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CAUSALITY, SYMMETRY

AND

VALIDITY OF LINEARIZED GRAVITY FOR WAVES

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Abstract

Based on causality, the validity of linearized gravity for waves is examined. It is pointed out that the applicability of the "gauge condition" for weak gravity actually has never been generally established. To analyze the linearized equations, symmetry is elucidated in connection with the principle of causality. It is found that, for gravitational waves, linearized gravity may not be valid. To illustrate this, a gravitational plane wave is provided as an example. Concurrently, it is also shown that implications of linearized gravity on plane waves are not valid. Thus, the term "gauge condition" is actually a misnomer.

Causality, Symmetry
and
Validity of Linearized Gravity for Waves

1. Introduction.

It was believed that linearized gravity were always valid for weak gravity [1-5]. On the other hand, in 1936 Einstein observed that linearizd gravity is not reliable for gravity waves [6]. It will be shown that, from the viewpoint of causality, Einstein is right again.

In fact, the proof for the general applicability of the gauge condition actually has never been established. Although there is an equation to calculate the required gauge vector, it remains to show that a physically valid solution would always exist [1-5]. Since the "gauge condition" is a pure mathematical condition, it seems, the root of the problem is the misconception that a mathematical equation would have a physically valid solution. This mistake is understandable because physicists are used to that a physical equation always has a physical solution.

For an equation to be valid in physics, there are a number of general physical requirements. Among them, causality is an important one. Although causality is usually satisfied, it can play a crucial role in examining the validity of a formula. For example, in classical electrodynamics, causality restricts the possible form of the radiation reaction force [7]. In this paper, it will be shown that causality implies compatibility of symmetries between causes and effects (see §2). As a corollary, for

gravitational waves, linearized gravity is not valid (see §3). This will be illustrated with an example (see §4).

It should be pointed out that the applicability of the "gauge" is distinct from the Cauchy problem. In a Cauchy problem, the initial satisfaction of the "gauge condition" is assumed. Since the harmonic coordinate condition is not necessarily valid in physics, such a Cauchy solution may not be physical either (see also §4). Thus, although the existence of a solution for a Cauchy problem has been confirmed by Bruhat [8], the applicability problem of the "gauge" remains (see §5).

2. The Principle of Causality and Symmetry.

The concept of causality describes the ideas of cause and effect. In time ordering, a cause event must happen before its effects. In relativity, the time order is further restricted because no cause event can propagate faster than the light speed. However, the time ordering aspect of causality does not include its own relevance. In scientific studies, the validity of causality is implicitly assumed.

The time-tested causality assumption that phenomena can be explained in terms of identifiable causes will be called the principle of causality. This principle is the most fundamental aspect of causality; otherwise scientific studies would be meaningless if things just happen without an identifiable cause(s). Here, the fundamental aspects of causality will be elucidated first in connection with symmetries, and then the validity of an equation in physics.

In practices, we assume certain properties (such as symmetries etc.) for a "normal" state whose existence is without any specific cause. Then, any deviation from the normal state must have a physically identifiable cause(s). Since the principle of causality implies that symmetry breaking must have a cause(s), a symmetry must be preserved if no cause breaks it. For example, in electrodynamics, the electromagnetic field is zero in a normal state. The implication of causality to symmetry is used in deriving the inverse square law from the Gauss's law.

In general relativity, matter is the cause of gravity. The normal state of a metric is the flat metric in special relativity. The flat metric is a specific constant matrix, and posses all the symmetry allowed by special relativity. Thus, if a non-constant metric does not posses a certain symmetry, then there must be a physical cause which has broken such a symmetry. In other words, the metric should have at least the same symmetry as its physical cause(s). For example, in the Schwarzschild solution, causality requires that the metric is spherically symmetric and asymptotically flat. Note that, without the principle of causality, the flat metric is not the only solution for Einstein's equation in an empty space.

However, the physical cause should not be confused with the mathematical source term in the field equation. Such a confusion would be possible because, for some situations, such a distinction does not seem to be meaningful. For instance, in electrodynamics, the physical cause and the source term are the same charged currents. (The media and the boundary values are also due to electromagnetic structure of matter.)

In general relativity, the source term in Einstein equation consists of energy-stress tensors which, as indicated by its dependence on the metric, do not represent the physical cause. For the accompanying gravitational wave of an electromagnetic wave, the physical cause is the electromagnetic wave. In the Schwartzschild case, the physical cause is the mass distribution. (Note that the perfect fluid model also depends on the metric.) Thus, it does not make sense, without directly using causality, to infer the symmetries of the metric from the source term although their symmetries are not unrelated.

Moreover, inferences based on the source term can be misleading. Sometimes, the source term may have higher symmetries than those of the cause and the metric. For instance, a transverse electromagnetic plane wave is not a rotational invariant with respect to the direction of propagation. But the related electromagnetic energy-stress tensor can be a rotational invariant and even be a constant [9]. In the literature [2,10,11], the metric is incorrectly assumed to be rotationally invariant. This assumption violates the principle of causality, and the resulting solutions also have theoretical difficulties.

The principle of causality supports Einstein's equation also indirectly because, for a solution, a violation of causality would lead to violation of other physical conditions. Classical electrodynamics and experiments imply that the flat metric is an accurate approximation of the metric which is caused by the presence of a weak electromagnetic plane wave. This physical requirement is supported by the principle of causality which implies that such a metric is a bounded periodic function.

However, this requirement is not satisfied by solutions in the literature [2,10,11], because they are not bounded, independent of how weak the electromagnetic plane waves are (see also §4). Another problem of these solutions is that they do not satisfy the principle of causality.

3. Symmetry and Validity of a Field Equation.

This compatibility of symmetry due to causality is independent of any specific field equation. On the other hand, since any field equation and its physical solutions must be compatible with the principle of causality, symmetry consideration can be used as a criterion, which is independent of the field strength.

For some mathematical equations, the symmetries of a solution can be very different from that of the source term (which may or may not be the physical cause). For example, consider the following linear equation,

$$\eta^{ab}\partial_a\partial_b\, g = f(u), \tag{1}$$

where η_{ab} is the flat metric and $u \equiv (t - z)$. If g is a function of only t and z, then the inhomogeneous solution of eq. (1) is

$$g(t,z) = \frac{v}{4} \int^{u} f(t)\, dt, \tag{2}$$

where $v \equiv (t + z)$. Solution (2) depends not only on u, but also v. Thus, a solution of eq. (1) cannot be a function of only u. However, one must first identify the physical cause before considering its symmetries.

Based on the above results, one can examine the validity of related field equations. The left-hand side of eq. (1) can be considered as a Maxwell's equation or an equation in linearized gravity. For the case of Maxwell's equation, the principle of causality implies that the source term may not be in the form of plane waves. This restriction is satisfied physically because, in nature, a charged particle is invariably massive. For linearized gravity, g represents an element of the metric. If the physical cause is an electromagnetic plane wave, then it can be a function of u [2,9,10,11]. Thus, the principle of causality implies that the metric element g is a function of u only [2,9]. On the other hand, the related energy-stress tensor, to the lowest order approximation, is a function of u, and thus the source term in linearized gravity would have the form f(u). Then, according to solution (2), it is mathematically impossible that g is a function of only u. This contradiction suggests that, for gravitational waves, eq. (1) is not an appropriate form. Thus, causality implies that there are weak gravity exact solutions, which cannot be approximated with linearized gravity. In other words, causality supports Einstein's observation that linearized gravity is not reliable [6].

In short, the principle of causality is a foundation of physics. Although its implication may appear to be questionable in mathematics, nature has a way to deal with the mathematics such that this principle is satisfied. In classical electrodynamics, it is well-known that causality restricts the possible form for the radiation reaction force. Here, causality explains physically that, even for weak gravity, the field equation of gravity must go beyond a linear equation of Maxwell type because the physical causes of gravity include electromagnetic waves.

4. A Counter Example for Linearized Gravity and Related Problems.

In this section, a plane wave will be provided to show that no solution of the linearized equation can be a first order approximation. This example illustrates that the "gauge condition" is actually a misnomer. Let us consider Einstein equation $G_{ab} = - K T_{ab}$ with a source tensor,

$$T_{lk} = P_l P_k \, g^{mn} A_m A_n - g^{mn}(\nabla_l A_m - \nabla_m A_l)(\nabla_k A_n - \nabla_n A_k) \tag{3a}$$

where

$$P_x = P_y = 0 \ , \ P_t = -P_z = \omega \ , \tag{3b}$$

g^{ab} is the inverse metric, and A_m represents a plane wave propagating in the z-direction. The only non-zero component of A_m is:

$$A_x = A_0 \cos\omega(t - z). \tag{3c}$$

Note that $\omega(t - z) = P_m x^m = \int P_m dx^m$ represents a scalar.

Let the g_{ab} be the metric and $\gamma_{ab}(\equiv g_{ab} - \eta_{ab})$ be the deviations from the flat metric η_{ab}. Then, a solution of the Einstein equation is:

$$\gamma_{xx} = - \frac{K}{2} A_0^2 [1 - \cos 2\omega(t - z)], \tag{4a}$$

$$\gamma_{tt} = - \gamma_{zz} = [1 - \gamma_{xx}]^{-\frac{1}{2}} - 1 \ ; \tag{4b}$$

and otherwise $\gamma_{ab} = 0$. Plane wave (4) does not satisfy the harmonic coordinate condition nor its linearized approximation. Both $(\gamma_{xx} + \gamma_{yy})$

and γ_{tt} are of first order deviations. The wave γ_{ab} has the transverse polarization γ_{xx}, and has a longitudinal component γ_{zz}. These are not allowed by linearized gravity [1-5].

It follows from eqs. (3) and (4) that the Einstein tensor is of first order deviations. In a linearized approximation, one has

$$T_{lk} = - P_l P_k A_0^2 \cos 2\omega u \ . \tag{5}$$

Then, a solution of the related linearized equation cannot be a function of only u. The linearized equation is an inhomogeneous Maxwell equation with a source of <u>first</u> order deviations. On the contrary, the metric elements γ_{ab}, as plane waves, satisfy a homogeneous Maxwell equation,

$$\partial^c \partial_c \bar{\gamma}_{ab} = 0 \ , \tag{6}$$

where $\bar{\gamma}_{ab} \equiv \gamma_{ab} - \frac{1}{2} \eta_{ab}\gamma$, and $\gamma \equiv \eta^{ab}\gamma_{ab}$. Thus, linearized gravity cannot provide a first order approximation. This also means that no gauge transformation could make the "gauge condition" applicable.

Physically, solution (4) is the accompanying gravitational wave of the electromagnetic plane wave (3c) [9]. This solution is consistent with special relativity since the flat metric is an accurate approximation if the electromagnetic wave is weak. Since wave (3c) is the physical cause of gravity, the principle of causality implies that such a metric is also a function of u [2], and is periodic. Mathematically, for a circularly polarized wave A_m, the periodicity has been proven [9].

Since the electromagnetic wave is propagating in the z-direction, the related photons should also have a momentum in the z-direction, i.e. $P^x = P^y = 0$, and $P^z = P^t$. It follows from eq. (3b) that [9]

$$g^{tt} - 2g^{tz} + g^{zz} = g_{tt} + 2g_{tz} + g_{zz} = 0 \ , \tag{7a}$$

and

$$g^{xt} - g^{xz} = g^{yt} - g^{yz} = g_{xt} + g_{xz} = g_{yt} + g_{yz} = 0 \ . \tag{7b}$$

Eq. (7) reduces Einstein equation to a differential equation of u,

$$G'' - g_{xx}'g_{yy}' + (g_{xy}')^2 - G'g'/2g = -2KGT_{tt} \tag{8}$$

where

$$G = g_{xx}\, g_{yy} - g_{xy}^2 \ ,$$

and g is the determinant of the metric. Eq. (7) implies also that

$$\partial^a\overline{\gamma}_{ax} = \partial^a\overline{\gamma}_{ay} = 0, \text{ and } \ \partial^a(\overline{\gamma}_{az} + \overline{\gamma}_{at}) = 0. \tag{9a}$$

Thus, the gauge condition is partially satisfied. However, the gauge condition is not yet complete unless

$$\partial^a(\overline{\gamma}_{az} - \overline{\gamma}_{at}) = 0. \tag{9b}$$

But, eq. (9b) may not be valid since the longitudinal wave component is not zero. Eq. (9b) and eq. (3b) would imply $P_z = P_t = 0$. Now, from the view point of physics, clearly the gauge condition cannot be completely satisfied, i.e. no gauge transformation can make the gauge valid.

To illustrates the difficulties of linearized gravity, let us consider some of the solutions. If the metric elements are not functions of x and y, then the inhomogeneous solution for (5) is

$$\gamma_{tt} = \gamma_{zz} = - \gamma_{tz} = \frac{v}{4} \omega^2 \int^u f(t)dt \ , \tag{10a}$$

where $f(u) = 2KA_0{}^2\cos 2\omega u$. (The detail form of f(u) does not matter.) Otherwise, γ_{ab} is zero. This solution is not consistent with weak gravity because the metric elements (10a) can be very large even the electromagnetic wave is very weak. If one assumed that the metric elements could be a function of x, y, then a possible inhomogeneous solution would be

$$\gamma_{ab} = - \frac{1}{4} (x^2 + y^2) P_a P_b f(u) \ . \tag{10b}$$

Although this solution satisfies the gauge condition, it may not be weak for large x or y. It depends on an arbitrary origin. This solution is, in fact, arbitrary. For instance, the x, y dependence factor can be a very different $(2x^2 + Cxy)$. Moreover, these solutions have no transverse components, and this is in dissonance with its cause, a transverse wave.

Both (10a) and (10b) are rotational invariants with respect to the z-axis. But, the exact solution plane wave (4) is not an invariant. The above solutions of the linearized equation are not a first order approximation of the exact solution. It is interesting to note that (10b) itself is also an exact solution. However, this solution is not physical since it is incompatible with causality and special relativity. Thus, solution (10b) shows also that a Cauchy solution may not be physically valid.

5. Conclusion and Discussion.

In linearized gravity, Einstein's field equation is reduced to

$$G_{ab}{}^{(1)} = - K\ T_{ab}\ , \tag{11a}$$

where

$$G_{ab}{}^{(1)} \equiv \frac{1}{2}\partial^c\partial_c\bar{\gamma}_{ab} - \partial^c\partial_{(b}\bar{\gamma}_{a)c} + \frac{1}{2}\eta_{ab}\partial^c\partial^d\bar{\gamma}_{cd}\ .$$

The gauge condition ($\partial^a\bar{\gamma}_{ab} = 0$) would reduce eq. (11a) to

$$\frac{1}{2}\partial^c\partial_c\bar{\gamma}_{ab} = - K\ T_{ab}\ . \tag{11b}$$

Since $\partial^a G_{ab}{}^{(1)} \equiv 0$, the linearized conservation law,

$$\partial^a\ T_{ab} = 0 \tag{11c}$$

is necessarily <u>exact</u>. Due to eq. (11c), linearized gravity is applicable only if self-gravitating is not important [4,5]. For example, the Newtonian potential of gravity can be supported by linearized gravity. However, eq. (11c) does not imply the validity of eq. (11b).

For the wave example, solution (4), eq. (11c) is satisfied. If the source takes the approximation (5), then equation (11a) is even equivalent to the exact eq. (8). However, eq. (11b) is not valid because neither the harmonic coordinate condition nor its linearized approximation is applicable. Thus, the mathematical harmonic coordinate condition, as predicted by causality, may not be valid for all physical situations.

Whereas the validity of the eq. (6) implies that a plane wave travels at the speed of light; the "gauge" excludes some possible polarizations. One can show alternatively that, if the Einstein tensor is of first order deviations, linearized gravity cannot be valid for plane waves. The order of the Einstein tensor depends on the wave polarization [9].

In conclusion, since the "gauge condition" is a fundamental equation of linearized gravity, it is perhaps well-advised to re-examine its results as indicated by Einstein's letter of 1936 [6]. In electrodynamics, the classical gauge invariance is proven to be inconsistent with experiment [12]. Naturally, one might ask whether classical gauge is indeed an indispensible feature of nature or rather an indication for the need of further studies. For the problem of motion, Damour [13] concludes that nearly all aspects of approximation methods need to be thoroughly re-investigated. Note that he uses the harmonic coordinate condition.

Satisfying causality is necessary for any solution in physics. It has been shown that causality can be used to examine the form of an equation. For a given equation, causality can be used to determine whether a source term is appropriate. Since the source term in general relativity would be subjected to modification as pointed out by Einstein, the principle of causality would continue to serve as a useful criterion.

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