$\mathrm{U}_A(1)$ anomaly and $\eta' \to 3\pi$ decay

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$\eta' \to 3\pi$ decay width is calculated in QCD taking in to account $\mathrm{U}_A(1)$ anomaly.

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It has been shown recently that [1] $\eta \rightarrow 3\pi$ decay width can be satisfactorily calculated in QCD by taking in to account $U_A(1)$ anomaly. The purpose of this note is to calculate the decay width of $\eta' \rightarrow 3\pi$ in QCD. Gell-Mann- Okubo mass formula gives mass of η_8-meson close to the mass of physical η-meson. This implies negligible mixing between η_8 and η_1 mesons (η_1 is $SU(3)$ singlet). On the other hand,the decay of η and η' into two photons implies a mixing angle between 15^o- 20^o. We discuss the η and η' decay to three pions taking into account the mixing and $U_A(1)$ anomaly

We start with the Σ-term for $\pi\eta_8$ and $\pi\eta_1$ scattering. The Σ-term is given by

$$\sum_{\pi A}(0) = \frac{2\bar{m}}{\sqrt{3}} < A|S_8 + \sqrt{2}S_o|A > \tag{1}$$

where $A = \eta_8$ ar η_1. From Gell-Mann-Okubo mass formula

$$< \eta_8|S_8|\eta_8 >= -\frac{1}{\sqrt{3}}\frac{(m_K^2 - m_\pi^2)}{m_s - \bar{m}} \tag{2}$$

$$< \eta_1|S_8|\eta_1 >= 0 \tag{3}$$

where m_s is the mass of strange quark and $\bar{m} = \frac{1}{2}(m_u + m_d)$; m_u and m_d are the masses of u and d quarks respectively. $S_i = \bar{q}\frac{\lambda_i}{2}q$, where λ_i's are Gell-Mann matrices and $\lambda_0 = \sqrt{2/3}I$. We assume nonet symmetry so that

$$\begin{aligned} < \eta_8|\sqrt{2}S_o|\eta_8 > &= \frac{2}{\sqrt{3}}\frac{(m_K^2 - m_\pi^2)}{m_s - \bar{m}} \\ &=< \eta_1|\sqrt{2}S_o|\eta_1 > \end{aligned} \tag{4}$$

Thus we get

$$\sum_{\pi\eta_8}(o) = \frac{2\bar{m}}{3}\frac{(m_K^2 - m_\pi^2)}{m_s - \bar{m}} \tag{5a}$$

$$\sum_{\pi\eta_1}(o) = \frac{4\bar{m}}{3}\frac{m_K^2 - m_\pi^2}{m_s - \bar{m}} \tag{5b}$$

Hence we get for the scattering amplitudes the values

$$T(\pi\eta_8 \rightarrow \pi\eta_8) = \frac{2\bar{m}(m_K^2 - m_\pi^2)}{F_\pi^2 3(m_s - \bar{m})} \tag{6a}$$

$$T(\pi\eta_1 \rightarrow \pi\eta_1) = \frac{4\bar{m}(m_K^2 - m_\pi^2)}{F_\pi^2 3(m_s - \bar{m})}, \tag{6b}$$

where F_π is the pion decay constant ($F_\pi = 93MeV$).

The eight pseudoscalar mesons are regarded as Nambu- Goldstone bosons. Their masses are given by well known relations [2,3]

$$F_\pi m_\pi^2 = 2\bar{m}v \tag{7a}$$

$$F_K^2 m_{K^+}^2 = (m_s + m_s)v \tag{7b}$$

$$F_K^2 m_{K^o}^2 = (m_d + m_s)v \tag{7c}$$

$$F_8^2 m_{\eta_8}^2 = \frac{2}{3}(\bar{m} + 2m_s)v \tag{7d}$$

where $-v =< \bar{u}u >=< \bar{d}d >=< \bar{s}s >$. Using $F_K = F_8 = F_\pi$, we get the relations $m_{\eta_8^2} = (4m_K^2 - m_\pi^2)/3$(the Gell-Mann- Okubo mass rlation) and

$$\frac{(m_K^2 - m_\pi^2)}{(m_s - \bar{m})} = \frac{m_\pi^2}{2\bar{m}} \tag{8}$$

where $m_K^2 = \frac{1}{2}(m_{K^+}^2 + m_{K^o}^2)$.

Using Eq. (8), we can express Eqs. (6)as

$$T(\pi\eta_8 \rightarrow \pi\eta_8) = \frac{m_\pi^2}{3F_\pi^2} \tag{9a}$$

$$T(\pi\eta_1 \rightarrow \pi\eta_1) = \frac{2m_\pi^2}{3F_\pi^2} \tag{9b}$$

Eq (9a) was first obtained in reference [4].

In QCD, the mass Hamiltonian H_M can be written

$$\begin{aligned} H_M &= m_u \bar{u}u + m_d \bar{d}d + m_s s \\ &= \sqrt{2/3}(2\bar{m} + m_s)S_o + \frac{2}{\sqrt{3}}(\bar{m} - m_s)S_8 + (m_u - m_d)S_3 \end{aligned} \tag{10}$$

We note that, using eqs. (2),(3) and (4)

$$< \eta_8 | H_M | \eta_8 >= \frac{(\bar{m} + 2m_s)}{\bar{m}} \frac{m_\pi^2}{3} \tag{11}$$

$$< \eta_1 | H_M | \eta_1 >= \frac{2\bar{m} + m_s}{\bar{m}} (\frac{m_\pi^2}{3}) \tag{12}$$

Comparing Eq. (11) with Eqs. (7), we see that Eq. (11) is exactly the same as Eq.(7d). This justifies the use of nonet symmetry. Moreover, we note that entire mass of η_8 meson arises from its quark content.

On the other hand using the values [3,5]

$$\frac{m_s}{\bar{m}} = 25.8, m_d/m_u = 1.80 \tag{13}$$

and $m_\pi^2 = 0.018 GeV^2$, we get from Eq. (12)

$$< \eta_1 | H_M | \eta_1 > \approx 0.167 GeV^2 \tag{14}$$

Let us define the physical mesons η and η'

$$\eta = \cos\theta \eta_8 - \sin\theta \eta_1$$

$$\eta' = \sin\theta \eta_8 + \cos\theta \eta_1 \tag{16}$$

Since $m_{\eta_1}^2 = m_{\eta'}^2 + m_\eta^2 - m_{\eta_8}^2 = 0.894 GeV^2$, it is clear that about 17 % of $M_{\eta_1}^2$ arises from the quark content of the η_1 meson; the rest of the contribution is from its gluon content.

This is expected since $SU(3)$singlet axial vector current $A_{o\mu}$ is not conserved in the chiral limit. In fact

$$\partial_\mu A_{o\mu} = \frac{2}{3}(2\bar{m} + m_s)P_o + \frac{\sqrt{2}}{3}(2\bar{m} - 2m_s)P_8 + \sqrt{2/3}(m_u - m_d)P_3 + \sqrt{6}\frac{\alpha}{4\pi}G\bar{G} \quad ((17))$$

where $P_i = i\bar{q}\frac{\lambda_i}{2}\gamma_5 q$ and $G\bar{G} \equiv Tr(G_{\nu\lambda}\bar{G}_{\nu\lambda})$. From Eq. (17), we get [5] (neglecting the term with $(m_u - m_d)$):

$$F_o m_{\eta_1}^2 = \frac{2}{3}(\frac{2\bar{m} + m_s}{2\bar{m}})\frac{F_\pi^2 m_\pi^2}{F_o} + \sqrt{6} < o|\frac{\alpha_s}{4\pi}G\bar{G}|\eta_1 > \quad (18)$$

$$0 = \frac{2\sqrt{2}}{3}(\frac{\bar{m} - m_s}{2\bar{m}})\frac{F_\pi^2 m_\pi^2}{F_8} + \sqrt{6} < 0|\frac{\alpha}{4\pi}G\bar{G}|\eta_8 > \quad (19)$$

We first note that if we use nonet symmetry vis $F_0 = F_8 = F_\pi$, the first term on the right hand side of Eq. (18) is just the same as given in Eq. (12)

We write $|\eta_1 >= |\eta_1 >_G + |\eta_1 >_q$, where G and q signify the gluon and quark contents of η_1. We note that $|\eta_8 >= |u\bar{u} + d\bar{d} - 2s\bar{s} >$ and $|\eta_1 >_q = |u\bar{u} + d\bar{d} + s\bar{s} >$. Then comparing Eqs. (19) and (18), we get

$$\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi}G\bar{G}|\eta_1 >_q = -\frac{2}{3}(\frac{2\bar{m} + m_s}{2\bar{m}})m_\pi^2 = -(0.167Gev^2) \quad (20)$$

$$\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi}G\bar{G}|\eta_1 >_G = m_{\eta_1^2} \quad (21)$$

$$\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi}G\bar{G}|\eta_8 >= \frac{2\sqrt{2}}{3}(\frac{m_s - \bar{m}}{2\bar{m}})m_\pi^2 = 0.210GeV^2 \quad (22)$$

We now discuss the decays $\eta \to 3\pi^0$ and $\eta' \to 3\pi^0$. The decay amplitude T_η can be written [1]

$$T_\eta = -\frac{i}{\sqrt{6}}(m_d - m_u)\frac{1}{\bar{m}}[4T(\eta\eta \to \pi\pi)\frac{1}{m_{\eta^2}}\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi}G\bar{G}|\eta >] \quad (23)$$

From Eqs. (1), (2),(3),(4),(8) and (16), we get

$$T(\eta\eta \to \pi\pi) = \frac{1}{F_{\pi^2}}\sum_{\pi\eta}(0) = \frac{m_\pi^2}{3F_\pi^2}(\cos\theta - \sqrt{2}\sin\theta)^2 \quad (24)$$

$$T(\eta'\eta' \to \pi\pi) = \frac{1}{F_{\pi^2}} \sum_{\pi\eta'}(0) = \frac{m_\pi^2}{3F_\pi^2}(\sin\theta + \sqrt{2}\cos\theta)^2 \tag{25}$$

Hence we get

$$\begin{aligned} T_\eta &= -\frac{i}{\sqrt{6}}\frac{m_d - m_u}{2\bar{m}}\frac{8m_\pi^2}{3F_\pi^2}(\cos - \sqrt{2}\sin\theta)^2\frac{1}{m_\eta^2}\times \\ &\qquad [\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta >] \end{aligned} \tag{26}$$

$$\begin{aligned} T_{\eta'} &= -\frac{i}{\sqrt{6}}\frac{m_d - m_u}{2\bar{m}}\frac{8m_\pi^2}{3F_\pi^2}(\sin\theta + \sqrt{2}\cos\theta)^2\frac{1}{m_{\eta'}^2}\times \\ &\qquad [\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta' >_q] \end{aligned} \tag{27}$$

It may be noted that Eq.(26) gives T_η as in reference [1] for $\theta = 0$ and $F_8 = F_\pi$). Also note that due to OZI rule; in Eq.(27) only the quark part of the anomaly contributes to $\eta' \to 3\pi^0$ decay. Using Eq.(16), and $\theta = -18^0$(see below), we get

$$\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta >= 0.148GeV^2 \tag{28}$$

$$\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta' >_q= -0.224GeV^2 \tag{29}$$

Hence we get

$$|T_\eta| = 0.61 \tag{30}$$

to be compared with the experimental value (0.69 ± 0.04) [6]and

$$|T'_\eta| = 0.17 \tag{31}$$

to be compared with the experimental value 0.20 ± 0.02[6].

We conclude that our calculation gives rather a good agreement with the experimental results both for η and η' decays.

In order to discuss $\eta_8 - \eta_1$ mixing, we make the assumption that mixing between η_8 and η_1 arises from quark content of these mesons viz mixing term arises from the commutator of type $[F_0^5, \partial_\mu A_{8\mu}]$ and $[F_8^5, \partial_\mu \tilde{A}_{0\mu}]$, where

$$\partial_\mu \tilde{A}_{0\mu} = \partial_\mu A_{0\mu} - \sqrt{6}\frac{\alpha_s}{4\pi} G\bar{G} \tag{32}$$

In this case, the mixing term is given by

$$F_0 F_8 m^2_{\eta_1 - \eta_8} = \frac{2\sqrt{2}}{3}(\bar{m} - m_s)\frac{F_\pi^2 m_\pi^2}{2\bar{m}} \tag{33}$$

Hence from Eqs. (7), (18) and (33), we get [5,7]

$$\tan 2\theta = -\frac{2\sqrt{2}(m_s - \bar{m})/F_0 F_8}{[\frac{2\bar{m}+m_s}{F_0^2} + \sqrt{6} < 0|\frac{3\alpha_s}{4\pi} G\bar{G}|\eta_1 > \frac{\bar{m}}{F_\pi^2 m_\pi^2}\frac{1}{F_0} - \frac{(\bar{m}+2m_s)}{F_8^2}]} \tag{34}$$

We note that with $F_0 = F_8 = F_\pi$ and with no anomaly, we get $\tan 2\theta = 2\sqrt{2}$ i.e. we get nonet mixing angle as is the case for $w - \phi$ mixing. We can put Eq.(28) in the form

$$\tan 2\theta = \frac{2\sqrt{2}(\frac{m_s}{\bar{m}} - 1)}{[(1 + 2\frac{m_s}{\bar{m}}) - 3\frac{m_{\eta_1}^2}{m_\pi^2}]} \tag{35}$$

Putting the numerical values for $m_s/\bar{m}$ and $m^2_{\eta_1}/m^2_\pi$, we get $\theta \approx -18^o$. This is the value of θ we have used in our calculation.

The mixing angle $\theta \approx -18^o$ is consistent with the value of θ obtaind from the decays $\eta, \eta' \to 2\gamma$. We conclude that taking into account $U_A(1)$ anomaly in QCD and η, η' mixing, reasonably good values for the decay widths $\eta \to 3\pi^0$ and $\eta' \to 3\pi^0$ can be obtained. I would like to thank Professor Riazuddin for useful discussions.

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