Quantum Integrable Systems

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1. Introduction

In 1965, the soliton as a new concept in nonlinear dynamics was introduced by Zabusky and Kruskal[1]. Exciting findings in the pioneer period were collected in an excellent review article by Scott, Chu and McLaughlin[2]. During the last decade, there continued to be many important developments[3]. One of them is a unification of exactly solvable models in physics. The quantum inverse scattering method places the theory of exactly solvable models in a unified framework and provides a powerful method for studying those models. Solvable models in $(1+1)$-dimensional quantum theory and in 2-dimensional classical statistical mechanics share a common property: to each model we can associate a family of commuting transfer matrices which are generators of an infinite number of conserved quantities. This property may correspond to the Liouville theorem for classical Hamiltonian systems. A sufficient condition for the commutability of the transfer matrices is called the Yang-Baxter relation[4]. The Yang-Baxter relation is a key of new ideas and new concepts in recent mathematical physics such as knot theory based on solvable models, and quantum groups.

This report presents some recent results related to the quantum integrable systems. It is known that there exists a correspondence between 1D quantum system and 2D classical system. The Baxter's formula relates a 2D solvable lattice model to a 1D integrable spin system. In section 2, we extend the Baxter's formula into the case of finite temperature. Combining this extension with the evaluation of finite size corrections, we obtain a systematic method to calculate low temperature expansions of thermodynamic quantities[5,6,7]. Sections 3 and 4 deal with integrable systems with long-range interactions. Some time ago, Gaudin introduced a class of such spin models using the Bethe ansatz method. In section 3, we give an algebraic formulation of the models. Introducing inhomogeneities into lattice models, we present a general method to construct quantum integrable spin Hamiltonians with long-range pairwise interactions. The last section is devoted to a reformulation of the quantum inverse scattering method for 1D quantum particle systems. In particular, construction of conserved operators for the Calogero-Moser system is explicitly shown. The expression in terms of the Lax operator is new. Further, we find that the Lax pair yields an interesting algebra.

2. Finite Temperature Baxter's Formula

We consider a square lattice with M rows and N columns under the periodic boundary condition ($M+1 \equiv 1, N+1 \equiv 1$), and define a model of classical statistical mechanics on the lattice. We assume that the Boltzmann weights of the lattice model satisfy the Yang-Baxter relation. Let $T_N(u)$ denote the transfer matrix, where u is the spectral parameter. The Yang-Baxter relation assures the commutability of the transfer matrices :

$$[T_N(u), T_N(v)] = 0. \tag{2.1}$$

It is known that there is a relationship between the row-to-row transfer matrix $T_N(u)$ and a 1D quantum Hamiltonian H :

$$H = -\, T_N(0)^{-1} \frac{\partial}{\partial u} T_N(u)\bigg|_{u=0} . \tag{2.2}$$

For instance, the Heisenberg XYZ (XXZ) model is related to the eight- (six-) vertex model by a formula (2.2). We call (2.2) Baxter's formula.

The relation (2.1) contains useful information. Expanding the transfer matrix in powers of the spectral parameter, we can easily verify that the transfer matrix is a generator of an infinite number of conserved quantities (operators). The first term in such an expansion is the shift operator :

$$T_N(0)^{b_1 b_2 \cdots b_N}_{a_1 a_2 \cdots a_N} = \delta(b_1, a_N) \cdots \delta(b_N, a_{N-1}), \tag{2.3}$$

where $a = \{a_j\}$ and $b = \{b_j\}$ denote state variables respectively in lower and upper rows, and $\delta(b_i, a_j)$ is the Kronecker's delta. A set of the conserved quantities, $\{I_k\}$, is given by

$$I_k = \frac{\partial^k}{\partial u^k} \log T_N(u)\bigg|_{u=0} . \tag{2.4}$$

The Baxter's formula (2.2) corresponds to I_1 up to the sign.

The above discussion indicates that $[T_N(u), I_k] = 0$ and that $T_N(u)$ and $\{I_k\}$ have common eigenstates. Thus, the Baxter's formula (2.2) leads to the relationship between the ground-state energy E_g of H and the maximum eigenvalue $\Lambda_{\max}(u)$ of $T_N(u)$:

$$E_g = -\, \frac{\partial}{\partial u} \log \Lambda_{\max}(u)\bigg|_{u=0} . \tag{2.5}$$

We have observed that the equivalence through the Baxter's formula between a 1D integrable quantum system and a 2D solvable lattice model is useful in particular when we discuss the zero-temperature properties (e.g. ground state energy) of the former. A challenging problem may be how we extend the Baxter's formula to study the finite temperature properties.

We look for a new approach to analyze thermodynamic properties of the 1D quantum system. The Baxter's formula (2.2) assures the expansion,

$$T_N(u) = T_N(0)[1 - uH + O(u^2)]. \tag{2.6}$$

Using the Trotter's formula, we have

$$\exp(-\beta H) = \lim_{M\to\infty} (T_N(0)^{-1} T_N(\beta/M))^M , \tag{2.7}$$

where $\beta = 1/k_B T$ as usual. Since $T_N(0)$ is the shift operator, we may interpret $T_N(0)^{-1}T_N(u)$ as a diagonal-to-diagonal transfer matrix and denote it by $T_N^{DTD}(u)$. Then, the free energy per site, f, of the 1D quantum system can be expressed in terms of the partition function Z_{MN} of a 2D classical model on the lattice:

$$f = -\frac{1}{N\beta} \lim_{N\to\infty} \lim_{M\to\infty} \log Z_{MN}(u), \tag{2.8a}$$

$$Z_{MN}(u) = \mathrm{Tr}(T_N^{DTD}(u))^M, \quad u \equiv \beta/M. \tag{2.8b}$$

The eigenvalues of the transfer matrix are infinitely degenerate as $u \to 0$. Therefore, (2.8) is not practical and remains to be formal. To avoid this situation, we introduce a trick: we look the lattice from the crossing channel, that is, from a 90° rotated frame.

Because of the crossing symmetry of the model, this manipulation is made by the change in the spectral parameter u into $\lambda - u$, where λ is called the crossing parameter[4,5]. Correspondingly, we introduce a notation $T_M^X(u) \equiv T_M^{DTD}(\lambda - u)$. Now, (2.8) is replaced by

$$f = -\frac{1}{\beta} \lim_{M\to\infty} \log \Lambda_M(u), \tag{2.9a}$$

$$Z_{MN}(u) = \mathrm{Tr}(T_M^X(u))^N, \quad u = \beta/M, \tag{2.9b}$$

where $\Lambda_M(u)$ is the largest eigenvalue of the crossing transfer matrix $T_M^X(u)$. Equation (2.9) is a finite temperature extension of the Baxter's formula (2.2), and has been named finite temperature Baxter's formula. Remark that the spectral parameter u, a fundamental quantity in soliton theory, now plays a role of intertwiner between finite size system and finite temperature system.

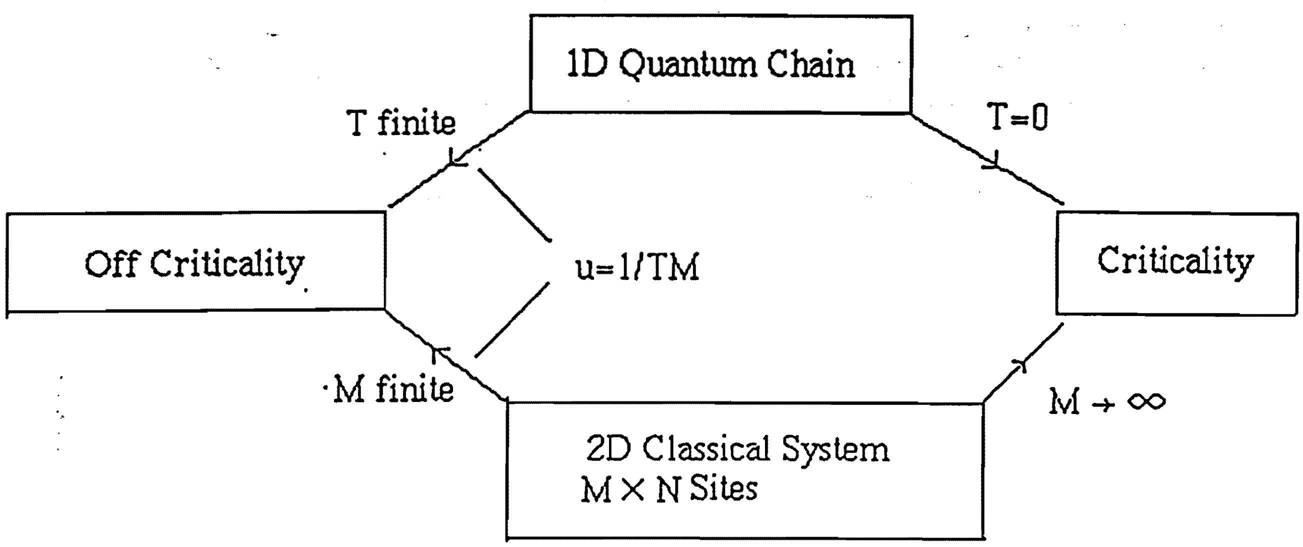

Fig 2.1. Relations between finite size system and finite temperature system

A caution is necessary for making use of the formula (2.9). Since the temperature dependence enters only through $u = \beta/M$, one should be careful in taking the limit $M \to \infty$. Otherwise, any information of finite temperature will be lost. Instead, a finite size calculation or a $1/M$-expansion gives a systematic method to obtain low temperature expansions of thermodynamic quantities. Figure 2.1 summarizes the significance of the limits $T \to 0$ and $M \to \infty$ for 1D quantum system and 2D classical system.

The theory of finite size corrections has attracted much attention of theoretical physicists since it gives central charge c and scaling dimension x which are essential quantities in the conformal field theory. To evaluate finite size corrections, we adopt a method by de Vega, Woynarovich and Eckle[8]. The calculation is involved but straightforward. We apply the Euler-Mclaurin expansion formula and the solution technique of the Wiener-Hopf integral equation to the Bethe ansatz equation.

A new approach to finite temperature quantum systems presented above has been successfully applied[5,6,7]. It should be stressed that this approach is valid to any quantum chain as long as it is solvable by the Bethe ansatz method. We only mention here the results for the XXZ spin chain in a gapless phase:

$$H = J_T \sum_{l=1}^{N} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y + \Delta \cdot \sigma_l^z \sigma_{l+1}^z), \tag{2.10}$$

$$\Delta = \cos\lambda. \tag{2.11}$$

The free energy f with the first thermal correction is

$$f = f_\infty - \frac{\lambda (k_B T)^2}{12 \sin\lambda \cdot J_T} + \cdots, \tag{2.12}$$

where f_∞ is the ground state energy. The result (2.12) is the same as the one obtained by the Bethe ansatz method with the string hypothesis.

Calculation of the higher terms in T can be done by making use of the perturbation theory of the Wiener-Hopf integral equation. In the limit $\lambda \to 0$ (the XXX model limit), the correction terms have rather simple expressions. The free energy f and the correlation length ξ are respectively given by

$$f = f_\infty - \frac{(k_B T)^2}{12 J_T} + \frac{\pi}{2 J_T} \frac{(k_B T)^2}{(\log(J_T / k_B T))^3} + \cdots, \tag{2.13}$$

$$\xi = \frac{a_1}{T} + \frac{a_2}{T \log(1/T)} + \cdots, \tag{2.14}$$

where a_1 and a_2 are constants. Two remarks should be made. First, the appearances of logarithmic terms in (2.13) and (2.14) are noteworthy. This is the first analytical derivation of such terms and has been confirmed by numerical calculations based on a different formalism. Second, we find that the calculation of the correlation length is quite straightforward, by making use of the results for the corresponding 2D classical models. This is remarkable since in string hypothesis approach we can obtain only the free energy and cannot get any information on the correlation length.

3. The Gaudin Model and Generalizations

Depending on the interaction ranges, completely integrable systems may be classified into two groups. One is a system with short-range interactions including the

We can regard $\{H_k\}$ as commuting Hamiltonians. In fact, substituting the expression $Z_k = I + \eta H_k + O(\eta^2)$ into (3.7), we see that

$$[H_k, H_l] = 0, \quad k, l = 1, 2, \cdots, N. \tag{3.14}$$

This indicates that a Hamiltonian system

$$H = \sum_{k=1}^{N} a_k H_k, \quad a_k : \text{constants} \tag{3.15}$$

is completely integrable.

It is interesting to compare a formula (3.13) with the Baxter's formula (2.2). While the Baxter's formula gives a spin Hamiltonian with nearest-neighbor interactions, the formula (3.13) yields a spin Hamiltonian with long-range interactions.

The above discussion is general. We have used only the Yang-Baxter relation and the properties of the R-matrix such as the regular condition, the quasi-classical condition and the inhomogeneity. Given an R-matrix, we have commuting Hamiltonians $\{H_k\}$ by the formula (3.13). For instance, from the R-matrix for the spin-1/2 XYZ model, we have

$$\begin{aligned} H_l = \frac{1}{2}\sum_{j\neq l} \frac{1}{\mathrm{sn}(x_l - x_j)} \{ &\left(1 + k\mathrm{sn}^2(x_l - x_j)\right) \sigma_l^x \sigma_j^x \\ &+ \left(1 - k\mathrm{sn}^2(x_l - x_j)\right) \sigma_l^y \sigma_j^y + \mathrm{cn}(x_l - x_j)\mathrm{dn}(x_l - x_j)(\sigma_l^z \sigma_j^z - 1) \}, \end{aligned} \tag{3.16}$$

where k is the modulus of Jacobi's elliptic functions. Applications to other models including the spin-1 model have been discussed[10,11].

4. Quantum Integrable Particle Systems

The quantum inverse scattering method for N-particles on a line may be introduced as follows. Let L and M be $N \times N$ matrix operators. We choose L, M (Lax pair) such that the Lax equation

$$\dot{L} = i[H, L] = i[L, M], \tag{4.1}$$

is equivalent to the equation of motion generated by a Hamiltonian H under consideration. We associate an evolution of "eigenstate" as

$$i\dot{U} = [U, H] = MU. \tag{4.2}$$

Then, from (4.1) and (4.2), we have

$$[H, U^{-1}LU] = 0. \tag{4.3}$$

Equation (4.3) is a quantum version of the unitary equivalence by P.Lax. For classical systems, a condition that $U^{-1}LU$ does not depend on time leads to the existence of N conserved quantities $\{I_n\}$,

$$I_n \equiv \frac{1}{n}\mathrm{Tr}(U^{-1}L^n U) = \frac{1}{n}\mathrm{Tr}L^n, \tag{4.4}$$

where Tr means the trace of matrix. For quantum systems, since L and U are operators, the last equality in (4.4) is not guaranteed.

To be specific, we restrict our discussion to the Calogero-Moser system whose Hamiltonian is given by

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2} g \sum_{j \neq k} \gamma(x_j - x_k), \tag{4.5}$$

$$p_j = -i\frac{\partial}{\partial x_j}, \quad \gamma(x) = \frac{1}{x^2}. \tag{4.6}$$

The Lax pair is found to be

$$L_{jk} = p_j \delta_{jk} + ia(1-\delta_{jk})\alpha(x_j - x_k), \tag{4.7}$$

$$M_{jk} = a(1-\delta_{jk})\beta(x_j - x_k) + a\delta_{jk}\sum_{l\neq j}\gamma(x_j - x_l), \tag{4.8}$$

where a is a constant related to the coupling constant g by $g = a^2 - a$, and

$$\alpha(x) = 1/x, \quad \beta(x) = -1/x^2. \tag{4.9}$$

Calogero, Ragnisco and Marchioro[12] presented a set of conserved operators (rigorously speaking, their potential is $1/\sinh^2 x$ instead of $1/x^2$, but the discussion is the same). They proved that with $g = a^2$,

$$[H, \det\Lambda] = 0, \tag{4.10}$$

where the matrix Λ is an $N \times N$ matrix defined by $\Lambda_{jk} = L_{jk} - \lambda\delta_{jk}$. The expansion of $\det\Lambda$ in powers of λ gives a set of conserved operators $\{J_n\}$:

$$\begin{aligned} \det[\Lambda_{jk}] &= \det[L_{jk} - \lambda\delta_{jk}] \\ &= (-\lambda)^N + \sum_{n=1}^{N}(-\lambda)^{N-n} J_n. \end{aligned} \tag{4.11}$$

Those conserved operators $\{J_n\}$ have the same functional forms as the classical ones. Only the difference is that p_j is the operator in the quantum case. To carry out the proof of (4.10), they modified (4.1) into

$$[H, L_{jk}] = \frac{1}{2}\sum_l \{L_{jl}M_{lk} + M_{lk}L_{jl} - M_{jl}L_{lk} - L_{lk}M_{jl}\}. \tag{4.12}$$

However, the symmetrized commutator in the r.h.s. of (4.12) needs a justification from a viewpoint of the quantum inverse scattering method. We therefore believe that their proof is not satisfactory, although the conserved operators $\{J_n\}$ are widely accepted.

An alternative method to find conserved operators may be the following. From (4.1), we have

$$[H, (L^n)_{jk}] = \sum_l \{(L^n)_{jl}M_{lk} - M_{jl}(L^n)_{lk}\}. \tag{4.13}$$

Explicit form of M_{jk} in (4.8) gives

$$\sum_j M_{jk} = 0, \quad \sum_k M_{jk} = 0. \tag{4.14}$$

Then, we find from (4.13) and (4.14) that

$$[H, \sum_{j,k}(L^n)_{jk}] = 0, \tag{4.15}$$

that is, conserved operators $\{I_n\}$ are given by

$$I_n = \frac{1}{n}\sum_{j,k}(L^n)_{jk}, \quad n = 1, 2, \cdots, N. \tag{4.16}$$

The expression (4.16) is new. First three of the conserved operators are

$$\begin{aligned}
I_1 &= \sum_{j,k} L_{jk} = \sum_j p_j, \\
I_2 &= \frac{1}{2}\sum_{j,k}(L^2)_{jk} = \frac{1}{2}\sum_j p_j^2 + \frac{1}{2}g\sum_{j\neq k}\frac{1}{(x_j - x_k)^2}, \\
I_3 &= \frac{1}{3}\sum_{j,k}(L^3)_{jk} \\
&= \frac{1}{3}\sum_j p_j^3 + \frac{1}{3}g\sum_{j\neq k}\{p_j\frac{1}{(x_j - x_k)^2} \\
&+ \frac{1}{(x_j - x_k)}p_k\frac{1}{(x_j - x_k)} + \frac{1}{(x_j - x_k)^2}p_k\}.
\end{aligned} \tag{4.17}$$

As we expected, g appears in $\{I_n\}$. While I_1 and I_2 are the total momentum operator and the Hamiltonian, I_3 is a non-trivial conserved operator. In general, $I_n = 1/n \cdot \sum p_j^n + \cdots$ is a polynomial in p_j and $1/(x_j - x_k)$.

We have found that the Lax pair gives an interesting algebra. Let us introduce a set of operators,

$$h_j^\dagger = \sum_k L_{kj} = p_j + ia\sum_{k\neq j}\frac{1}{x_k - x_j}, \tag{4.18}$$

$$h_j = \sum_k L_{jk} = p_j + ia\sum_{k\neq j}\frac{1}{x_j - x_k}. \tag{4.19}$$

The operators $h_j^\dagger$ and h_j are hermitian conjugate each other. They satisfy the commutation relations

$$\begin{aligned}
&[h_l, h_m] = 0, \quad [h_l^\dagger, h_m^\dagger] = 0, \\
&[h_l, h_m^\dagger] = 2M_{lm}, \qquad l, m = 1, 2, \cdots, N.
\end{aligned} \tag{4.20}$$

A similarity between (3.14) and (4.20) is intriguing. In terms of these operators, the total momentum operator I_1 and the Hamiltonian I_2 are simply expressed as

$$\begin{aligned}
I_1 &= \sum_{k=1}^{N} h_k^\dagger = \sum_{k=1}^{N} h_k, \\
I_2 &= \frac{1}{2}\sum_{l=1}^{N} h_l^\dagger h_l.
\end{aligned} \tag{4.21}$$

Further detail analysis of long-range integrable Hamiltonian systems, in particular, the Calogero-Moser system with spins will be discussed elsewhere[13].

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