TOPICS ON KALUZA-KLEIN THEORY

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Abstract

A modern review of Kaluza-Klein theories is presented. We adopt the version where the whole space is a principal fiber bundle with the four-dimensional spacetime as base, and as typical fiber a G Lie group. It is a natural generalization of gauge theories which metric is just the Kaluza-Klein metric. For the five dimensional theory we give an invariant formulation of the axisymmetric-stationary case. Some techniques for obtaining exact solutions and cosmology in specific dimensions are studied. Finally the method of spontaneus compactification is outlined.

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In 1919 Th. Kaluza presented to Albert Einstein a new idea for unifing all till that moment known interactions based on the Einstein geometrization theory. It consisted in a generalization of the general relativity theory into a five-dimensional riemannian space interpreting part of the five-dimensional metric as the four-electromagnetic potential. In a letter from Einstein to Kaluza, Einstein expressed his view of Kaluza's idea with the comment:"Ihr Gedanke gefelt mir zunächst ausserordentlich. "(I like your idea at first sight very much).Indeed Einstein was enthusiastic with Kaluza's idea and presented it at the Sitzungsberichte der Preussiscchen Akademie der Wissenschaften at 8th December 1921, in a paper entitled "Zur Unitätsproblem der Physic" [1]. This paper contained some inconsistences with the theory of Quanta as remarked by Einstein himself.The first important step in ascribing physical reality to the fifth dimension was taken by de Broglie and Schrödinger for the treatment of quantum problems.By starting with a generalized wave equation, he discovered in the equation surprising solutions which were periodic in the fifth dimension with a period related to the Planck constant. However, the first serious attempt to assign physical meaning to the fifth dimension was made by Einstein and Bergmann [76]. These authors introduced the remarkable assumption that the space is closed in a very small circle in the direction of the fifth dimension. Through this change not only was the Kaluza theory generalized, but also a justification for the four dimensional appearence of the "real" world was obtained. In a subsequent article, Einstein and Pauli [77] argued that the theory is still unsatisfactory with respect to the group of admissible coordinate transformations, because the fifth dimension is treated differently from the other four dimensions. However they made the following remark: "When one tries to find a unified theory of the gravitational and electromagnetic fields,one cannot help feeling that there is some truth in Kaluza's 5- dimensional theory".

Kaluza-Klein theory (KK) consisted basically in associate to the $\gamma_{\mu\nu}, (\mu, \nu = 1...4)$ components of the five-dimensional metric the gravitational interaction and to $\gamma_{5\mu}$ the electromagnetic one, while γ_{55} remained constant, the x^5 coordinate was a circle and all components of $\gamma_{AB}, A, B = 1...5$ depended only on $x^1...x^4$ but not on x^5. These assumptions appeared rather artificial and therefore unsatisfactory. Jordan [3] proposed a modification of the KK theory by assuming that the component γ_{55} varies like the other components depending only on $x^1...x^4$. Jordan found that this function γ_{55}, behaves like a scalar field without mass,being the Jordan's theory a gravitation, electromagnetism and scalar fields one. Nevertheless the assumptions of x^5 was a circle and γ_{AB} depended only on $x^1...x^4$ remained yet artificial. In the 1960's E. Schmutzer [5] constructed the KK theory supposing only invariance of the five metric under the action of a one-dimensional group i.e. he supposes the existence of a Killing vector field in the five manifold and projected all the physical quantities into the four space using the Killing vector field. The projective theory reproduces all the four-dimensional physics excepting that the geodesic motion in five-dimensions does not project into the usual four dimensional one. Kovacs [6] has shown more recently that there are many possible forms of motion in such a projection.

A reneved interst in the Kaluza-Klein theory arise with number of interesting obsevations made by Rayaki [78]. He pointed out that the 5- dimensional theory yields a geometrical interpretation of the electromagnetic field and of the electric charge. It provides a connection between the gravitational constant G the radius of the circle of the fifth dimension l, and the electric charge e of the form $G = e^2 l^2$. Morever, increasing the dimensionality beyond five dimensions may be a plausible way to include the isospin space, and in this way to obtain a unified theory involving strong interactions. Trautman [7] was the first to relate five-dimensional KK theory with the structure of fiber bundles. The relationship between principal fiber bundles and higher dimensional theories is clearer now [8].

The generalization of KK theories to more than five-dimensions was first mentioned by deWitt [9] and further developed by many others. In this generalization, the Yang-Mills fields became part of the metric in 4+n-dimensions in a similar context as the electromagnetic field did in the 5-dimensional theory. After these works many attempts to clarify the higher dimensional theory have been made [10],[18],[39] but the microscopical interpretation of the theory remains yet unclear.

In the 1970's, interest in higher dimensional Kaluza-Klein theory arose thanks to the introduction of supergravity [63] and string theory [41]. In fact, these two theories naturally lead physicists to consider higher dimensional field theories. As a result of this combination, new ideas appeared such as dimensional reduction and the process called "spontaneous compactification ". At the present time, Kaluza-Klein theory is a very promising theory in connection with the superstring one [41]. It seems that any future unified theory must be related in some way to the suggestion made by Kaluza in 1919.

In this course we plan to give a exposition of the main ideas of the KK theory. The central idea here is not to give a general review of the subject (since at the present time it has a tremendous extension) but rather to point out what we consider the main and more interesting feature of the Kaluza-Klein theory. In order to do so, the first part of this work is devoted to the five-dimensional theory, as explained in most of the literature. In the second part we give a definition of the physical quantities in a covariant manner, and explain how it can be used for understanding better the theory. In the third part, we will generalize the five-dimensional theory, and explain it consequences and problems. The fourth part pretend to give a clear explanation of the geometry of the theory begining with purely geometrical suppositions. Finally, in the last part, we will discuss the method of spontaneous compactification. We will see that such a method is closed related with supergravity [63], [40] and superstrings [41].

I The Five-dimensional Theory

The Phylosophy of modern physics supposes the existence of two kinds of symmetries: the geometric and the inner one. Geometric symmetries refer to the existence of privileged directions in the spacetime in which the physics remains invariant. Inner symmetries refer to the invariance of the action under certain transformations. The first one depends on the phenomenal but the second one depends of the sort of interaction we are studing. It is well-known that invarance of the action under the group $U(1)$ refers to electromagnetic interactions [12], or invariance of the action under the group $SU(2)\times U(1)$ refers to electroweak interaction. Let us start supposing that the group $U(1)$ is acting on a M^5 riemannian space. This implies the existence of a Killing vector field X in M^5. If we choose a local coordinate system on M^5 such that $X=\frac{\partial}{\partial x^5}$ the components of the metric tensor $d\hat{S}^2$ do not depend on θ. Observe that the non θ-dependence of $d\hat{S}$ is a consequence of the action of the group on M^5. Of course if we choose any other coordinate system the metric could depend explicitly on θ. This is because the presence of the group symmetry allows to choose a gauge for the five-metric.

In part four we will deduce the explicit form of the metric in terms of purely geometrical suppositions. But now let us begin with the so called K K ansatz for the five metric:

$$d\hat{S}^2 = g_{\mu\nu}dx^\mu dx^\nu + I^2(B_\mu dx^\mu + d\theta)(B_\nu dx^\nu + d\theta) \qquad (1.1)$$

$$\mu,\nu = 1,...,4$$

where $d\hat{S}^2$ is the five-dimensional metric, $g_{\mu\nu}$ are the components of the four-dimensional metric, I is the scalar potential and B_μ the electromagnetic potential. $g_{\mu\nu}$, I and B_μ depend only on $x^1...x^4$ but not on $\theta = x^5$. Observe also that $I^2 = X^A X_A = \hat{g}_{55}$ (A=1,...,5) is the radius of the five dimension. It is easy to check that a coordinate transformation of the fifth dimension

$$\theta \rightarrow \theta + \Lambda(x^\mu) \qquad (1.2)$$

i.e. a local transformation of the group U(1) in M^5 is equivalent to a gauge transformation of the four electromagnetic potential $B_\mu \rightarrow B_\mu + \partial_\mu\Lambda$, because of the transformation rule

$$\hat{g}_{AB} \rightarrow \hat{g}'_{CD}\frac{\partial x'^C}{\partial x^A}\frac{\partial x'^D}{\partial x^B}. \qquad (1.3)$$

The field equations can be deduced from the Einstein-Hilbert action in five dimensions

$$S = -\frac{1}{16\pi G_k}\int d^5x\sqrt{-\hat{g}_5}\hat{R} \qquad (1.4)$$

where $\hat{g}_5$ is the determinant of the metric components $\hat{g}_{AB}$ and $\hat{R}$ is the five-dimensional curvature scalar. If we substitute the metric (1.1) into (1.4) and integrate over the θ coordinate, one gets the four- dimensional action

$$S = -\frac{2\pi}{16\pi G_k}\int d^4x\sqrt{-g_4}I[R + \frac{1}{4}I^2B_{\mu\nu}B^{\mu\nu}] \qquad (1.5).$$

Here g_4 is the determinant of the four-dimensional metric. R the four-dimensional curvature scalar and $B_{\mu\nu}$ the Maxwell tensor $B_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu}$. Variation of (1.5) with respect to the metric yields the Einstein's equations coupled with the Maxwell stress tensor for $B_{\mu\nu}$ and a scalar stress tensor for I as sources. Variation with respect to B_μ gives the Maxwell equations for the potential B_μ and with respect to I one finds a field equation for the scalar potential I where the currents are electromagnetic and gravitational.

In order to have a direct coparison with the standard electromagnetic potential, one expects to recover the Einstein-Maxwell theory when I is constant for some x^μ. Then we take the limit when I $(x^\mu) = I_0$=cte and comparing with the Einstein-Maxwell action [13]

$$S_{EM} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g_4}[R + \frac{16\pi G}{4}F_{\mu\nu}F^{\mu\nu}];$$

thus the association with the constant G_k and the Maxwell tensor $B_{\mu\nu}$

$$\frac{2\pi I_0}{16\pi G_k} = \frac{1}{16\pi G}$$

and

$$F_{\mu\nu} = (I_0/\sqrt{16\pi G}B_{\mu\nu}) \tag{1.6}$$

holds

An interesting observation is that a redefinition of the 4-dimensional components of the metric like

$$g_{\mu\nu} \rightarrow \frac{1}{I}g_{\mu\nu} \tag{1.7}$$

gives rise to the creation of a scalar field in the Lagrangian (1.5)

$$S \rightarrow S = \frac{-1}{16\pi G_k}\int d^4x\sqrt{-g_4}[R + \frac{1}{4}\phi B_{\mu\nu}B^{\mu\nu} - \frac{1}{6}\varphi^{-2}\partial_\alpha\phi\partial^\alpha\phi] \tag{1.8}$$

being $\varphi^{1/3} = I$. This transformation eliminates the factor I of R in the Lagrangian (1.5). Now ϕ is like the scalar potential of the Brans-Dicke theory [4]. This scalar potential is a feature of KK theories and is very important in the analysis of the geodesic motion.

Following the phylosophy of general relativity, (see for example [13]) the free particle motion must be a geodesic, in this case a five-dimensional one. Because of the presence of the Killing vector $X = \frac{\partial}{\partial\theta}$, the geodesic equation can be separated in two parts

a)

$$[\frac{I^2dx^5}{d\hat{S}} + I^2B_\mu\frac{dx^\mu}{d\hat{S}}]\hat{m} = cte = \hat{p},$$

b)

$$\frac{d^2x^\mu}{d\hat{S}^2} + \Gamma^\mu_{\nu\alpha}\frac{dx^\nu}{d\hat{S}}\frac{dx^\alpha}{d\hat{S}} = \frac{\hat{p}}{\hat{m}}\hat{B}^\mu_\nu\frac{dx^\nu}{d\hat{S}} + \frac{\hat{p}^2}{\hat{m}^2}g^{\mu\nu}\frac{I_{,\nu}}{I^3}. \tag{1.9}$$

Here, $\hat{m}$ is the mass of a test particle and $\hat{p}$ is the momentum of the particle on the fifth direction $\hat{p} = \hat{m}g_{5A}\frac{dx^A}{d\hat{S}}$. If we want to reproduce the Lorentz force in (1.9) we have to associate to the charge of the test particle the quantity

$$\hat{p} = qI_0/\sqrt{16\pi G} \tag{1.10}$$

Neverthelees an observer will measure dS and not $d\hat{S}$ as in equation (1.9). Therefore if we write the geodesic equations in terms of dS, from (1.1) and (1.9a) one gets

$$-\epsilon d\hat{S}^2 = -dS^2 + (\hat{p}^2/I^2\hat{m}^2)d\hat{S}^2,$$

$\epsilon = -1, 0, +1$ for a space, null and time-like geodesic)and then equation (1.9) transform to ([6] and [14])

$$\frac{d^2x^\mu}{dS^2} + \Gamma^\mu_{\nu\alpha}\frac{dx^\nu}{dS}\frac{dx^\alpha}{dS} = \frac{q}{m_{eff}}F^\mu_\nu\frac{dx^\nu}{dS} + \frac{q^2}{m^2_{eff}}\frac{I_0^2}{16\pi G}\frac{I_{,\mu}}{I^3}(g^{\mu\nu} + \frac{dx^\mu}{dS}\frac{dx^\nu}{dS}) \tag{1.11}$$

(a study of the geodesic equation (1.9) and (1.11), and a clear deduction of (1.11) is given in [6]). Here we have defined the quantity

$$m_{eff} = \hat{m}\frac{dS}{d\hat{S}} = [\epsilon\hat{m}^2 + \frac{\hat{p}^2}{I^2}]^{\frac{1}{2}}$$

which is the effective mass of the test particle projected into four-dimensions.

It is not possible to determine the radius of the circle so far. However the radius I_0 can be estimated requiring quantization of charge $q = ne$ and momentum $\hat{p} = n\hbar$ of the fifth dimension. One obtains from (1.10)

$$I_0 = \hbar\sqrt{16\pi G}/e \sim 10^{-32}cm$$

i.e. in equation (1.11) the KK radius I_0 is of Plank length order. In the general case when one does not choose a local coordinate system with killing vector $X = \frac{\partial}{\partial\theta}$ the five-metric might depend on x^5 too, but then the metric can be expanded in Fourier series

$$\hat{g}_{AB} = \hat{g}_{AB}(x,\theta) = \sum_{n=-\infty}^{\infty} \hat{g}^{(n)}_{AB}(x)e^{in\theta}. \tag{1.12}$$

The KK ansatz does not make sense in this coordinate system because a function B_μ like in (1.1) would not be a vector potential fulfilling the Maxwell equations and the I potential would not be a scalar potential. If we take only the modes with $n = 0$ in (1.12), we recover the spatial coordinate system with Killing vector $X = \frac{\partial}{\partial\theta}$. This is because g_{AB} is a gauge potential in M^5 due to the action of G on M^5, (G=U(1) in this case), all the $n \neq 0$ modes are only a consequence of a gauge. Then we can interpret this gauge fixing as a dimensional reduction.

There must exist a certain limit for which we recover the flat space time with a isometry U(1), i.e. we have to recover the Poincaré symmetries and the U(1) group in this limit where the M^5 space is of the form $M^4 \times S^1$. But in general one expects to have a more general symmetry acting on the M^5 space. In order to see this fact we make a general infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \zeta^\mu(x,\theta)$$
$$\theta \to \theta + \zeta^5(x,\theta)$$

where

$$\zeta^A(x,\theta) = \sum_{n=-\infty}^{\infty} \zeta^{(n)A}(x)e^{in\theta} \tag{1.13}$$

The Fourier series expansion of $\zeta^A(x,\theta)$ can be make because of the periodicity in the θ coordinate. Now we proceed like in four-dimensional gravity for recovering the Poincaré invariance (see [15] or [11] and [16]). In four dimensional gravity one restricts the $\zeta^\mu(k)$ function to be linear in x^μ : $\zeta^\mu = a^\mu + \omega^\mu_\nu x^\nu$, with a^μ and ω^μ_ν constants. In five-dimensions we restrict the functions $\zeta^{(n)A}(k)$ to be linear in x^νin analogous manner , i.e.

$$\zeta^{(n)\mu}(x) = a^{(n)\mu} + \omega^{(n)}_{\mu\nu}x^\nu$$
$$\zeta^{(n)5} = C^{(n)} \tag{1.14}$$

where $a^{(n)\mu}, \omega^{(n)\mu}_\nu$ and $c^{(n)}$ are constants. Now we want to indentify the generator of this transformation.If we take, for example, all $\omega^{(n)}$ and c^n zero and a single non-zero $a^{(n)\mu}$, we have

$$x^\mu \to x^\mu + \zeta^{(n)\mu}e^{in\theta}$$

$$\theta \to \theta.$$

Then, a function $\phi(x, \theta)$ would transform like

$$\phi(x,\theta) \to \phi(x,\theta) + e^{in\theta}\zeta^{(n)\mu}\partial_\mu\phi = (1 - i\zeta^{(n)\mu}P_\mu^{(n)})\phi$$

where we have defined

$$P_\mu^{(n)} = ie^{in\theta}\partial_\mu \tag{1.15}$$

Similarly we find

$$M_{\mu\nu}^{(n)} = ie^{in\theta}(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{1.16}$$

and

$$Q^{(n)} = ie^{in\theta}\partial_\theta \tag{1.17}$$

corresponding to the generators of the Lorentz transformations and the S^1 translations. These quantities generate a (non-compact) infinite parameter Lie algebra containing the usual Poincaré algebra

$$\begin{aligned}
[P_\mu^{(n)}, P_\nu^{(m)}] &= 0\\
[M_{\mu\nu}^{(m)}, P_x^{(n)}] &= i(\eta_{\lambda\nu}P_\mu^{(m+n)} - \eta_{\lambda\mu}P_\nu^{(m+n)})\\
[M_{\mu\nu}^{(n)}, M_{\rho\sigma}^{(m)}] &= i(\eta_{\nu\rho}M_{\mu\sigma}^{(n+m)} + \eta_{\mu\sigma}M_{\nu\rho}^{(m+n)} - \eta_{\mu\rho}M_{\nu\sigma}^{(m+n)} - \eta_{\nu\sigma}M_{\mu\rho}^{(m+n)})\\
[Q^{(n)}, Q^{(m)}] &= (n-m)Q^{(n+m)}\\
[Q^{(n)}, P_\mu^{(m)}] &= -mP_\mu^{(n+m)}\\
[Q^{(n)}, M_{\mu\nu}^{(m)}] &= -mM_{\mu\nu}^{(n+m)}
\end{aligned} \tag{1.18}$$

The algebra (1.18) represents the full Kac-Moody symmetry [17] and contains for $m = n = 0$ the usual Poincaré $\otimes U(1)$ algebra. This means that the full Kac-Moody symmetry is spontaneously broken when one takes only the modes n = m = 0, or equivalently when one chooses the local coordinate system with Killing vector $X = \frac{\partial}{\partial\theta}$. Actually, we get the algebra Poincaré $\otimes$ SO(1,2) for $m = n = 0$, because the generators $P_\mu^{(0)}, M_{\mu\nu}^{(0)}, Q^{(1)}, Q^{(-1)}$ form this closed algebra (see ref.[18]).

To conclude this part, we determine the four-dimensional classical mass spectrum of the KK theory. In order to do so, we vary the metric components $\hat{g}_{AB}$ around flat spacetime [5]

$$\hat{g}_{AB}(x,\theta) = \eta_{AB} + h_{AB}(x,\theta) \tag{1.19}$$

where now $\eta_{AB} = diag(1,1,1,-1,l_0)$. Analogously as in four-dimensional gravity we expand the field equations $R_{AB} = 0$ to first order in h_{AB}. One arrives at

$$\partial_A\partial_B h_C^C - \partial_C\partial_A h_B^C - \partial_C\partial_B h_A^C + \partial^C\partial_C h_{AB} = 0 \tag{1.20}$$

Observe that equation (1.20) is invariant under the gauge transformations of h_{AB}

$$h_{AB} \to h_{AB} + \partial_A\zeta_B + \partial_B\zeta_A \tag{1.21}$$

so that we may choose the gauge (see [19])

$$\begin{aligned}
\partial^\mu h_{\mu 5} &= 0\\
\partial^5 h_{\mu 5} &= 0\\
\partial^5 h_{55} &= 0 \ ,
\end{aligned} \tag{1.22}$$

Again, because of the $U(1)$ symmetry acting on M^5, we can expand the h_{AB} functions in Fourier series. Then the gauge (1.22) can be written as

$$\partial^\mu h^{(0)}_{\mu 5}(x) = 0$$
$$h^{(n)}_{\mu 5}(x) = 0 \quad n \neq 0$$
$$h^{(n)}_{55} = 0 \quad n \neq 0 \tag{1.23}$$

We substitute (1.23) into (1.20). The 55 component yields

$$a) \qquad \partial^\alpha \partial_\alpha h^{(0)}_{55} = 0$$
$$b) \quad h^{(n)\alpha}_{\alpha} = 0 \quad n \neq 0 \tag{1.24}$$

From the $\mu 5$ component one arrives at

$$\partial^\alpha \partial_\alpha h^{(0)}_{\mu 5} = 0 \tag{1.25}$$

and using (1.24) one obtains

$$\partial^\alpha h^{(n)}_{\mu\alpha} = 0 \quad (n \neq 0) \tag{1.26}$$

The $\mu\nu$ components reduce to

$$\partial^\alpha \partial_\alpha h^{(0)}_{\mu\nu} + \partial_\mu \partial_\nu (h^{(0)\alpha}_{\alpha} + h^{(0)5}_{5}) - \partial_\alpha \partial_\mu h^{(0)\alpha}_{\nu} - \partial_\alpha \partial_\nu h^{(0)\alpha}_{\mu} = 0 \tag{1.27}$$

and using (1.24) and (1.26) one arrives at

$$(\partial^\alpha \partial_\alpha + \frac{n^2}{I_0^2}) h^{(n)}_{\mu\nu} = 0 \quad n \neq 0 \tag{1.28}$$

Therefore the $n \neq 0$ tensor modes $h^{(n)}_{\mu\nu}$ are massive with masses n^2/I_0^2. Equations (1.25) are the Maxwell equations for the four vector $h^{(0)}_{\mu 5}$ in the gauge $\partial_\mu h^{(0)\mu} = 0$, i.e. $h^{(0)}_{\mu 5}$ are massless. Finally equation (1.24.a) is the equation for the massless scalar potential $h^{(0)}_{55}$. To recover the massless graviton from equation (1.28) we make the transformation

$$\bar{h}^{(0)}_{\mu\nu} = h^{(0)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} h^{(0)5}_{5}$$

then equation (1.28) can be rewritten as

$$\partial^\alpha \partial_\alpha \bar{h}^{(0)}_{\mu\nu} + \partial_\mu \partial_\nu \bar{h}^{(0)\alpha}_{\alpha} - \partial_\alpha \partial_\mu \bar{h}^{(0)\alpha}_{\nu} - \partial_\alpha \partial_\nu \bar{h}^{(0)\alpha}_{\mu} = 0$$

which is just the equation for the massless spin-2 gravitation field. (see also ref.[20])

II The Potential Formalism.

In this part we want to give a covariant definition of the physical quantities of the KK theory. We shall proceed like in the Einstein's theory of relativity defining a sort of Ernst potentials in five-dimensions. But first we discuse shortly the Ernst and the electromagnetic potential in four dimensions.

The Ernst potential [21] is defined in the stationary case, i.e. when there exist a time-like Killing vector field ξ with

$$\xi^\alpha \xi_\alpha < 0 \quad \alpha = 1...4 \tag{2.1}$$

Then the Lie derivative with respect to ξ of the metric and the electromagnetic potential vanishes.

$$\mathcal{L}_\xi B_{\alpha\beta} = 0, \quad \mathcal{L}_\xi g_{\alpha\beta} = 0 \tag{2.2}$$

where $B_{\alpha\beta}$ is the electromagnetic field tensor, which fulfills the Maxwell's equations

$$\tilde{B}^{\mu\alpha}{}_{;\alpha} = 0 \Leftrightarrow \xi^{\alpha}\tilde{B}_{[\alpha\mu;\nu]} = 0. \tag{2.3}$$

Here $\tilde{B}_{\alpha\beta}$ is the complex self-dual electromagnetic field tensor (see ref. [22] and [23])

$$\tilde{B}_{\alpha\beta} = B_{\alpha\beta} + \frac{i}{2}\epsilon_{\alpha\beta\mu\nu}B^{\mu\nu} \tag{2.4}$$

It follows that Φ defined by

$$\Phi_{\mu} = \sqrt{\frac{\kappa_o}{2}}\xi^{\alpha}\tilde{B}_{\alpha\mu} \tag{2.5}$$

is a gradient $\Phi_{\mu} = \Phi_{,\mu}$. Because of (2.1) and (2.3) the integrability condition for the potential Φ

$$2\Phi_{,[m;n]} = \sqrt{\frac{\kappa_o}{2}}(\xi^{\alpha}\tilde{B}_{[\alpha\mu;\nu]} - \mathcal{L}_{\xi}B_{\mu\nu}) = 0$$

holds. The real and imaginary part of the complex potential Φ describe the electrostatic and magnetostatic potentials, respectively. Now we make use of the Einstein equations and of the Ricci identity observing that

$$(R^{\mu}_{\nu} - \frac{\kappa_o}{2}B^{\mu\alpha}\bar{\tilde{B}}_{\alpha\nu})\xi^{\nu} = 0 \tag{2.6}$$

we find that

$$\frac{1}{2}\tilde{K}^{\mu\alpha}{}_{;\alpha} = -\tilde{\xi}^{\mu;\alpha}{}_{;\alpha} - \sqrt{\frac{\kappa_o}{2}}\bar{\Phi}_{,\alpha}\tilde{B}^{\mu\alpha} = 0 \tag{2.7}$$

where we have defined the vector

$$\tilde{\xi}_{\mu;\nu} = \xi_{\mu;\nu} + \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}\xi^{\alpha;\beta}$$

(a bar denotes complex conjugation). Other way it is easy to show that the Lie derivative of $\tilde{K}_{\mu\nu}$ with respect to ξ vanishes, i.e. we have the same situation as for the tensor (2.4) and allows us to define the complex potential

$$\mathcal{E}_{,\mu} = \mathcal{E}_{\mu} = \xi^{\alpha}\tilde{K}_{\alpha\mu} \tag{2.8}$$

called the Ernst potential. When electromagnetism vanishes, the real and complex parts of $\mathcal{E}$ are the gravitational and rotational potentials, respectively. The Einstein equations in terms of the Ernst and Electromagnetic potentials (2.5) and (2.8) respectively, are the Ernst equations (see ref.[21] [22] and [23]).

We proceed in the same way for the five-dimensional KK theory. In the stationary case we have a second Killing vector Y which commutes with the Killing vector X. Stationarity means:

$$Y^{A}Y_{A} < 0. \tag{2.9}$$

With these two conmmutating Killing vectors X and Y we can define in a covariant manner five potentials [24]

$$\begin{gathered} \kappa^{4/3} = I^2 = X^A X_A \quad ; f = -IY^A Y_A + I^{-1}(X^A Y_A)^2 \\ \psi = -I^{-2}X_A Y^A \quad \chi_{,E} = \epsilon_{ABCDE}X^A Y^B X^{C;D} \\ \epsilon_{,E} = \epsilon_{ABCDE}X^A Y^B Y^{C;D} \end{gathered} \tag{2.10}$$

In the spatial coordinate system where $X^A = \delta^A_5$ and $Y^A = \delta^A_4$ one finds that κ, f, ψ, χ and ϵ respectively are the scalar, gravitational, electrostatic, magnetostatic and rotational potentials. We can write now the five dimensional field equations in terms of the potentials (2.10), but better than this we can write down the

Lagrangian L from which we can derive the field equations

$$L = \frac{\alpha}{2f^2}[f_{,A}f^{,A} + (\epsilon_{,A} + \psi\chi_{,A})(\epsilon^{,A} + \psi\chi^{,A})] \tag{2.11}$$
$$+\frac{\alpha}{2f}(\kappa^2\psi_{,A}\psi^{,A} + \frac{1}{\kappa^2}\chi_{,A}\chi^{,A}) + \frac{2}{3}\frac{\alpha}{\kappa^2}\kappa_{,A}\kappa^{,A}$$

and look for its invariant transformations. This is important because if we have a solution of the field equation Φ^A an invariant transformation of (2.11) $\Phi^A \rightarrow \Phi'^A(\Phi^B)$ will give us a new solution. All the invariant transformations of (2.11) were found in ref. [24], by Neugebauer. In order to do so he defined a metric derived from (2.11)

$$dS^2 = \frac{1}{2f^2}[df^2 + (d\epsilon + \psi d\chi)^2] + \frac{1}{2f}(\kappa^2 d\psi^2 + \frac{1}{\kappa^2}d\chi^2) + \frac{2}{3}\frac{d\kappa^2}{\kappa^2} \tag{2.12}$$

which is the metric of a symmetric Riemannian space V^5 i.e. the covariant derivative of the curvature tensor of V^5 with respect to each coordinate, vanishes. The group of motion of the metric (2.12) will give us the invariant transformations of (2.11). This group has 8 parameters. It was found in reference [25] that the group of isometries of the metric (2.12) is SL(3,**R**) and the invariant transformations of (2.12) can be cast in a very simple form as

$$g \rightarrow cgc^T \tag{2.13}$$

where the matrix c is a constant matrix of the same groups SL(3,**R**). The matrix $g \epsilon$ SL(3,**R**) can be parametrized as

$$g = \frac{-2}{f\kappa^{2/3}}\begin{pmatrix} f^2 + \epsilon^2 - f^2\kappa^2\psi^2 & -\epsilon & -\frac{1}{2\sqrt{2}}(\epsilon\chi + f\kappa^2\psi) \\ -\epsilon & 1 & \frac{1}{2\sqrt{2}}\chi \\ -\frac{1}{2\sqrt{2}}(\epsilon\chi + f\kappa^2\psi) & \frac{1}{2\sqrt{2}}\chi & \frac{1}{8}(\chi^2 - \kappa^2 f) \end{pmatrix} \tag{2.14}$$

Observe that if we set $\chi = \psi = 0$ and $\kappa = 1$, the matrix g transforms to:

$$g = \frac{-2}{f}\begin{pmatrix} f^2 + \epsilon^2 & -\epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & \frac{-1}{8}f \end{pmatrix}$$

which is just the potential matrix for the Einstein theory where $\mathcal{E} = f + i\epsilon$ is the Ernst potential, g belongs in this case to the group SL(2,**R**), but these group is homomorphic to SU(1,1), therefore it is possible using the complex transformation $\mathcal{E} = f + i\epsilon$ to obtain

$$g = \frac{-2}{Re\mathcal{E}}\begin{pmatrix} \mathcal{E}\bar{\mathcal{E}} & Im\mathcal{E} & 0 \\ Im\mathcal{E} & 1 & 0 \\ 0 & 0 & -\frac{1}{8}Re\mathcal{E} \end{pmatrix}$$

which belongs to the group SU(1,1) of the Einstein's equations (see ref. [23] cap. 30). This means that solutions of the Einstein equations in vacuum will be also solutions of the KK theory. If we start with the Kerr-NUT solution of the Einstein's theory [23] and make a transformation (2.13) we get a 7 parametric new solution which contains for example, the Belinsky-Ruffini solitonic solution [26], the Kramer [27], the Neugebauer [24] and the "Kerr-NUT" solutions as limits (see ref. [25]).

Axialsymmetry is represented by the existence of a third Killing Vector field Z which is space like. One can choose then a coordinate system in which the components of the five metric depend only on two coordinates. In this case the field equations for the matrix (2.14) are the chiral equations

$$(\rho g_{,z}g^{-1})_{,\bar{z}} + (\rho g_{,\bar{z}}g^{-1})_{,z} = 0 \tag{2.15}$$

being $z = \rho + i\zeta, g = g(\rho, \zeta)$. The generalized inverse scattering method was applied to equation (2.15) for finding exact solitonic solutions of g [28].

One dimensional subgroups (one parametric subgroups) of SL(3,**R**) will give us two equivalent classes of solutions of equations (2.15). In this case equation (2.15) becomes to

$$g_{,\lambda} = Ag \tag{2.16}$$

where λ is the parameter of the group. These two classes are divided in subclasses depending on the eigenvalues of A. All classes and subclasses are given in [29] and shown in Table 1. (the subclass b =0 in table 1 is given also in [30]).

There are three two-dimensional subgroups of SL(3,**R**). One is abelian, the second is SL(2,**R**) $\cong$ SU(1,1) to which the Einstein theory in vacuum belongs and the group 0(2,1) which contains a correspondence with the Einstein Maxwell theory where the scalar potential $\kappa = 1$. (See ref. [29])

From this technique many exact solutions of the KK axisymmetric stationary field equations have been generated. In the literature one can find gravitational fields coupled with monopoles [3], [31]., [32]; dyons [33],[34] ; dipoles [35];monopoles and dipoles [36]; strange potentials [36] etc. We want now to study some of these solutions in order to establish some of the main features of the theory.

The first one was found by a simple extention of the Taub-NUT Euclidean solution [23] in four dimensions [13][31]. We follow these papers now. This solution of the five-dimensional KK field equations is of a pure magnetic monopole described by the metric (Killing Vectors $X^A = \delta^A_5; Y^B = \delta^B_4$)

$$dS^2 = -dt^2 + I^2(dx^5 + 4m(1 - cos\theta)d\phi)^2 + I^{-2}(dr^2 + r^2 d\theta^2 + r^2 sen^2\theta d\phi^2)$$

$$X^A X_A = (1 + \frac{4m}{r})^{-1} \tag{2.17}$$

If we compare it with the metric (1.1) we find that the electromagnetic potential and the scalar potential respectively are

$$A_\mu = (0, 0, 4m(1 - cos\theta), 0), \quad I^2 = (1 + 4m/r)^{-1} \tag{2.18}$$

Observe that if we set $t = cte$ in (2.17) we recover the Taub-NUT metric, which is regular if x^5 is periodic with period $16\pi m$ [37]. Thus we have to identify 16πm with 2 to obtain $m = 1/8R$. The flat space limit is found to be $r >> 1$, in this limit we recover the Minkowski metric in spherical coordinates. Then $I_0 = 1$ and comparing (2.18) with (1.6) we find that the magnetic charge is

$$g = \frac{4m}{\sqrt{16\pi G}}$$

Of course in this coordinates the gravitational mass obtained for the asymptotic limit of $g_{44} = -1$ for $r >> 1$ vanishes, but the inertial mass, deduced from the energy momentum pseudo-tensor

$$-\hat{g}t^{AB} = \partial_C h^{ABC} \quad , \quad h^{ABC} = \frac{1}{16\pi G_k}\partial_D[-\hat{g}(\hat{g}^{AB}g^{CD} - g^{AC}g^{BD})] \tag{2.19}$$

is m/G [31]. What it is perhaps really happen is that the scalar potential I acts gravitationally and cancels identically the gravitational contribution of the monopole. In order to see this, one should solve the geodesic equations of the metric (2.17), but this is actually rather complicated (see ref. [14]). We will consider this issue in an other easier solution. Let us write the four metric and the scalar potential (ref. [32] solution (16) with $\delta_0 = -1$ and $I_0 = 1$)

$$dS^2 = -\sqrt{1 - \frac{2m}{r}}dt^2 + \frac{dr^2}{\sqrt{1 - \frac{2m}{r}}} + \sqrt{1 - \frac{2m}{r}}r^2 d\Omega^2$$

$$\phi = (1 - \frac{2m}{r})^{3/2} \qquad f = \sqrt{1 - \frac{2m}{r}} \qquad \psi = \chi = \epsilon = 0 \tag{2.20}$$

where we have used the transformation (1.7) and the notation of (1.8). The gravitational mass is $m/2$, and in this case equals the inertial mass. But a test particle would interact with the mass and the scalar potential.

To see this we write the geodesic equation (1.11) for the metric (2.20) as

$$\frac{d}{ds}(m_{eff}u^{\mu}) + m_{eff}\Gamma^{\mu}_{\nu\alpha}u^{\nu}u^{\alpha} = \frac{\tilde{p}^2}{m_{eff}}g^{\mu\nu}\frac{I_{,\nu}}{I^3}$$

$$m_{eff} = \tilde{m}(\epsilon + \frac{\tilde{p}^2}{I^2\tilde{m}^2})^{1/2} \qquad u^{\mu} = \frac{dx^{\mu}}{ds}$$

If we substitute the expression $g_{\mu\rho,\alpha}\frac{dx^{\alpha}}{ds} = \frac{dg_{\mu\rho}}{ds}$ we obtain

$$\frac{d}{ds}(m_{eff}u^{\mu}g_{\mu\rho}) - \frac{1}{2}m_{eff}g_{\nu\alpha,\rho}u^{\nu}u^{\alpha} + m_{eff,\rho} = 0 \tag{2.21}$$

One solution of these equations for a photon (with $\epsilon = 0$) is $u^{\mu} = \delta^{\mu}_4(1 - \frac{2m}{\gamma})^{-1/4}$ Then the newtonian force will be

$$\frac{1}{2}m_{eff}g_{\nu\alpha,\rho}u^{\nu}u^{\alpha} = -\frac{1}{2}m_{eff}g_{44,\rho}u^4u^4 = m_{eff,\rho}$$

this means that it is exactly cancelled by the interaction with the scalar field [31], which gives rise to a repulsive force cancelling exactly the newtonian force. Thus a test particle traveling around the mass $\frac{m}{2}$ will have a constant momentum

$$P_{\mu} = m_{eff}u_{\mu} = \tilde{m} = m_{eff}u^4g_{44}$$

in other words, it does not feel the attraction of the mass $\frac{1}{2}m$. Such a situation appears again in solutions of higher dimensions. (See the next section)

The last solution we want to deal with is again a magnetic monopole, but with a Schwarzschild-like gravitational potential, and a scalar potential given by [32]

$$dS^2 = (1 - \frac{2m}{r})^{1/2}[1 - \eta ln(1 - \frac{2m}{r})^{1/2}]\{[1 - \frac{m^2sin^2\theta}{r^2(1 - \frac{2m}{r})}]^{1/4}[\frac{dr^2}{(1 - \frac{2m}{r})} + r^2d\theta^2]$$

$$+r^2sin^2\theta d\phi^2\} - (1 - \frac{2m}{r})dt^2$$

$$A_{\mu} = (0, 0, \eta m(1 - cos\theta), 0) \qquad I^2 = \frac{1}{[1 - \frac{2m}{r}]^{\frac{1}{2}}[1 - \eta ln(1 - \frac{2m}{r})^{\frac{1}{2}}]} \tag{2.22}$$

This is an asymptotically flat solution of the five-dimensional field equations. For $r >> 2m$, (2.22) approaches flat space in spherical coordinates. The gravitational mass M and the magnetic charge g_{Mg} are given by

$$M = m \;\; and \;\; g_{Mg} = \frac{\eta m}{\sqrt{16\pi G}}$$

Observe that if $m = 0$ (2.22) becomes flat. But if $\eta = 0$, only the monopole charge vanishes. The four metric (2.22) has a singularity at $r = 2m$, which is an horizon and if $r_m = m + m\sqrt{1 + sin^2\theta}$ the factor

$$1 - \frac{m^2sin^2\theta}{r^2[1 - \frac{2m}{r}]} = 0.$$

The $r = 2m$ singularity is perhaps not essential, but one would expect that the four spacetime will be really singular at $r = 0$.

For a test particle of mass μ and charge e within the monopole field (2.22) we have

$$-\mu^2 = g^{AB}P_AP_B$$

In the equatorial plane with $\theta = \pi/2$ we can write the energy equation in terms of the components of the five-momentum P_A. We obtain (see ref.[38])

$$E = \frac{g^{11}}{g^{44}}(\frac{\partial r}{\partial \phi})^2 \dot{\phi}^2 + V_e(r)$$

where the effective potential function $V_e(r)$ is given by

$$V_e(r) = \frac{L_e^2 \chi^{1/2}}{r^2(1 - \eta ln \chi^{1/2})} + \chi(\mu^2 + \frac{L^2{}_5}{I^2})$$

L_5 is the angular momentum in the fifth dimension and L_e is the angular momentum about the axis of symmetry. Results for $V_e(r)$ vs r are calculated for the particles with $\mu^2 = 1$. $m = 1$, $L_e = 10$ and $\eta = .09$ for different L_5 in ref. [38] and plotted in fig.1. and also for $L_5 = 2$ and different η are plotted in fig 2. The orbits on the equatorial and polar planes are also calculated in ref. [38] and shown in figs. 3 and 4.

Observe that the scalar potential I approaches $I \sim 1$ very quickly for $r > 2m$ and approaches also very quickly infinity when r approaches the horizon. This means that the effects of the scalar potential I become important only near the horizon but desapear far away of it. This can be a reason why we can not detect I. The behaviour of I^2 is shown in fig. 5

III The n-dimensional theory

In this section we present the $d = (n + 4)$-dimensional KK theory and give its main results.

As in the first section let us begin with the so called KK ansatz, now for a d-dimensional riemannian space, we have [8],[42].

$$\check{dS}^2 = g_{\mu\nu}dx^\mu dx^\nu + \tilde{g}_{ab}(\omega^a + ekB^a_\mu dx^\mu)(\omega^b + ekB^b_\nu dx^\nu) \tag{3.1}$$

$$\mu, \nu = 1...4, \qquad a, b = 5...d$$

where the d-dimensional space $\check{g} = \check{dS}^2$ contains a n-dimensional group G of motion ;$g = g_{\mu\nu}dx^\mu dx^\nu$ is the four dimensional metric, i.e. the spacetime metric ; $\tilde{g} = \tilde{g}_{ab}\omega^a\omega^b$ is the metric of a Lie group which will be called the inner space and $B = B^a_\mu t_a dx^\mu$ is the Yang Mills gauge potential, t_a being the generators of the group G. e is the couplings constant and k a scale parameter.

In general it is not necessary to take the inner space as the group of motion, but they are closely related. For instance it is possible to take it as the homogeneous space G/H, where H is a normal subgroup of G. However for the moment andin order to obtain the field equations we will consider the inner space as the group itself. In the next part we will clarify this point and from geometrical assumtions we will derive the metric (3.1). In most of the literature [11],(3.1) is taken as the "ground state metric" and is put in by hand. The spacetime metric and the Yang Mills gauge potentials are suppossed to depend only on the four spacetime coordinates and G is assumed to be compact. Here we take a more general d-metric and suppose only that in $\check{g}$ are acting n Killing vectors which form an $\mathcal{G}$-algebra, corresponding to the G-group.

Let $\{\xi_a\}$ be a left-invariant basis of the tangent space to G, dual to $\{\omega^a\}$. Of course there exists a canonical isomorfism between the basis $\{\xi_a\}$ at the identity and the Lie algebra $\mathcal{G}$

$$i : T_e G \rightarrow \mathcal{G}$$

$$\xi_a \rightarrow i(\xi_a) = t_a \tag{3.2}$$

where the $\{\xi_a\}$ vectors form the Lie algebra

$$[\xi_a, \xi_b] = k^{-1} f^c_{ab}\xi_c \quad a, b, c = 1...n \tag{3.2a}$$

k^{-1} is a scale factor and f^c_{ab} are the structure constants of $\mathcal{G}$. It is easy to show that a local coordinate transformation of the internal coordinates in the direction of the Killing vector, i.e.

$$y^n \rightarrow y^n + \xi^n_a(y)\epsilon^a(x)$$

the gauge potentials transform like Yang Mills fields

$$B^a_\mu \to B^a_\mu + f^a_{bc} A^b_\mu \epsilon^c + \partial_\mu \epsilon^a \qquad (3.3b).$$

The symmetry of the whole space means that the Lie derivative with respect to the vectors $\{\xi_a\}$ of the metric $\hat{g}$ vanishes:

$$\mathcal{L}_{\xi_a} \hat{g} = 0 \qquad (3.4)$$

From this, the internal dependence of the fields [8]

$$\partial_a g_{\mu\nu} = 0 \quad \partial_a B^b_\mu = -k^{-1} f^b_{ac} B^c_\mu$$

$$\partial_a \tilde{g}_{bc} = k^{-1} f^d_{ab} \tilde{g}_{dc} + k^{-1} f^d_{ac} \tilde{g}_{bd} \qquad (3.5)$$

holds.

The field equations of the unification are derived from the d-dimensional Einstein-Hilbert action

$$I_d = \frac{1}{16\pi G_k} \int \sqrt{-\hat{g}_d}(\hat{R} + \Lambda) d^d x \qquad (3.6)$$

where $\hat{R}$ is the curvature scalar of the d-dimensional space, $\hat{g}_d$ is the determinant of the $\hat{g}$ matrix coefficients, Λ is the d-dimensional cosmological constant and G_k is the gravitational constant.

The next step is to obtain an explicit expression for (3.6) using the metric given by (3.1) . In order to do so we write the components of (3.1) in matrix notation

$$\hat{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + e^2 k^2 \tilde{g}_{ij} B^i_\mu B^j_\nu & ek B^i_\mu \tilde{g}_{ij} \\ ek B^i_\nu \tilde{g}_{ij} & \tilde{g}_{ij} \end{pmatrix} \qquad (3.7)$$

with inverse

$$\hat{g}^{AB} = \begin{pmatrix} g^{\mu\nu} & -keg^{\mu\nu} B^j_\mu \\ -keg^{\mu\nu} B^i_\nu & \tilde{g}^{ij} + e^2 k^2 g^{\mu\nu} B^i_\mu B^j_\nu \end{pmatrix}$$

being $g^{\mu\nu}$ and $\tilde{g}^{ij}$ the inverse of $g_{\mu\nu}$ and $\tilde{g}_{ij}$ respectively.

With this expressions we can calculate the curvature scalar $\hat{R}$ of the whole space. We arrive at [8],[45]

$$\hat{R} = R + \tilde{R} + \frac{e^2 k^2}{4} \tilde{g}_{ab} B^a_{\mu\nu} B^{b\mu\nu} + \frac{1}{4} \tilde{g}^{ab} \tilde{g}^{cd} [(D_\mu \tilde{g}_{ac})(D^\mu g_{bd}) - (D_\mu \tilde{g}_{ab})(D^\mu \tilde{g}_{cd})]$$
$$+ \nabla_\mu (\tilde{g}^{ab} D^\mu \tilde{g}_{ab}) \qquad (3.8)$$

where ∇_μ is the covariant derivative defined in B^4, the spacetime; $B^a_{\mu\nu}$ is the field strength of the Yang Mills potentials

$$B^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + e f^a_{bc} B^b_\mu B^c_\nu \qquad (3.9)$$

R and $\tilde{R}$ are the curvature scalars of B^4 and G, respectively and D_μ is the gauge covariant derivative

$$D_\mu = \partial_\mu - ek B^a_\mu \partial_a \qquad (3.10).$$

So the action (3.5) can be now written down in the following form its

$$I = \frac{1}{16\pi G_k} \int \sqrt{-g_4} \sqrt{g_G} \{ R + \tilde{R} + \frac{1}{4} e^2 k^2 \tilde{g}_{ab} B^a_{\mu\nu} B^{b\mu\nu} +$$

$$+\frac{1}{4}\widetilde{g}^{ab}\widetilde{g}^{cd}[(D_\mu\widetilde{g}_{ac})(D^\mu\widetilde{g}_{bd})-(D_\mu\widetilde{g}_{ab})(D^\mu\widetilde{g}_{cd})]+\Lambda\} \quad (3.11)$$

up to a total divergence. The unification means then that [10]

$$e^2k^2/16\pi G_k = 1 \quad (3.12)$$

and we have again a unified Lagrangian of gravitation, Yang Mills, and scalar interactions, with cosmological constant.

In this approach we can derive the field equations from the action (3.11) or directly from the d-dimensional Einstein equations. In general we can add a energy momentum tensor and start from

$$\hat{R}_{AB}-(\frac{1}{2}\hat{R}+\Lambda)\hat{g}_{AB} = -8\pi G_k T_{AB} \quad A,B=1...d. \quad (3.13)$$

the result (up to the T_{AB} tensor) will be the same.

The field equations (3.13) must be compatible with the vacuum or Minkowski metric and the group G. Let us suppose that the 4-dimensional Ricci tensor vanishes, and $B^a_\mu = 0$. It follows that [46] $\partial_\mu\widetilde{g}_{ab} = 0$. Then

$$\hat{g}_{\mu\nu} = g_{\mu\nu} \quad , \hat{R}_{\mu\nu} = 0, \quad -\frac{1}{2}(\hat{R}+\Lambda)g_{\mu\nu} = -8\pi G_k T_{\mu\nu}$$

which implies

$$T_{\mu\nu} = \frac{1}{4}T_4 g_{\mu\nu} \quad (3.14)$$

On the other hand $\hat{R}_{ab} = \widetilde{R}_{ab}$, because of the last conditions, and thus

$$\widetilde{R}_{ab} = -8\pi G_k(T_{ab}-\frac{1}{4}T_4\widetilde{g}_{ab}) \quad (3.15)$$

from which the expression for the cosmological constant

$$\Lambda = 8\pi G_k(\widetilde{T}-\frac{1}{4}(n-2)T_4) \quad (3.16)$$

holds. Equation (3.14), (3.15) and (3.16) are the flatness conditions. An interesting case is when the internal space G is an Einstein space, i.e. $R_{ab} = c\widetilde{g}_{ab}$. It follows that [47]

$$T_{ab} = -(\frac{c}{8\pi G_k}+\frac{1}{4}T_4)\widetilde{g}_{ab} \quad (3.17)$$

i.e., T_{ab} is determined by the four dimensional part of the energy momentum tensor and the internal metric. We will come back to this point in part V.

We want now to show some of the results of this theory. In order to do so, we follow the work of [43], who defines the inner metric as

$$\widetilde{g}_{ab} = \phi^{1/n}\rho_{ab}$$

in order to have det $\widetilde{g}_{ab} = \phi$ for an unimodular G group. If we make this transformation, the unified lagrangian transforms to

$$Lo = -\frac{1}{16\pi G_k}\sqrt{g_4}\sqrt{\phi}[R+\widetilde{R}+4\pi G_k\phi^{1/n}\rho_{ab}B^a_{\mu\nu}B^{b\mu\nu}$$

$$-\frac{n-1}{4n}\frac{(\partial_\mu\phi)^2}{\phi^2}$$

$$+\frac{1}{4}\rho^{ab}\rho^{cd}(D_\mu\rho_{ac})(D_\mu\rho_{db})+\Lambda+\lambda(|det\rho_{ab}|-1)] \quad (3.18)$$

where λ is a Lagrangian multiplier. Nevertheless the volume element should be $\sqrt{-g_4}$ and not $\sqrt{-g_4\varphi}$ as in (3.18). Furthermore the φ field appears with a negative kinetic energy. We can remove these defects by performing the conformal transformation (see also eq.(7))

$$g_{\mu\nu} \rightarrow \sqrt{\varphi}g_{\mu\nu} \tag{3.19}$$

Considering this transformation the Lagrangian (3.18) becomes

$$Lo = -\frac{\sqrt{-g_4}}{16\pi G_k}[R + 4\pi G_k e^{-c\sigma}\rho_{ab}B^a_{\mu\nu}B^{b\mu\nu} + \frac{1}{2}(\partial_\mu\sigma)^2$$

$$+\tilde{R}e^{+c\sigma} + \Lambda C^{\frac{1}{c}\sigma} - \frac{1}{4}(D_\mu\rho^{ab})(D^\mu\rho_{ab}) + \lambda(|\bar{d}et\rho| - 1) \tag{3.20}$$

where σ is the dilaton field defined by

$$\sigma = \frac{1}{2}[\frac{n+2}{n}]^{1/2}ln\phi = -\frac{1}{2}cln\varphi \tag{3.21}$$

Observe that the couplings constant of the gauge fields is $G_k = G_k e^{-c\sigma}$.

Now let us briefly study the cosmology of this theory. We start from the field equations (3.13) and the natural ansatz (see ref.[44])

$$T_{\mu\nu} = e^{-c\sigma}[(\rho + p)u_\mu u_\nu + pg_{\mu\nu}]$$

$$T_{ab} = \sqrt{\phi}e^{-c\sigma}p'\tilde{g}_{ab} \qquad T_{\mu a} = 0 \tag{3.22}$$

which corresponds to a perfect fluid energy-momentum tensor in d-dimensions (here we have used the transformation (3.19)). One chooses $g_{\mu\nu}$ as the Robertson-Walker form, with ρ_{ab} time-independent and $\sigma = \sigma(t)$. We substitute all of this in (3.13) and obtain

$$H^2 + \frac{k}{r^2} = \frac{1}{6}(\frac{1}{2}\dot{\sigma}^2 + V) + \frac{8\pi G_k}{3}\rho e^{-c\sigma}$$

$$\ddot{\sigma} + 3H\dot{\sigma} + \frac{dV}{d\sigma} = 8\pi G_k b(\rho - 3p + 2p')e^{-c\sigma}$$

$$\dot{\rho} + 3H(\rho + p) + \dot{\sigma}[\frac{b}{2}(\rho - 3p + 2p') - c\rho] = 0 \tag{3.23}$$

$$V = \tilde{R}e^{-a\sigma} + \Lambda e^{-b\sigma}$$

being $a = \sqrt{(n+2)/n}$, $b = \sqrt{n/(n+2)}$. a, b and c will be taken as free parameters for the moment. H is the Hubble field defined in standard from $H = \dot{r}/r$. Equations (3.23) corresponds to the standard cosmology with $\sigma = 0$ and cosmological constant $\tilde{R} + \Lambda$. These equations have two symmetries. They are invariant under the transformation

a)

$$a \leftrightarrow b, R \leftrightarrow \Lambda$$

and under the scaling

b)

$$\sigma \rightarrow \sigma + \sigma_o, \quad \tilde{R} \rightarrow \tilde{R}e^{a\sigma_o}, \quad \Lambda \rightarrow \Lambda e^{b\sigma_o}, \quad G_k \rightarrow G_k e^{c\sigma_o}.$$

An exhaustive study of equations (3.23) and their interpretation is made in ref. [44] and plotted in fig. 6. Solutions with and without big bang or horizon are shown, for an universe expanding or recontracting. In other words, with this theory it is possible to solve the horizont problem of standard cosmology, because there are some solutions without it. Also the missing mass problem could be solved, because equations (3.23) show that there exist a contribution of the σ field to the density of the Universe. To see this point, we observe that if we identify the density and the presure of the σ field as

$$\rho_\sigma = \frac{1}{16\pi G_k}(\frac{1}{2}\dot\sigma^2 + V) \quad p_\sigma = \frac{1}{16\pi G_k}(\frac{1}{2}\dot\sigma^2 - V)$$

and substitute that in (3.23), we arrive at

a)

$$H^2 + \frac{k}{r^2} = \frac{8\pi G_k}{3}(\rho_\sigma + \rho e^{-c\sigma})$$

b)

$$\dot\rho_\sigma + 3H(\rho_\sigma + p_\sigma) + (\dot\rho + 3H(\rho + p))e^{-c\sigma} = c\dot\sigma\rho e^{-c\sigma} \tag{3.24}$$

The first equation shows how ρ_σ contributes to the whole density of the universe, i.e. the existence of the σ field explain the stationarity of the galaxies. The second equation describes the conservation of the energy-momentum tensor (see ref. [44]).

All these results make clear (together with the dependence of G_N on σ, the coupling constant of the gauge fields), that the dilaton or scalar field plays a very important role in this theory.

In order to study it in the context of axisymmetric or spherically symmetric solutions we suppose that the four-metric contains two commuting Killing vectors, one of them time like and the other space like. Furthermore we suppose that the only field acting in the whole space is electromagnetism. In such a case we have n + 2 commutating Killing vectors and the metric can be cast in the form [48],

$$dS^2 = f(\rho,\zeta)(d\rho^2 + d\zeta^2) + g_{ij}(\rho,\zeta)dx^i dx^j \tag{3.25}$$

$$i, j = 3...\mathrm{d}$$

Whith this metric, the Einstein equations (3.13) can be written as [49]

a)

$$(lnf)_{,z} = \frac{1}{(ln\rho)_{,z}}[(ln\rho)_{,zz} + \frac{1}{4}tr(g_{,z}g^{-1})^2]$$

b)

$$(\rho g_{,z}g^{-1})_{,\bar z} + (\rho g_{,\bar z}g^{-1})_{,z} = 0$$

$$z = \rho + i\zeta \tag{3.26}$$

where the $(n+2)x(n+2)$ matrix $(g)_{ij} = g_{ij}$. If we want to solve equations (3.26) we need first to find a solution of the chiral equations b) in order to solve a). There are some techniques for doing so: the solitonic (inverse scatering) method for finding exact solutions of (3.26b) developed in refs. [48] and [50]; the "sub-space ansatz" consisting in to parametrizing the g matrix as

$$g = g(\lambda^i) \quad \lambda^i = \lambda^i(z,\bar z) \tag{3.27}$$

is developed in general in ref. [51] and for the one and two subspaces in refs. [53] and [54], (see also [52]).

We want shortly to outline the one dimensional subspace ansatz.

Let us take the ansatz

$$g = g(\lambda) \quad \lambda = \lambda(z,\bar z) \tag{3.28}$$

where λ fulfills the Laplace equation

$$(\rho\lambda_{,\bar z})_{,z} + (\rho\lambda_{,z})_{,\bar z} = 0 \tag{3.29}$$

The chiral equations reduce to

$$g_{,\lambda} = Ag \tag{3.30}$$

...eing A a constant matrix. (see also equation (2.15)). In this case the integration of the function f in (3.25) is determined by λ only.

$$(ln\rho^{1-\frac{1}{N}}f)_{,z} = b\rho(\lambda_{,z})^2, \quad b = \frac{1}{2}trA^2. \tag{3.31}$$

The integrability conditions of this last equation follows from the chiral equations [53]. The solution of the matrix equation (3.30) depends on the classification of the matrix A. A classification under eigenvectors is given in ref. [53] and shown in table 2. We get magnetic monopoles with gravitational potentials like the Newtonian potential, dipoles, monopoles and dipoles etc. Let us give one example [55]

$$g = A(r)dt^2 - B(r)(dr^2 + r^2 d\Omega^2)$$
$$\rho = \sqrt{r^2 - 2mr}$$
$$z = (r - m)cos\theta$$

with the electromagnetic potential given by

$$A_\mu = (0, 0, 0, I(r)), \quad \widetilde{g}_{ab} = \phi^{1/3}diag(h^{-1}, h^{-1}, h^2)$$

and

$$A(r) = \left(\frac{r-m}{r+m}\right)^\alpha, \quad \phi(r) = \left(\frac{r-m}{r+m}\right)^\beta, \quad h(r) = \left(\frac{r-m}{r+m}\right)^\gamma. \tag{3.32}$$

If we calculate, like in part II, the inertial mass from (2.18) we obtain [55]

$$m_I = \alpha\frac{m}{G} \tag{3.33}$$

But now remember that we have taken $\sqrt{\phi}g_{\mu\nu}$ as the physical quantities. In this case, from the asymptotical behavior of g_{44}, we get the gravitational mass to be

$$m_g = \frac{(2\alpha - \beta)m}{2G} \tag{3.34}$$

This solution allows us to take $\beta = 0$ (that is no alweys so), in which case the gravitational and inertial mass coincide. But then the scalar field (dilaton) vanishes.

If we accept that particles move in geodesic in the d-space, we can proced like in the first part (equation (1.10)) to obtain the geodesic equation in four dimensions. If λ is a dimensional geodesic parameter and S the four- dimensional one, (but now using $\sqrt{\phi}g_{\mu\nu}$ as the physical quantity) we obtain [55]

$$\frac{dp^\mu}{dS} + \Gamma^\mu_{\alpha\beta}p^\alpha p^\beta = \sqrt{16\pi G_k}\frac{\sqrt{\phi}}{\chi}g^{\mu\alpha}q_a B^a_{\alpha\beta}p^\beta$$
$$+\frac{1}{2}(\epsilon_4 g^{\mu\alpha} + p^\mu p^\alpha)(\frac{\partial_\alpha\phi}{\phi} - 2\frac{\partial_\alpha\chi}{\chi})$$

with

$$dS = \chi d\lambda \quad , \chi = (\frac{\sqrt{\phi}(\epsilon_d + \widetilde{g}^{ab}q_a q_b}{\epsilon_4}))^{\frac{1}{2}}$$
$$p^\mu = \frac{dx^\mu}{dS} \quad , q^A = \frac{dx^A}{d\lambda} \tag{3.35}$$

where now we have to take as effective mass

$$m_{eff} = \frac{1}{\sqrt{16\pi G_k}}(\frac{\epsilon_d + \widetilde{g}^{ab}q_a q_b}{\epsilon_a\sqrt{\phi}})^{\frac{1}{2}}$$

The presence of $\sqrt{\varphi}$ in the effective mass is because of the transformation $g_{\mu\nu} \rightarrow \sqrt{\phi} g_{\mu\nu}$. ϵ_α is = -1, 0, 1 for the d-dimensional geodesic and ϵ_4 is the same one for the four-dimensional geodesic. Observe that even when $\epsilon_d = -1$(space like geodesic) the effective mass can be associated to the mass of an ordinary particle.

If we substitute the solution (3.32) in the geodesic equation (3.35) we observe that the scalar potential takes an important part in the gravitational interaction. In the newtonian limit (3.35) reduces to

$$\frac{d^2x^i}{dS^2} = g^{ij}\partial_j(\frac{1}{2}\varphi^{-1/2}g_{44}) = g^{ij}\partial_j(\frac{-(2\alpha-\beta)m}{2r})$$

which means that the dilaton interacts with a repulsive or attractive force with the gravitational potential depending whether the dilatonic charge β is negative or positive [55].

It is now clear that the dilaton plays a very important role in the KK theory and its existence could decide whether the KK theory could be taken as a realistic theory or not.

IV Geometrical Formulation of KK

In this section we want to show how the new n-dimensional KK theory is the unification of three theories: the mathematical theory of fiber bundles; gauge theory and the old KK theory.

Let us start by showing the analogy between General Relativity with the standard gauge theory.

In general relativity one formulates the theory using geometrical principles. The interactions between particles or fields, i.e. between matter is because of the curvature provoke by them in the spacetime. The curvature of spacetime determines how matter interacts. On the other hand, interactions in gauge field theory is understood as exchanging of virtual particles. Interaction, fundamental escences in the formulation of physics, is rather diferent in the two theories of the 20th Century. How can we make both theories compatible?

We know that in a geometrical formulation the curvature and the covariant derivative play an important role. In general relativity, one starts with a metric and determines the Christoffel symbols. These are in fact the affine connection in the spacetime where the concept of force makes contact with the newtonian theory. In other words: the knowlege of the connection is the fundamental point in the formulation of Einstein's theory. We could start with the Lagrangian formulation in four dimensions (3.6) and ask for the connection which makes this Lagrangian extrem without torsion. The result is the Einstein theory of relativity.

On the other hand gauge theory is constructed with a fundamental piece: the minimal coupling principle, consisting in substituting the momentum p_μ with

$$p_\mu \rightarrow p_\mu + eA_\mu^a t_a \tag{4.1}$$

where now the A_μ^a are the Yang-Mills gauge potentials. In a coordinate representation, one changes the partial derivative by the covariant derivative D_μ. But with this we are defining a connection in coordinate space and with it we can formulate the geometrization of the gauge fields. The curvature is also defined in the same way as in general relativity (see eg. [12])

$$[D_\mu, D_\nu] = -eB_{\mu\nu} \tag{4.2}$$

where $B_{\mu\nu}$ is given in (3.9). The difference here is that we have no metric and the connection B_μ^a diffieres from the Christoffel symbols

Let us now comment about Fiber bundles. A fiber bundle is a mathematical structure which generalized the concept of cartesian product between sets. For example, a cylinder is the cartesian product of the circle S^1, and [a,b], a closed set of the line (fig. 7) but the Möbius strip is not a cartesian product of S^1 and $[a, b]$, only locally, i.e. it takes two rotations of the circle (fig. 8). We can define a projection π from the fiber bundle (from the cylinder or Möbius strip for instance) to the base set, in the above example, for the circle (fig. 9):

$$\pi : P \rightarrow S^1 \tag{4.3}$$

$$p \rightarrow \pi(p) = (x, y)$$

taking each point of P to a point of S^1. In the case of the cylinder this projection can be defined as the "first projection"

$$\pi_1(x,y,z) = (x,y) \quad , x^2+y^2 = 1 \quad a \leq z \leq b$$

but in the second example the projection π_1 can be taken only localy, for each loop.

The bundle structure in which we are interested is a fiber bundle endowed with a connection. In the two examples above we can project the tangent vectors of P into the base space. Of course, a vertical vector has zero projection in the circle. Therefore the vertical vectors are well defined throught the projection: they are the vectors with zero projection in the base set (see fig. 9) But the horizontal vector can be defined in many ways, they are actually free. To define a connection in a fiber bundle is to define the horizontal vectors in it which remain invariant over the set $\pi^{-1}(x) = F_x$. The set $\pi^{-1}(x) = F_x = \{p\epsilon P|\pi(p) = x\}$ is called the fiber, and note that in the examples above it is always the same for all $x : F_x \simeq [a,b]$. Finally, principal fiber bundles are fiber bundles which fiber is a Lie group G and it is defined a product between points in P and elements of G. The product between them is called a right action of G on P

$$R : P \times G \longrightarrow P$$

$$(p,a) \longrightarrow R(p,a) = p_1$$

such that

$$a) \quad R(p,e) = p$$

$$b) \quad R(R(p,a),b) = R(p,ab) \tag{4.4}$$

The theory of fiber bundles is presented in many books references. For example well-know principal fiber bundles are the Hoff fibering. They are interesting because they are made of spheres, for instance:

$$\begin{array}{c} S^1 \\ | \\ S^3 \\ |^{\pi} \\ S^2 \end{array} \tag{4.5}$$

In a principal fiber bundle the connection defines a one-form ω in P with valaues in the corresponding Lie algebra $\mathcal{G}$ of G. That can be done in the following way: for each horizontal vector the one form relates it with the zero vector of the algebra. Because there exists a one to one relation between the vectors of the tangent space of G and the Lie algebra $\mathcal{G}$ of G, the one-form relates to each vertical vector on P a vector of $\mathcal{G}$. For example if G=SU(2) in one point the relation between the tangent space of G and the Lie algebra $\mathcal{G}$ could be given by

$$z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$x\frac{\partial}{\partial z} - z\frac{\partial}{\partial z} \longrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$y\frac{\partial}{\partial y} - x\frac{\partial}{\partial y} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.6}$$

and the one form ω is defined with relation (4.6) plus the association of the zero matrix to each horizontal vector of P, in each point p of P. In general the association varies from point to point, because each linear combination of (4.6) is equally good as an other. The main point is the projection of this one form ω in the base space. In order to do so we take a cross section σ, that is, a local function from the base set to the fiber bundle P such that the projetion from P comes back to the original point (see fig. 10). σ defines obviously a "high" in the bundle. An other σ_1 would define an other high (see fig. 10). With σ we can project (because it determines a high) the connection one form into the base set

$$A = \sigma^{*}\omega \tag{4.7}$$

If we take an other σ, say σ_1 we had $A^{'} = \sigma_1^{*}\omega$, but the relation between A and $A^{'}$ is [56], [59]

$$A^{'} = aAa^{-1} + ada^{-1} \tag{4.8}$$

with a the transition elements of G. But this is the well-known relation (3.3). For example if $G = U(1)$, topologically $U(1) = S^1$, an element of S^1 can be written as $a = e^{i\phi}$, then

$$A^{'} = e^{i\phi}A\bar{e}^{i\phi} - e^{i\phi}de^{-i\phi} = A + d\phi. \tag{4.9}$$

In components

$$A^{'}_{\mu} = A_{\mu} + \partial_{\mu}\phi \tag{4.10}$$

which are just the gauge transformations of electromagnetism. Furthermore we can write the one form of connection of P in the form [56]

$$\omega = a^{-1}Aa + a^{-1}da \tag{4.11}$$

of course, under the right action of the group $a \to a' = ab$, ω remains invariant $\omega = a'^{-1}Aa' + a'^{-1}da'$. The curvature is defined as

$$\Omega = d\omega + \omega \wedge \omega = a^{-1}Ba$$

being

$$B = dA + A \wedge A = \frac{1}{2}B^{a}_{\mu\nu}t_a dx^{\mu} \wedge dx^{\nu} \tag{4.12}$$

Ω obeys the Bianchi identity

$$d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0. \tag{4.13}$$

Let us return to the example (4.5). Here the base space is S^2, the sphere. We cover S^2 with two recubriments and write the 1-form of connection in each half of the bundle as

$$A_{+} = A_{-} + d\phi$$

One finds that a gauge potential satisfying the Maxwell's equations is

$$A_{\pm} = \frac{1}{2}(\pm 1 - cos\theta)d\phi$$

which is just the Dirac monopole. The curvature is given by

$$F = dA_{\pm} = \frac{1}{2}sin\theta d\theta \wedge d\phi \tag{4.14}$$

which is just the corresponding strengh tensor.

Another example is the instanton, which is a connection of the fiber bundle [56]

$$SU(2)$$

$$|P$$

$$\downarrow^{\pi}$$
$$S^4 \tag{4.15}$$

So far we have done mathematics. On the other hand the Yang-Mills theory in the formalism of principal fiber bundle over the Minkowski space is well-known so as the theory of general relativity with fiber the group $O(3,1)$ [60]. Wath about a Yang-Mills theory over an arbitrary four dimensional riemannian space?

Let us take a principal fiber bundle P with connection whose fiber is a paracompact group G, and base a four-dimensional riemannian space. These assumptions define a metric in P because the connection separates the vectors on P in their vertical and horizontal parts, so the metric in P can be defined as (see eg. [8] and [42])

$$\begin{aligned} \hat{g}(U_v, V_v) &= \tilde{g}(U_v, V_v) \\ \hat{g}(U_H, V_v) &= 0 \\ \hat{g}(U_H, V_H) &= g(d\pi(U_H), d\pi(V_H)) \end{aligned} \tag{4.16}$$

where $\tilde{g}$, g and $\hat{g}$ are the metrics on G, the base space B^4 and P, respectivelly. If $\{\hat{\omega}^A\}$ is a base of the one-forms defined over P, the metric (4.16) can be written as

$$\hat{g} = g_{\alpha\beta}\omega^{\alpha} \otimes \hat{\omega}^{\beta} + I_{ab}\hat{\omega}^{a} \otimes \hat{\omega}^{b}, \tag{4.17}$$

which is defined in all P. In what follows we will write $\hat{g}$ in local coordinates. P is a fiber bundle, this means that it is locally a cartesian product of an open set U of B^4 and G, the fiber, i.e., there exist an homomorphism ϕ called trivialization from P to $U\times G$, $\phi : P \to U \times G$ (see fig. 11). As before the vertical space in P is well defined because the projection π of these vector is zero. Let $\{\hat{e}_a\}$ be a base of the vertical space and $\{\hat{e}_\alpha\}$ a base of the complement, the horizontal one. Of course the projection of the horizontal vectors is non zero, furthermore their form a base $\{e_\alpha\}$ of the tangent space of U, ie

$$\begin{aligned} d\pi(\hat{e}_\alpha) &= e_\alpha \\ d\pi(\hat{e}_a) &= 0 \end{aligned} \tag{4.18}$$

Now we project the vectors $\{\hat{e}_\alpha, \hat{e}_a\}$ to the tangent space $U \times G$ throught the trivialization. Observe that the projection from $U \times G$ into U, is the canonical projection $\pi_1 : U \times G \to U, (x, a) \to x$ in such a form that

$$\pi = \pi_1 \circ \phi \tag{4.19}$$

Let be the projection of $\{\hat{e}_\alpha, \hat{e}_a\}$ into $T(U \times G)$

$$\begin{aligned} d\phi(\hat{e}_\alpha) &= B^{\beta}_{\alpha} e_\alpha - A^{m}_{\alpha} e_m \\ d\phi(\hat{e}_a) &= C^{\beta}_{a} e_\beta + D^{m}_{a} e_m \end{aligned} \tag{4.20}$$

where $\{e_m\}$ is a right invariant basis of the tangent space of G, such that $\{e_\alpha, e_m\}$ is a base of the tangent space of $U \times G$. But from (4.19) we have

$$\begin{aligned} d\pi(\hat{e}_\alpha) &= d\pi_1 \circ d\phi(\hat{e}_\alpha) = B^{\beta}_{\alpha} e_\beta = e_\alpha \\ d\phi(\hat{e}_a) &= d\pi_1 \circ d\phi(\hat{e}_a) = C^{\beta}_{a} e_\beta = 0 \end{aligned} \tag{4.21}$$

i.e. $B^{\beta}_{\alpha} = \delta^{\beta}_{\alpha}$ and $C^{\beta}_{a} = 0$. The set $d\phi(\hat{e}_a) = D^{m}_{a} e_m$ is again a basis of the tangent space of G and we can rewrite them as $D^{m}_{a} e_m \to e_a$. So we have

$$\begin{aligned} d\phi(\hat{e}_\alpha) &= e_\alpha - A^{m}_{\alpha} e_m \\ d\phi(\hat{e}_a) &= e_a. \end{aligned} \tag{4.22}$$

It is easy to find the dual basis of (4.22). we arrive at

$$\bar{e}_A = \begin{cases} e_\alpha - A^{m}_{\alpha} e_m \\ e_m \end{cases}$$

$$\hat{\omega}^A = \begin{cases} \omega^\alpha \\ \omega^a + A^a_\alpha \omega^\alpha \end{cases} \tag{4.23}$$

where $\{\omega^\beta\}$ is the dual of $\{e_\alpha\}$ and $\{\omega^m\}$ is the dual of $\{e_m\}$. With this basis we can write the metric $\bar{g}$ in the trivialization,

$$\bar{g} = g_{\alpha\beta}\omega^\alpha \otimes \omega^\beta + I_{nm}(\omega^n + A^n_\alpha\omega^\alpha) \otimes (\omega^m + A^m_\beta\omega^\beta) \tag{4.24}$$

of course this is the metric (3.1) if we write

$$\omega^\alpha = dx^\alpha$$

and

$$A^a_\alpha = ekB^a_\alpha, \qquad I_{ab} = \tilde{g}_{ab} \tag{4.25}$$

i.e. if we write (4.23) in a coordinate basis. To obtain $\hat{g}$ we take the pull-back of ϕ observing that the pull-back is in this case

$$\phi^*(\Omega) = \Omega_\alpha\hat{\omega}^\alpha + \Omega_n(\hat{\omega}^n - A^n_\beta\hat{\omega}^\beta)|_{\phi^{-1}} \tag{4.26}$$

The pull-back of the cotangent basis of $U \times G$ is

$$\phi^*(\omega^\alpha) = \hat{\omega}^\alpha \quad , \; \phi^*(\omega^a + A^a_\alpha\omega^\alpha) = \hat{\omega}^a \tag{4.27}$$

so for the pull-back of the metric we obtain

$$\hat{g} = \phi^*\bar{g} = g_{\alpha\beta}\hat{\omega}^\alpha \otimes \hat{\omega}^\beta + I_{mn}\hat{\omega}^n \otimes \hat{\omega}^m|_{\phi^{-1}} \tag{4.28}$$

i.e. (4.17). It is clear that $g = g_{\alpha\beta}\omega^\alpha \otimes \omega^\beta$ is the space-time metric i.e.,the metric on B^4 and $\tilde{g} = I_{nm}\omega^n \otimes \omega^n$ the metric in G.

Finally we want to show that $A^a_\beta\omega^\beta t_a$ is the connection component projected into B^4 of the bundle. Remember that the one-form of connection in P is a one form ω which assigns zero to the horizontal vectors and an element of the corresponding Lie algebra to the vertical vectors. We can write ω as:

$$\omega = \hat{\omega}^a t_a \tag{4.29}$$

It fulfills these condition, because

$$\omega(\hat{e}_\alpha) = \hat{\omega}^a(\hat{e}_\alpha)t_a = 0$$
$$\omega(\hat{e}_b) = \hat{\omega}^a(\hat{e}_b)ta = \delta^a_b t_a = t_b.$$

Let us now project it into $U \subset B^4$. For doing so we define a local cross section using the trivialization ϕ

$$S = \phi^{-1} \circ \bar{Id} : U - P \tag{4.30}$$

where $\bar{Id}$ is a identity function defined as $\bar{Id} : U \longrightarrow U \times G, x - (x, e)$ e being the identity in G. Then the pull-back of S applied to ω is given by (see equation(4.27))

$$\begin{aligned} S^*(\omega) &= (\bar{Id}^* \circ \phi^{-1*})(\hat{\omega}^a t_a) \\ &= \bar{Id}^*((\omega^a + A^a_\beta\omega^\beta)t_a) = A^a_\beta\omega^\beta t_a \end{aligned} \tag{4.31}$$

that is $A = A^a_\beta\omega^\beta t_a$ is the projection of the one-form of connection to U and is the Yang-Mills-gauge potential

We have shown that the KK ansatz **(3.1)** is actually not an ansatz but **the metric of the natural generalization of gauge theory.** Some remarks must be done:

1) The decomposition (4.16) or (4.17) can be only done if $\tilde{g}$ is right invariant on the group G. Nevertheless we can take as fiber the quotient set G/H where H is a normal group of G (see [18] and [45]). For

example if $G = SU(3) \times SU(2) \times U(1)$, the maximal normal group is $H = SU(2) \times U(1) \times U(1)$, so as dim G = 12 and dim H = 5, we have dim G/H = 7. We conclude that the minimal dimension of P in order to have the metric g invariant under $SU(3) \times SU(2) \times U(1)$ is 7 + 4 = 11, i.e. the maximal dimension of supergravity [61][18].

2) There is no way to decide which group G or G/H must be taken. Perhaps a quantum mechanics treatment (QM) could do so.

3) There is no satisfactory quantum mechanics treatment of general relativity and of KK theory either. This is a classical theory of fields but in order to have interactions of all kind we have to work with the Planck scale, but then an QM treatment becomes important. We think that at this level the structure of the elemental particles take an important rolle. Geodesics suppose particles as points. We should suppose that particles have indeed structure at this level being the geodesic motion only an aproximation of the real one. One could suppose that partilcles are strings insteat of points. Then a treatment of string (i.e. that they move over minimal surfaces) instead of geodesic motion could be better for the understanding of the microworld.

V The method of spontaneus compactification

We have seen how by considering the spacetime to be of the form $M^4 \times G$ the Kaluza-Klein formalism elegantly unites the Gravitational and Yang-Mills theories in one framework. In this chapter, by using the method called "Spontaneus Compactification" [62] we will now adress the problem of how a $d-$manifold breaks into a 4−manifold and small compactified (d-4)-manifold. In other words, here we are interested to understand how the transition

$$M^d - M^4 \times G$$

takes place.

The purpose of this chapter, however, is not to give a complete review of the subject, but rather to mention their main features. For this reason instead to consider a general dimensionality d most of the time we will consider d=11. We will show that in such dimensionality the method of Spontaneous Compactification leads naturaly to consider Supergravity [63] and Superstring theories [41].

Let us start with the Einstein-Hilbert action in d dimentions;

$$S = \frac{1}{4\pi G_k} \int d^dx\{\sqrt{-g}R + other \quad fields\} \tag{5.1}$$

The field equations obtained from this action are

$$R_{MN} - \frac{1}{2}g_{MN}R = 8\pi G T_{MN} \tag{5.2}$$

and

$$other\ field\ equation = 0 \tag{5,3}$$

Here T_{MN} is the energy momentum tensor in d-dimentions due to matter fields; scalar, external Yang-Mills fields and other fields.

The central idea in the method of Spontaneus Compactification is to look for solutions of the field equations (5.2) and (5.3) wich allow the metric g_{MN} to be written in the form

$$g_{MN}(x,y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix} \tag{5.4}$$

where the metrics $g_{\mu\nu}$ and g_{mn} satisfy the reduced field equations

$$R_{\mu\nu} = C_1 g_{\mu\nu} \qquad C_1 \leq 0 \tag{5.5}$$

$$R_{mn} = C_2 g_{mn} \qquad C_2 > 0 \tag{5.6}$$

respectively. Here C_1 and C_2 are constants. With $C_2 > 0$ assure that the internal space is compact [64] , and with $C_1 < 0$, we expected that the four dimensional space-time satisfy the Positive Energy Theorem [65]. If we assume that all the matter fields vanish then the field equations (5.2) and (5.3) reduce to the field equation

$$\hat{R}_{MN} = 0 \tag{5.7}$$

Therefore, in this case $C_1 = 0$ and $C_2 = 0$. The vanishing of C_1 is fine, but a vanishing of C_2 does not agree with the field equation (5.6). So, pure gravity with zero energy-momentum tensor seems to be not very satisfactory. We can still try the case

$$\hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} + \Lambda \hat{g}_{MN} = 0 \tag{5.8}$$

where Λ is a cosmological constant term. This equations imply

$$R_{MN} = \frac{2\Lambda}{d-2}\hat{g}_{MN} \tag{5.9}$$

and hence either

$$C_1 > 0 \qquad C_2 > 0$$

or

$$C_1 < 0 \qquad C_2 < 0$$

So, both conditions are not in agreement with the field equations (5.5) and (5.6). Thus, we conclude, like many others [66], that in Kaluza-Klein theory it is necesary to have gravity plus "matter" fields. The natural question is what kind of "matter" fields. We could cosider as a matter fields, for instance, scalar fields Φ or external Yang-Mills fields A^a_M. However, the completely antisymmetric gange field A_{MNP} provides the simplest and more interesting object to produce spontaneous compactification.

Let us first write the fields equations associated to the metric $\hat{g}_{MN}$ and the gauge field A_{MNP};

$$R_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} = \frac{(1}{6}F_{MABC}F_N^{ABC} - \frac{1}{48}g_{MN}F^{ABC}F_{ABCD}) \tag{5.10}$$

$$F^{MNPQ}{}_{;Q} = 0 \tag{5.11}$$

where F_{MNPQ} is the curl of the gange field A_{MNP}, that is,

$$F_{MNPQ} = \partial_{[M}A_{NPQ]} \tag{5.12}$$

Let us now to show that the following solution of the field equation (5.11),

$$F^{\mu\nu\alpha\beta} = F_4\epsilon^{\mu\nu\alpha\beta},\ F_4 = const \tag{5.13}$$

$$= 0 \qquad otherwise$$

produces a spontaneus compactification
First notice that

$$F_{\mu MNP}F_\nu^{MNP} = 6F_4^2 g_{\mu\nu} \tag{5.14}$$

$$F_{mMNP}F_n^{MNP} = 0 \tag{5.15}$$

and

$$F_{MNPQ}F^{MNPQ} = 24F_4^2 \tag{5.16}$$

Therefore (5.10) reduces to

$$R_{\mu\nu} = -\frac{1}{2}F_4^2 g_{\mu\nu} \tag{5.17}$$

$$R_{mn} = \frac{1}{5} F_4^2 g_{mn} \tag{5.18}$$

Thus, $C_1 = -F_4^2 < 0$ and $C_2 = F_4^2 > 0$ so, if $F_4 \neq 0$ the appearance of the gauge field A_{MNP} causes a spontaneous compactification.

At this stage, although we have had success in producing a spontaneous compactification, at least three new important problems we need to face. The first one is that the gauge field A_{MNP} has been introduced by hand. The second one is that the assumption that the compact space has a radius of the order of the Planck lenght leads to a very large cosmological constant for the ordinary space-time, finally, the third problem is that by introducing the gauge field A_{MNP} we lost the nice geometrical original idea of Kaluza. The two first problems are presumible solved by Supergravity theory in eleven dimentions [67]. While the third problem seems to be solved by Superstring theory [41]. In this work we will briefly explain how Supergravity in $d = 11$ solves the first problem and we leave the reader to consult the literature about the problem of the cosmological constant [66]. We will briefly explain how Superstring solves the third problem.

Let is first start recalling the main aspects of Supergravity. We need first to clarify the meaning of the "super" of the word Supergravity. Before 1974 the symmetries of bosons (particles with integer spin) and fermions (particles with half integer spin) were studied separately. Bosons were transformed into bosons and fermions were transformed into fermions. But at that year an important symmetry was descovered which units a boson and fermion in only one superparticle [69]. This symmetry is now called supersymmetry [70].

If we associate a generator operator $\hat{Q}$ to such a supersymmetry then $\hat{Q}$ will change fermionic states into bosonic ones and vice versa.

$$\hat{Q}|boson >= |fermion > \tag{5.19}$$

$$\hat{Q}|fermion >= |boson > \tag{5.20}$$

Now, normally a generator, let say $\hat{J}$,of a usual symmetry determines an element of the group Λ trough the formula $\Lambda = e^{\theta\hat{J}}$, where θ is an infinitesimal parameter. In supersymmetry a similar construction is possible. In fact the formula $g = e^{\epsilon\hat{Q}}$ defines an element of a "super" group. The infinitesimal parameter ϵ may or may not be a function of the space-time coordinates. If ϵ is constant (independent of the space-time coordinates) the supersymmetry is called global and local if $\epsilon = \epsilon(x)$. Supergravity is the theory of local supersymmetry. Since according to supersymmetry bosons and fermions accur always in pairs we expect that there must be one fermionic companion to the ordinary spin-2 gravitational field $g_{\mu\nu}$. The metric in terms of the tetrad e_μ^m is .

$$g_{\mu\nu} = e_\mu^m e_\nu^n \eta_{mn} \tag{5.21}$$

where $\eta_{\mu\nu}$ is the Minkowsti metric. The infinitesimal supermetric transformation of the tetrad e_μ^m turns out to be

$$\delta e_\mu^m = \frac{1}{2}\bar{\epsilon}^a \gamma_{ab}^m \Psi_\mu^b \tag{5.22}$$

where $\epsilon = \epsilon(x)$ is the infinitesimal parameter and γ^m are the Dirac matrices. The field Ψ_m^a called the gravitino has spin 3/2 and is the fermion companion to the gravitational field $g_{\mu\nu}$ (spin 2).

If there are N gravitinos in the theory we have $\Psi_{\mu i}^a$ with $i = 1, ..., N$ and $N \leq 8$. Theories with $N > 8$ seem to be unconsistent [63]. Of course if we have more gravitinos, in addition to the bosonic graviton, we need to introduce more bosonic degrees of freedom.

Let us count the degrees of freedom of $N = 1$ supergravity in eleven dimensions.

$$e_M^a \ transversal \ and \ traceless : \frac{9(9+1)}{2} - 1 = 44 \tag{5.23}$$

$$\Psi_M^a \ transversal \ in \ gauge \ \gamma^M \Psi_M^a = 0 : \frac{9 \times 32 - 32}{2} = 128 \tag{5.24}$$

Here Ψ_M^a is a Majorana spinor. Thus in order to match the number of fermionic degrees of freedom with the number of bosonic degrees of freedom we need additional 128-44=84 bosonic degrees of freedom. Since

$$A_{MNP}; \ transversal \ : \ \binom{9}{3} = 84 \tag{5.25}$$

The gauge field A_{MNP} provides such an extra bosonic degrees of freedom.

Returning now to the problem of spontaneous compactification we first notice that the field equation (5.10) and (5.11) corresponds to the bosonic sector of N=1 supergravity in $d = 11$. Thus, from this point of view the gauge field A_{MNP} is not a field put it by hand, but rather is a bosonic field that comes from $N = 1$ supergravity in $d = 11$.

Let us now see if it is possible to give a geometrical interpretation to the gauge field A_{MNP}. In electromagnetism the gauge field A_M is the source of a point particle with charge g_0. The relevant term in the Lagranian is

$$g_0 \frac{d\chi^M}{d\tau} A_M \tag{5.26}$$

Suppose we have an antisymmetric gauge field A_{MN}. Because A_{MN} is antisymmetric a term of the form

$$g_1 \frac{d\chi^M}{d\tau}\frac{d\chi^N}{d\tau} A_{MN} \tag{5.27}$$

vanishes. So if we want to construct the analog of (5.26) for A_{MN} we need to make an important change. The problem is solved if we introduce another parameter σ such that $\chi^M = \chi^M(\tau,\sigma) = \chi(\xi^a)$, $a, b = 0, 1$, because now we can make the combination

$$\frac{1}{2!} g_1 \epsilon^{ab} \frac{\partial\chi^M}{\partial\xi^a}\frac{\partial\chi^N}{\partial\xi^b} A_{MN} \tag{5.28}$$

Thus we conclude that the gauge field A_{MN} can not be the source of a point particle, but rather is the source of a string parametrized by σ.

Similarly, the gauge field A_{MNP} will be the source of a membrane. In this case the analog of (5.26) and (5.27) will be

$$\frac{1}{3!} g_2 \epsilon^{abc} \frac{\partial\chi^M}{\partial\xi^a}\frac{\partial\chi^N}{\partial\xi^b}\frac{\partial\chi^Q}{\partial\xi^c} A_{MNQ} \tag{5.29}$$

Therefore in general a completely antisymmetric gauge field $A_{M_1,\ldots M_{P+1}}$ will be the source of a p-brane with interaction term of the form

$$\frac{1}{(p+1)!} g_p \epsilon^{a_1\ldots a_{p+1}} \frac{\partial\chi^{M_1}}{\partial\xi^{a_1}}\cdots\frac{\partial\chi^{M_{p+1}}}{\partial\xi^{a_{p+1}}} A_{M_1,\ldots M_{p+1}} \tag{5.30}$$

where g_p is the "charge" of the system and the indices a,b = 0,...,p.

Let us introduce the induced metric

$$h_{ab} = \frac{\partial\chi^M}{\partial\xi^a}\frac{\partial\chi^N}{\partial\xi^b} g_{MN}(\chi^Q) \tag{5.31}$$

and the notation

$$h = det(h_{ab}) \tag{5.32}$$

The action of a point particle moving in a gravitational field g_{MN} and electromagnetic field A_M is

$$S_0 = -\Omega_0 \int \sqrt{-h}\, d\tau + g_0 \int \frac{d\chi^M}{d\tau} A_M d\tau \tag{5.33}$$

where $a, b = 0$ and Ω_0 measures the inertial of the system; Ω_0 is the rest mass of the particle.

The analog of (5.33) for the string is the action [41]

$$S_1 = -\Omega_1 \int \sqrt{-h}\, d^2\xi + \frac{g_1}{2!} \int d^2\zeta \epsilon^{ab} \frac{\partial\chi^M}{\partial\xi^a}\frac{\partial\chi^N}{\partial\xi^b} A_{MN} \tag{5.34}$$

for the membrane we have the action [71]-[72]

$$S_2 = -\Omega_2 \int d^3\xi \sqrt{-h} + \frac{g_2}{3!} \int d^3\xi \epsilon^{abc} \frac{\partial \chi^M}{\partial \xi^a} \frac{\partial \chi^N}{\partial \xi^b} \frac{\partial \chi^Q}{\partial \xi^c} A_{MNQ} \tag{5.35}$$

and in general for any p-brane we have

$$S_p^I = -\Omega_p \int d\xi^{p+1} \sqrt{-h} + \frac{g_p}{(p+1)!} \int d\xi^{p+1} \epsilon^{a_1 \ldots a_{p+1}} \frac{\partial \chi^{M_1}}{\partial \xi^{a_1}} \cdots \frac{\partial \chi^{M_{p+1}}}{\partial \xi^{a_{p+1}}} A_{M_1,\ldots M_{p+1}} \tag{5.36}$$

Therefore from these actions we see that the gauge field A_{MN} leads naturally to consider string theory, A_{MNQ} to membrane theory, and in general the gauge field $A_{M_1,\ldots M_{p+1}}$ to p-brane theory.

There are other two alternative classical equivalent actions associated to p-branes. One is [71]

$$S_p^{II} = -\frac{\Omega_p}{2} \int d\xi^{p+1} \sqrt{-\gamma} (\gamma^{ab} h_{ab} + 1 - p) +$$
$$\frac{g_p}{(p+1)!} \int \epsilon^{a_1 \ldots a_{p+1}} \frac{\partial \chi^{M_1}}{\partial \xi^{a_1}} \cdots \frac{\partial \chi^{M_{p+1}}}{\partial \xi^{a_{p+1}}} A_{M_1} \ldots M_{p+1} \tag{5.37}$$

and the other one is [72-73]

$$S_p^{III} = -\Omega_p (p+1)^{-\frac{p+1}{2}} \int d^{p+1}\xi \sqrt{-f} (f^{ab} h_{ab})^{\frac{p+1}{2}} +$$
$$+\frac{g_p}{(p+1)!} \int d^{p+1}\xi \epsilon^{a_1 \ldots a_{p+1}} \frac{\partial \chi^{M_1}}{\partial \xi^{a_1}} \cdots \frac{\partial \chi^{M_{p+1}}}{\partial \xi^{a_{p+1}}} A_{M_1,\ldots M_{p+1}} \tag{5.38}$$

where γ_{ab} and f_{ab} are auxiliary metrics.

Under the Weyl transformation

$$\gamma'_{ab} = \Lambda(\xi) \gamma_{ab} \tag{5.39}$$

we notice that S_p^{II} is Weyl invariant only for the string (p=1), while under the Weyl transformation

$$f'_{ab} = \Lambda(\xi) f_{ab} \tag{5.40}$$

S_p^{III} is Weyl invariant for any p-branes

Let us now return to the problem of spontaneous compactification. We notice before that the gauge field A_{MNP} is an important object in N=1 d=11 supergravity. Now from (5.35) we observed that the gauge field A_{MNP} is also an important object in membranes theory. The question arises weather N=1, d=11 supergravity and membranes theory are related.

In connection with such a question the following picture is known [74]

$$\begin{array}{ccc} \text{Supermembrane} & \cdots\cdots\!\!> & N = 1,\ d = 11 \text{ Supergravity} \\ \wr & & \wr \\ \text{Type IIA Superstring} & \text{——} & N = 2,\ d = 10 \text{ Supergravity (non-chiral)} \end{array}$$

Picture [A]

To explain in detail this picture will make this work very long. So instead we will try first to clarify such a picture and then for completeness we will explain how string can be obtained from membranes. We will also briefly explain how gravity arises in strings theory.

Let us first briefly clarify the picture [A]. With the arrows $\leadsto$ we mean that from Kaluza-Klein procedure we can obtain type IIA superstring, from supermembranes and that N = 2, d = 10 nonchiral supergravity can be obtained from N = 1, d = 11 supergravity . The arrow —— means that N = 2. d = 10 nonchiral supergravity is the field theory limit of Type IIA superstring. Finally the arrow ......> mean that there is not still a direct prove that from supermembranes we can obtain N = 1, d = 11 supergravity.

The central idea to show that strings can be obtained from membranes is to apply the Kaluza-Klein procedure simultaneously to the world-volume and to the space-time. This procedure is called double dimensional reduction [7.4].

In order to perform such a procedure let us first rewrite the membrane action in the form

$$S_2^{II} = -\frac{1}{2}\int d^3\xi\sqrt{-\hat{\gamma}}(\gamma^{\hat{a}\hat{b}}h_{ab} - 1)+$$

$$\frac{g_2}{3!}\int d^3\xi\epsilon^{\hat{a}\hat{b}\hat{c}}\frac{\partial\chi^{\hat{M}}}{\partial\xi^{\hat{a}}}\frac{\partial\chi^{\hat{N}}}{\partial\xi^{\hat{b}}}\frac{\partial\chi^{\hat{Q}}}{\partial\xi^{\hat{c}}}A_{\hat{M}\hat{N}\hat{Q}} \tag{5.41}$$

where a,b.., etc. $= 0, 1, 3$ and $MN = 0,1,...,$d-1. Now it is convenient to split the coordinates as

$$\chi^{\hat{M}} = (\chi^M, y), M = 0, 1, ..., d-2, \tag{5.42}$$

$$\xi^{\hat{a}} = (\xi^a, \rho), \quad a = 0, 1. \tag{5.43}$$

The procedure of double dimensional reduction is determined by the following ansatz

$$\partial_\rho\chi^M = 0, \tag{5.44}$$

$$y = \rho, \tag{5.45}$$

$$\partial_\rho\hat{\gamma}_{\hat{a}\hat{b}} = 0, \tag{5.46}$$

$$\partial_y\hat{g}_{\hat{M}\hat{N}} = 0. \tag{5.47}$$

This ansatz allows to write the space-time metric $\hat{g}_{\hat{M}\hat{N}}$ and the world volume metric $\hat{\gamma}_{\hat{a}\hat{b}}$ in the Kaluza-Klein form;

$$\hat{g}_{\hat{M}\hat{N}} = \Phi^{-2/3}\begin{pmatrix} g_{MN} + \Phi^2 A_M A_N & \Phi^2 A_M \\ \Phi^2 A_N & \Phi^2 \end{pmatrix}, \tag{5.48}$$

$$\hat{\gamma}_{\hat{a}\hat{b}} = \phi^{-2/3}\begin{pmatrix} \gamma_{ab} + \phi^2 G_a G_b & \phi^2 G_a \\ \phi^2 G_b & \phi^2 \end{pmatrix} \tag{5.49}$$

Using these two ansätze the action (5.41) becomes

$$S_2^{II} = -\frac{1}{2}\int d^2\xi\sqrt{-\gamma}\{\left(\frac{\phi}{\Phi}\right)^{\frac{2}{3}}\gamma^{ab}h_{ab}+$$

$$\phi^{2/3}\Phi^{4/3}\gamma^{ab}(G_a - \partial_a\chi^M A_M)(G_b - \partial_b\chi^M A_N) \tag{5.50}$$

$$+[(\frac{\Phi}{\phi})^{4/3} - 1]\} + \frac{g_2}{2!}\int d^2\xi\epsilon^{ab3}\frac{\partial\chi^M\partial\chi^N\partial y}{\partial\xi^a\partial\xi^b\partial\rho}A_{MNy}$$

where an overall factor of $\int d\rho$ has been dropped. Making variations of this action wich respect to ϕ and G_a we learn that

$$\phi = \Phi \tag{5.51}$$

$$G_a = \partial_a\chi^M A_M \tag{5.52}$$

where we used the equation $\gamma_{ab} = h_{ab}$ which is obtained from (5.50) when we make variations with respect to γ_{ab}.

Now, subtituting these results back into (5.50) we get

$$S_1^{II} = -\frac{1}{2}\int d^2\xi\sqrt{-\gamma}\gamma^{ab}\partial_a\chi^M\partial_b\chi^N g_{MN}+$$

$$\frac{g_2}{2!}\int d^2\xi\epsilon^{ab}\frac{\partial\chi^M}{\partial\xi^a}\frac{\partial\chi^N}{\partial\xi^b}A_{MN} \tag{5.53}$$

which is the string action. Here $\epsilon^{ab} = \epsilon^{ab3}$ and $A_{MN} = A_{MNy}$. Similar procedure can be applyed to super-membranes. In this case which such a procedure a type II A superstring is obtained [74].

Let us now make some few comments about how gravity arises from strings.
From (5.37) we have that the action of a string moving in a flat Minkowski spacetime is

$$S_1^{II} = -\frac{1}{2\alpha'}\int d^2\xi\sqrt{-\gamma}\gamma^{ab}\partial_a\chi^M\partial_b\chi^N\eta_{MN}, \tag{5.54}$$

where

$$\alpha' = \frac{1}{\Omega_1}.$$

Varing this action with respect to γ^{ab} gives the two-dimensional energy-momentum tensor

$$T_{ab} = \partial_a\chi^M\partial_b\chi^N\eta_{MN} - \frac{1}{2}\gamma_{ab}(\gamma^{cd}\partial_c\chi^M\partial_d\chi^N\eta_{MN}) = 0 \tag{5.55}$$

and varing it with respect to χ^M leads to the Euler-Lagange Equation

$$\frac{1}{\sqrt{-\gamma}}\frac{\partial}{\partial\xi^a}\{\sqrt{-\gamma}\gamma^{ab}\frac{\partial}{\partial\xi^b}\}\chi^M = 0 \tag{5.56}$$

Two reparametrizations and one Weyl invariance allow to choose the three independent elements of γ_{ab} so that

$$\gamma_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

the two dimensional Minkowski metric. Making this choice (5.55) and (5.56) simplify to

$$T_{ab} = \partial_a\chi^M\partial_b\chi^N\eta_{MN} - \frac{1}{2}\eta_{ab}(\eta^{cd}\partial_c\chi^M\partial_d\chi^N\eta_{MN}) = 0 \tag{5.57}$$

and

$$\diamond^2\chi^M = \eta^{ab}\partial_a\partial_b\chi^M = 0 \tag{5.58}$$

The general solution of this equation is

$$\chi^M = \chi_R^M(\tau-\sigma) + \chi_L^M(\tau+\sigma) \tag{5.59}$$

Let us consider closed strings. For them the appropriate boundary condition is just periodicity of the coordinates

$$\chi^M(\tau,\sigma) = \chi^M(\tau,\sigma+2\pi) \tag{5.60}$$

The general solution of (5.58) compatible with the periodicity requirement is

$$\chi_R^M = \frac{1}{2}\chi^M + \frac{1}{2}l^2p^\mu(\tau-\sigma) + \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}\alpha_n^M e^{-2in(\tau-\sigma)}, \tag{5.61}$$

$$\chi_L^M = \frac{1}{2}\chi^M + \frac{1}{2}l^2p^\mu(\tau+\sigma) + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^M e^{-2in(\tau+\sigma)}, \tag{5.62}$$

where α_n^M and $\tilde{\alpha}_n^M$ are oscillator coordinates. The constant $l = \sqrt{2\alpha'}$. We have the following poisson brachets

$$\begin{aligned}
\{\alpha_m^\mu, \alpha_n^\nu\} &= im\delta_{m,-n}\eta^{\mu\nu}, \\
\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} &= im\delta_{m,n}\eta^{\mu\nu}, \\
\{\alpha_n^\mu, \tilde{\alpha}_m^\nu\} &= 0. \\
\{p^\mu, \chi^\nu\} &= \eta^{\mu\nu},
\end{aligned} \tag{5.63}$$

The oscillator α_n^μ and $\tilde{\alpha}_n^\mu$ are not all independent since we have the constraints (5.57). In fact using the light cone coordinate $\sigma^+ = \tau + \sigma$ and $\sigma^- = \tau - \sigma$ we have that

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) = \dot{\chi}_R^2 = 0 \tag{5.64}$$

$$T_{--} = \frac{1}{2}(T_{00} - T_{01}) = \dot{\chi}_L^2 = 0 \tag{5.65}$$

Using (5.61) and (5.62) we have

$$L_m = \frac{1}{2}\int e^{-2im\sigma}T_{--}d\sigma = \frac{1}{2}\sum_{-\infty}^{\infty}\alpha_{m-n}\alpha_n = 0, \tag{5.66}$$

$$\tilde{L}_m = \frac{1}{2}\int e^{+2im\sigma}T_{++}d\sigma = \frac{1}{2}\sum_{-\infty}^{\infty}\tilde{\alpha}_{m-n}.\tilde{\alpha}_n = 0. \tag{5.67}$$

L_m and $\tilde{L}_m$ satisfy the Virasoro algebra

$$\{L_m, L_n\} = i(m-n)L_{m+n} \tag{5.68}$$

$$\{\tilde{L}_m, \tilde{L}_n\} = i(m-n)\tilde{L}_{m+n} \tag{5.69}$$

At the quantum level the Poisson brackets (5.63) become quantum relations and the constraints L_m must apply to physical states.

$$(L_0 - 1)|\phi>= 0 \tag{5.70}$$

$$L_m|\phi>= 0 \quad m = 1, ...etc. \tag{5.71}$$

An important change at the quamtum level is that due to normal ordering of the oscilators α_n^M and $\tilde{\alpha}_n^M$ the Virasoro relation (5.68) would introduce a c-number. In fact we get

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}(m^3 - m)\delta_{m,-n}. \tag{5.72}$$

This anomaly is the responsable that in order to have a consisten quantum theory we should fix d=26 for the boson string and d=10 for the superstrings. Surprisingly these two numbers can be written in terms of "sacred" mesoamerican numbers; 13 and 5.

Let us consider the state

$$|\Omega^{ij}>= \alpha_{-1}^i\tilde{\alpha}_{-1}^j|0> \tag{5.73}$$

where $i, j = 0, ..., 23$. The state $|\Omega^{ij}>$ corresponds to the tensor product of a massless vector of $SO(24)$ from left moving modes with a massless vector of SO(24) from right-moving modes. The state $|0>$ is the ground state of the bosonic open string. The part of $|\Omega^{ij}>$ which is symmetric and traceless in i and j transforms under $SO(24)$ as a massless spin two particle. Therefore this part of $|\Omega^{ij}>$ can be associated to the graviton. The trace term $\delta_{ij}|\Omega^{ij}>$ is a massless scalar. Finally the antisymmetric part $|\Omega^{ij}>= -|\Omega^{ij}>$ tranforms under $SO(24)$ as an antisymmetric second rank tensor. What we would like to emphatize here is that the graviton is part of the spectrum of closed strings. At this stage however is difficult to understand how a curved space-time could be built from the "graviton" string spectrum.

Suppose we consider a closed string propagating in a curved spacetime. The action (5.53) can be generalized to include scalar fields in the form

$$S = \int d^2\xi\sqrt{-\gamma}\Phi(\chi) + \frac{1}{4\pi}\int d^2\xi\sqrt{-\gamma}RC(x)$$
$$+\frac{1}{2\pi\alpha'}\int d^2\xi[\frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a\chi^M\partial_b\chi^N g_{MN} + \frac{1}{2}\epsilon^{ab}\partial_a\chi^M\partial_b\chi^N A_{MN}] \tag{5.74}$$

where we taked $g_2 = 1$ and introduce the constant α' According to Fradkin and Teythin [75] the effective action

$$\Gamma(\Phi, g_{MN}, A_{MN}, ..) = \int \mathcal{D}\gamma_{ab} \int \mathcal{D}\chi^M e^{\frac{-iS}{\hbar}} \tag{5.75}$$

can be written in first approximation in α' as. (for Φ=cte)

$$\Gamma_2 \sim \int d^{26}\chi\sqrt{-g}e^{-2C}[1 + \frac{1}{4}\alpha'\{R + 4(\partial_M C)^2 - \frac{1}{12}F^2_{MNP} + 0(\alpha'^2)] \tag{5.76}$$

Considering $C = const.$ then the classical field equations corresponding to this action will be

$$R_{MN} - \frac{1}{2}g_{MN}R \simeq F_{MPQ}F_N^{PQ} - g_{MN}F^2$$
$$F^{MPQ}_{\ \ \ \ Q} = 0 \tag{5.77}$$

where α and β are numerical constants.
Here we notice that solution with maximal symmetry are obtained in the case $F_{MNP} \sim \epsilon_{MNP}$. But this kind of compactification would lead to spaces of the form $M^d = \tilde{S}^3 \times B$ where $\tilde{S}^3$ is an anti-de Sitter three-dimensional space time.

There is a conjecture that a similar procedure can be applied to any p-brane. In that cases the general structure of Γ will be

$$\Gamma(\Phi, g, A) \sim \int d^d\chi\sqrt{-g}\{V(\Phi) - \frac{1}{\Omega_p^2}f_1(\Phi)\partial_M\Phi\partial_N\Phi g^{MN}$$
$$+\frac{1}{\Omega_p^2}f_2(\Phi)R+$$
$$\frac{1}{(p+1)!\Omega_p^2}f_3(\Phi)F_{M_1...M_{p+1}}F^{M_1..+...M_{p+1}} + 0(\frac{1}{\Omega_p^4})$$

If we consider a solution $F_{M_1...M_{p+1}} \sim \epsilon_{M_1...M_{p+1}}$ then compactification of four dimensions is preferred in the case of membranes.

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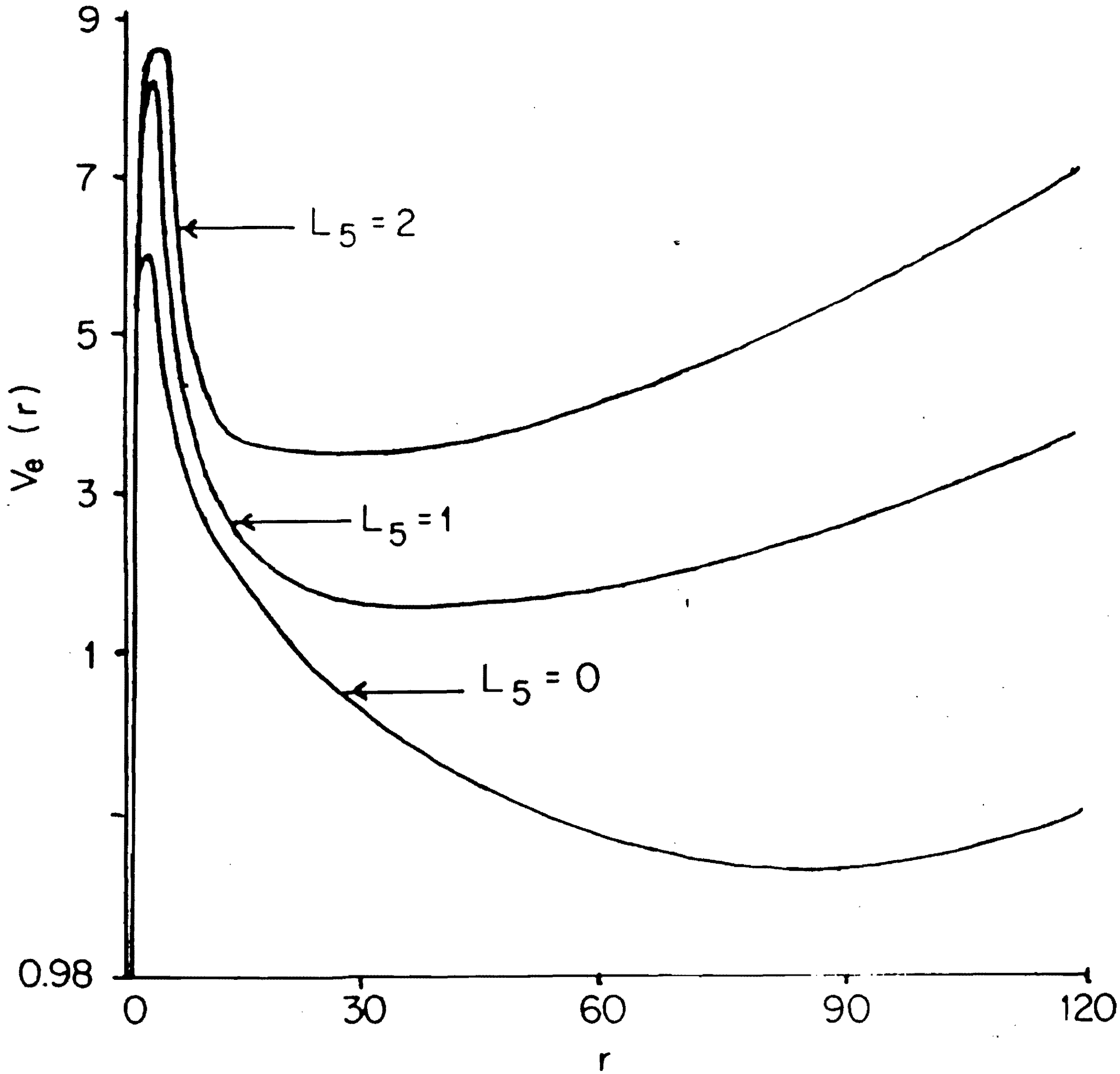

FIG. 1. Plot of $V_e(r)$ for various values of L_5 with $\mu^2 = 1$, $m = 1$, $L_e = 10$, and $\eta = 0.09$.

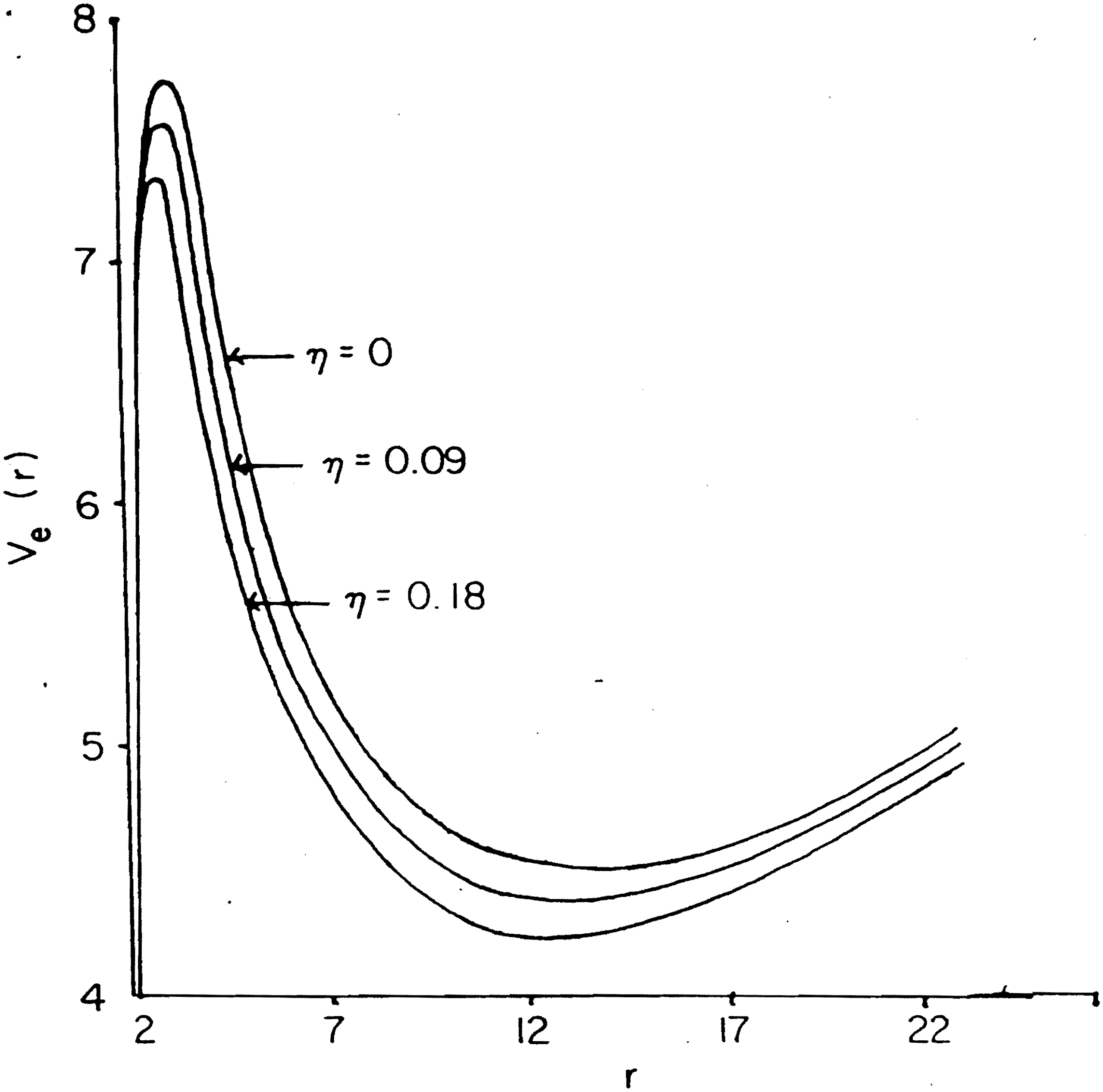

FIG. 2. Plot of $V_e(r)$ for various values of η with $\mu^2 = 1$, $m = 1$, $L_e = 10$, and $L_5 = 2$.

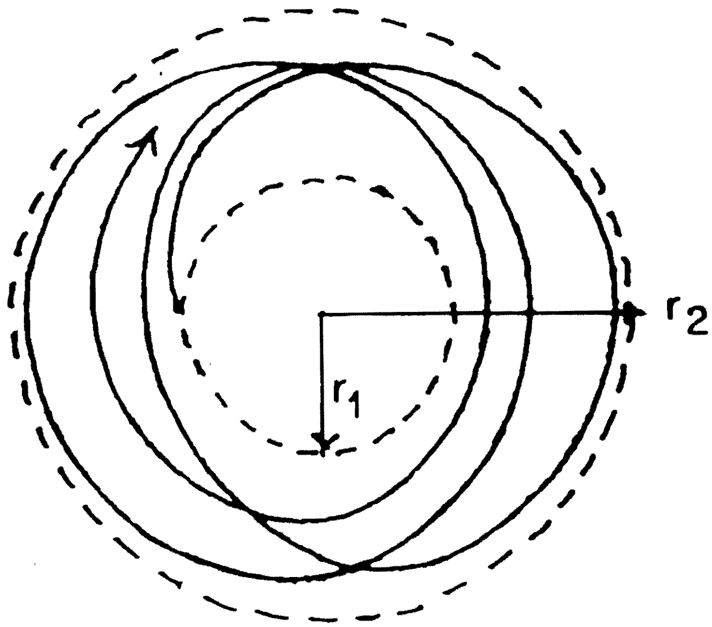

FIG. 3. Nature of equatorial orbits of the test particles.

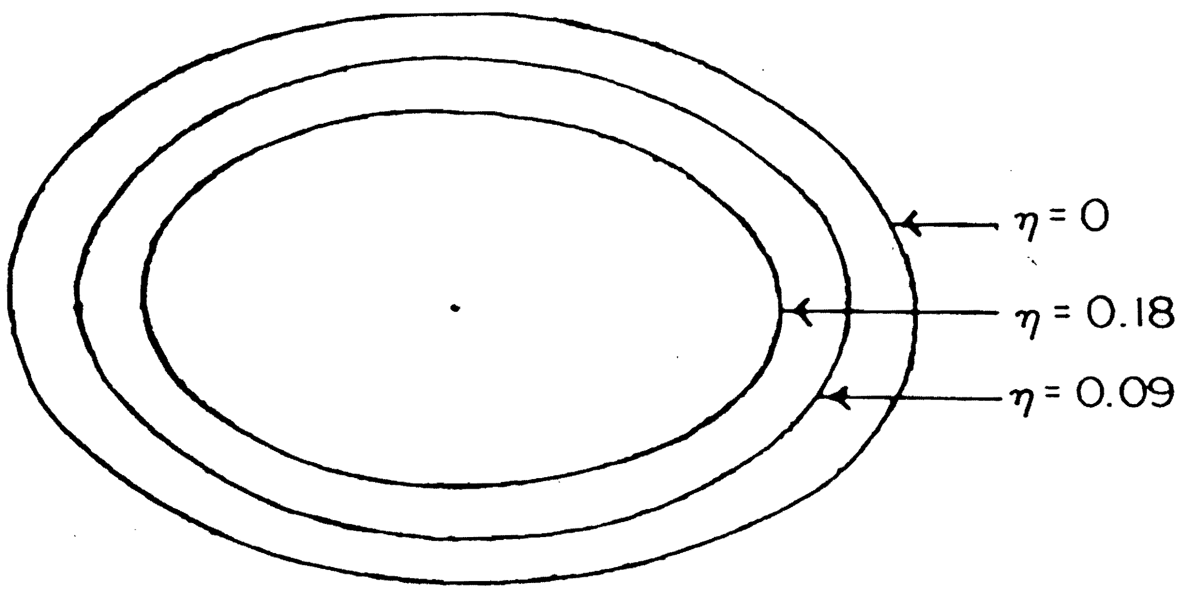

FIG. 4. Polar orbits for various values of η with $\mu^2 = 1$, $m = 1$, $L_p = 10$, and $L_5 = 2$.

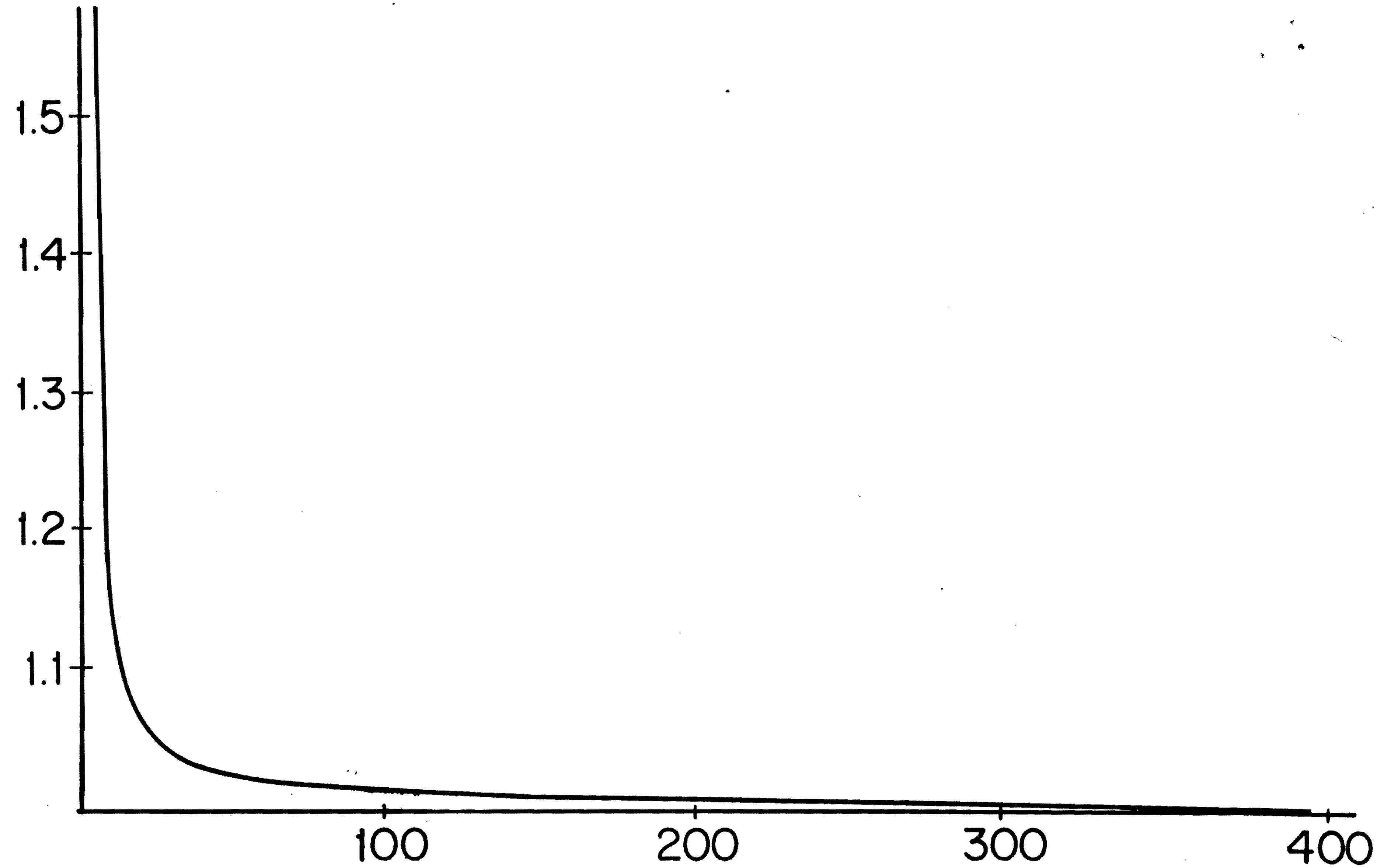

Fig. 5 Plot of the I^2 potential with $\mu^2 = 1$, m=1 and =0.09

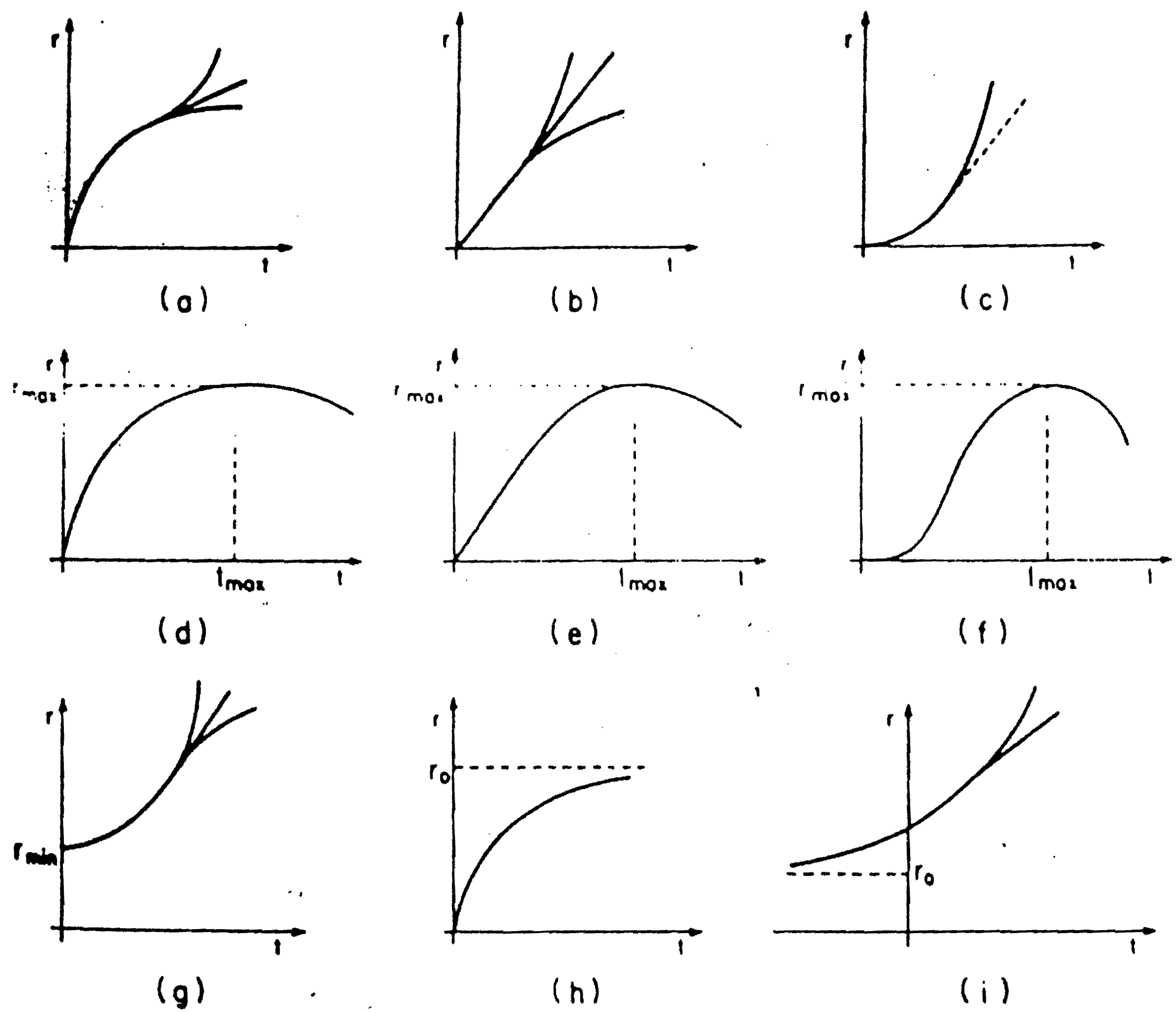

FIG. 6 The cosmological solutions in the absence of a vacuum condensation. All the solutions except (a), (d), and (h) have no particle horizon.

$$\{(x,y)\in \mathbb{R}^2 \mid x^2+y^2=1\}\times[a,b] = S^1\times[a,b] =$$

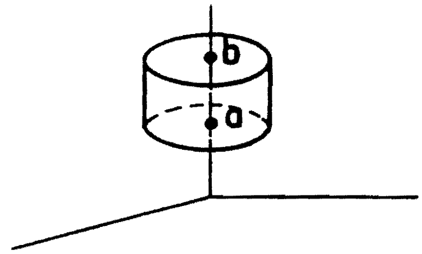

Fig. 7 The cylinder

$$S^1\times[a,b] \cup S^1\times[b,a] =$$

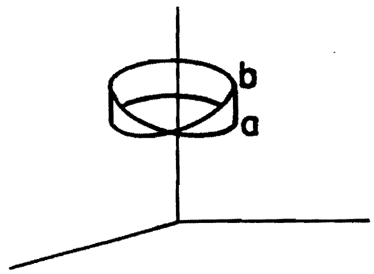

Fig. 8 The Möbius strip

Fig. 9 The proyection

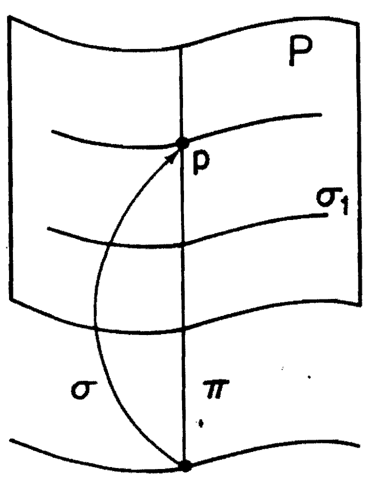

Fig. 10 Cross section in principal fibred bandle.

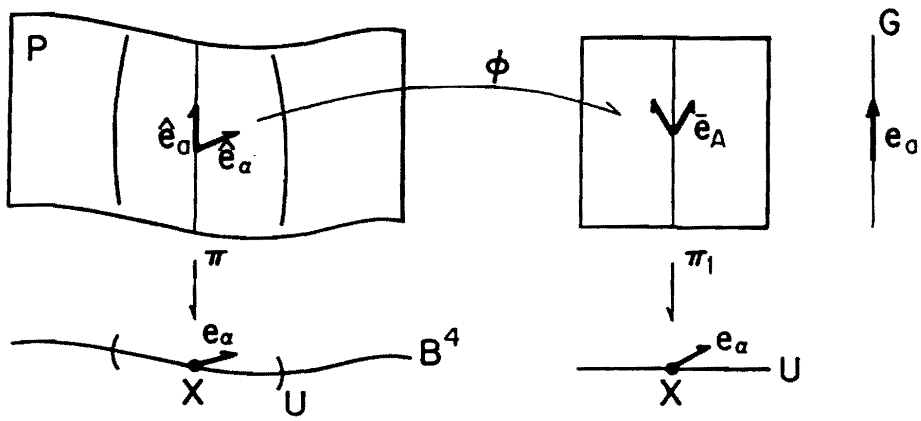

Fig. 11. The trivialization of a Fibred bandle.