho_F}{\Gamma} \, \langle \rho_\xi/\rho \rangle \,. \tag{E4}$$

Since we are mostly interested in how well the model equation works for soliton solutions, we will use the soliton solutions of Figures 3 and 4 for comparison (we are thus checking these soliton solutions for consistency). For the soliton in Figure 3, the quantity $A\rho_F/\Gamma = 3/10$ and the average $\langle \rho_\xi/\rho \rangle = 2.7$. If we use a representative value of $\chi = 10^{-6}$ and $n_0 = 100$ cm^{-3}, we obtain $\mathcal{R} = 0.6$, which is reasonably small and thus fairly gratifying (recall that we have overestimated the effect of the factor $\langle \rho_\xi/\rho \rangle$). For the soliton in Figure 4, the quantity $A\rho_F/\Gamma = 1/10$ and the average $\langle \rho_\xi/\rho \rangle = 1.3$. If we use the same values of $\chi = 10^{-6}$ and $n_0 = 100$ cm^{-3}, we obtain $\mathcal{R} = 0.1$, which suggests that this second soliton is even more consistent with the use of the model equation.

Notice that in the above discussion our assumed value of the ionization fraction (10^{-6}) is somewhat small for a number density of $n = 100$ cm^{-3} (see Elmegreen 1979). Keep in mind, however, that the reference density n_0 sets the scale of the density (i.e., it defines what $\rho = 1$ means), whereas the mean ionization fraction appearing in equation [E4] is that appropriate for the mean density of the soliton wave (and this density can be much larger, a factor of $\sim$20 higher in this particular case). In any case, we are living fairly close to the edge of the limit of validity of our model equation. If the mean ionization fraction (or the reference density scale) is much larger than the values given above, then the ratio $\mathcal{R}$ can approach unity (or larger) and the model equation becomes a poorer representation of the real physical system. On the other hand, the available parameter space is sufficiently large that the ratio $A\rho_F/\Gamma$ will often be much smaller than in these examples. When we adopt the choice $\rho_F = \Gamma/\kappa$ (which eliminates the logarithmic term from $\mathcal{F}$), the ratio $A\rho_F/\Gamma = A/\kappa$. Soliton solutions can thus arise from the model equation in a consistent fashion for small ratios A/κ, i.e., relatively slow wave motions with a large component of "turbulent" pressure support.

APPENDIX F: TRANSFORMATIONS OF THE MODEL EQUATION

In this Appendix we find a Lie group of scaling transformations which leave our model equation of motion invariant. We begin with the model equation itself, which can be written in the form

$$\rho\rho_{\xi\xi}\left[\rho^2 + \kappa\rho - A^2\right] + \rho_\xi\rho_\xi\left[3A^2 - \rho^2 - 2\kappa\rho\right] + \rho^5 + \Gamma\rho^3\left[1 - \frac{\rho}{\rho_F}\right] = 0, \qquad \text{(F1)}$$

where ρ_F is considered to be a constant. Now consider transforming the scale of the density and the variable ξ according to

$$\rho \rightarrow \Lambda\rho, \qquad \text{(F2)}$$

$$\xi \rightarrow \gamma\xi. \qquad \text{(F3)}$$

If we insert these forms into the equation of motion [F1], we find that the equation of motion remains invariant under the transformation provided that

$$\gamma = \Lambda^{-1/2}, \qquad \text{(F4)}$$

and the constants scale according to

$$A \rightarrow \Lambda A, \tag{F5}$$

$$\kappa \rightarrow \Lambda \kappa, \tag{F6}$$

$$\rho_F \rightarrow \Lambda \rho_F, \tag{F7}$$

$$\Gamma \rightarrow \Lambda^2 \Gamma. \tag{F8}$$

If we now let $a \equiv \Lambda^{-1/2}$, and let G_a represent the transformation given above (for a given a), then the set

$$\left\{ G_a \,\middle|\, a \in \mathbf{R}, a \neq 0 \right\} \tag{F9}$$

forms a Lie group of transformations which leave the model equation of motion invariant. The group operation is defined by $G_a G_b = G_{ab}$. Transformations of this type show us how to rescale solutions to different densities (or for different values of the other parameters). Notice that if we set $\Gamma = 0$ in equation [F1], we obtain the equation of motion appropriate for clouds with no magnetic fields. If we ignore equations [F7] and [F8], we are left with the scaling tranformations for the field-free equation of motion.

REFERENCES

Adams, F. C. 1992, *ApJ*, **387**, 572

Arons, J., & Max, C. 1975, *ApJ Letters*, **196**, L77

Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton: Princeton University Press)

Blitz, L. 1992, in Protostars and Planets III, ed. E. Levy and M. S. Mathews (Tucson: University of Arizona Press), in press

Coleman, S. 1985, Classical Lumps and their Quantum Descendants, in Aspects of Symmetry (Cambridge: Cambridge University Press)

Dame, T. M., Elmegreen, B. G., Cohen, R. S., & Thaddeus, P. 1986, *ApJ*, **305**, 892

de Geus, E. J., Bronfman, L., & Thaddeus, P. 1990, *A & A*, **231**, 137

Dewar, R. L. 1970, *Phys. Fluids*, **13**, 2710

Draine, B. T., Roberge, W., & Dalgarno, A. 1983, *ApJ*, **270**, 519

Drazin, P. G., & Johnson, R. S. 1989, Solitons: An Introduction (Cambridge: Cambridge University Press)

Elmegreen, B. G. 1979, *ApJ*, **232**, 729

Elmegreen, B. G. 1990, *ApJ*, **361**, L77

Fatuzzo, M., & Adams, F. C. 1992, submitted to *ApJ*

Goodman, A. A., Crutcher R. M., Heiles, C., Myers, P. C., & Troland, T. H. 1989, *ApJ*, **338**, 161

Houlahan, P., & Scalo, J. 1992, *ApJ*, **393**, 172

Infeld, E., & Rowlands, G. 1990, Nonlinear Waves, Solitons, and Chaos (Cambridge: Cambridge University Press)

Langer, W. D. 1978, *ApJ*, **225**, 95

Larson, R. B. 1981, *MNRAS*, **194**, 809

Lizano, S., & Shu, F. H. 1989, *ApJ*, **342**, 834

McBreen, B., Fazio, G. G., Stier, M., & Wright, E. L. 1979, *ApJ*, **232**, L183

Mouschovias, T. Ch. 1976, *ApJ*, **206**, 753

Mouschovias, T. Ch. 1978, in Protostars and Planets, ed. T. Gehrels (Tucson: University of Arizona Press), p. 209

Myers, P. C. 1983, *ApJ*, **270**, 105

Myers, P. C. 1987, in Interstellar Processes, ed. D. Hollenbach and H. Thronson, (Dordrecht: Reidel), p. 71

Myers, P. C., & Goodman, A. A. 1988, *ApJ*, **329**, 392

Myers, P. C. 1991, in Fragmentation of Molecular Clouds and Star Formation (IAU

Symposium 147), eds. E. Falgarone and G. Duvert (Dordrecht: Kluwer), in press

Myers, P. C., & Fuller, G. A. 1992, *ApJ*, Sept. 10, in press

Nakano, T. 1985, *Pub. A.S.J.*, **37**, 69

Norman, C., & Silk, J. 1980, *ApJ*, **238**, 158

Pego, R. L., & Weinstein, M. I. 1992, *Phys. Lett. A*, **162**, 263

Pudritz, R. E. 1990, *ApJ*, **350**, 195

Rajaraman, R. 1987, Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory (Amsterdam: North Holland)

Riemann, B. 1858, Göttingen Abhandlungen, Vol. viii, p. 43

Russell, J. S. 1844, 14th Meeting of the British Association Report (York), p. 311

Saffman, P. G., & Yuen, H. C. 1985, in Advances in Nonlinear Waves, ed. L. Debnath (Boston and London: Pitman), p. 1

Scalo, J. 1990, in Physical Processess in Fragmentation and Star Formation, eds. R. Capuzzo-Dolcetta et al. (Dordrecht: Kluwer), p. 151

Shu, F. H. 1977, *ApJ*, **214**, 488

Shu, F. H. 1983, *ApJ*, **273**, 202

Shu, F. H. 1992, Gas Dynamics (Mill Valley: University Science Books)

Shu, F. H., Adams, F. C., & Lizano, S. 1987, *ARA&A*, **25**, 23

Solomon, P. M., Rivolo, A. R., Barrett, J. W., & Yahil, A. 1987, *ApJ*, **319**, 730

Strittmatter, P. A. 1966, *MNRAS*, **132**, 359

Terebey, S., Shu, F. H., & Cassen, P. 1984, *ApJ*, **286**, 529

Whitham, G. B. 1974, Linear and Nonlinear Waves (New York: Wiley)

Wiseman, J. J., & Adams, F. C. 1992, in preparation

Wood, D.O.S., Myers, P. C., & Daugherty, D. A. 1999, to be submitted to *ApJ Suppl.*

Zinnecker, H., McCaughrean, M. J., & Wilking, B. A. 1992, in Protostars and Planets III, ed. E. Levy and M. S. Mathews (Tucson: University of Arizona Press), in press

FIGURE CAPTIONS

Figure 1. Phase diagram for nonlinear waves in molecular clouds without magnetic fields. All of the cases considered here use $\kappa = 10$ and $A = 3$. The family of curves corresponds to varying $\beta = 10$, 15, 20, 25, and 30. The function $\rho_\xi(\rho)$ increases with increasing β.

Figure 2. Wave profiles for nonlinear waves in molecular clouds without magnetic fields. We have set $\kappa = 10$ and $A = 3$. (a) Wave profile for $\beta = 10$. (b) Wave profile for $\beta = 20$. (c) Wave profile for $\beta = 40$.

Figure 3. Soliton solution to the model equation of motion for stationary waves in a molecular cloud with magnetic fields. The parameters for this case are $\kappa = 10$, $A = 3$, $\Gamma = 9$, $\beta = 38$, and $\rho_F = \Gamma/\kappa = 0.9$. (a) The first integral function $\mathcal{F}(\rho)$. (b) The wave profile $\rho(\xi)$.

Figure 4. A second soliton solution to the model equation of motion for stationary waves in a molecular cloud with magnetic fields. The parameters for this case are $\kappa = 10$, $A = 1$, $\Gamma = 16$, $\beta = 23\frac{11}{12}$, and $\rho_F = \Gamma/\kappa = 1.6$. (a) The first integral function $\mathcal{F}(\rho)$. (b) The wave profile $\rho(\xi)$.

Figure 5. Kink solution (also known as a topological soliton) to a limiting form of the model equation for stationary waves in a molecular cloud with magnetic fields. This limiting form of the equation has $\Gamma \to 0$ and $\rho_F \to 0$, but $\Gamma/\rho_F \to constant = \mu$. The parameters for this case are $\kappa = 10$, $\mu = 10$, $A = 10$, and $\beta = 40$. (a) The first integral function $\mathcal{F}(\rho)$. (b) The wave profile $\rho(\xi)$.

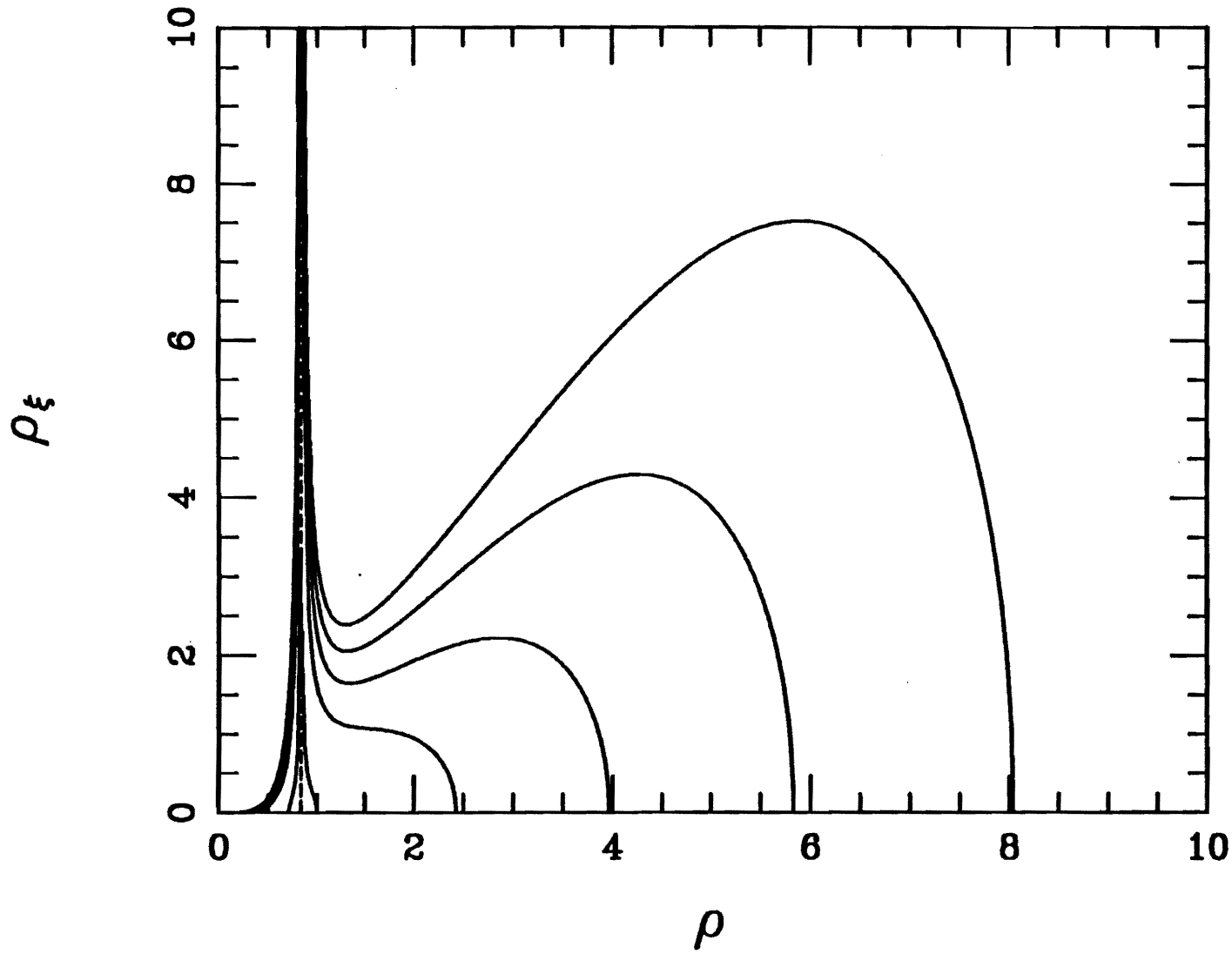

Fig. 1

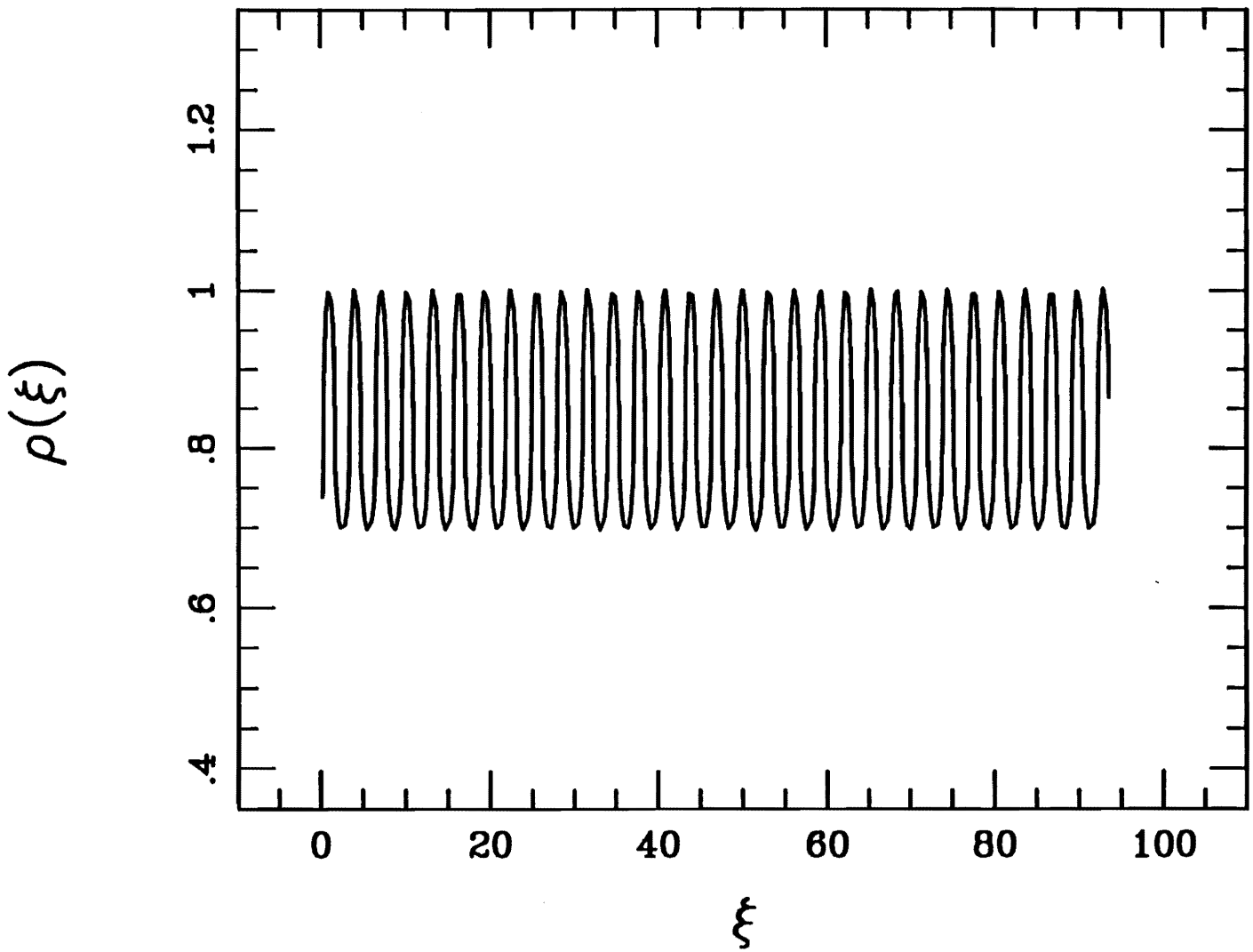

Fig. 2a

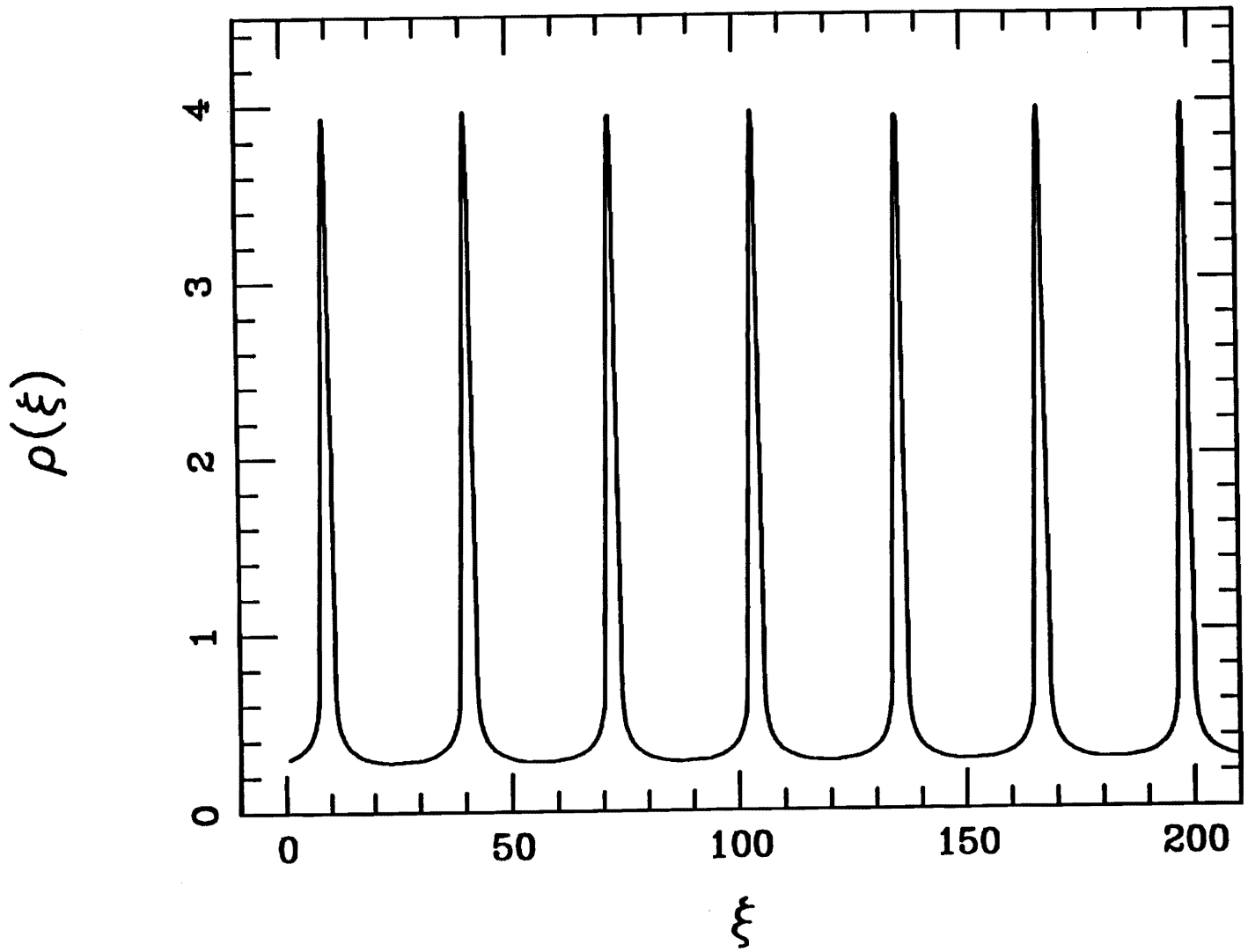

Fig. 2b

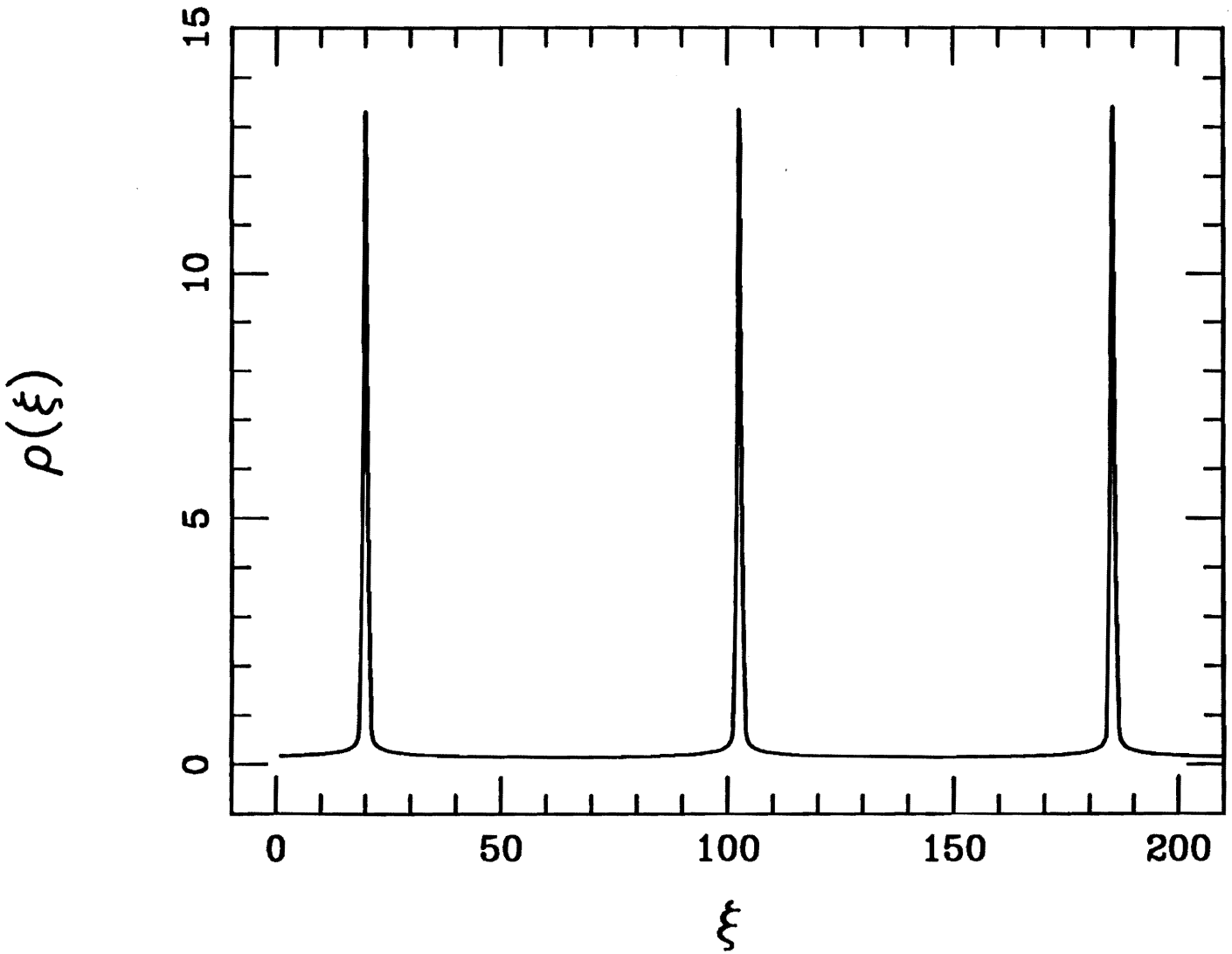

Fig. 2c

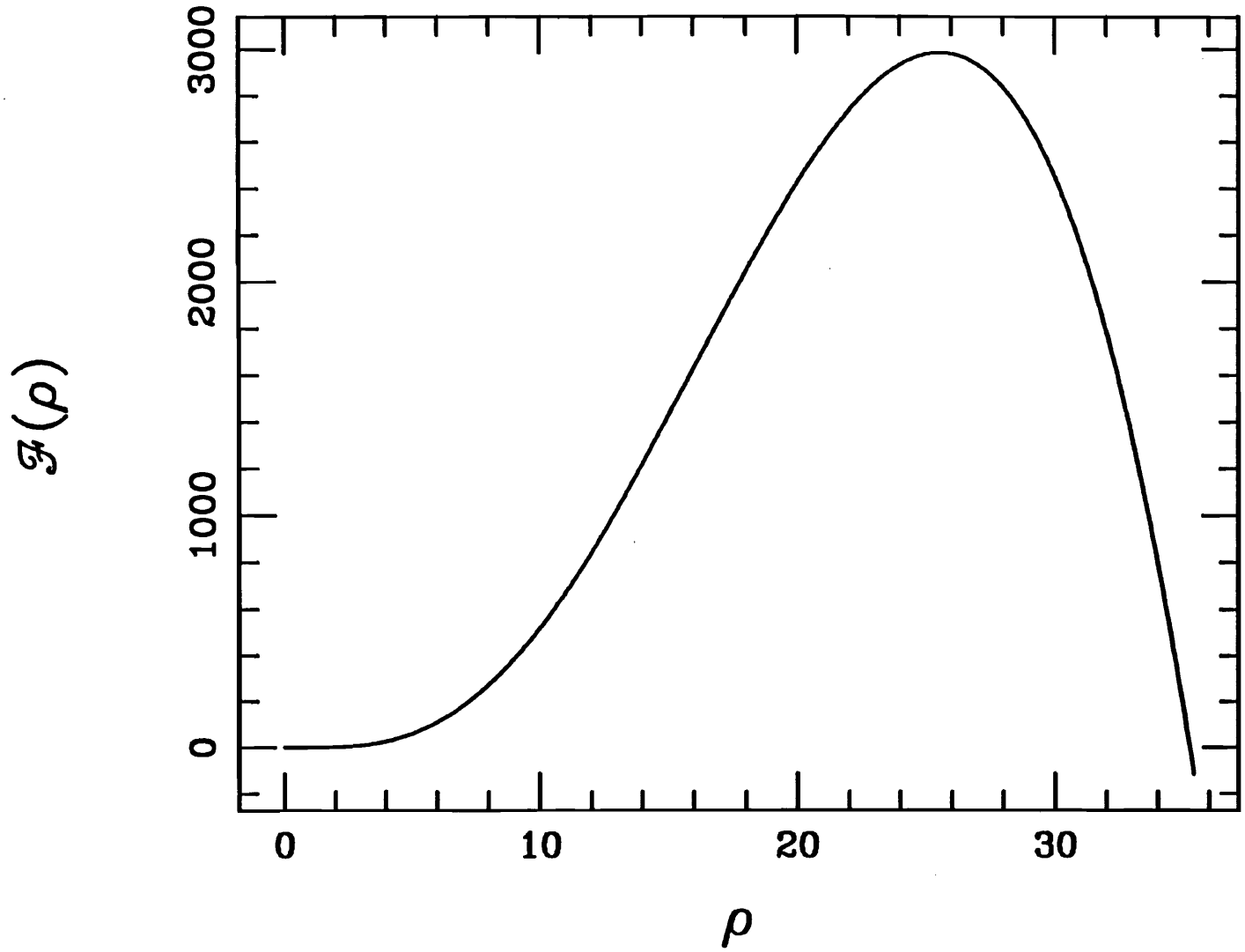

Fig. 3a

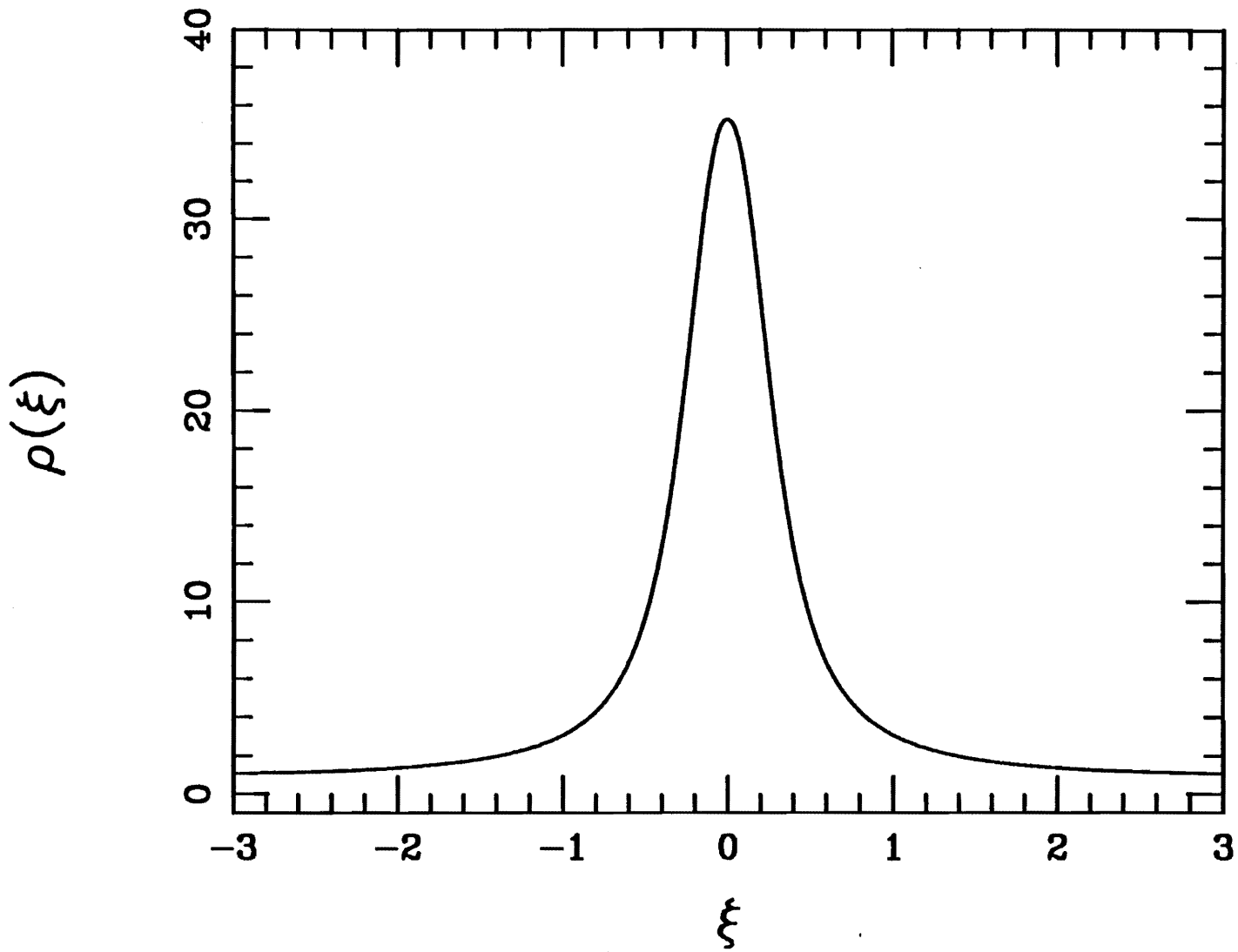

Fig. 3b

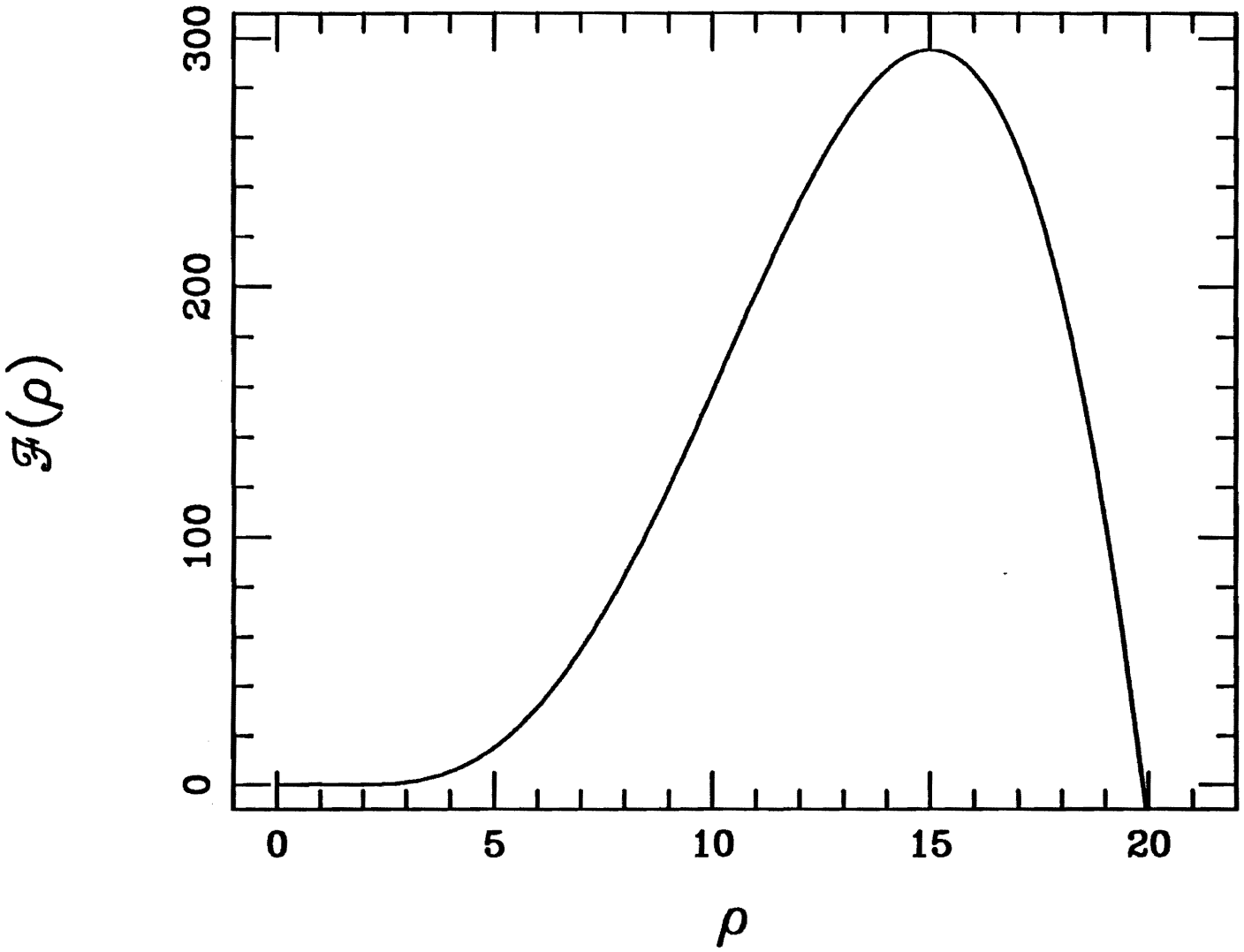

Fig. 4a

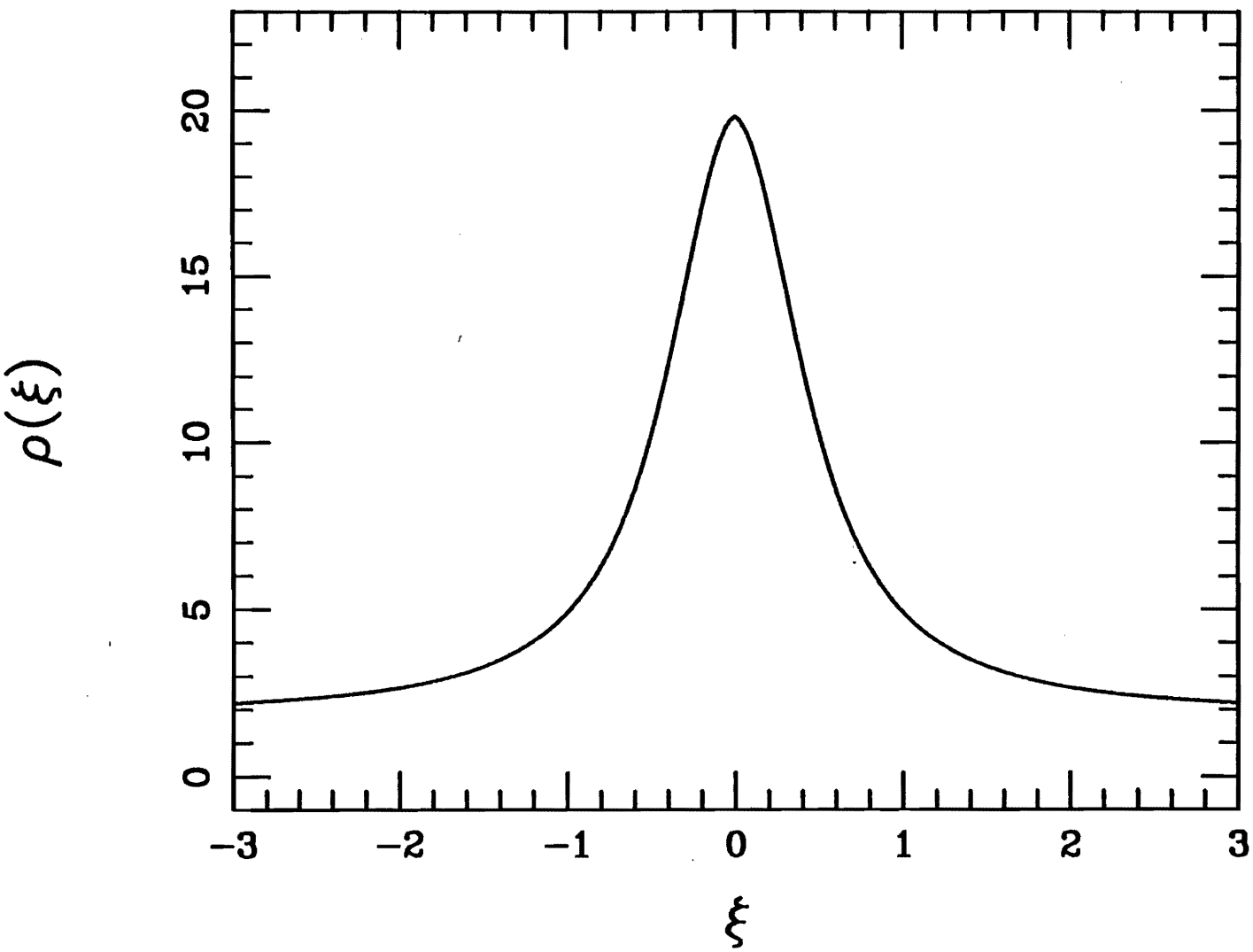

Fig. 4b

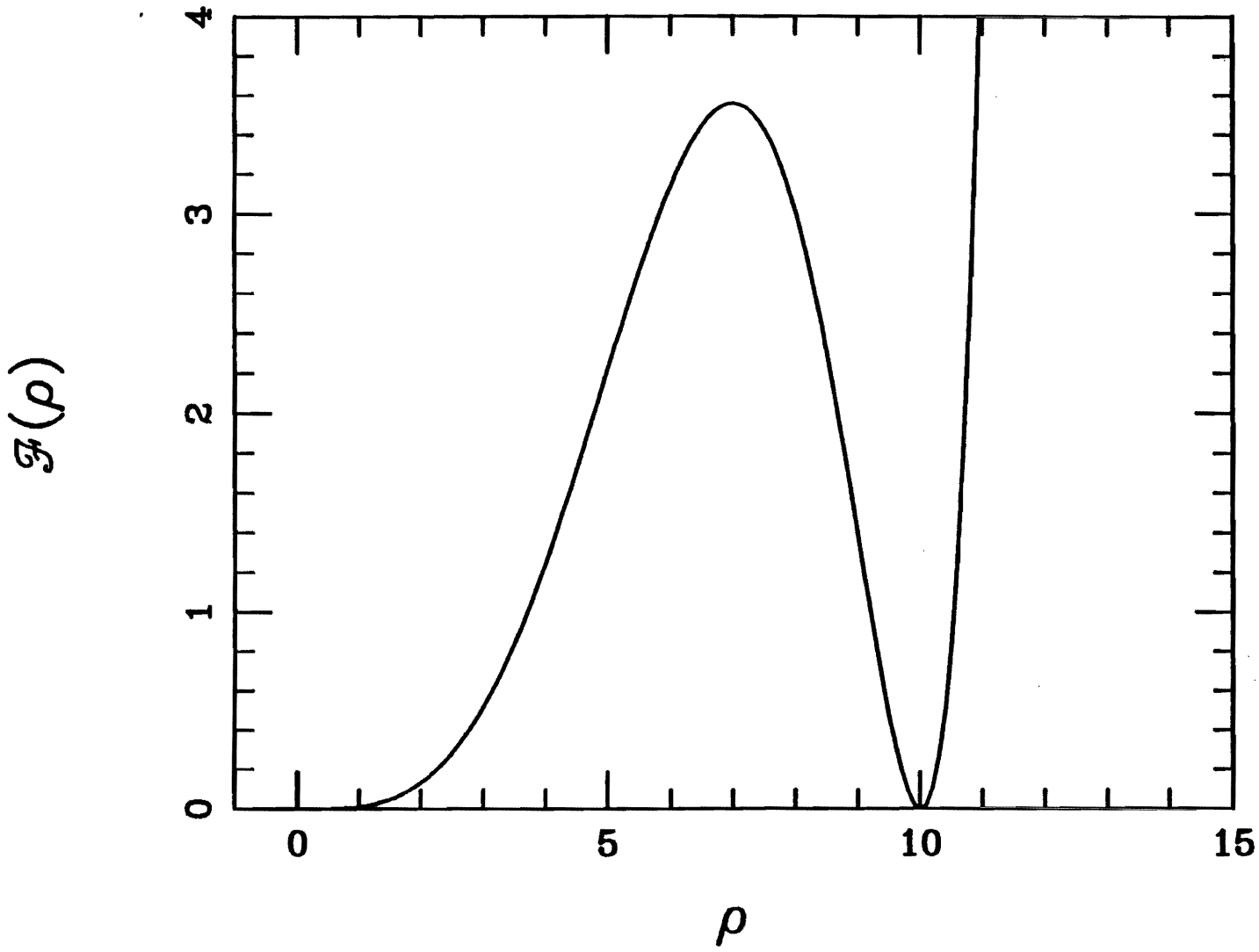

Fig. 5a

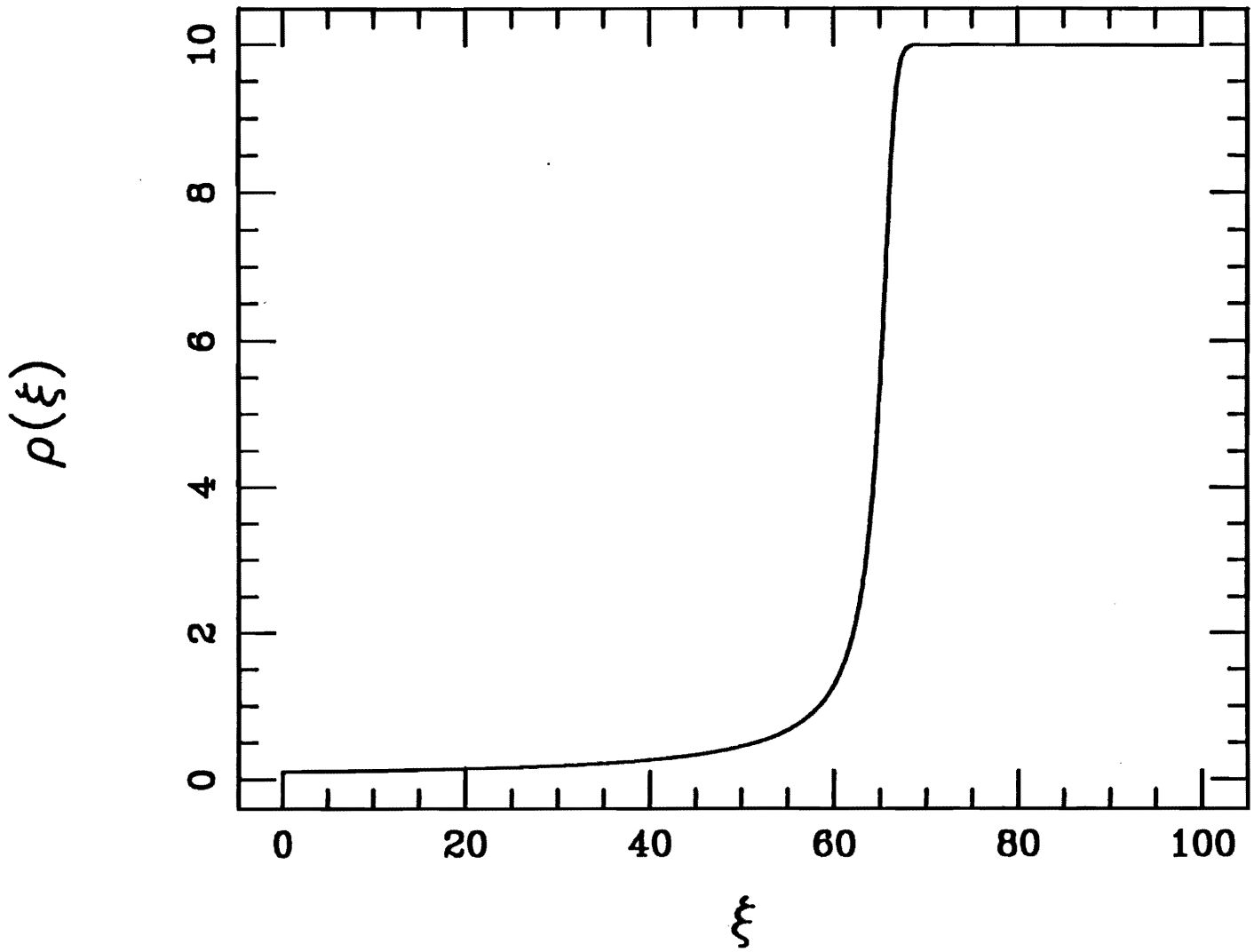

Fig. 5b